First Passage Time of Filtered Poisson Process with Exponential Shape Function[1]

A.Novikov[2], R.E.Melchers[3], E.Shinjikashvili[2], N.Kordzakhia[4]

Abstract

Solving some integro-differential equation we find the Laplace transformation of the first passage time for Filtered Poisson Process generated by pulses with uniform or exponential distributions. Also, the martingale technique is applied for approximations of expectations and distributions for the first passage times. The approximations accuracy is verifying with the help of Monte-Carlo simulations.

Keywords: first passage times, Laplace transformation, martingales, integro-differential equations, Filtered Poisson process, Ornstein-Uhlenbeck process.

1. Introduction

We study the distribution of first passage time over a given level by a process $\{X_t, t \geq 0\}$ which solves the linear stochastic equation:

$$X_t = x - \beta \int_0^t X_s ds + Y_t, \quad t \geq 0, \quad \beta > 0, \tag{1}$$

governed by compound Poisson's process

$$Y_t = \sum_{k=1}^{N_t(\lambda)} \xi_k + mt. \tag{2}$$

Here $\{\xi_k, k \geq 1\}$ is i.i.d. sequence of random variables (pulses) appearing at arrival times $\{T_k, k \geq 1\}$ of the Poisson process $\{N_t(\lambda), t \geq 0\}$ with the intensity parameter $\lambda > 0$.

Due to (1) and (2) (here $I(\cdot)$ is the indicator function),

$$X_t = \frac{m}{\beta} + (x - \frac{m}{\beta})e^{-\beta t} + \sum_{k=1}^{N_t(\lambda)} \xi_k e^{-\beta(t-T_k)} I(T_k \leq t). \tag{3}$$

---

[1]Supported by ARC Large Grant A0010474

[2]University of Technology, Sydney, Department of Mathematical Sciences, University of Technology, PO Box 123, Broadway, Sydney, NSW 2007, Australia; e-mail:Alex.Novikov@uts.edu.au

[3]University of Newcastle, Department of Civil Engineering, Australia; e-mail: Rob.Melchers@newcastle.edu.au

[4]Commonwealth Bank of Australia

In the literature related to engineering applications, X_t is called Filtered Poisson Process (FPP) with the exponential shape function ([1], [2]). In finance and physics, X_t is called Generalized Ornstein-Uhlenbeck process ([3],[4]) or the shot noise process ([5]).
The first passage time of X_t over b is defined as follows:

$$\tau_b = \inf\{t \geq 0 : X_t \geq b\}, \quad x < b.$$

Note that

$$P(\tau_b \leq T) = P(\sup_{t\leq T} X_t \geq b).$$

The problem of finding the distribution of τ_b or, equivalently, the distribution of $\sup_{t\leq T} X_t$ is of great importance in engineering applications (e.g. reliability analysis [1]), finance applications (e.g. for pricing of exotic options [6]), dam theory ([7]) etc.
This paper is concerned with the derivation of exact formulas for the distribution of τ_b and also for its approximation. As a tool for this study we use integro-differential equations for the Laplace transform of τ_b, the martingale technique and Monte-Carlo simulation.

Denote the Laplace transformation of τ_b by

$$q_\alpha(x) = E_x(I\{\tau_b < \infty\}e^{-\alpha\tau_b}),\ \alpha > 0$$

where $E_x(\cdot) = E(\cdot|X_0 = x)$.
It was shown in [8] (and in a more general form in [9]) that under the assumptions

$$P\{\xi_1 > 0\} > 0, \quad E|\xi_1| < \infty \tag{4}$$

the first passage time τ_b possesses a finite exponential moment and, therefore, $q_\alpha(x)$ is the analytical function in the region $\{\alpha : Re(\alpha) > -c\}$ with some $c > 0$. Throughout the paper, condition (4) is assumed to be valid.
The explicit formula of $q_\alpha(x)$ is known for negatively distributed pulses ξ_k. This case was studied in [11] and [9] with the help of the martingale technique. An analysis of nonnegative pulses is more difficult as it involves the "overshoot" problem. In Section 2 we show that a solution of the integro-differential equation for $q_\alpha(x)$ (known as Dynkin's formula) can be found in an explicit form for the following two special cases of ξ_1-distributions: 1) exponential and 2) uniform. The exponential distribution in a different setting was studied in [7] and [10].
In this paper, we use the martingale approach to find bounds for $E_x(\tau_b)$ and for $q_\alpha(x)$. These bounds might be useful for estimating of $P(\tau_b < T)$, e.g., via the Chebyshev inequality. The same approach is used in [21] for computing the low bounds for $E_x(\tau_b)$ (see Section 3.3). In Section 3.4 we propose

asymptotic approximations for $E_x(\tau_b)$ and $P(\tau_b < T)$ for large values of b. In Section 4 we compare simulation results with the derived approximations for Gaussian, exponential and uniformly distributed ξ_1.

