AN ADAPTIVE MODEL ON ASSET PRICING AND WEALTH DYNAMICS WITH HETEROGENEOUS TRADING STRATEGIES

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ABSTRACT. This paper develops an adaptive model on asset pricing and wealth dynamic of a financial market with heterogeneous agents and examines the profitability of momentum and contrarian trading strategies. In order to characterize asset price, wealth dynamics and rational adaptiveness arising from the interaction of heterogeneous agents with CRRA utility, an adaptive discrete time equilibrium model in terms of return and wealth proportions (among heterogeneous representative agents) is established. Taking trend followers and contrarians as the main heterogeneous agents in the model, the profitability of momentum and contrarian trading strategies is analyzed. Our results show the capability of the model to characterize some of the existing evidence on many of anomalies observed in financial markets, including the profitability of momentum trading strategies over short time intervals and of contrarian trading strategies over long time intervals, rational adaptiveness of agents, overconfidence and underreaction, overreaction and herd behavior, excess volatility, and volatility clustering.

Date: First Version: Dec.10, 2001; Second Version: Jan. 15, 2002; Latest Version: May 22, 2002.
Key words and phrases. Asset pricing, wealth dynamics, heterogeneity, adaptiveness, profitability, momentum trading strategies, contrarian trading strategies.
Acknowledgments. Financial support from AC3 and Capital Markets CRC for the second author is greatly acknowledged.

1. Introduction

The traditional asset-pricing models—such as the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), the arbitrage pricing theory (APT) of Ross (1976), or the intertemporal capital asset pricing model (ICAPM) of Merton (1973)—have as one of their important assumptions, investor homogeneity. In particular the paradigm of the representative agent assumes that all investors are homogeneous with regard to their preferences, their expectations and their investment strategies. However, as already argued by Keynes in the 1930s, agents do not have sufficient knowledge of the structure of the economy to form correct mathematical expectations that would be held by all agents.

The other important paradigm underpinning these models, the efficient market hypothesis, assumes that the current price contains all available information and past prices cannot help in predicting future prices. However there is evidence that markets are not always efficient and there are periods when real data show significantly higher than expected autocorrelation of returns. Over the last decade, a large volume of empirical work has documented a variety of ways in which asset returns can be predicted based on publicly available information and many of the results can be thought of as belonging to one of two broad categories of phenomena[1]. On the one hand, returns appear to exhibit continuation, or momentum, over short to medium time intervals, which may imply the profitability of momentum trading strategies over short to medium time intervals. On the other hand, there is also tendency toward reversals over long time intervals, leading to possible profitability of contrarian strategies. The traditional models of finance theory seem to have difficulty in explaining this growing set of stylized facts. As a result, there is a growing dissatisfaction with (i) models of asset price dynamics based on the representative agent paradigm, as expressed for example by Kirman (1992), and (ii) the extreme informational assumptions of rational expectations.

As an alternative to the traditional models of finance theory, studies over the last decade have involved some departure from the classical assumptions of strict rationality and unlimited computational capacity, and turned to heterogeneity and bounded rationality of agents. This line of research has attempted to incorporate aspects of investor psychology into the standard asset pricing theory in finance and characterize the interaction of heterogeneous agents, and to show that many of anomalies observed in financial markets are due to these effects.

Analytical models of how mistaken beliefs cause momentum (over short time intervals) and reversal (over long time intervals) have been considered by Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998) and Hong and Stein (1999). All these models generate an impulse response function to a new information signal in which there is gradual rise in the average reaction to a positive signal and a gradual average process of correction.

Both Barberis et al (1998) and Daniel et al (1998) assume that prices are driven by a single representative agent. They posit a small number of cognitive biases that this representative agent might have, and then investigate the extent to which these are sufficient to simultaneously deliver both short-horizon continuation and long-horizon reversals.

[1] A detailed discussion and references on the related empirical work is provided in Section 2.

Hong and Stein (1999) pursue the same goal as Barberis et al (1998) and Daniel et al (1998), but adopt a fundamentally different approach, and focus on the interaction between heterogeneous agents. They consider a model of two types of boundedly rational agents—newswatchers and momentum traders. The newswatchers make forecasts based on signals that they privately observe about future fundamentals, not on the current and past prices, while the momentum traders make simple forecasts based on past prices. Under an assumption that private information diffuses gradually across the newswatcher population, they provide a unified account of under- and overreactions. If information diffuses gradually across the population, prices underreact in the short run. Therefore "early" momentum buyers using simple trend-chasing strategies can profit from the price underreaction. On the other hand, "early" momentum buyers impose a negative externality on "late" momentum buyers, leading inevitably to overraction at long horizons.

In financial markets, individuals are imperfectly rational. It is believed that, as they learn from their trading outcomes, the market will progress eventually toward the fully rational equilibrium. A number of recent models use this evolutionary approach to characterize the interactions of heterogeneous agents in financial markets (e.g. Brock and Hommes (1997), (1998), Bullard and Duffy (1999), Chiarella (1992), Chiarella and He (2002*a*), (2002*b*), Day and Huang (1990), Farmer (1999), Farmer and Lo (1999), Franke and Nesemann (1999), Frankel and Froot (1987), Hommes (2001), LeBaron (2000), Lux (1995), and Lux and Marchesi (1999)). To avoid the constraints of analytical tractability, many of these authors use computer simulations to explore a wider space of economic settings. A general finding in many of these studies that long-horizon agents frequently do not drive short-horizon agents out of financial markets, and that populations of long- and short-horizon agents can create patterns of volatility and volume similar to actual empirical patterns.

In their framework, Brock and Hommes (1997), (1998) propose to model economic and financial markets as adaptive belief system (ABS), which is essentially an evolutionary competition among trading strategies. A key aspect of these models is that they exhibit expectations feedback and adaptiveness of agents. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. Agents are boundedly rational, in the sense that most traders choose strategies with higher fitness. The evolutional model generates endogenous price fluctuations with similar statistical properties as those observed in financial markets. The model of Brock and Hommes has been extended in Chiarella and He (2002*b*) by allowing agents to have different risk attitudes and different expectation formation schemes for both first and second moments of the price distribution.

Levy and Levy (1996) and Levy, Levy and Solomon (1994) consider a more realistic model where investors' optimal decisions depend on their wealth (resulting from the underlying CRRA utility function) and both price and wealth processes are thus growing. Using numerical simulations and comparing the stock price dynamics in models with homogeneous and heterogeneous expectations, they conclude that the homogeneous expectation assumption leads to a highly inefficient market with periodic (and therefore predictable) booms and crashes while introduction of heterogeneous expectations leads to much more realistic dynamics.

Chiarella and He (2001) develop a model of interaction of portfolio decisions and wealth dynamics with heterogeneous agents. A growth equilibrium model of both the asset price and wealth is first obtained. To characterize the interaction of heterogeneous agents in financial markets and conduct a theoretical analysis, stationary models in terms of return and wealth proportions (among different types of agents) are then developed. As a special case of the general heterogeneous model, these authors consider models of homogeneous agents and of two heterogeneous agents without switching among them.

In this paper, we establish an adaptive model on asset pricing and wealth dynamics of heterogeneous agents, which extends the model in Chiarella and He (2001) further by adding a switching mechanism among different types of trading strategies. Based on certain fitness measures, such as the realized wealth, the agents are allowed to switch from one strategy to another from time to time. Consequently a model with adaptive beliefs is established where an evolutionary dynamics across predictor choice is coupled to the dynamics of the endogenous variable.

Empirical studies provide some evidence that momentum trading (or trend following) strategies are more profitable over short time intervals, while contrarian trading strategies are more profitable over long time intervals (e.g. Arshanapali et al (1998), Asnee (1997), Capaul et al (1993), Fama and French (1998), Jegadeesh and Titman (1993), (2001), Lee and Swaminathan (2000), Levis and Liodakis (2001), Moskowitz and Grinblatt (1999) and Rouwenhorst (1998)). To characterize the profitability of momentum and contrarian trading strategies, quasi-homogeneous model is introduced, in which agents use exactly the same trading strategies except for having different time intervals. Our results in general support the empirical findings on the profitability of the momentum and contrarian trading strategies. In addition, the model also exhibits underlying rationale for various anomalies observed in financial markets, including, overconfidence and underreaction, overreaction and herd behavior, excess volatility, and volatility clustering.

This paper is organized as follows. Section 2 establishes an adaptive asset pricing and wealth dynamic model with heterogeneous beliefs from agents. It is shown how the distributions of the wealth and population across heterogeneous agents are measured. As a simple case, a model of two types of agents is then considered in Section 3. To characterize the profitability of momentum and contrarian trading strategies, a quasi-homogeneous model is also introduced as a special case of the model of two types of agent in Section 3. The profitability of momentum and contrarian trading strategies is then analyzed in Sections 4 and 5, respectively. Section 6 concludes.

2. Adaptive Model with Heterogeneous Agents

This section is devoted to establish an adaptive asset pricing and wealth dynamic model with heterogeneous beliefs among agents. The model can been treated as a generalization of some recent asset pricing models on the interaction between heterogeneous agents, say for example, Barberis et al (1998), Brock and Hommes (1998), Chiarella and He (2001), Daniel et al (1998), Hong and Stein (1999) and Levy and Levy (1996). The key characteristics of this modelling framework are the adaptiveness, the heterogeneity and the interaction of the economic agents. The heterogeneity is expressed in terms of different views on expectations of the distribution of future returns on the risky asset. The modelling framework of this paper extends that of the cited work by focusing on both asset price and wealth dynamics (Brock and Hommes

(1998) considered only asset price dynamics) and by allowing a mechanism of adaptiveness of heterogeneous agents (Chiarella and He (2001) considered fixed proportion of heterogeneous agents).

The setting of the following adaptive model is similar to the one in Chiarella and He (2001) and Levy and Levy (1996). Our hypothetical financial market contains two investment choices: a stock (or index of stocks) and a bond. The bond is assumed to be a risk free asset and the stock is a risky asset. The model considered in the following is discrete time model, in which investors are allowed to revise their portfolios at each time interval.

Following the standard portfolio optimization approach, a growth model in terms of price and wealth is established first in this section. Then, to reduce the growth model to a stationary model, the return on the risky asset and the wealth proportions (among heterogeneous investors), instead of price and wealth, are used as state variables to form a stationary model. Based on a certain performance (or fitness) measure, an adaptive mechanism is finally introduced, leading to a general adaptive model. This is a model of asset price and wealth dynamics that characterizes three important and related issues in financial markets: heterogeneity, adaptiveness, and interaction.

2.1. **Notations.** Denote

p_t : Price (ex dividend) per share of the risky asset at time t;
y_t : Dividend at time t;
R : Risk free return with risk free rate $r = R - 1$;
N : Total number of shares of the risky asset;
H : Total number of investors;
$N_{i,t}$: Number of shares acquired by agent i at time t;
$W_{i,t}$: Wealth of agent i at time t;
$W_{i,0}$: Initial wealth of agent i;
$\pi_{i,t}$: Proportion of wealth of agent i invested in the risky asset at time t;
ρ_t : The return on the risky asset at period t.

It is assumed that[2] all the investors have the same attitude to risk with the same utility function $U(W) = \log(W)$. Following the above notation, the return on the risky asset at period t is then defined by[3]

$$\rho_t = \frac{p_t - p_{t-1} + y_t}{p_{t-1}}. \tag{2.1}$$

[2]To make the following analysis more tractable and transparent, the assumption that all investors have the same utility function $U(W) = \log(W)$ is maintained in this paper. However, the analysis can be generalized to the case of the utility functions that allow agents to have different risk coefficients, say, $U_i(W) = (W^{\gamma_i} - 1)/\gamma_i$ with $0 < \gamma_i < 1$. The dynamics generated by different risk aversion coefficient γ_i is an interesting and important issue that is left for future work.

[3]The return can also be defined by the difference of logarithms of the prices. It is known that the difference between these two definition becomes smaller and smaller as the time interval is reduced (say, from monthly to weekly or daily).

