A Structure for General and Specific Market Risk

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Summary. The paper presents a consistent approach to the modeling of general and specific market risk as defined in regulatory documents. It compares the statistically based beta-factor model with a class of benchmark models that use a broadly based index as major building block for modeling. The investigation of log-returns of stock prices that are expressed in units of the market index reveals that these are likely to be Student t distributed. A corresponding discrete time benchmark model is used to calculate Value-at-Risk for equity portfolios.

Key words: Risk measurement, general market risk, specific market risk, Value at Risk, benchmark model, growth optimal portfolio.

1 Introduction

Trading portfolios of financial institutions are characterized by non-linear instruments, tied to complex trading strategies. The nominal volume of such positions is in general not proportional to the risk that is taken. Financial institutions can run *internal models* for calculating *regulatory capital*, see Basle (1996a, 1996b). In this context it is important to see how regulatory terms are translated into quantitative risk modeling.

Market risk, which is due to fluctuations of market prices, plays an essential role in determining regulatory capital. It is understood as the core risk that an institution is exposed to through its trading portfolio. Market risk is split into *general* and *specific market risk.* For an equity portfolio general market risk denotes the risk exposure of the portfolio against the equity market as a whole. On the other hand, specific market risk relates to the

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risk of holding an individual security, which is not covered by general market risk. Specific market risk can be decomposed into *idiosyncratic* and *event risk.* This distinction is used because events like mergers, earnings surprises, bankruptcies and rating migrations are key inputs for the individual security dynamics.

The separation of market risk into its general and specific components has significant impact on the amount of regulatory capital required to cover the market risk of a trading book. In the framework of internal models this capital charge is determined by means of a risk measure, the *Value-at-Risk* (VaR). This paper addresses issues arising from the application of the current regulatory approach. The rich literature on VaR comprises, for instance, RiskMetrics (1996), Alexander (1996), Duffie & Pan (1997, 2001), Jorion (2000) and Embrechts et al. (2002).

As prices are relative, a modeling structure should define an appropriate reference unit to be used as numeraire or *benchmark* in establishing a corresponding metric for measuring risk. In this paper we suggest a *benchmark approach*, where we consistently use a *broadly based index* (BBI) as *benchmark.* This defines a natural model structure, which goes beyond the regression based *beta-factor model* and considerably improves the measurement of market risk. Furthermore, as shown in Platen (2003), a BBI approximates under general conditions the *growth optimal portfolio* (GOP), see Kelly (1956). Using the GOP as reference unit the resulting benchmark model has a number of useful properties, see Platen (2001, 2002).

We analyze in this paper log-returns of equity prices when these are expressed in units of the equity market index. Strong evidence is shown that these are Student t distributed with degrees of freedom that range typically between 3 and 5. This leads to the specification of a Student t benchmark model that can be shown to yield VaR numbers consistent with empirical findings.

2 Discrete Time Market

Let us consider a *discrete time equity* and *fixed income market.* Prices are assumed to change their values only at the given discrete, equidistant times $0 \leq t_0 < t_1 < \ldots < t_n < \infty$ for $n \in \{0, 1, \ldots\}$. The time step size is denoted by $\Delta = t_{i+1} - t_i$, which is assumed to be small, $i \in \{0, 1, \ldots, n-1\}$. We consider $d+1$ primary assets, $d \in \{1, 2, \ldots\}$ and denote by $S_i^{(j)}$ the strictly positive value at time t_i of the jth *primary security account*, which is typically an equity or bond with all dividends and coupon payments reinvested, $j \in \{0, 1, \ldots, d\}$, $i \in \{0, 1, \ldots, n\}$. We assume that $S_i^{(0)}$ is the riskless *savings account* at time t_i. The *return* $R_{i+1}^{(j)}$ of the jth primary security account at time t_{i+1} is defined as

$$R^{(j)}_{i+1} = \frac{S^{(j)}_{i+1} - S^{(j)}_i}{S^{(j)}_i} = H^{(j)}_{i+1} - 1 \tag{2.1}$$

for $i \in \{0, 1, \dots, n-1\}$ and $j \in \{0, 1, \dots, d\}$, where $H^{(j)}_{i+1}$ is the corresponding *growth ratio*. This leads us naturally to the introduction of the jth *log-return* at time t_{i+1} in the form

$$L^{(j)}_{i+1} = \log\left(H^{(j)}_{i+1}\right), \tag{2.2}$$

where we assume that

$$E_i\left(L^{(j)}_{i+1}\right) \approx \eta^{(j)}_i \, \Delta \tag{2.3}$$

for $i \in \{0, 1, \dots, n-1\}$, $j \in \{0, 1, \dots, d\}$ and some finite random variable $\eta^{(j)}_i$. Here E_i denotes the conditional expectation given the information up until time t_i. Note, for typical daily price movements the return $R^{(j)}_{i+1}$ approximates well the log-return $L^{(j)}_{i+1}$.