2. Integro-differential equation

Notice that the infinitesimal generator for the FPP process X_t is defined as follows: for any continuously differentiable and bounded functions $f(z)$

$$L[f(z)] = -(\beta z - m)f'(z) + \lambda \int_{-\infty}^{\infty} [f(z+u) - f(z)]dP(\xi_1 < u). \tag{5}$$

By Dynkin's formula, [12] (see also the martingale method in Section 3), the Laplace transform $q_\alpha(x)$ is defined by the integro-differential equation subject to the boundary condition:

$$\begin{aligned} L[q_\alpha(x)] - \alpha q_\alpha(x) &= 0 \quad \text{for} \quad x < b; \\ q_\alpha(x) &= 1 \quad \text{for} \quad x \geq b \end{aligned} \tag{6}$$

Similarly, $Q(x) = E_x(\tau_b)$ solves

$$\begin{aligned} L[Q(x)] &= -1 \quad \text{for} \quad x < b; \\ Q(x) &= 0 \quad \text{for} \quad x \geq b. \end{aligned} \tag{7}$$

We show that (6) can be transformed to a second order differential equation (see (8) and (13) below) provided that ξ_1 is exponentially or uniformly (over interval $(0, c), c > b$) distributed random variable. This differential equation is valid for $x < b$; for $x \geq b$, $q_\alpha(x) = 1$.

2.1. Exponentially distributed pulses

Assume ξ_1 is exponentially distributed random variable with a positive parameter ν and the parameter $m = 0$. With a natural hypothesis that $q_\alpha(x)$ is twice differentiable for $x < b$, (6) is equivalent to

$$-\beta x q_\alpha''(x) + (\nu\beta x - (\lambda + \alpha + \beta))q_\alpha'(x) + \nu\alpha q_\alpha(x) = 0 \tag{8}$$

(see details in Appendix 1). It is well known that any solution of (8) is expressed in terms of the Kummer series:

$$\Phi(a, d; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(d)_k} \frac{x^k}{k!}, \quad (a)_k = a(a+1)\cdots(a+k-1), \ (a)_0 = 1,$$

which is the particular solution of the degenerate hypergeometric equation ([13]).

Moreover, the general solution of (8) is a linear combination

$$q_\alpha(x) = C_1\Phi_1(x) + C_2\Phi_2(x),$$

where

$$\Phi_1(x) = \Phi\left(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu x\right), \ \Phi_2(x) = x^{-\frac{\lambda+\alpha}{\beta}} \Phi\left(-\frac{\lambda}{\beta}, 1-\frac{\lambda+\alpha}{\beta}; \nu x\right)$$

are independent solutions of (8) and constants C_1 and C_2 are defined from some additional conditions. The first one is concerned to the boundedness of $q_\alpha(x)$. The function $\Phi_1(x)$ increases from $\Phi_1(-\infty) = 0$ and so it is bounded for $x \in (-\infty, b)$. Since $\Phi_2(x)$ is unbounded at $x = 0$, we chose $C_2 = 0$.
Thus, with an arbitrary parameter C_1, we fix

$$\begin{aligned} q_\alpha(x) &= C_1 \Phi\left(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu x\right) \quad \text{for} \quad x < b; \\ q_\alpha(x) &= 1 \quad \text{for} \quad x \geq b. \end{aligned}$$

The use of

$$\lim_{x \uparrow b} E_x(q_\alpha(x + \xi_k)) = 1,$$

provided by $P(\xi_1 > 0) = 1$, and the analysis of (6) under $x \uparrow b$, gives the following equation for C_1 determination:

$$-\beta b C_1 \Phi'\left(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu b\right) - $$
$$(\lambda+\alpha) C_1 \Phi\left(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu b\right) + \lambda = 0.$$

Moreover, the basic properties of hypergeometric functions allows to get

$$q_\alpha(x) = \frac{\lambda}{\lambda+\alpha} \frac{\Phi(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu x)}{\Phi(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}; \nu b)}, \quad x < b \tag{9}$$

(for more details see Appendix 2).
To derive the formula for $E_x(\tau_b)$ with the help of (9), we apply well known properties of Kummer's series as $\alpha \to 0$:

$$\frac{1 - q_\alpha(x)}{\alpha} \sim \frac{1}{\lambda} + \frac{\Phi(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}; \nu b) - 1}{\alpha} - \frac{\Phi(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu x) - 1}{\alpha},$$

$$\frac{\Phi(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}; \nu b) - 1}{\alpha} \sim \frac{\nu b}{\lambda} \sum_{k=0}^{\infty} \frac{(\nu b)^k}{(k+1)(\frac{\lambda}{\beta}+1)_k},$$

and

$$\frac{\Phi(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu x) - 1}{\alpha} \sim \frac{\nu x}{\lambda+\beta} \sum_{k=0}^{\infty} \frac{(\nu x)^k}{(k+1)(\frac{\lambda}{\beta}+2)_k}.$$

Hence, we get

$$E_x(\tau_b) = \lim_{\alpha\to 0} \frac{1-q_\alpha(x)}{\alpha} = \frac{1}{\lambda} + \frac{\nu b}{\lambda}\ \widetilde{\Phi}(\lambda/\beta+1;\nu b) -$$

$$-\frac{\nu x}{\beta+\lambda}\ \widetilde{\Phi}(\lambda/\beta+2;\nu x), \quad x<b, \tag{10}$$

where

$$\widetilde{\Phi}(b;x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)(b)_k}$$

Notice that (10) can be derived from (7) which is reduced to a second order differential equation too.