2.2. **Portfolio Optimization Problem of Heterogeneous Agents.** Following the standard portfolio optimization approach, the wealth of agent (or investor) i at time period $t+1$ is given by

$$\begin{aligned} W_{i,t+1} =& (1-\pi_{i,t})W_{i,t}R + \pi_{i,t}W_{i,t}(1+\rho_{t+1}) \\ =& W_{i,t}[R + \pi_{i,t}(\rho_{t+1} - r)]. \end{aligned} \tag{2.2}$$

As in Brock and Hommes (1998) and Levy and Levy (1996), a Walrasian scenario is used to derive the demand equation, i.e., each trader is viewed as a price taker and the market is viewed as finding (via the Walrasian auctioneer) the price p_t that equates the sum of these demand schedules to the supply. That is, the agents treat the period t price, p_t, as parametric when solving their optimisation problem to determine $\pi_{i,t}$. Denote by $F_t = \{p_{t-1}, \cdots ; y_t, y_{t-1}, \cdots\}$ the information set[4] formed at time t. Let E_t, V_t be the conditional expectation and variance, respectively, based on F_t, and $E_{i,t}, V_{i,t}$ be the "beliefs" of investor i about the conditional expectation and variance. Then it follows from (2.2) that

$$\begin{array}{ll} E_{i,t}(W_{i,t+1}) & = W_{i,t}[R + \pi_{i,t}(E_{i,t}(\rho_{t+1}) - r)], \\ V_{i,t}(W_{i,t+1}) & = W_{i,t}^2 \pi_{i,t}^2 V_{i,t}(\rho_{t+1}). \end{array} \tag{2.3}$$

Consider investor i, who faces a given price p_t, has wealth $W_{i,t}$ and believes that the asset return is conditionally normally distributed with mean $E_{i,t}(\rho_{t+1})$ and variance $V_{i,t}(\rho_{t+1})$. This investor chooses a proportion $\pi_{i,t}$ of his/her wealth to be invested in the risky asset so as to maximize the expected utility of the wealth at $t+1$, as given by

$$\max_{\pi_{i,t}} E_{i,t}[U(W_{i,t+1})].$$

It follows that[5] the optimum investment proportion at time t, $\pi_{i,t}$ is given by

$$\pi_{i,t} = \frac{E_{i,t}(\rho_{t+1}) - r}{V_{i,t}(\rho_{t+1})}. \tag{2.4}$$

Heterogeneous beliefs are introduced via the assumption that

$$E_{i,t}(\rho_{t+1}) = f_i(\rho_{t-1}, \cdots, \rho_{t-L_i}), \qquad V_{i,t}(\rho_{t+1}) = g_i(\rho_{t-1}, \cdots, \rho_{t-L_i})$$

for $i = 1, \cdots, H$, where L_i are integers, f_i, g_i are some deterministic functions which can be differ across investors. Under this assumption, both $E_{i,t}(\rho_{t+1})$ and $V_{i,t}(\rho_{t+1})$ are functions of the past prices up to $t-1$, which in turn implies the optimum wealth proportion $\pi_{i,t}$, defined by (2.4), is a function of the history of the prices $(p_{t-1}, p_{t-2}, \cdots)$[6].

[4]Because of the Walrasian scenario, the hypothetical price p_t at time t is included in the information set to determine the market clearing price. However, agents form their expectations by using the past prices up to time $t-1$.

[5]See Appendix A.1 in Chiarella and He (2001) for details.

[6]In Levy and Levy (1996), the hypothetical price p_t is included in the above conditional expectations on the return and variance. In this case, the market clearing price is solved implicitly and is much more involved mathematically. The approach adopted here is the standard one in deriving the price via the Walrasian scenario and also keeps the mathematical analysis tractable. A similar approach has been adopted in Brock and Hommes (1997), (1998) and Chiarella and He (2002*b*).

2.3. **Market Clearing Equilibrium Price—A Growth Model.** The optimum proportion of investment in the risky asset, $\pi_{i,t}$, determines the number of shares at price p_t that investor i wishes to hold:

$$N_{i,t} = \frac{\pi_{i,t} W_{i,t}}{p_t},$$

Summing the demands of all investors gives the aggregate demand. The total number of shares in the market, denoted by N, is assumed to be fixed. Hence the market clearing equilibrium price p_t is determined by

$$\sum_{i=1}^{H} N_{i,t} = \sum_{i=1}^{H} \frac{\pi_{i,t} W_{i,t}}{p_t} = N,$$

i.e.,

$$\sum_{i=1}^{H} \pi_{i,t} W_{i,t} = N p_t. \tag{2.5}$$

Thus, in this model, as in real markets, the equilibrium price p_t and the wealth of investors, $W_t \equiv (W_{1,t}, \cdots, W_{H,t})$, are determined simultaneously by (2.2) and (2.5). The optimum demands of investors are functions of the price and their wealth. Also, as observed in financial markets, the model implies that both the price and the wealth are growing processes in general.

2.4. **Population Distribution Measure.** Now suppose all the investors can be grouped in terms of their conditional expectations of mean and variance. That is, within a group, all the investors follow the same expectation schemes on the conditional mean and variance of the return ρ_{t+1}, and hence the optimum wealth proportion ($\pi_{i,t}$) invested in the risky asset for the investors are the same. Assume all the investors can be grouped as h types (or groups) and group j has $\ell_{j,t}$ investors at time t with $j = 1, \cdots, h$, then $\ell_{1,t} + \cdots + \ell_{h,t} = H$. Denote $n_{j,t}$ as the proportion of the number of investors in group j, at time t, relative to the total number of the investors, H, that is, $n_{j,t} = \ell_{j,t}/H$, so that $n_{1,t} + \cdots + n_{h,t} = 1$.

Some simple examples on return and wealth dynamics when proportions of different types of agents $n_{j,t}$ are fixed over time are illustrated in Chiarella and He (2001). This is a highly simplified assumption. It would be more realistic to allow agents to adjust their beliefs from time to time, based on some performance or fitness measures (say, for example, the realized returns or errors, as in Brock and Hommes (1998)). In this way, one can account for investor psychology and herd behavior[7]. As a consequence, the proportions of different types of agents become endogenous state variables. Therefore $(n_{1,t}, n_{2,t}, \cdots, n_{h,t})$ measures the population distribution among different types of heterogeneous agents. The change in the distribution over time can be used to measure herd behavior among heterogeneous agents, in particular, during highly volatile periods in financial markets.

2.5. **Heterogeneous Representative Agents and Wealth Distribution Measure.** For investors within the group j, the optimum demands on wealth proportions to be invested in the risky asset are the same, and are denoted by $\bar{\pi}_{j,t}$. Let $\bar{W}_{j,t}$ be the average wealth of each investor within group j (so that $\ell_{j,t}\bar{W}_{j,t}$ gives the total wealth of group j).

[7]See more discussion on this aspect in the next section.

To measure the average wealth distribution among heterogeneous representative agents, introduce $\bar{w}_{j,t}$ to be the average wealth proportion of group j relative to the total average wealth $\bar{W}_t$ at time t, that is,

$$\bar{w}_{i,t} = \frac{\bar{W}_{i,t}}{\bar{W}_t}, \qquad \text{with} \qquad \bar{W}_t = \sum_{j=1}^{h} \bar{W}_{j,t}. \tag{2.6}$$

One can see that the average wealth proportion vector $(\bar{w}_{1,t}, \bar{w}_{2,t}, \cdots, \bar{w}_{h,t})$ corresponds to the wealth distribution among *representatives of heterogeneous agents* of different types, it measure the average wealth levels used in the different trading strategies.

2.6. **Performance Measure, Population Evolution and Adaptiveness.** Following Brock and Hommes (1997), (1998), a performance measure or *fitness function*, say $(\Phi_{1,t}, \cdots, \Phi_{h,t})$, is publicly available to all agents. Based on the performance measure agents make a (boundedly) rational choice among the predictors. This results in the *Adaptive Rational Equilibrium Dynamics*, introduced by Brock and Hommes (1997), an evolutionary dynamics across predictor choice which is coupled to the dynamics of the endogenous variables. In the limit as the number of agents goes to infinity, the probability that an agent j chooses trading strategy j is given by the well known *discrete choice model* or 'Gibbs' probabilities[8]

$$n_{j,t} = \exp[\beta(\Phi_{j,t-1} - C_j)]/Z_t \qquad Z_t = \sum_{j=1}^{h} \exp[\beta(\Phi_{j,t-1} - C_j)], \tag{2.7}$$

where $C_j \geq 0$ measures the cost of the strategy j for $j = 1, 2, \cdots, h$.

The crucial feature of (2.7) is that the higher the fitness of trading strategy j, the more traders will select strategy j. The parameter β, called *intensity of choice* or *switching intensity*, plays an important role and can be used to characterize various psychological effects, as discussed by Hirshleifer (2001).

- *Overconfidence and Underreaction.*
  Edwards, (1968) identified the phenomenon of "*conservatism*", that under appropriate circumstances individuals are overconfident. They do not change their beliefs as much as would a rational Bayesian in the face of new evidence. One explanation for conservatism is that processing new information and updating beliefs is costly. Another is that, in a stable environment, *self-deception* can cause conservatism because an individual who has explicitly adopted a belief may be reluctant to admit to having made a mistake. Conservatism implies under-weighting of new evidence. Both overconfidence and underreaction can be partially measured when the switching intensity parameter β is small. In an extreme case when $\beta = 0$, there is no switching among strategies and agents populations are evenly distributed across all trading strategies[9].
- *Overreaction and Herd Behavior.*
  If the environment is volatile, or agents are less conservative and less confident about their beliefs, there may be no dishonor in recognizing that different

[8]See Manski and McFadden (1981) and Anderson, de Palma and Thisse (1993)) for extensive discussion of discrete choice models and their applications in economics.

[9]See Chiarella and He (2001) for models with fixed, but not evenly distributed, population proportion among different types of trading strategies.

beliefs are called for and investors are more willing to switch to beliefs which generate better outcomes. This can be measured when the switching intensity parameter β is high. An increase in the switching intensity β represents an increase in the degree of rationality with respect to evolutionary selection of trading strategies. In an extreme case when β is very large (close to infinity), a large proportion of traders are willing to switch more quickly to successful trading strategies. Consequently, market overreaction and herd behavior may be observed.

A natural performance measure or fitness function can be taken as a weighted average of the realized wealth return on the proportion invested in the risky asset, given by

$$\Phi_{j,t} = \phi_{j,t} + \gamma\Phi_{j,t-1};$$

for $j = 1, \cdots, h$, where $0 \leq \gamma \leq 1$ and

$$\phi_{j,t} = \bar{\pi}_{j,t-1}\frac{\bar{W}_{j,t} - \bar{W}_{j,t-1}}{\bar{W}_{j,t-1}} = \bar{\pi}_{j,t-1}[r + (\rho_t - r)\bar{\pi}_{j,t-1}]$$

is the realized wealth return invested in the risky asset in period t. Here γ is a *memory* parameter measuring how fast past realized fitness is discounted for strategy selection.

2.7. **An Adaptive Model.** The above growth model is rendered stationary by formulating it in terms of the stock return and the relative proportions of the wealth among the investors, instead of the wealth W_t and the stock price p_t.

Proposition 2.1. *The average wealth proportions evolve according to*

$$\bar{w}_{i,t} = \frac{\bar{w}_{i,t-1}[R + (\rho_t - r)\bar{\pi}_{i,t-1}]}{\sum_{j=1}^{h} \bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}]} \qquad (i = 1, \cdots, h) \tag{2.8}$$

with return ρ_t given by[10]

$$\rho_t = r + \frac{\sum_{i=1}^{h} \bar{w}_{i,t-1}[(1+r)(n_{i,t-1}\bar{\pi}_{i,t-1} - n_{i,t}\bar{\pi}_{i,t}) - \alpha_t n_{i,t-1}\bar{\pi}_{i,t-1}]}{\sum_{i=1}^{h} \bar{\pi}_{i,t-1}\bar{w}_{i,t-1}(n_{i,t}\bar{\pi}_{i,t} - n_{i,t-1})}, \tag{2.9}$$

where α_t corresponds to dividend yield defined by $\alpha_t = y_t/p_{t-1}$. The population proportions $n_{j,t}$ evolve according to

$$n_{i,t} = \exp[\beta(\Phi_{i,t-1} - C_j)]/Z_t, \tag{2.10}$$

where the fitness functions are defined by

$$\Phi_{i,t} = \phi_{i,t} + \gamma\Phi_{i,t-1}; \qquad 0 \leq \gamma \leq 1,$$
$$\phi_{i,t} = \bar{\pi}_{i,t-1}\frac{\bar{W}_{i,t} - \bar{W}_{i,t-1}}{\bar{W}_{i,t-1}} = \bar{\pi}_{i,t-1}[r + (\rho_t - r)\bar{\pi}_{i,t-1}],$$
$$Z_t = \sum_{i=1}^{h} \exp[\beta(\Phi_{i,t-1} - C_i)],$$

and $C_i \geq 0$ measure the cost of the strategy for $i = 1, 2, \cdots, h$.

Proof. See Appendix A.1. □

[10]It is easy to check that $\rho_t \equiv r$ is a trivial solution. As a necessary condition for investing in risky asset, we assume that $E(\rho_t) > r$.

It is easy to see that, when $h \leq H$, $\ell_j \geq 1$ and $\beta = 0$ for $j = 1, \cdots, h$, Proposition 2.1 leads to the model in Chiarella and He (2001) with fixed proportion $n_{j,t} = n_j$ $(j = 1, \cdots, h)$ among heterogeneous types of agents.