Equivalent forms for (9) and (10) are derived in Section 3 with the help of martingale technique. For adaptation of these results to each other, notice that $q_\alpha(x)$ and $E_x(\tau_b)$ can be rewritten in the integral forms provided by the integral representation of the Kummer series:

$$\Phi(a,d;x) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)} \int_0^1 e^{xu} u^{a-1}(1-u)^{d-a-1} du, \quad d>a>0,$$

where $\Gamma(\cdot)$ is the gamma function (see, e.g., [13]). It follows that

$$q_\alpha(x) = \frac{\int_0^1 e^{\nu x u} u^{\frac{\alpha}{\beta}-1}(1-u)^{\frac{\lambda}{\beta}} du}{\int_0^1 e^{\nu b u} u^{\frac{\alpha}{\beta}-1}(1-u)^{\frac{\lambda}{\beta}-1} du}\ , \tag{11}$$

and

$$E_x(\tau_b) = \lim_{\alpha\to 0} \frac{1-q_\alpha(x)}{\alpha} = \frac{1}{\beta} \int_0^1 \frac{e^{\nu b u} - e^{\nu x u}(1-u)}{u}\ (1-u)^{\frac{\lambda}{\beta}-1} du. \tag{12}$$

Remark 1. Tsurui and Osaki, [10], derived an integral equation for $q_\alpha(x)$ and found its explicit solution for $\beta = 1/n$, $n = 1, 2, ...$

Kella and Stadje, [7], found the Laplace transformation and expectation of the first *hitting* time $T_b = \min\{t \geq 0 : X_t = b\}$ by solving an integro-differential equation similar to (8). Their results for T_b follow from ours by applying the memoryless property of the exponentially distributed pulses and the independence of random variables τ_b and $T_b - \tau_b$.

Remark 2. Under $\beta \to \infty$ or $\beta \to 0$ the limiting distributions of τ_b are derived form (9). For $\beta \to \infty$, the limiting distribution of τ_b is exponential with parameter $\lambda e^{-\nu b}$ while for $\beta \to 0$ the limiting distribution has the Laplace transformation $\frac{\lambda}{\lambda+\alpha} e^{-\nu(b-x)\frac{\alpha}{\lambda+\alpha}}$ for $x < b$. These results are completely compatible with Tsurui and Osaki, [10] (see also [14]). Notice that the above asymptotic results can be derived directly from (9).

2.2. Uniformly distributed pulses

Let ξ_1 be uniformly distributed on $(0, c)$ and the parameter m in (5) is zero. Assuming the function $q_\alpha(x)$ is twice differentiable for $x < b$ we transform (6) to

$$-\beta x q_\alpha''(x) - (\lambda + \alpha + \beta) q_\alpha'(x) + \frac{\lambda}{c}\,(q_\alpha(x+c) - q_\alpha(x)) = 0, \quad 0 < x < b; \quad (13)$$

$$q_\alpha(x) = 1\ ,\ x \geq b.$$

Notice that $q_\alpha(x + c) = 1$, $c \geq b$. Then the solution of (13) can be expressed in terms of the Bessel functions:

$$q_\alpha(x) = 1 + x^{-\frac{\lambda+\alpha}{2\beta}} \left(C_1 J_{\frac{\lambda+\alpha}{\beta}} \left(2\sqrt{\frac{\lambda x}{c\beta}} \right) + C_2 Y_{\frac{\lambda+\alpha}{\beta}} \left(2\sqrt{\frac{\lambda x}{c\beta}} \right) \right).$$

where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second types respectively, [13]:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu + k + 1)}, \quad Y_\nu(x) = \frac{J_\nu(x)\cos \pi\nu - J_{-\nu}(x)}{\sin \pi\nu}.$$

The boundedness of $q_\alpha(x)$ provides the use of the first type Bessel function only. Therefore,

$$q_\alpha(x) = 1 + C_1\, x^{-\frac{\lambda+\alpha}{2\beta}}\ J_{\frac{\lambda+\alpha}{\beta}} \left(2\sqrt{\frac{\lambda x}{c\beta}} \right).$$

Now, C_1 is determined from (6) by substituting in it the above expression for $q_\alpha(x)$ and computing the limit in $x \uparrow b$. As previously (Section 2.1) $P(\xi_1 > 0) = 1$ and so

$$\lim_{x \uparrow b} E_x(q_\alpha(x + \xi_k)) = 1.$$

This and some properties of the Bessel function of the first type provide

$$C_1 = -\frac{\alpha\, b^{\frac{\lambda+\alpha}{2\beta}}}{\beta\sqrt{\frac{\lambda b}{c\beta}}\ J_{\frac{\lambda+\alpha}{\beta}-1}\left(2\sqrt{\frac{\lambda b}{c\beta}}\right)}.$$

Finally, for $c \geq b$ (13) possesses the explicit solution:

$$q_\alpha(x) = 1 - \frac{\alpha\ \left(\frac{x}{b}\right)^{-\frac{\lambda+\alpha}{2\beta}}\ J_{\frac{\lambda+\alpha}{\beta}}(2\sqrt{\frac{\lambda x}{c\beta}})}{\beta\ \sqrt{\frac{\lambda b}{c\beta}}\ J_{\frac{\lambda+\alpha}{\beta}-1}(2\sqrt{\frac{\lambda b}{c\beta}})}, \quad 0 < x < b. \quad (14)$$

We emphasize that the right-hand side of (14) is not defined for $x = 0$. For the definition of $q_\alpha(0)$ the fact is used that $(x/2)^{-\nu} J_\nu(x) \to \frac{1}{\Gamma(\nu+1)}$ as $x \downarrow 0$. It implies

$$q_\alpha(0) = 1 - \frac{\alpha}{\beta\,\Gamma(\frac{\lambda+\alpha}{\beta}+1)\; J_{\frac{\lambda+\alpha}{\beta}-1}\left(2\sqrt{\frac{\lambda b}{c\beta}}\right)} \left(\frac{\lambda b}{c\beta}\right)^{\frac{\lambda+\alpha}{2\beta}-\frac{1}{2}}.$$

The moments of τ_b can be obtained from (14) in the usual manner. For example, with the help of Bessel's functions we get

$$E_x(\tau_b) = \lim_{\alpha\to 0} \frac{1-q_\alpha(x)}{\alpha} = \frac{(\frac{x}{b})^{-\frac{\lambda}{2\beta}}\; J_{\frac{\lambda}{\beta}}(2\sqrt{\frac{\lambda x}{c\beta}})}{\beta\,\sqrt{\frac{\lambda b}{c\beta}}\; J_{\frac{\lambda}{\beta}-1}(2\sqrt{\frac{\lambda b}{c\beta}})}, \quad 0 < x < b,$$

$$E_0(\tau_b) = \frac{(\lambda b/c\beta)^{\frac{\lambda}{2\beta}-\frac{1}{2}}}{\beta\,\Gamma(\frac{\lambda}{\beta}+1)\; J_{\frac{\lambda}{\beta}-1}\left(2\sqrt{\frac{\lambda b}{c\beta}}\right)}.$$

If $c < b$, then $q_\alpha(x)$ is found recursively over the intervals $x \in (b-c, b),\ x \in (b-2c, b-c), ..., x \in (0, b-kc)$, with $k = [b/c]$ with the rule: for new interval the solution from the previous one is used. For the first interval when $x \in (b-c, b)$ we have $q_\alpha(x+c) = 1$ and so $q_\alpha(x)$ is given by (14).

3. Martingales and first passage times

By the definition, the process $\{M_t, t \geq 0\}$ with finite expectation is called the martingale if for any $t \geq s$

$$E(M_t|F_s) = M_s,$$

where the symbol $E(.|\mathrm{F}_s)$ means conditioning with respect to an information flow F_s generated (usually) by the observed process X_t (see details e.g. in [15]). An usefulness of martingales for analyzing the first passage time distributions is due to the fact that for any bounded stopping time τ (with respect to given F_s) the Wald identity holds:

$$E(M_\tau) = E(M_0). \tag{15}$$

With a properly chosen martingale M_t this identity is useful, as we will see below, for getting some properties of τ.
From the point of view of modern theory of random processes (see, e.g. [15]) equations (6) and (7) are simple results of application the Itô formula and the identity (15). Indeed, by Itô's formula applied to $e^{-\alpha t} f(X_t)$ with a smooth function $f(x)$ for any $t \geq 0$ we find

$$e^{-\alpha t} f(X_t) = f(x) + \int_0^t e^{-\alpha t}(L[f(X_s)] - \alpha f(X_s))ds + M_t \tag{16}$$

where M_t is the martingale. Now, (15) and (16) provide

$$E_x e^{-\alpha \min(t,\tau_b)} f(X_{\min(t,\tau_b)}) =$$

$$f(x) + E_x \int_0^{\min(t,\tau_b)} e^{-\alpha t}(L[f(X_s)] - \alpha f(X_s))ds. \qquad (17)$$

If f solves (6), then the integral in (17) vanishes. Passing to the limit as $t \to \infty$ we obtain

$$E_x e^{-\alpha \tau_b} f(X_{\tau_b}) = f(x). \qquad (18)$$

Since the boundary condition in (6) implies $f(X_{\tau_b}) = 1$, we finally get $E_x e^{-\alpha\tau_b} = f(x)$.
Similarly, if (7) holds for a smooth function Q(x) then by (15) we have

$$E_x Q(X_{\min(t,\tau_b)}) = Q(x) - E_x \min(t, \tau_b). \qquad (19)$$

Recall that $Q(x) = 0$, $x \geq b$. Now, assuming that

$$\lim_{t\to\infty} E_x(\cdot) = E_x \lim_{t\to\infty}(\cdot)$$

in the above equation, we get $Q(x) = E_x(\tau_b)$.
On the other hand, if the Laplace transformation $q_\alpha(x)$ is known, its derivative at the point $\alpha = 0$ provides $Q(x)$.

3.1. Martingale families for filtered Poisson processes. The Wald identity

Set $K = \sup\{u \geq 0 : Ee^{uY_1} < \infty\}$ and, for $u < K$, $\psi(u) = \log\left(Ee^{uY_1}\right)$ and define

$$\varphi(u) = \frac{1}{\beta}\int_0^u v^{-1}\psi(v)dv,\ u < K.$$

For $\alpha > 0$, set

$$H_\alpha(z) = \int_0^K e^{uz-\varphi(u)}u^{\alpha/\beta-1}du,$$

and

$$G(z) = \frac{1}{\beta}\int_0^K (e^{uz} - e^{ux})u^{-1}e^{-\varphi(u)}du.$$

Assume the distribution function of ξ_1 has all exponential moments, that is:

$$K = \infty. \qquad (20)$$

It is shown in [16] (see, [9]) that, under assumptions (20) and (4), both processes

$$e^{\alpha t}H_\alpha(X_t) \quad \text{and} \quad G(X_t) - t$$

are martingales. Notice that in [16] the mentioned-above martingale property is provided by a corresponding discrete time approximation for X_t while in [9] a completely different technique (stochastic calculus) is exploited. In principle, it is readily to check that $H_\alpha(z)$ and $G(z)$ solve