2.8. **Trading Strategies.** The adaptive model established in Proposition 2.1 is incomplete unless the conditional expectations of agents on the mean and variance of returns are specified. Different trading strategies can be incorporated into this general adaptive model. To illustrate various features of the model, this paper considers only three simple, but well-documented, types of agents, termed *fundamentalist*, *momentum traders* and *contrarians*. Neither type is fully rational in the usual sense. The information on the dividends and realized prices is publicly available to all trader types.

2.8.1. *Fundamental traders.* The fundamentalists make forecasts on the risk premium level based on both public and their private information about future fundamentals. It is assumed that

$$E_{F,t}(\rho_{t+1}) = r + \delta_F, \tag{2.11}$$

where $E_{F,t}$ denotes the fundamentalists' expected return on ρ_{t+1} for the next period $t+1$ and δ_F is the risk premium estimated[11] of the fundamental traders. That is, the fundamentalists believe that the excess conditional mean for the risky asset (from the risk-free rate) is given by the risk premium δ_F.

2.8.2. *Momentum Traders.* Momentum traders, in contrast to the fundamental traders, do condition on the past prices. Momentum, or positive feedback, trading has several possible motivations, one being that investors form expectations of future prices by extrapolating trends. They buy into price trends and exaggerate trends, leading to overshooting. As a result there may be excess volatility.

In an efficient market, a stock having good growth prospects does not necessarily have good prospects for future risk-adjusted returns (which are on average zero). If people mistakenly extend their favorable evaluation of a stock's earnings prospects to its return prospects, growth stocks will be overpriced (see Lakonishok et al (1994) and Shefrin and Statman (1995)).

Empirical studies have subscribed to the view that momentum trading strategies yield significant profits over short time intervals (e.g. Asnee (1997), Jegadeesh and Titman (1993), (2001), Lee and Swaminathan (2000), Moskowitz and Grinblatt (1999) and Rouwenhorst (1998)). Although these results have been well accepted, the source of the profits and the interpretation of the evidence are widely debated. In addition, there does not exist in the literature a quantified model to clarify and justify such evidence. As a first step, this issue is discussed in the next section within the framework of the adaptive heterogeneous model outlined in Proposition 2.1.

For momentum traders, it is assumed in this paper that their forecasts are "simple" functions of the history of past returns. More precisely, it is assumed that

$$E_{M,t}(\rho_{t+1}) = r + \delta_M + d_M \bar{\rho}_{M,t}, \qquad \bar{\rho}_{M,t} = \frac{1}{L_M} \sum_{k=1}^{L_M} \rho_{t-k}, \tag{2.12}$$

where $E_{M,t}(\rho_{t+1})$ denotes the expected return of momentum agents on ρ_{t+1} for the next period $t+1$ and δ_M is their risk premium estimated and $d_M > 0$ corresponds to the

[11]A constant risk premium is a simplified assumption. In practice, the risk premium is not necessarily constant but could also be a function of the variance, for example.

extrapolation rate of the momentum trading strategy. The integer $L_M \geq 1$ corresponds to the memory length of momentum traders. In other words, the expected excess return (from risk-free rate) of momentum agents has two components: their estimated risk premium δ_M and trend extrapolation $d_M \bar{\rho}_{M,t}$, which is positively proportional to the moving average of the returns over the last L_M time periods.

2.8.3. *Contrarian Traders.* The profitability of contrarian investment strategies is now a well-established empirical facts in the finance literature (see, for example, Levis and Liodakis (2001)). Empirical evidence suggests that over long time intervals, contrarian strategies generate significant abnormal returns (see, for example, Arshanapali et al (1998), Fama and French (1998), and Capaul et al (1993)). Some evidence has shown that overreaction can use aggregate stock market value measures such as dividend yield to predict future market returns, so that contrarian investment strategies are on average profitable. In spite of the apparent robustness of such strategies, the underlying rationale for their success remains a matter of lively debate in both academic and practitioner communities.

In the following section, the role of expectational errors in explaining the profitability of contrarian strategies is examined. For contrarian strategy, it is assumed that

$$E_{C,t}(\rho_{t+1}) = r + \delta_C - d_C \bar{\rho}_{C,t}, \qquad \bar{\rho}_{C,t} = \frac{1}{L_C} \sum_{k=1}^{L_C} \rho_{t-k}, \tag{2.13}$$

where $E_{C,t}(\rho_{t+1})$ denotes the expected return of contrarian agents on ρ_{t+1} for the next period $t+1$ and δ_C is their estimated risk premium and $d_C > 0$ corresponds to their extrapolation rate. The integer $L_C \geq 1$ corresponds to the memory length of contrarian agents. In other words, contrarian agents believe that the difference of excess conditional mean and the risk premium $[E_{C,t}(\rho_{t+1}) - r] - \delta_C$ is negatively proportional to the moving average of the returns over the last L_C time periods.

3. An Adaptive Model of Two Types of Agents

In the rest of this paper, the focus will be on a simple model of just two types of agents—momentum traders and contrarian traders. In this case, the adaptive model developed in Section 2 can be reduced to a simple form, as indicated below. To examine profitability of momentum and contrarian trading strategies over different time intervals, a special case of the model, termed *quasi-homogeneous* model, is then considered. Detailed discussion on the dynamics of such quasi-homogeneous models, including profitability, herd behavior, price overshooting, statistical patterns of returns, is then undertaken in the following sections.

3.1. **Notation.** Assume that there are only two different types trading strategies. Let $\bar{w}_t$, $\bar{n}_t$ be the difference of the average wealth proportions and population proportions of type 1 and type 2 agents; that is

$$\bar{w}_t = \bar{w}_{1,t} - \bar{w}_{2,t}, \qquad n_t = n_{1,t} - n_{2,t}. \tag{3.1}$$

Then it follows from $\bar{w}_{1,t} + \bar{w}_{2,t} = 1$ and $n_{1,t} + n_{2,t} = 1$ that

$$\bar{w}_{1,t} = \frac{1 + \bar{w}_t}{2}, \qquad \bar{w}_{2,t} = \frac{1 - \bar{w}_t}{2}$$

and

$$n_{1,t} = \frac{1+n_t}{2}, \qquad n_{2,t} = \frac{1-n_t}{2}.$$

Correspondingly, the adaptive model in Proposition 2.1 can be reduced to a simple form.

3.2. **The Model for Two Types of Agents.** Given the above notations, the adaptive model for two types of agents following different trading strategies assumes the form give by Proposition 3.1.

Proposition 3.1. *The difference of the average wealth proportions $\bar{w}_t$ evolves according to*

$$\bar{w}_{t+1} = \frac{f_1 - f_2}{f_1 + f_2} \tag{3.2}$$

with return ρ_t given by

$$\rho_{t+1} = r + \frac{g_{11} + g_{12}}{g_{21} + g_{22}}, \tag{3.3}$$

where

$$\begin{aligned}
f_1 &= (1+\bar{w}_t)[1 + r + (\rho_{t+1} - r)\bar{\pi}_{1,t}],\\
f_2 &= (1-\bar{w}_t)[1 + r + (\rho_{t+1} - r)\bar{\pi}_{2,t}],\\
g_{11} &= (1+\bar{w}_t)[(1 + r - \alpha_{t+1})(1+\bar{n}_t)\bar{\pi}_{1,t} - (1+r)(1+\bar{n}_{t+1})\bar{\pi}_{1,t+1}],\\
g_{12} &= (1-\bar{w}_t)[(1 + r - \alpha_{t+1})(1-\bar{n}_t)\bar{\pi}_{1,t} - (1+r)(1-\bar{n}_{t+1})\bar{\pi}_{2,t+1}],\\
g_{21} &= (1+\bar{w}_t)\bar{\pi}_{1,t}[(1+\bar{n}_{t+1})\bar{\pi}_{1,t+1} - (1+\bar{n}_t)],\\
g_{22} &= (1-\bar{w}_t)\bar{\pi}_{2,t}[(1-\bar{n}_{t+1})\bar{\pi}_{2,t+1} - (1-\bar{n}_t)]
\end{aligned}$$

and $\bar{\pi}_{j,t}\,(j = 1, 2)$ are defined by (2.4). The difference of population proportions n_t evolves according to

$$n_{t+1} = \tanh[\frac{\beta}{2}((\Phi_{1,t} - \Phi_{2,t}) - (C_1 - C_2))], \tag{3.4}$$

where the fitness functions are defined as

$$\Phi_{j,t+1} = \bar{\pi}_{j,t}[r + (\rho_{t+1} - r)\bar{\pi}_{j,t}] + \gamma\Phi_{j,t}, \tag{3.5}$$

and $C_j \geq 0$ measure the cost of the strategy for $j = 1, 2$.

3.3. **Wealth distribution and profitability of trading strategies.** The average wealth distribution among two types of agents (following different trading strategies) is now characterized by $\bar{w}_t$, the difference of the average wealth proportions. Over a certain time period, if $\bar{w}_t$ stays above (below) the initial value $\bar{w}_o$ and increases (decreases) significantly as t increases, then, on average, type 1 agents accumulate more (less) wealth than type 2 agents, and one may say type 1 trading strategy is more (less) profitable than type 2 trading strategy. Otherwise, if the difference is not significantly different from $\bar{w}_o$, then there is no evidence that on average either trading strategies is more profitable than the other.

3.4. **Population distribution and herd behavior.** The distribution on populations using different types trading strategies is now characterized by the difference of the population proportions n_t. At time period t, if n_t is positive (negative), then this indicates that there are more (less) agents using type 1 trading strategy than type 2 trading strategy. Moreover, if n_t is significantly different from zero, then this could be taken as an indication of herd behavior. This is, in particular, frequently observed to be the case when the switching intensity $\beta > 0$ is high.

When there is evidence on the profitability of type 1 (type 2) trading strategy and a clear indication on herd behavior using type 1 (type 2) trading strategy over the time period, we say type 1 (type 2) trading strategy *dominates the market*.

3.5. **A Quasi-Homogeneous Model.** As a special case of the adaptive model with two types of agents, consider the case, termed *quasi-homogeneous* model, where both types of agents use exactly the same trading strategies except that they use different memory lengths.

The trading strategies for all three types of agents can be unified in the following form:

$$E_{i,t}(\rho_{t+1}) = r + \delta_i + d_i \bar{\rho}_{i,t}, \qquad \bar{\rho}_{i,t} = \frac{1}{L_i} \sum_{k=1}^{L_i} \rho_{t-k}, \tag{3.6}$$

for $i = 1, 2$, where $L_i \geq 1$ is integer, $r(> 0), \delta_i(> 0)$ and $d_i \in \mathbb{R}$ are constants. For quasi-homogeneous model, $\delta_1 = \delta_2 = \delta, d_1 = d_2 = d$ but $1 \leq L_1 \leq L_2$.

In the following discussion, assume that the conditional variances of agents are given by a constant σ^2. One can standardize both the risk premium δ and extrapolation rate d as follows:

$$\bar{\delta} = \frac{\delta}{\sigma^2}, \qquad \bar{d} = \frac{d}{\sigma^2}.$$

Correspondingly, the optimal demand of type j agents in terms of the wealth proportion invested in the risky asset is given by

$$\pi_{j,t} = \bar{\delta}_j + \bar{d}_j \bar{\rho}_{j,t}.$$

We also assume that the dividend yield process has the form

$$\alpha_t = \alpha_o + q\mathcal{N}(0, 1), \tag{3.7}$$

where $\mathcal{N}(0, 1)$ is the standard normal distribution.

Because of the highly nonlinear nature of the adaptive model theoretical analysis (even of the steady states) seems intractable and thus the model is analyzed numerically.

3.5.1. *Existence of steady-state returns.* If $\alpha_t = \alpha_o$ is a constant, the system (3.2)-(3.6) becomes a deterministic system. How distributions of noise processes, such as the dividend yield process in our model, can be affected by the underlying dynamics of nonlinear deterministic systems is an important and interesting issue, but one for which there is little theory to serve as a guide. Understanding the underlying dynamics of the deterministic systems, such as existence of steady-states, their stability and bifurcation, is naturally a first step towards the goal of understanding the noise perturbed system.

When $\alpha_t = \alpha_o$ is a constant, in terms of steady-state of return and wealth proportions, it is easy to see that the quasi-homogeneous model has the same steady-state as the homogeneous model (with $L_1 = L_2$) does. The existence of such steady-state is studied in Chiarella and He (2001) and the results are summarized in the following.

- The steady-state of the wealth proportions stays at their initial level, while the steady-state of the return depends on the extrapolation rate $\bar{d}$.
- There is a unique steady-state return when $\bar{d} = 0$. In other words, when agents are fundamentalists, there is a unique steady-state return and, for convenience of discussion, it is called the *fundamental steady-state return*. Moreover, high risk premia δ correspond to high levels of the steady-state return.
- There exist two steady-state returns when $\bar{d} < 0$, that is when agents are contrarians. One of the steady-state returns is negative while the other is positive, the positive steady-state return is called the *contrarian steady-state return*. More importantly, with the same risk premium, when agents act as contrarians, the contrarian steady-state return is pushed below the fundamental steady-state return.
- There exist two steady-state returns when $\bar{d} > 0$ small. That is, when agents are momentum traders and they extrapolate weakly, the return has two positive fixed steady-states. However, when $\bar{d}$ is close zero, only one of the steady-state is bounded and this steady-state return is called the *momentum steady-state return*. Furthermore, given the same risk premium, compared to the fundamental equilibrium, a weakly homogeneous momentum trading strategy (i.e. $\bar{d} > 0$ small) leads to a higher level of steady-state return.