$$L[H_\alpha(z)] - \alpha H_\alpha(z) = 0$$

and

$$L[G(z)] = -1$$

respectively (comp. (6) and (7)). Taking into the consideration the above equations with the help of the Itô formula we may claim that $e^{\alpha t} H_\alpha(X_t)$ and $G(X_t) - t$ are martingales.
This fact and (15) implies that (see details in [9])

$$E_x G(X_{\tau_b}) = E_x(\tau_b) < \infty \tag{21}$$

and, similarly,

$$E_x(e^{-\alpha\tau_b} H_\alpha(X_{\tau_b})) = H_\alpha(x), \quad \alpha > 0. \tag{22}$$

Assumption (20) fails for the exponentially distributed pulses and some other distributions with $0 \leq K < \infty$. However, for $0 < K < \infty$, the truncation technique (for large positive jumps) allows to extend the Wald identity for (21) and (22) (see, [20]).
Denote by

$$\Delta_b(\beta) = X_{\tau_b} - b$$

the overshoot of X_t over the level b. Then, it makes sense to rewrite (21) and (22) to the form

$$E_x(\tau_b) = \frac{1}{\beta} \int_0^K (E_x(e^{u\Delta_b(\beta)}) - e^{u(x-b)}) u^{-1} e^{ub-\varphi(u)} du, \tag{23}$$

$$E_x(e^{-\alpha\tau_b} \int_0^K e^{u\Delta_b(\beta)} u^{\alpha/\beta-1} e^{ub-\varphi(u)} du) = \int_0^K u^{\alpha/\beta-1} e^{ux-\varphi(u)} du, \quad \alpha > 0 \tag{24}$$

which are useful for the corresponding bounds (see Section 3.2).

Remark 3. Notice that (11) and (12) are provided by (23) and (24). Indeed, if ξ_1 is exponentially distributed with the parameter ν, then $K = \nu$ and

$$\varphi(u) = -\frac{\lambda}{\beta} \log(1 - u/\nu), \quad u < \nu.$$

Moreover, the memoryless property of exponential distribution allows readily to check that $\Delta_b(\beta)$ and τ_b are independent random variables and $\Delta_b(\beta)$ shares the distribution with ξ_1. The latter provides

$$E_x(e^{u\Delta_b(\beta)}) = \frac{1}{1-u/\nu}, \quad u < \nu.$$

Both $\varphi(u)$ and $E_x(e^{u\Delta_b(\beta)})$, being substituted in (23) and (24), give (11) and (12) under obvious change of variables. For $m \neq 0$, similar to (11) formula can be found too.

3.2. Bounds for $E_x(\tau_b)$ and $P\{\tau_b < T\}$

In general, the distribution function of $\Delta_b(\beta)$ is unknown. Owing to $\Delta_b(\beta) \geq 0$, from identities (22) and (21) ((23) and (24)) it follows

$$q_\alpha(x) \leq \frac{H_\alpha(x)}{H_\alpha(b)}, \quad \alpha > 0 \tag{25}$$

and

$$E_x(\tau_b) \geq G(b). \tag{26}$$

With the help of Chebyshev's inequality we derive from (25) the following upper bound:

$$P\{\tau_b < T\} \leq \inf_{\alpha>0} \left\{ e^{\alpha T} \frac{H_\alpha(x)}{H_\alpha(b)} \right\} \tag{27}$$

and, in turn, we get

$$P\{\tau_b < T\} \geq 1 - \frac{E_x(\tau_b)}{T}. \tag{28}$$

Typically, the bounds in (27) and (28) are not effective because of essentially over-under-estimating the probability $P\{\tau_b < T\}$.

3.3 Approximations for $E_x(\tau_b)$ and $P\{\tau_b < T\}$

Notice that

$$\Delta_b(\beta) \xrightarrow{d} \Delta_b(0) \quad \text{as } \beta \to 0 \tag{29}$$

and

$$\Delta_b(0) \xrightarrow{d} R_\infty \quad \text{as } b \to \infty, \tag{30}$$

where symbol $\xrightarrow{d}$ denotes convergence in distribution; (29) is obvious and (30) is well-known from [17].

Under the exponentially boundedness of ξ_1, we have the fast convergence in (30). Moreover, under $P\{\xi_k \geq 0\} = 1$ and $0 < E(\xi_k) < \infty$, we have

$$E(R_\infty) = \frac{E(\xi_k^2)}{2E(\xi_k)} \quad \text{and} \quad E(R_\infty^2) = \frac{E(\xi_k^3)}{3E(\xi_k)}. \tag{31}$$

For $(0,1)$-Gaussian distribution of ξ_1, the first and second moments of R_∞ are expressed in terms of Riemann's zeta-function $\zeta(x)$. In particular, (see [18] or [19])

$$E(R_\infty) = -\frac{\zeta(1/2)}{\sqrt{2\pi}} = 0.5826 \quad \text{and} \quad E(R_\infty^2) = 3.5366. \tag{32}$$

For $\beta \to 0$ and $b \to \infty$, (23), (29) and (30) imply

$$E_x(\tau_b) = \frac{1}{\beta} \int_0^K (E(e^{uR_\infty}) - e^{u(x-b)})u^{-1}e^{ub-\varphi(u)}du \; (1 + o(1)). \tag{33}$$