An aim of the following analysis is to determine to what extent the adaptive model for two types of agents reflects these characteristics.

3.5.2. *Parameters and initial values selection.* Using data for the United States during the 1926-94 period, as reported by Ibbotson Associates, the annual risk-free interest rate, $r = 3.7\%$, corresponds to the average rate during that period. The initial history of rates of return on the stock consists of a distribution with a mean of 12.2% and a standard derivation of 20.4%. A mean dividend yield of $\alpha_o = 4.7\%$ corresponds to the historical average yield on the $S\&P500$. The initial share price is $p_o = \$10.00$.

The analysis in the following sections, selects the annual risk-free rate r, standard derivation σ and the mean dividend yield α_o as indicated above. For the simulations, the time period between each trade is one day and simulations are conducted over 20 years. Parameters and initial values are selected as follows, unless state otherwise,

$$\bar{\delta} = 0.6, \beta = 0.5, \gamma = 0.5, C_1 = C_2 = 0 \tag{3.8}$$

and

$$\bar{w}_o = 0, n_o = 0, \Phi_{1,o} = \Phi_{2,0} = 0.5, p_o = \$10. \tag{3.9}$$

Furthermore, annual rates of risk-free rate and returns of the risky asset are used in the fitness functions $\Phi_{j,t}$ for $j = 1, 2$.

4. Wealth Dynamics of Momentum Trading Strategies

This section considers the quasi-homogeneous model with $d_1 = d_2 = d > 0$ and $1 \leq L_1 < L_2$, that is both types of agents follow the same momentum trading strategy except for having different memory lengths. An interesting question is which type of agent dominates the market over the time.

As discussed in section 2, some empirical studies seem to support a view that momentum traders are profitable over short time intervals, but not over long time intervals. The following discussion examines different combination of (L_1, L_2) and analyzes the effect of lag length on the wealth dynamics[12]. The results indicate in general that, the strategy with short memory length will dominate the market by accumulating more wealth and attracting more population. Thus the adaptive model outlined in this paper is capable of characterizing some broad features found in empirical studies.

4.1. **Case:** $(L_1, L_2) = (3, 5)$**.** The following subsection considers first the dynamics of the underlying deterministic system, that is, when $q = 0$. The impact of the noise processes on the dynamics is then considered in the subsequent subsection.

4.1.1. *No-noise Case.* Taken $q = 0$. For $\bar{d} = 0.5$, initial population proportion $n_o = 0$ and any initial wealth proportion $\bar{w}_o$, numerical simulations show that

$$\rho_t \to \rho^* = 15.45\% \quad \text{(annual)}, \qquad \bar{w}_t \to \bar{w}_o, \quad n_t \to 0.$$

By changing various parameters and initial values, we obtain the following results on the momentum trading strategies from the quasi-homogeneous model presented in section 3.

(i) *Risk premium and over-pricing.*
It is found that, ceteris paribus, for $\bar{\delta} = \delta/\sigma^2 = 0.35$, $\rho_t \to \rho^* = 10.94\%$, while for $\bar{\delta} = 0.53$, $\rho^* = 15.45\%$. In general, a high level of risk-adjusted premium leads to a high return, and a high price as well. In fact, for the given parameters, there exists $\delta_o \in (0.69, 0.7)$, so called bifurcation value[13], such that the returns converge to fixed values for $\bar{\delta} < \delta_o$ and diverge for $\bar{\delta} > \delta_o$, leading to price explosion.

(ii) *Over-extrapolation and overshooting.*
Momentum traders form expectations of future prices by extrapolating trends. However, when the prices or returns are over-extrapolated, stocks are over-priced, and as a result, overshooting takes place. Based on the parameters selected, there exists $d_o \in (0.573, 0.574)$ such that, ceteris paribus, returns converge to fixed values for $\bar{d} < d_o$ and diverge for $\bar{d} > d_o$, leading prices to exhibit overshooting.

(iii) *No noise, no effects on population and wealth distribution and no herd behavior.*
For either fixed $n_o \neq 0$ and a range of $\bar{w}_o$ (say $n_o = -0.3$ and $\bar{w}_o \in (-0.5, 0.3)$), or fixed $\bar{w}_o \neq 0$ and a range of n_o (say, $\bar{w}_o = -0.3$ and $n_o \in (-1, 0.3)$),

$$\rho_t \to \rho^* = 15.45\% \quad \text{(annual)}, \qquad \bar{w}_t \to \bar{w}_o + \epsilon, \quad n_t \to 0$$

with $\epsilon \approx 10^{-6}$. Also, the switching intensity parameter β has almost no effect on the results (as long as the returns series converge to constants). This implies that, without the noise from the dividend yield process, in terms of profitability, type 1 trading strategy is slightly better than type 2, but not significantly. In addition, the populations of agents using different strategies become evenly distributed. In other words, no one of the momentum trading strategies dominates the market, even though both wealth and population are not evenly distributed initially. Therefore, when there is no noise from the dividend yield process and

[12]The selection of various combinations of lag lengths is arbitrary. However other simulations (not reported) indicate some robustness of the results presented in this paper.

[13]As in Brock and Hommes (1998), the dynamics of the system through various types of bifurcation can be analyzed and one of interest. However, in this paper, we focus on the dynamics of the stochastic system when the return process of the underlying deterministic system is stable.

the returns converge to constants, the average wealth proportions, as expected, stay at their initial level, while the average population proportions are evenly distributed, and there is no herd behavior.

4.1.2. *Effect of Noise.* Select the annualized standard derivation of the noisy dividend yield process, $q = 0.03 = 3\%$. When adding a noisy dividend process to the adaptive system, the general features of the corresponding deterministic system (without the noise), such as the results (i)-(ii) above, still hold. However, it has a significant impact on the dynamics of the system, such as wealth and population distributions, autocorrelation of returns, volatility of returns and prices etc., as indicated below. In particular, the dynamics of the model is greatly affected by agents' behavior, which is measured by their extrapolation rate, $\bar{d}$, and switching intensity, β. The following discussion is focused on the dynamics of the system for various combinations of these two parameters $\bar{d}$ and β.

The following results are based on the parameters selected above, unless otherwise indicated.

- *Wealth distribution.*
  Wealth distribution is largely influenced by agents' extrapolation and strategy switching activity. It is found that, in general, a strong extrapolation leads type 1 agents (with lag 3) accumulate more wealth than type 2 agents do. In other words, type 1 trading strategy(with lag of 3) is more profitable than type 2 (with lag of 5) under the noisy dividend process. Furthermore, as the switching intensity β increases, the profitability of type 1 trading strategy is improved significantly. This result is unexpected and interesting. This result is optimal in terms of the initial wealth and population distributions.
  - *Effect of the initial wealth distribution.*
    When the wealth and population are evenly distributed across the two types of agents initially (i.e. $\bar{w}_o = 0, n_o = 0$), on average, type 1 agents accumulate more wealth (about 5% to 6%) than type 2 agents over the whole period, as indicated by the time series plot for the wealth ($\bar{w}_t$) in Figure 4.1. Also, as the extrapolations rate increases (i.e. as $\bar{d}$ increases), type 1 agents accumulate more wealth than type 2 agents (say, about 2% to 3% more for $\bar{d} = 0.5$, compared to 5-6% more for $\bar{d} = 0.53$). This suggests that, when both types of agents start with the same level of wealth and have the same number of traders, agents using short memory length accumulate more wealth than ones using long memory span. In other words, type 1 trading strategy is more profitable under the noisy dividend process. This result still holds when the initial wealth is not so evenly distributed. However, on average, when type 1 agents start with more wealth than type 2 (say $\bar{w}_o = 0.2$, that is type 1 agents have 20% more initial wealth than type 2 agents do on average), the prices can be pushed immediately to very high levels so that any further trend chasing from type 1 agents can cause price to overshoot, leading an explosion of price.
  - *Effect of the initial population distribution*
    For a fixed initial wealth proportion $\bar{w}_o$ (say $\bar{w}_o = 0$) and a range of n_o (say, $n_o \in (-1, 0.5)$), $\bar{w}_t$ increases in t. But for large n_o (say $n_o = 0.6$), the prices are pushed to explosion. This indicates that type 1 agents accumulate more wealth over the period, even when the population of type

2 agents is high initially. However, an initial over concentration of type 1 agents can lead to overshooting of price.

- *Herd behavior.*
  Herd behavior is measured by the population proportion difference n_t and the switching intensity parameter β. For $\beta = 0$, there is no switching between the two trading strategies. However, when agents are allowed to switch (i.e., $\beta > 0$), as indicated by the time series plot for the population (n_t) in Figure 4.1, agents switch between two strategies frequently. In general, because of the profitability of type 1 strategy, more agents switch from type 2 to type 1, as indicated by the mean and standard deviation of the population n_t in Table A.2.1. Also, as the switching intensity β increases, the frequency of such switching increases too. Furthermore, as β increases, both prices and returns become more volatile, as indicated by the time series plots on returns (ρ_t) and prices (p_t) in Figure 4.1.
- *Excess volatility and volatility clustering.*
  As indicated by the time series plot of returns ρ_t in Figure 4.1, adding the noisy dividend process causes an otherwise stable return series to fluctuate. This fact itself is not unexpected. What is of interest is the contrast between the simply normally distributed dividend process that is input to the system and the return process that is the output of the system. With the increase of either the standard derivation of the noise process q, or agents extrapolation rate $\bar{d}$, or switching intensity β, both returns and prices become more volatile. Moreover, volatility clustering is also observed.
- *Autocorrelation.*
  Significant positive autocorrelation (AC) for lags 1 and 2, negative for lags 3 to 8, positive for lags 9-14, are founded, as indicated by Table A.2.4. However, as lag length increases, the ACs become less significant.
- *Overshooting.*
  Related simulations indicate that either strong extrapolation (corresponding to high $\bar{d}$), or high volatility of the dividend yield process (q), or high switching density (β) can cause price to overshoot and lead to price explosion. Numerical simulations also show that, to avoid price overshooting, a minimum level of risk premium ($\bar{\delta}$) is required.

4.2. **Other Lag Length Combinations.** How the above results are affected by different lag length combinations is interesting and important, and is addressed in the following.

- $(L_1, L_2) = (3, 7)$: The general dynamic features are similar to the case when $(L_1, L_2) = (3, 5)$, except for the following differences.
  - The underlying deterministic system is stable over a wide range of extrapolation rates $\bar{d} \in [0, d^*)$. $d^* \approx 0.57$ for $L_2 = 5$ and $d^* \approx 0.773$ for $L_2 = 7$.
  - The trading strategy with short lag dominates the market. Similar impacts of initial wealth and population distributions, the switching intensity, and the standard derivation of the noisy process on the returns and wealth dynamics are also observed over a wider range of the parameters and initial values. However, compared with the previous case, for the same set of

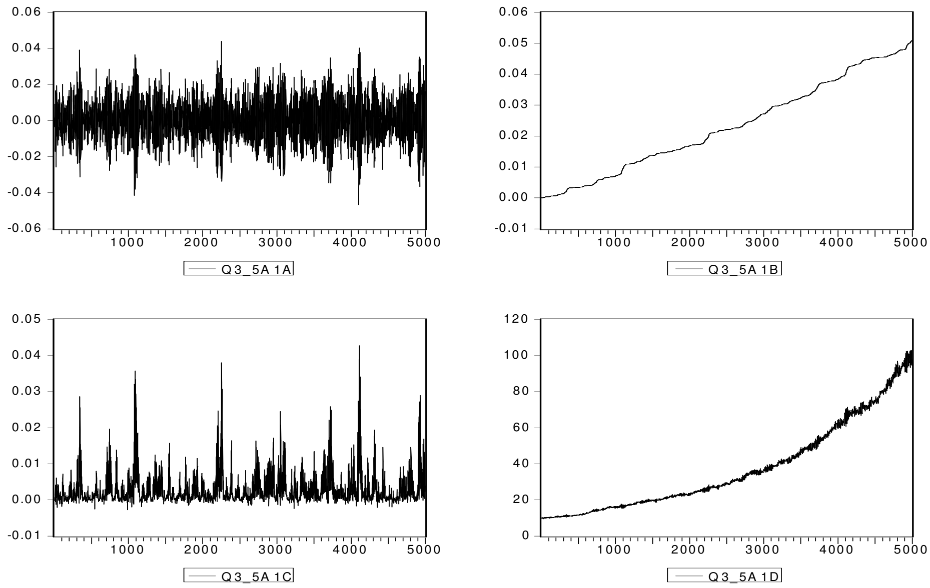

FIGURE 4.1. Time series plots for returns (top left), wealth (top right) and population (bottom left) distributions, and prices (bottom right) when the same momentum trading strategies with different lags $(L_1, L_2) = (3, 5)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = 0.53$ and $q = 0.03$.

parameters and initial values, both the profitability and herd behavior increase, as indicated by the time series plots for wealth and population in Figure A.2.1 and the corresponding statistical result in Table A.2.1.
- ACs are significantly positive for lags 1 and 2, either positive or negative for lag 3, but not significantly, negative for lags 4 to 9, positive for lags 10 to 15, as indicated in Table A.2.4 .