Then, by applying the Taylor expansion we give the following approximation for $E_x(\tau_b)$:

$$E_x(\tau_b) \approx \frac{1}{\beta} \int_0^K (1 + uE(R_\infty) + u^2 E(R_\infty^2)/2 - e^{u(x-b)})u^{-1}e^{-\varphi(u)}du. \tag{34}$$

Exponential approximation. Let pulse ξ_1 be Gaussian or bounded, let

$$b \to \infty \quad \text{or} \quad \beta \to 0, \;\; b \to \infty, \;\; b > \frac{\lambda}{\beta}\, E(\xi_k),$$

then

$$P\left(\frac{\tau_b}{E_x(\tau_b)} < T\right) \to 1 - e^{-T} \;\; \text{for all} \;\; T > 0.$$

This fact is verified by the technique from [8] adapted to the continuous time case (see also [20]).

4. Numerical results

To speed up the Monte-Carlo simulations of τ_b and X_{τ_b} we have used the following approach. Observe from (3) that the paths of the process X_t are determined by the jump values located at $\{T_k, k \geq 1\}$. The jump values are defined by the recursion:

$$X_{T_0} = X_0 = x, \quad X_{T_k} = \frac{m}{\beta} + (X_{T_{k-1}} - \frac{m}{\beta})e^{-\beta(T_k - T_{k-1})} + \xi_k, \quad k = 1, 2, \ldots$$

Direct Monte-Carlo method works sufficiently fast for the small and moderate values of $E_x(\tau_b)$. If $E_x(\tau_b)$'s are large, we use the method of *control variates* for the variance reduction in our simulations. As the control variate we used relation (21).

Figures 1 and 2 illustrate the accuracy of the approximation (34). We considered the process X_t given by (1) and (2) with the initial value $x = 0$, $m = 0$ and the intensity of the corresponding Poisson process $\lambda = 10$. Figure

1 shows the case of $(0,1)$-Gaussian pulse, ξ_1, with two small values of β, $\beta = 0.1$ and 0.01, the level b ranges from $b = 1$ to 15. The approximation of $E_x(\tau_b)$ is computed using (32) along with (31).
These approximations are tabulated and compared with the results of the Monte-Carlo simulations (the number of realizations of the process was $n = 10^5$). The corresponding lower bounds from (26) are also provided for the comparison.
Figure 2 is similar to figure 1, only it shows the case of uniform over $(0,1)$ pulse ξ_1 . The first two moments of R_∞ are calculated as in (31).
Figures 3 and 4 illustrate that, in fact, the exponential distribution approximation for τ_b is valid with the high accuracy when $b > \frac{\lambda}{\beta}\ E(\xi_1)$. The plots represent the distribution function $P(\tau_b < T)$ obtained by the Monte-Carlo simulation (the number of realizations of the process X_t is $n = 10^5$, X_t is given by (1) and (2) with the initial value $x = 0$ and $m = 0$) and its approximation:

$$P(\tau_b < T) \approx 1 - e^{-\frac{T}{E_x(\tau_b)}}.$$

$E_x(\tau_b)$ is calculated by using the Monte-Carlo method. Figure 3 corresponds to the case of $(3,1)$-Gaussian pulses, level $b = 50$, $\beta = 0.1$ and $\lambda = 1$. Figure 4 corresponds to the case of $(0,1)$-uniform pulses, level $b = 16$, $\beta = 0.5$ and $\lambda = 10$.

References

[1] Grigoriu, M. (1995), Applied Non-Gaussian Processes. PTR Prentice Hall.

[2] Soong, T.T. and M. Grigoriu (1993), Random Vibration of Mechanical and Structural Systems, PTR. Prentice Hall.

[3] Barndorff-Nielsen, Ole E. and Shephard, N. (2001), Modelling by Lévy processes for financial econometrics. In : Lévy processes, 283–318, Birkhäuser Boston, Boston.

[4] Cáceres, M. O.; Budini, A. A. (1997), The generalized Ornstein-Uhlenbeck process. J. Phys. A 30 , no. 24, 8427–8444.

[5] Perry, D., Stadje, W. and Zacks, S. (2001), First-exit timed for Poisson shot noise. Stoch. Models, v. 17, no. 1, 25-37.

[6] Leblanc, B., Scaillet, O. (1998), Path dependent options on yields in the affine term structure model. Finance Stoch., v. 2, no. 4, 349–367.

[7] Kella, O. and Stadje, W. (2001), On hitting times for compound Poisson dams with exponential jumps and linear release rate. J. Appl. Probab., v. 38, no. 3, 781-786.

[8] Novikov, A. A.; Èrgashev, B. A. (1993), Limit theorems for the time of crossing a level by an autoregressive process. (Russian) Trudy Mat. Inst. Steklov. 202 (1993), Statist. i Upravlen. Sluchain. Protsessami, 209–233; translation in Proc. Steklov Inst. Math. 1994, no. 4 (202), 169–186.

[9] Novikov, A. (2003) Martingales and first passage times for Ornstein-Uhlenbeck processes with a jump component. Theory of Probability and its Applications, v. 49 , no. 3, in print.

[10] Tsurui, Akira. and Osaki, Shunji., (1976), On a first-passage problem for a cumulative process with exponential decay, Stochastic Processes Appl., v. 4, no. 1, 79-88.

[11] Hadjiev, Dimitar I. (1983), The first passage problem for generalized Ornstein-Uhlenbeck processes with nonpositive jumps. Séminaire de probabilités, XIX, 1983/84, 80–90, Lecture Notes in Math., 1123, Springer, Berlin, 1985.

[12] Dynkin, E.B. (1965), Markov Processes I, Springer, Berlin.

[13] Polyanin Andrei D. and Zaitsev, Valentin F. (1995), Handbook of exact solutions for ordinary differential equations, CRC Press, Boca Raton, FL.

[14] Cox, D. R. (1962), Renewal Theory. Methuen, London.

[15] Jacod, J., Shiryaev, A.N. (1987), Limit Theorem for Stochastic Processes. Springer, Berlin

[16] Novikov, A. A. (1990), On the First Passage Time of an Autoregressive Process over a Level and an Application to a Disorder Problem, Theory of Probability and its Applications, v. 35, no. 2, 269-279.

[17] Chang, Joseph T. (1994), Inequalities for the overshoot, in Ann. Appl. Probab., v. 4, no. 4, 1223-1233.

[18] Chang, Joseph T. and Peres, Yuval (1997), Ladder heights, Gaussian random walks and the Riemann zeta function, in Ann. Probab., v. 25, no. 2, 787-802.

[19] Lotov, V.I. (1996), On some boundary crossing problems for Gaussian random walks, in Ann. Probab., v. 24, no. 4, 2154-2171.

[20] Novikov, A., Borovkov, K. and Shinjikashvili E., (2003), Approximations for the maximum of the Ornstein-Uhlenbeck process with a jump component, preprint of the University of Technology, Sydney.

[21] Kordzakhia, N., Melchers, R.E. and Novikov, A. (2000), First passage of filtered Poisson processes with exponential peaks, 8th ASCE Speciality Conf. Prob. Mechanics and Structural Reliability, CD-ROM, paper 024.

Appendix 1. Differential equation for $q\ \ (x)$.

Let ξ_1 have the exponential distribution with parameter $\nu > 0$. Then

$$dP(\xi_1 < u) = \nu e^{-\nu u} du, \quad u > 0$$

and the equation (6) becomes

$$-(\beta x - m) q'_\alpha(x) + \lambda \left(\int_0^\infty q_\alpha(x+u) \nu e^{-\nu u} du - q_\alpha(x) \right) - \alpha q_\alpha(x) = 0. \quad (A1.1)$$

Observe that

$$\begin{aligned} \left(\int_0^\infty q_\alpha(x+u) \nu e^{-\nu u} du \right)' &= \int_0^\infty q'_\alpha(x+u) \nu e^{-\nu u} du \\ (Integrating\ by\ parts) \quad &= \nu \left(\int_0^\infty q_\alpha(x+u) \nu e^{-\nu u} du - q_\alpha(x) \right) \\ (from \quad (A1.1)) \quad &= \frac{\nu}{\lambda} ((\beta x - m) q'_\alpha(x) + \alpha q_\alpha(x)). \end{aligned}$$

Differentiate (A1.1) and substitute the above expression. We get

$$-(\beta x - m) q''_\alpha(x) + (\nu(\beta x - m) - (\lambda + \alpha + \beta)) q'_\alpha(x) + \nu \alpha q_\alpha(x) = 0, \quad (A1.2)$$

which becomes the equation (8) when $m = 0$.

Appendix 2. Finding C_1.

$$\begin{aligned} \frac{\lambda}{C_1} &= \beta b\ \Phi' \left(\frac{\alpha}{\beta}, \frac{\lambda + \alpha}{\beta} + 1; \nu b \right) + (\lambda + \alpha) \Phi \left(\frac{\alpha}{\beta}, \frac{\lambda + \alpha}{\beta} + 1; \nu b \right) \\ &= \nu b\ \frac{\alpha}{\frac{\lambda + \alpha}{\beta} + 1}\ \Phi \left(\frac{\alpha}{\beta} + 1, \frac{\lambda + \alpha}{\beta} + 2; \nu b \right) \\ &\quad + (\lambda + \alpha) \Phi \left(\frac{\alpha}{\beta}, \frac{\lambda + \alpha}{\beta} + 1; \nu b \right) \end{aligned}$$

(we have used differentiation formula $\Phi'(a,d;x) = \frac{a}{d}\Phi(a+1,d+1;x)$)

$$\begin{aligned}\frac{\lambda}{C_1(\lambda+\alpha)} \quad &= \quad \nu b \, \frac{\frac{\alpha}{\beta}}{\frac{\lambda+\alpha}{\beta}\ (\frac{\lambda+\alpha}{\beta}+1)} \ \Phi\left(\frac{\alpha}{\beta}+1, \frac{\lambda+\alpha}{\beta}+2; \nu b\right) \\ &+ \ \Phi\left(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}+1; \nu b\right) = \Phi\left(\frac{\alpha}{\beta}, \frac{\lambda+\alpha}{\beta}; \nu b\right).\end{aligned}$$