• $(L_1, L_2) = (10, 14)$: Compared with the previous two cases, the following differences have been observed.
  - The upper bound d^* for returns of the underlying deterministic system to be stable increases to $d^* \approx 1.57$.
  - By adding the noisy process, the trading strategy with memory length 10 accumulates more wealth than the one with lag length 14. However, comparing with the previous cases, for the same set of parameters and initial values, the profitability and herd behavior of the strategy of lag 10 over the one with lag 14 is much less significant (at about 0.1% to 0.2%), as indicated by the time series plots in Figure A.2.2 and the corresponding statistical results in Table A.2.2; although an increase of extrapolation improves the profitability of the strategy with lag 10, as indicated by the time series plots in Figure 4.2 and the corresponding statistics in Table A.2.2 for $\bar{d} = 1.2$.

- Indicated by Table A.2.4, ACs oscillate and become less significant when agents extrapolate weakly (say, for $\bar{d} = 0.5$), but become more significant when agents extrapolate strongly (say, $\bar{d} = 1.1$).
- Comparing with the previous cases, there is less herd behavior. This is partially because of the less significant profitability of one strategy with lag 10 over the other with lag 14.

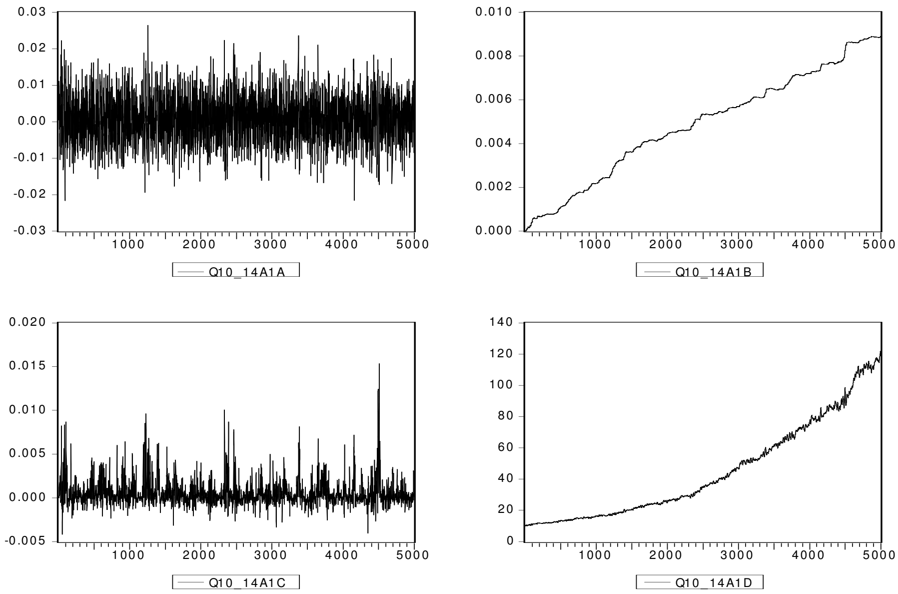

FIGURE 4.2. Time series plots for wealth and population distributions, returns and prices when the same momentum trading strategies with different lags $(L_1, L_2) = (10, 14)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = 1.2$ and $q = 0.03$.

- $(L_1, L_2) = (10, 26)$:
  - The upper bound d^* for returns of the underlying deterministic system to be stable increases to $d^* \in (2.2, 2.3)$.
  - By adding the noisy process, with the same parameters and initial values, profitability of type 1 trading strategy (with lag length 10) becomes questionable, as indicated by the time series plots in Figures A.2.3 and 4.3 and the corresponding statistics in Table A.2.3.
  - As demonstrated by Table A.2.4, the ACs have less patterns and do not die out as lags increase.

4.2.1. *Summary.* In summarizing, we obtain the following results when both types of agents follow the same momentum trading strategy, but with different memory lengths.

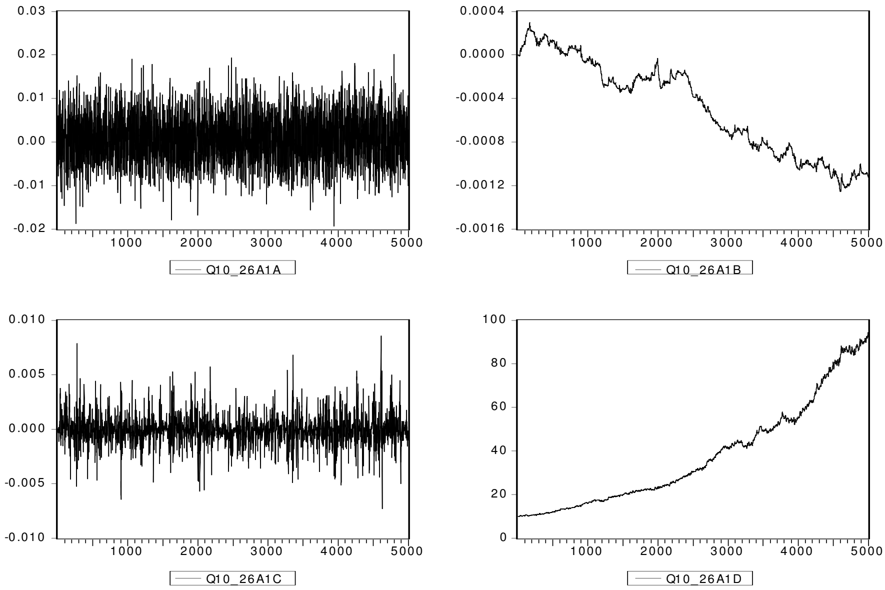

FIGURE 4.3. Time series plots for wealth and population distributions, returns and prices when the same momentum trading strategies with different lags $(L_1, L_2) = (10, 26)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = 1.2, q = 0.03$ and $\bar{w}_o = 0.6$.

- Without the noisy process, an increase in lag length from either one of the trading strategies stabilizes the return series of the underlying deterministic system, and enlarges the range of the extrapolation. However, for the same set of parameters, the profitability of the trading strategies and herd behavior become less significant.
- Adding the noisy dividend process in general improves the profitability of the trading strategies with short lag length L_1 (say, $(L_1, L_2) = (3, 5)$ and $(3, 7)$). However, such profitability becomes less significant when the short lag length L_1 increase, and is even questionable (say $(L_1, L_2) = (10, 14)$ and $(10, 26)$).
- When trading strategies become profitable, agents tend to adopt a herd behavior — more agents switch to the more profitable strategy over the time period. However, over concentration (in terms of the initial average wealth, population proportion), or over extrapolation (in terms of high extrapolation rates and improper risk premium levels) can cause overshooting of price and push prices to explosion, leading to a market crash.
- Momentum trading strategies can push the prices to a very high level and lead the returns to be more volatile, exhibiting volatility clustering.
- ACs follow certain patterns when one of the trading strategies becomes profitable and die out as lags increase. However, such patterns become insignificant

when the profitability of the trading strategy becomes less significant, and even questionable.

- Price levels are more determined by the risk premium levels rather than other parameters (say extrapolation rate and switching intensity).

5. WEALTH DYNAMICS OF CONTRARIAN TRADING STRATEGIES

In this section, we consider the quasi-homogeneous model with $d_1 = d_2 = d < 0$ and $1 \leq L_1 < L_2$, that is both types of agents follow the same contrarian trading strategy except for having different memory lengths. As discussed in Section 3, some empirical studies suggest that contrarian trading strategies are more profitable over long periods. Our results in this section provide some support for this view and show that the adaptive model presented in this paper is capable of characterizing some features founded in empirical studies. Furthermore, similar to the previous section, wealth and population distributions, statistical properties of returns (such as volatility clustering, autocorrelations), and herd behavior are discussed.

5.1. **Case:** $(L_1, L_2) = (3, 5)$**.** With the selection of the parameters and initial values in (3.8)-(3.9), consider first in the next subsection the dynamics of the underlying deterministic system, that is, when $q = 0$. The impact of the noisy processes on the dynamics is then studied in the subsequent subsection.

5.1.1. *No-noise Case.* Let $q = 0$. For $\bar{d} = -0.4$, initial difference of population proportions $n_o = 0$ and any initial wealth proportion $\bar{w}_o$, it is found that

$$\rho_t \to \rho^* = 15.45\% \quad \text{(annual)}, \qquad \bar{w}_t \to \bar{w}_o, \quad n_t \to 0.$$

By changing parameters and initial values, the following results are obtained.

- *Risk premium and over-pricing*
  It is found that, ceteris paribus, for $\bar{\delta} = \delta/\sigma^2 = 0.4$, $\rho_t \to \rho^* = 11.52\%$, while for $\bar{\delta} = 0.6$, $\rho^* = 15.45\%$. In general, a high level of risk-adjusted premium leads to a high return and a high price correspondingly. In fact, for the given parameters, there exists $\delta_o \in (0.6, 0.7)$, a so called *bifurcation* value, such that the returns converge to fixed values for $\bar{\delta} < \delta_o$ and diverge for $\bar{\delta} > \delta_o$, leading prices to explode.
- *Over-reaction and price shooting*
  Based on the parameters selected, there exists $d_o \in (-0.53, -0.52)$ such that, ceteris paribus, returns converge to fixed values for $(0 >)\bar{d} > d_o$ and diverge for $\bar{d} < d_o$, leading prices to overshoot. Like the momentum trading strategies, over-extrapolation from contrarian trading strategies also cause overshooting of prices.
- *Wealth distribution*
  Unlike the case of the momentum trading strategies, wealth distributions of the deterministic system are affected differently by the extrapolation rate $\bar{d}$, switching intensity, initial wealth and population distributions.
  - In general, as $\bar{d}(< 0$ decreases and is near the bifurcation value, the profitability of trading strategy with long lag ($L = 5$) is improved significantly, say from 5% for $\bar{d} = -0.454$, to 25% for $\bar{d} = -0.48$, and to 50% for $\bar{d} = -0.5$. However, for fixed $\bar{d} < 0$, say $\bar{d} = -0.5$, as β increases, the profitability of trading strategy 2 becomes less significantly, say from

45% for $\beta = 0.1$ to 20% for $\beta = 2$. This is different from the case of using momentum trading strategy.

  - For fixed $n_o \neq 0$ and a range of $\bar{w}_o$ (say, $n_o = 0.3$ and $\bar{w}_o \in (-0.5, 0.5)$),

$$\rho_t \to \rho^* = 15.45\% \quad \text{(annual)}, \qquad \bar{w}_t \to \bar{w}_o - \epsilon, \quad n_t \to 0$$

    with $\epsilon \approx 10^{-6}$. This implies that agents' wealth are distributed according to their initial wealth distribution, although populations are not evenly distributed initially.
  - For fixed $\bar{w}_o < 0$ (say, $\bar{w}_o = -0.3$) and $n_o \in [-1, 1]$, type 2 agents accumulated more wealth than type 1 agents over a very short period, but the difference is not significant (about 1%). In other words, when the initial average wealth for type 2 agents is more than average wealth for type 1 agents, no one of the contrarian trading strategies can make significant profit over the other, no matter how the initial populations are distributed.
  - For fixed $\bar{w}_o > 0$ (say, $\bar{w}_o = 0.3$) and $n_o \in [-1, 1]$, type 2 agents accumulated more wealth than type 1 agents over a very short period, and the difference becomes more significant (up to 37%) as more agents use type 2 trading strategy initially. This implies that, when the initial average wealth of type 1 agents is higher than the one of type 2 agents, agents using contrarian strategy with long memory length ($L_2 = 5$) are able to accumulate more wealth over a very short period than agents using the same strategy but with short memory length ($L_1 = 3$). In addition, the profitability becomes more significant when there are more agents using the strategy with long memory length initially. This is different from the case when agents use momentum strategies.

- Herd behavior
  The dynamics have no significant difference for different switching intensity parameter β when the returns of the underlying deterministic system is stable. However, as $\bar{d}$ near the bifurcation value, herd behavior is also observed.

5.1.2. *Effect of Noise.* Let the annualized standard derivation of the noisy dividend yield process $q = 3\%$. The following results are based on the parameters selected above, unless the difference is indicated.

- *Wealth distribution*

  - *Effect of the initial wealth distribution* — When the wealth and population are evenly distributed among two types of trading strategies initially (i.e. $\bar{w}_o = 0, n_o = 0$), type 2 agents accumulate more average wealth (about 5% to 7%) more than type 2 agents over the whole period, as indicated in Figure 5.1 for the time series plots on wealth and population. Also, as extrapolations increase (i.e. as $\bar{d}$ decreases), such extrapolations help type 2 agents accumulate more wealth than type 1 agents (say, about 5% for $\bar{d} = -0.45$, and about 45% for $\bar{d} = -0.5$). This suggests that, when both types of agents start with the same level of wealth and have equal number of traders, agents using long memory length ($L_2 = 5$) accumulate more wealth than ones using short memory length ($L_1 = 3$). In other words, type 2 agents benefit significantly from the noisy dividend noisy process.

This result still holds when the initial wealth are not so evenly distributed (say, $\bar{w}_o \in (-0.6, 0.3)$ for $\bar{d} = -0.5$).

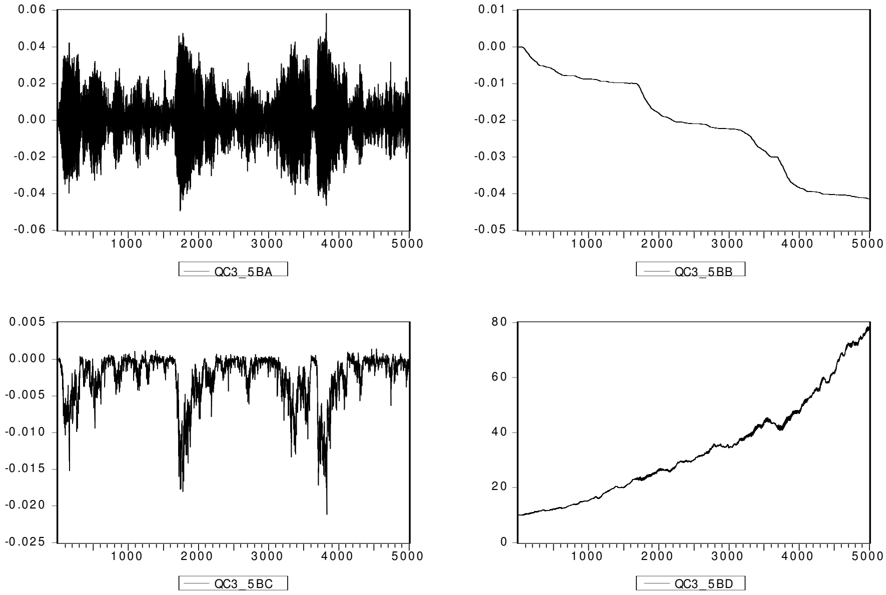

FIGURE 5.1. Time series plots for wealth and population distributions, returns and prices when the same contrarian trading strategies with different lags $(L_1, L_2) = (3, 5)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = -0.45$ and $q = 0.03$.

– *Effect of the initial population distribution* —Similar to the case without noise, the wealth distribution is affected differently as a function of different initial wealth levels. For fixed $\bar{w}_o < 0$ and a range of n_o (say, $\bar{w}_o = -0.3$ and $n_o \in (-0.8, 0.65)$), the profitability of trading strategy 2 does not change much for different n_o. However, for fixed $\bar{w}_o > 0$ and a range of n_o (say, $\bar{w}_o = 0.3$ and $n_o \in (-0.9, 0.9)$), the profitability of trading strategy 2 increases significantly as more and more agents use trading strategy 2. Price overshooting is possible when the populations are over concentrated in use of one of the trading strategies.

• *Herd behavior*

Herd behavior is also observed for changing parameter β. Given the profitability of the trading strategy over the long memory span, more agents tend to switch to this more profitable strategy, as indicated by the time series plot of population in Figure 5.1 and the corresponding statistic in Table A.2.5. Furthermore, as β increases, both prices and returns become more volatile, leading to excess volatility.

- *Excess volatility and volatility clustering*
  The addition of a noisy process cause an otherwise stable return series to be exhibit fluctuations. Similar to the case of using momentum trading strategies, an increase of either the standard derivation of the dividend yield noisy process q, or agents extrapolation $\bar{d}$, leads both returns and prices to be more volatile. Moreover, volatility clustering is also observed, as illustrated by the time series plot on the return in Figure 5.1 and the corresponding statistics in Table A.2.5.
- *Autocorrelation*
  ACs are significantly negative for odd lags and positive for even lags for all lags, as indicated in Table A.2.6.
- *Overshooting* — Similar to the momentum trading strategies discussed in Section 4, the noisy process has a significant impact on prices. An increase of q can push prices to significantly high levels. This can also result from either strong extrapolation (corresponding to low $\bar{d}$), or high risk premia $\bar{\delta}$, or high switching density β and causes prices to explode.

5.2. **Other Cases.**

- $(L_1, L_2) = (3, 7)$: The general dynamic features are similar to the case when $(L_1, L_2) = (3, 5)$, except for the differences indicated below.
  - The underlying deterministic system is stable over a wide range of extrapolation rates $\bar{d} \in (d^*, 0]$ with $d^* \in (-0.53, -0.52)$ for $L_2 = 5$, while with $d^* \in (-0.65, -0.6)$ for $L_2 = 7$.
  - Similar to the previous case, the trading strategy with long lag $L_2 = 7$ dominates the market,in particular, when $\bar{d}$ is near the bifurcation value. However, for the same set of parameters, compared with the case of $L_2 = 5$, the profitability is reduced slightly. On the other hand, agents can extrapolate over a wide range (of the parameter $\bar{d}$). Similar impacts of initial wealth and population distributions, the switching intensity, and the standard derivation of the noisy process on the dynamics can be observed over a wider range of the parameters.
- $(L_1, L_2) = (10, 14)$:
  - The lower bound d^* for returns of the underlying deterministic system to be stable decreases to $d^* \in (-1.6, -1.5)$. By adding the noisy process, with the same parameters and initial values, the profitability of type 1 trading strategy (with lag length 14) becomes questionable, as indicated by the time series plots of the wealth in Figure 5.2 and the corresponding statistics in Table A.2.5.
  - The patterns of the ACs are maintained, but they become less significant (for the same parameter $\bar{d} = -0.45$), as shown in Table A.2.6.
  - Compared with the previous cases, there is less herd behavior, as illustrated by the time series plot for the population in Figure 5.2 and the corresponding statics in Table A.2.5. This is partially because of the less significant (even no) profitability of the strategy with lag 14 over the other with lag 10.
- $(L_1, L_2) = (10, 26)$: The following differences have been observed.
  - In this case the lower bound d^* (on $\bar{d}$) such that returns of the underlying deterministic system be stable decreases to $d^* \in (-2.2, -2.1)$. By adding the noisy process, the trading strategy with memory length 26 accumulates

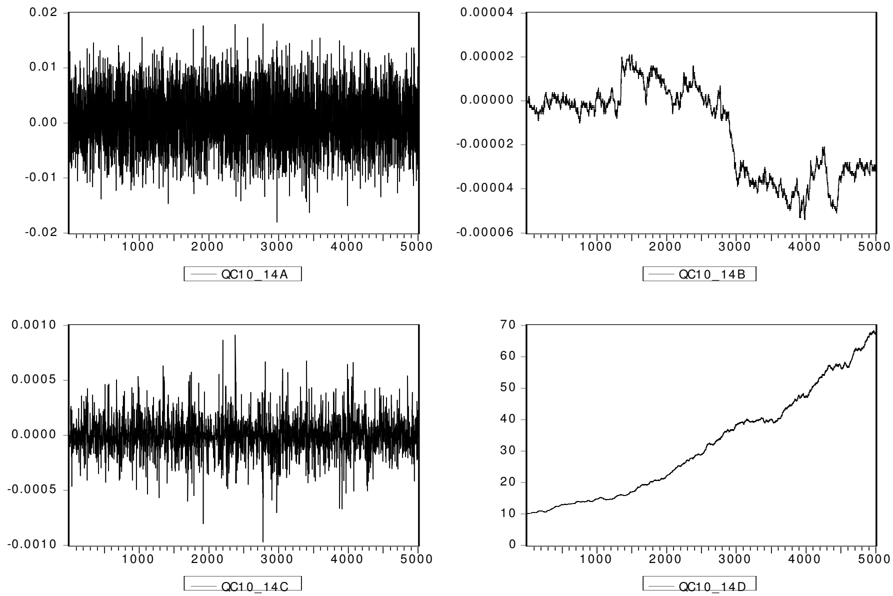

FIGURE 5.2. Time series plots for wealth and population distributions, returns and prices when the same contrarian trading strategies with different lags $(L_1, L_2) = (10, 14)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = -0.45$ and $q = 0.03$.

more wealth than the one with lag length 10, as shown in Figure 5.3 and Table A.2.5. However, comparing with the previous cases $(L_1, L_2) = (3, 5), (3, 7)$, the profitability of the strategy of lag 26 over the one with lag 10 is much less significant (at about 0.01% to 0.04% more for $\bar{d} = -0.45$), although a strong extrapolation can improve the profitability of the strategy with lag 26 (at about 5% to 7% for $\bar{d} = -2.0$).

– The ACs become less significant when agents extrapolate weakly (say, $\bar{d} = -0.45$), as indicated in Table A.2.6 , and more significant when agent extrapolate strongly (say, $\bar{d} = -2.0$).

5.2.1. *Summary.* In summarizing, we obtain the following results when both types of agents follow the same contrarian trading strategy, but with different memory lengths.

– Without the noisy process and a given set of parameters, an increase in lag lengths of the trading strategies stabilizes the return series of the underlying deterministic system. As both $\bar{d}$ and β are near their bifurcation values, the profitability of trading strategies and herd behavior are observed, in general.
– Adding a noisy dividend process, in general, improves the profitability of the trading strategies with long lag lengths (say, $L_2 = 5, 7, 26$). However,

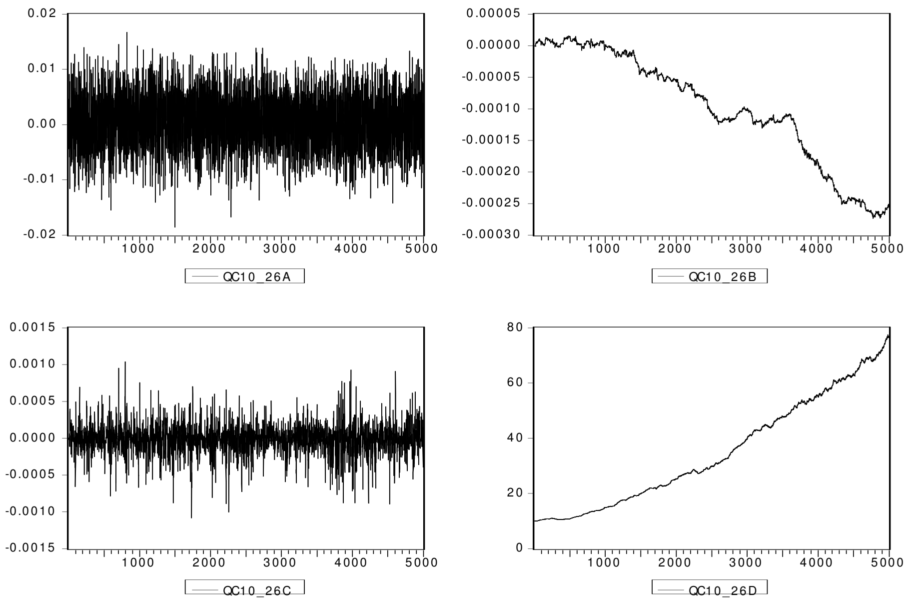

FIGURE 5.3. Time series plots for wealth and population distributions, returns and prices when the same contrarian trading strategies with different lags $(L_1, L_2) = (10, 26)$ are used. Here, $\bar{\delta} = 0.5, \bar{d} = -0.4$ and $q = 0.01$.

such profitability becomes less significant when the relative difference between the two lag lengths is small (say, $L_1 = 10, L_2 = 14$).

– Similar to the case of using momentum trading strategies, herd behavior is observed when one of the trading strategies becomes (significantly) profitable. Also, over-concentration (in terms of the initial average wealth and population proportion), or over extrapolation (in terms of low extrapolation rates and improper risk premium levels) can cause overshooting of price and push prices to explode, leading to a market crash. Price levels are more determined by the risk premium levels than other parameters (say extrapolation rate and switching intensity).
– The ACs are significantly negative for odd lags and positive for even lags when the memory lengths of the contrarian trading strategies are small. However, they become less significant when the memory lengths increase.