(Denote $x = \nu b$, $a = \frac{\alpha}{\beta}$, $d = \frac{\lambda+\alpha}{\beta} + 1$. Recall $\Phi(a,d;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(d)_k}\frac{x^k}{k!}$. Then the right hand side of the above equation becomes

$$\begin{aligned}&x\frac{a}{(d-1)d}\Phi(a+1,d+1;x) + \Phi(a,d;x) = \\ &x\frac{a}{(d-1)d}\sum_{k=0}^{\infty}\frac{(a+1)_k}{(d+1)_k}\frac{x^k}{k!} + \sum_{k=0}^{\infty}\frac{(a)_k}{(d)_k}\frac{x^k}{k!} = \\ &\sum_{k=1}^{\infty}\frac{(a)_k}{(d-1)_{k+1}}\frac{x^k}{(k-1)!} + 1 + \sum_{k=1}^{\infty}\frac{(a)_k}{(d)_k}\frac{x^k}{k!} = \\ &1 + \sum_{k=1}^{\infty}(a)_k x^k \frac{k+b-1}{(d-1)_{k+1}k!} = \\ &1 + \sum_{k=1}^{\infty}\frac{(a)_k}{(d-1)_k}\frac{x^k}{k!} = \\ &= \Phi(a, d-1; x).\end{aligned}$$

= 0.1, = 10					= 0.01, = 10				
b	**M-C**	**Approx**	**Low Bound**	**Error Appr%**	**b**	**M-C**	**Approx**	**Low Bound**	**Error Appr%**
1	3.10	3.10	1.87	0.21	1	9.30	9.15	5.70	1.60
2	5.38	5.37	3.98	0.21	2	15.30	15.20	11.62	0.68
3	7.99	7.97	6.38	0.23	3	21.37	21.48	17.76	-0.49
4	11.00	10.97	9.12	0.31	4	28.19	28.00	24.14	0.68
5	14.41	14.46	12.30	-0.30	5	34.89	34.79	30.77	0.29
6	18.62	18.56	16.02	0.31	6	42.12	41.85	37.66	0.65
7	23.37	23.45	20.41	-0.32	7	49.54	49.20	44.85	0.68
8	29.28	29.34	25.66	-0.19	8	57.06	56.88	52.33	0.32
9	36.57	36.53	32.02	0.13	9	65.16	64.88	60.13	0.43
10	45.54	45.42	39.82	0.26	10	73.71	73.24	68.28	0.63
11	56.46	56.58	49.52	-0.21	11	82.38	81.98	76.80	0.48
12	70.63	70.80	61.76	-0.24	12	91.94	91.13	85.70	0.88
13	89.33	89.19	77.45	0.15	13	100.52	100.72	95.03	-0.19
14	113.00	113.38	97.86	-0.34	14	111.24	110.77	104.80	0.43
15	145.59	145.71	124.87	-0.08	15	122.18	121.32	115.05	0.70

Figure 1: Approximation for $E_0\tau_b$,
$(0,1)$-Gaussian pulses.

= 0.1, = 10				
b	**M-C**	**Approx**	**Low Bound**	**Error Appr%**
30	9.13	9.16	9.00	-0.34
35	11.91	11.94	11.73	-0.29
40	15.67	15.72	15.42	-0.30
45	21.37	21.47	20.98	-0.43
50	31.91	32.22	31.18	-1.00
51	35.43	35.74	34.46	-0.88
52	39.72	40.10	38.51	-0.96
53	45.28	45.66	43.60	-0.83
54	52.29	52.93	50.21	-1.22
55	61.96	62.74	59.03	-1.27
56	75.13	76.45	71.19	-1.76
57	93.89	96.31	88.58	-2.58
58	122.59	126.22	114.40	-2.96
59	167.93	173.16	154.32	-3.11
60	239.71	249.97	218.65	-4.28
61	364.27	381.18	326.84	-4.64
62	586.49	615.17	516.77	-4.89
63	993.07	1050.80	864.83	-5.81
64	1804.20	1897.05	1530.43	-5.15
65	3380.69	3611.30	2857.94	-6.82

Figure 2: Approximation for $E_0\tau_b$ $(0,1)$-Uniform pulses.

$b = 50, \quad = 0.1, \quad = 1$							
T	**M-C**	**Exp Approx**	**Err Exp Appr%**	**T**	**M-C**	**Exp Approx**	**Err Exp Appr%**
50	0.081	0.137	-68.24	550	0.809	0.802	0.94
100	0.215	0.255	-18.56	600	0.837	0.829	0.94
150	0.330	0.357	-8.19	650	0.862	0.852	1.12
200	0.428	0.445	-4.03	700	0.883	0.873	1.14
250	0.509	0.521	-2.30	750	0.899	0.890	0.96
300	0.580	0.586	-1.05	800	0.914	0.905	0.93
350	0.644	0.643	0.07	850	0.928	0.918	1.02
400	0.696	0.692	0.61	900	0.937	0.929	0.83
450	0.742	0.734	1.08	950	0.946	0.939	0.71
500	0.778	0.770	0.97	1000	0.954	0.947	0.71

Figure 3: Distribution of τ_b, $P(\tau_b < T)$
$(3, 1)$-Gaussian pulses.

$b = 16$, $= 0.5$, $= 10$			
T	**M-C**	**Exp Approx**	**Err Exp Appr%**
50	0.234	0.253	-7.88
100	0.432	0.442	-2.28
150	0.581	0.583	-0.37
200	0.692	0.688	0.62
250	0.773	0.767	0.80
300	0.832	0.826	0.68
350	0.875	0.87	0.58
400	0.909	0.903	0.64
450	0.933	0.927	0.57
500	0.950	0.946	0.47

Figure 4: Distribution of τ_b, $P(\tau_b < T)$
$(0, 1)$-Uniform pulses