6. CONCLUSIONS

To incorporate investor psychology into the standard asset pricing theory in finance and characterize the interaction of heterogeneous agents, an adaptive model on asset pricing and wealth dynamics with agents using various trading strategies is developed.

As a special case, a quasi-homogeneous model of two types of agents using either momentum or contrarian trading strategies is introduced to analyze the profitability of the trading strategies over different time intervals. It is found that agents with different time-horizon coexist. Our results shed light on the empirical finding that momentum trading strategies are more profitable over short time intervals, while contrarian trading strategies are more profitable over long time intervals. It should be pointed out that this is an *unexpected* result from the set up of the adaptive model. Even though the quasi-homogeneous model is one of the simplest cases of the adaptive model, it generates various phenomena observed in financial markets, including rational adaptiveness of agents, overconfidence and underreaction, overreaction and pricing shooting, herd behavior, excess volatility, and volatility clustering. The model also displays the essential characteristics of the standard asset price dynamics model assumed in continuous time finance in that the asset price is fluctuating around a geometrically growing trend.

Our analysis in this paper is based on a simplified quasi-homogeneous model and further analysis on the adaptive model is necessary to explore the potential explanatory power of the model. One of the extensions is to consider models of two or three different types of trading strategies, to analyze the profitability of different trading strategies, and to examine the stylized facts of the return distribution. Secondly, the attitudes of investors towards the extrapolation and risk premium change when the market environment changes and this change should be made endogenous. Thirdly, there should be a more extensive simulation study of these richer models once they are developed. In fact a proper Monte-Carlo analysis is required to determine whether the models can generate with a high frequency the statistical characteristics of major indices such as the S&P500. These extensions are interesting problems which are left to future research work.

Appendix A.1. Proof of Proposition 2.1

Proof. It follows from (2.2) and (2.6) that

$$\bar{w}_{i,t} = \frac{\bar{W}_{i,t}}{\bar{W}_t} = \frac{\bar{W}_{i,t-1}[R + (\rho_t - r)\bar{\pi}_{i,t-1}]}{\bar{W}_t} = \frac{\bar{w}_{i,t-1}[R + (\rho_t - r)\bar{\pi}_{i,t-1}]}{\bar{W}_t/\bar{W}_{t-1}}. \tag{A.1.1}$$

Note that

$$\frac{\bar{W}_t}{\bar{W}_{t-1}} = \frac{\sum_{j=1}^h \bar{W}_{j,t}}{\bar{W}_{t-1}} = \sum_{j=1}^h \bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}]. \tag{A.1.2}$$

Then both (A.1.1) and (A.1.2) lead to (2.8).

With the notations introduced in Section 2, the market clearing equilibrium price equation (2.5) can be rewritten as:

$$\sum_{j=1}^h n_{j,t}\bar{\pi}_{j,t}\bar{W}_{j,t} = Np_t/H. \tag{A.1.3}$$

Note that

$$W_t = \sum_{j=1}^H W_{j,t} = \sum_{j=1}^h \ell_{j,t}\bar{W}_{j,t} = H\sum_{j=1}^h n_{j,t}\bar{W}_{j,t}. \tag{A.1.4}$$

It follows from (A.1.3) and (A.1.4) that the market clearing price equilibrium equation (A.1.3) becomes

$$W_t \sum_{j=1}^h n_{j,t}\bar{\pi}_{j,t}\bar{w}_{j,t} = Np_t \sum_{j=1}^h n_{j,t}\bar{w}_{j,t}, \tag{A.1.5}$$

From (A.1.5)

$$\frac{W_t}{W_{t-1}} \frac{\sum_{j=1}^h n_{j,t}\bar{\pi}_{j,t}\bar{w}_{j,t}}{\sum_{j=1}^h n_{j,t-1}\bar{\pi}_{j,t-1}\bar{w}_{j,t-1}} = (1 + \rho_t - \alpha_t)\frac{\sum_{j=1}^h n_{j,t}\bar{w}_{j,t}}{\sum_{j=1}^h n_{j,t-1}\bar{w}_{j,t-1}}. \tag{A.1.6}$$

Note that

$$\frac{W_t}{W_{t-1}} = \frac{\sum_{j=1}^h n_{j,t}\bar{W}_{j,t}}{\sum_{j=1}^h n_{j,t-1}\bar{W}_{j,t-1}} = \frac{\sum_{j=1}^h n_{j,t}\bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}]}{\sum_{j=1}^h n_{j,t-1}\bar{w}_{j,t-1}}. \tag{A.1.7}$$

Substituting (A.1.7) into (A.1.6),

$$\sum_{j=1}^h n_{j,t}\bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}] \sum_{j=1}^h n_{j,t}\bar{\pi}_{j,t}\bar{w}_{j,t} = (1 + \rho_t - \alpha_t)\sum_{j=1}^h n_{j,t}\bar{w}_{j,t}\sum_{j=1}^h n_{j,t-1}\bar{\pi}_{j,t-1}\bar{w}_{j,t-1}. \tag{A.1.8}$$

Also, using (2.8),

$$\sum_{j=1}^h n_{j,t}\bar{\pi}_{j,t}\bar{w}_{j,t} = \frac{\sum_{j=1}^h n_{j,t}\bar{\pi}_{j,t}\bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}]}{\sum_{k=1}^h \bar{w}_{k,t-1}[R + (\rho_t - r)\bar{\pi}_{k,t-1}]}, \tag{A.1.9}$$

$$\sum_{j=1}^h n_{j,t}\bar{w}_{j,t} = \frac{\sum_{j=1}^h n_{j,t}\bar{w}_{j,t-1}[R + (\rho_t - r)\bar{\pi}_{j,t-1}]}{\sum_{k=1}^h \bar{w}_{k,t-1}[R + (\rho_t - r)\bar{\pi}_{k,t-1}]}. \tag{A.1.10}$$

Substitution of (A.1.9) and (A.1.10) into (A.1.8) and simplification of the corresponding expression leads to equation

$$\sum_{j=1}^{h} n_{j,t}\bar{w}_{j,t-1}\bar{\pi}_{j,t}[R+(\rho_t-r)\bar{\pi}_{j,t-1}]$$
$$=[(\rho_t-r)+(1+r-\alpha_t)]\bigg(\sum_{j=1}^{h} n_{j,t-1}\bar{\pi}_{j,t-1}\bar{w}_{j,t-1}\bigg). \qquad \text{(A.1.11)}$$

Solving for ρ_t from (A.1.11), one obtains equation (2.9) for the return ρ_t. □

Appendix A.2. TIME SERIES PLOTS, STATISTIC AND AUTOCORRELATION RESULTS

For both momentum and contrarian trading strategies with different combinations of lag lengths (L_1, L_2), this appendix provide

- Time series plots for wealth ($\bar{w}_t$, the difference of wealth proportions), population (n_t, the difference of population proportions), returns (ρ_t), and prices (p_t);
- Numerical comparative statics for wealth (WEA), population (POP), and returns (RET);
- Autocorrelation coefficients (AC) for return series with lags from 1 to 36.

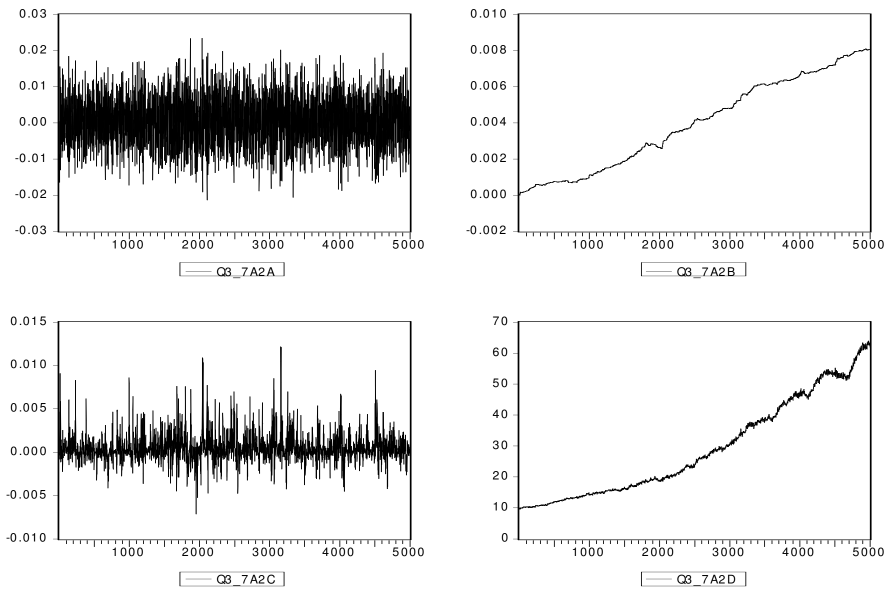

FIGURE A.2.1. Time series plots for wealth and population distributions, returns and prices when the same momentum trading strategies with different lags $(L_1, L_2) = (3, 7)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = 0.53$ and $q = 0.03$.

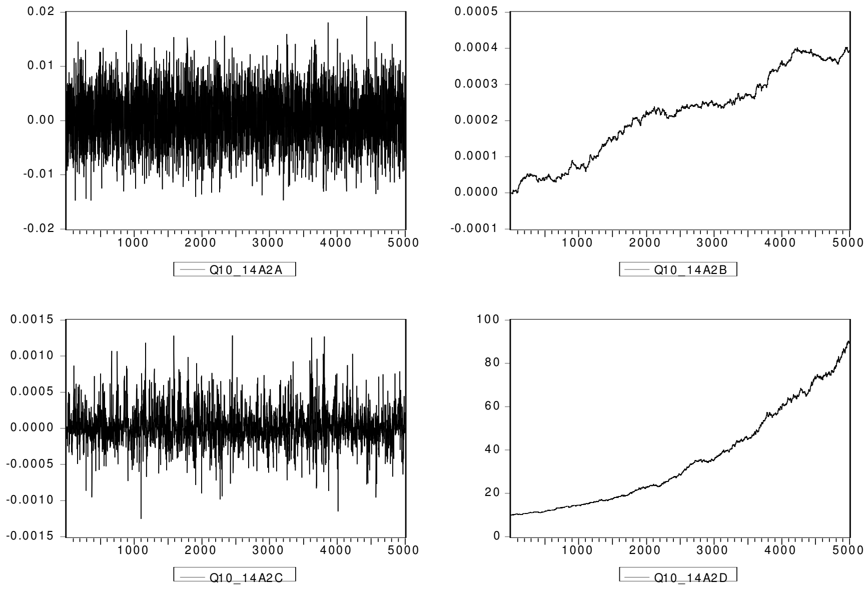

FIGURE A.2.2. Time series plots for wealth and population distributions, returns and prices when the same momentum trading strategies with different lags $(L_1, L_2) = (10, 14)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = 0.53$ and $q = 0.03$.

		(3, 5)				(3, 7)	
	RET	WEA	POP		RET	WEA	POP
Mean	0.000721	0.023248	0.003062		0.000525	0.003973	0.000484
Median	0.000691	0.022135	0.001505		0.000542	0.004138	0.000219
Maximum	0.043964	0.050955	0.042773		0.023386	0.008098	0.012162
Minimum	-0.046662	-1.00E-06	-0.002759		-0.021407	-4.00E-06	-0.00715
Std. Dev.	0.011343	0.014838	0.004475		0.006339	0.002526	0.001512
Skewness	-0.020851	0.155153	2.998613		0.023471	0.032782	1.6045
Kurtosis	3.241269	1.815871	15.84896		2.9332	1.589621	10.42554
Jarque-Bera	12.49201	312.2396	41896.41		1.389	415.389	13635.28
Probability	0.001938	0	0		0.499324	0	0

TABLE A.2.1. Statistics of time series of wealth, population and returns for momentum trading strategies with $(L_1, L_2) = (3, 5), (3, 7)$ and $\bar{\delta} = 0.6, \bar{d} = 0.53$ and $q = 0.03$.

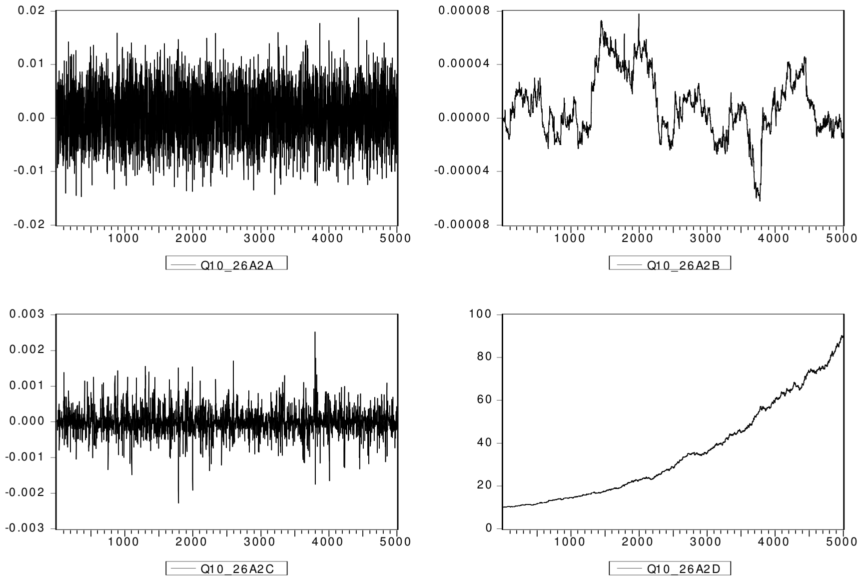

FIGURE A.2.3. Time series plots for wealth and population distributions, returns and prices when the same momentum trading strategies with different lags $(L_1, L_2) = (10, 26)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = 0.53$ and $q = 0.03$.

		(10, 14)(a)				(10, 14)(b)	
	RET	WEA	POP		RET	WEA	POP
Mean	0.000641	0.000215	2.36E-05		0.000739	0.004879	0.000536
Median	0.000682	0.000233	6.00E-06		0.000685	0.005311	0.000173
Maximum	0.019241	0.000403	0.001285		0.02642	0.008898	0.015336
Minimum	-0.014757	-3.00E-06	-0.001252		-0.021672	0	-0.004178
Std. Dev.	0.004793	0.000118	0.000232		0.006001	0.002481	0.001424
Skewness	-0.016961	-1.52E-01	0.506932		0.069921	-0.211947	2.629638
Kurtosis	3.018244	1.900675	6.319931		3.119651	2.01417	15.73849
Jarque-Bera	0.309123	270.9888	2510.89		7.058159	239.9537	39576.5
Probability	0.856791	0	0		0.029332	0	0

TABLE A.2.2. Statistics of time series of wealth, population and returns for momentum trading strategies with $(L_1, L_2) = (10, 14)$ and $\bar{\delta} = 0.6, q = 0.03$, $\bar{d} = 0.53$ for (a) and $\bar{d} = 1.2$ for (b).

		(10, 26)(a)				(10, 26)(b)	
	RET	WEA	POP		RET	WEA	POP
Mean	0.000641	8.66E-06	-7.90E-07		0.000651	-0.000483	-6.68E-05
Median	0.000678	5.00E-06	-1.30E-05		0.000521	-0.000365	-8.70E-05
Maximum	0.01877	7.80E-05	0.002522		0.020166	0.000299	0.008564
Minimum	-1.48E-02	-6.20E-05	-0.002285		-0.019343	-0.001257	-0.007335
Std. Dev.	0.00473	2.33E-05	0.000328		0.005295	0.000433	0.001248
Skewness	-2.04E-02	0.217143	0.159003		0.090161	-0.091597	0.23871
Kurtosis	3.001559	3.132038	7.858779		2.949879	1.620711	7.002653
Jarque-Bera	0.346695	42.93333	4940.334		7.298971	403.4138	3385.919
Probability	0.840845	0	0		0.026004	0	0

TABLE A.2.3. Statistics of time series of wealth, population and returns for momentum trading strategies with $(L_1, L_2) = (10, 26)$, $q = 0.03$, and $\bar{\delta} = 0.6, \bar{d} = 0.53$ for (a), and $\bar{\delta} = 0.6, \bar{d} = 1.2$ for (b).

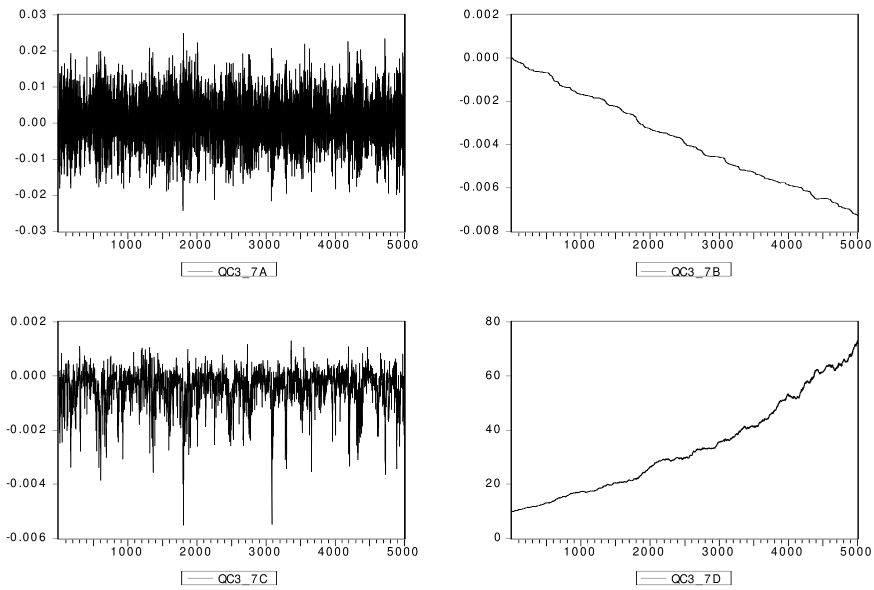

FIGURE A.2.4. Time series plots for wealth and population distributions, returns and prices when the same contrarian trading strategies with different lags $(L_1, L_2) = (3, 7)$ are used. Here, $\bar{\delta} = 0.6, \bar{d} = -0.45$ and $q = 0.03$.

Lag	(3,5)	(3,7)	(10,14)	(10, 14)(a)	(10, 26)	(10, 26)(a)
1	0.775	0.567	0.21	0.519	0.17	0.369
2	0.412	0.265	0.063	0.293	0.048	0.157
3	-0.04	0.009	0.025	0.178	0.019	0.076
4	-0.492	-0.317	-0.015	0.146	-0.023	0.041
5	-0.753	-0.366	-0.018	0.105	-0.021	0.025
6	-0.824	-0.314	0.006	0.06	0.006	-0.016
7	-0.643	-0.257	0.01	0.005	0.01	-0.03
8	-0.287	-0.182	-0.009	-0.036	-0.009	-0.048
9	0.118	-0.057	0.002	-0.1	0.005	-0.075
10	0.489	0.037	-0.017	-0.191	-0.013	-0.113
11	0.696	0.108	-0.095	-0.35	-0.094	-0.284
12	0.704	0.189	-0.044	-0.308	-0.033	-0.202
13	0.514	0.185	-0.016	-0.258	-0.005	-0.094
14	0.187	0.135	-0.026	-0.239	-0.005	-0.037
15	-0.176	0.084	-0.114	-0.304	-0.029	-0.009
16	-0.478	-0.007	-0.041	-0.245	-0.002	0.017
17	-0.628	-0.065	-0.016	-0.167	0	0.034
18	-0.602	-0.077	-0.007	-0.106	-0.002	0.03
19	-0.411	-0.084	0.019	-0.05	0.017	0.028
20	-0.114	-0.084	0.015	-0.028	0.013	0.035
21	0.2	-0.057	-0.009	0.023	-0.012	0.045
22	0.448	-0.033	-0.007	0.09	-0.006	0.066
23	0.56	-0.022	-0.002	0.107	-0.002	0.063
24	0.509	0.018	-0.014	0.12	-0.017	0.053
25	0.322	0.038	0.013	0.15	0.009	0.014
26	0.05	0.04	0.031	0.191	0.006	-0.019
27	-0.227	0.065	0.018	0.192	-0.036	-0.106
28	-0.424	0.048	0.017	0.172	-0.001	-0.091
29	-0.499	0.018	-0.002	0.147	-0.011	-0.055
30	-0.428	0	0.014	0.115	0.001	-0.03
31	-0.241	-0.029	0.003	0.099	-0.004	-0.008
32	0.006	-0.044	0.036	0.054	0.034	0.008
33	0.241	-0.038	0.014	0.027	0.015	0.007
34	0.399	-0.033	-0.032	-0.003	-0.03	0.008
35	0.436	-0.03	-0.014	-0.016	-0.011	-0.015
36	0.348	0.001	-0.023	-0.041	-0.023	0.011

TABLE A.2.4. Autocorrelation coefficients (AC) of returns for momentum trading strategies with $(L_1, L_2) = (3,5), (3,7), (10,14)$ and $(10,26)$. The parameters are: $\bar{\delta} = 0.6, \bar{d} = 0.53$ and $q = 0.03$ for $(3,5), (3,7), (10,14)$ and $(10,26)$; $\bar{\delta} = 0.6, \bar{d} = 1.2$ and $q = 0.03$ for $(10,14)(a)$; and $\bar{\delta} = 0.6, \bar{d} = 1.2$ and $q = 0.03$ and $(10,26)(a)$.

		(3, 5)				(3, 7)	
	RET	WEA	POP		RET	WEA	POP
Mean	0.000732	-0.021357	-0.002489		0.000586	-0.003781	-0.000436
Median	0.000565	-0.020961	-0.00115		0.00061	-0.003947	-0.000253
Maximum	0.058371	1.00E-06	0.001417		0.024943	1.00E-06	0.001301
Minimum	-0.049512	-0.041469	-0.02122		-0.024361	-0.007294	-0.005522
Std. Dev.	0.0172	0.012664	0.003256		0.007258	0.002071	0.000648
Skewness	3.94E-02	-0.23075	-1.79741		-2.38E-02	0.128596	-1.706059
Kurtosis	2.623585	1.776767	6.184726		2.73569	1.77624	7.727226
Jarque-Bera	30.8166	356.1715	4806.214		15.02887	325.8435	7082.505
Probability	0	0	0		0.000545	0	0

		(10, 14)				(10, 26)	
	RET	WEA	POP		RET	WEA	POP
Mean	0.000546	-1.36E-05	-1.74E-06		0.000587	-9.89E-05	-1.53E-05
Median	0.000481	-5.00E-06	-5.00E-06		0.000674	-1.00E-04	-4.00E-06
Maximum	0.018059	2.10E-05	0.000912		0.016719	1.60E-05	0.001042
Minimum	-0.018072	-5.40E-05	-0.000972		-0.018608	-2.74E-04	-0.001083
Std. Dev.	0.004875	2.01E-05	0.000136		0.004787	8.84E-05	0.000182
Skewness	0.067356	-0.269859	0.137827		-0.084361	-0.498882	-0.30024
Kurtosis	3.073003	1.604405	7.232651		2.928112	2.074143	6.742584
Jarque-Bera	4.891911	466.5478	3748.941		7.008679	386.0658	2993.831
Probability	0.086643	0	0		0.030067	0	0

TABLE A.2.5. Statistics of time series of wealth, population and returns for contrarian trading strategies with parameters $\bar{\delta} = 0.6, \bar{d} = -0.45$ and $q = 0.03$ and lag length combinations (L_1, L_2) are (3, 5) for (a), (3, 7) for (b), (10, 14) for (c), and (10, 26) for (d).

Lag	(3,5)	(3,7)	(10,14)	(10, 26)
1	-0.944	-0.678	-0.144	-0.155
2	0.922	0.53	0.056	0.047
3	-0.914	-0.529	-0.014	-0.025
4	0.93	0.636	-0.01	0.003
5	-0.924	-0.563	-0.001	0.001
6	0.926	0.479	0	-0.003
7	-0.914	-0.462	0.016	0.01
8	0.909	0.533	-0.003	0.004
9	-0.902	-0.51	0.009	0.001
10	0.906	0.452	0.011	-0.003
11	-0.9	-0.428	0.097	0.107
12	0.896	0.451	-0.016	-0.039
13	-0.891	-0.427	0.026	0.009
14	0.889	0.393	-0.013	0.006
15	-0.884	-0.371	0.051	-0.027
16	0.882	0.37	-0.017	0.009
17	-0.876	-0.366	0.014	-0.004
18	0.875	0.351	-0.013	0.013
19	-0.869	-0.337	0.009	-0.001
20	0.867	0.34	-0.008	-0.01
21	-0.863	-0.322	0.01	0.004
22	0.861	0.297	0.025	-0.001
23	-0.858	-0.282	-0.008	-0.016
24	0.857	0.277	0.016	-0.014
25	-0.851	-0.271	-0.02	0
26	0.849	0.245	0.012	-0.002
27	-0.844	-0.232	-0.002	0.043
28	0.844	0.228	0.014	-0.014
29	-0.841	-0.224	-0.001	0.021
30	0.838	0.215	-0.002	-0.019
31	-0.833	-0.202	-0.019	0.013
32	0.83	0.193	-0.01	-0.009
33	-0.828	-0.186	-0.012	0.007
34	0.825	0.181	0.021	0.001
35	-0.82	-0.162	0.01	0.008
36	0.817	0.162	0.001	0.01

TABLE A.2.6. Autocorrelation coefficients (AC) of returns for contrarian trading strategies with parameters $\bar{\delta} = 0.6, \bar{d} = -0.45, \beta = 0.5$ and $q = 0.03$ and lag length combinations $(L_1, L_2) = (3,5), (3,7), (10,14)$ and $(10,26)$.

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