By $a^\top$ we denote the transpose of a vector or a matrix. For the characterization of a portfolio at time t_i it is sufficient to describe the vector of proportions $\pi_i = (\pi^{(1)}_i, \dots, \pi^{(d)}_i)^\top$, with $\pi^{(j)}_i \in (-\infty, \infty)$ denoting the proportion of the value of the portfolio at time t_i that is invested in the jth primary security account, $j \in \{0, 1, \dots, d\}$. Obviously, the proportions sum to one, that is

$$\sum_{j=0}^{d} \pi^{(j)}_i = 1 \tag{2.4}$$

for all $i \in \{0, 1, \dots, n\}$. The value of the corresponding portfolio at time t_i is denoted by $S^{(\pi)}_i$. The number δ^j_i of units of the j-th primary security account at time t_i is then given by the relation

$$\delta^j_i = \pi^{(j)}_i \frac{S^{(\pi)}_i}{S^{(j)}_i} \tag{2.5}$$

for $j \in \{0, 1, \dots, d\}$ and $i \in \{0, 1, \dots, n\}$. The corresponding *portfolio return* is

$$R^{(\pi)}_{i+1} = H^{(\pi)}_{i+1} - 1 = \sum_{j=0}^{d} \pi^{(j)}_i \, R^{(j)}_{i+1} \tag{2.6}$$

with *portfolio growth ratio* $H^{(\pi)}_{i+1}$ for $i \in \{0, 1, \dots, n-1\}$. Under assumption (2.3) it follows for small Δ

$$E_i\Big(\log\left(H^{(\pi)}_{i+1}\right)\Big) \approx \eta^{(\pi)}_i \, \Delta \tag{2.7}$$

with

$$\eta_i^{(\pi)} = \sum_{j=0}^{d} \pi_i^{(j)} \, \eta_i^{(j)}$$

for $i \in \{0, 1, \dots, n-1\}$.

3 Regulatory Terminology and Framework

We denote by $\mathrm{VaR}_h(S_i^{(\pi)}, \alpha)$ the VaR number at time t_i of a given portfolio $S^{(\pi)}$ with proportions π, a given level of significance α and a forecast horizon of h trading days, typically $h \in \{1, 10\}$ and $\alpha = 99\%$. More precisely, this VaR number denotes the α-quantile of the distribution function of the random variable

$$\Delta \, S_{i,h}^{(\pi)} = S_i^{(\pi)} - \tilde{S}_{i+h}^{(\pi)},$$

where $S_i^{(\pi)}$ is known at time t_i. The variable $\tilde{S}_{i+h}^{(\pi)}$ denotes at time t_{i+h} the random value of the at the time t_i fixed portfolio, that is

$$\tilde{S}_{i+h}^{(\pi)} = \sum_{j=0}^{d} \delta_i^j \, S_{i+h}^{(j)},$$

where the number of units δ_i^j to be held in the jth primary security account at time t_i is given in (2.5). The VaR number can be interpreted as an upper bound of losses that might only be surpassed with probability $1 - \alpha$.

Next we quote several official regulatory definitions, see Basle (1996a, 1996b), that have to be implemented appropriately: *Market risk is defined as the risk of losses in on and off-balance-sheet positions arising from movements in market prices. The risks subject to this requirement are: the risks pertaining to interest rate related instruments and equities in the trading book; foreign exchange risk and commodity risk throughout the bank. ... General market risk covers the risk of holding long or short positions in interest rate or equity risk against the market as a whole. ... The market should be identified with a single factor that is representative for the market as a whole, for example, a widely accepted broadly based stock index for the country concerned. ... Specific risk includes the risk that an individual debt or equity security moves by more or less than the general market in day-to-day trading, including periods when the whole market is volatile. Specific risk includes the risk that an individual debt or equity security moves by more or less than the general market in day-to-day trading, including periods when the whole market is volatile. Specific risk covers that risk in holding long or short positions in an individual equity or debt security. Event risk covers the risk, where the price of an individual debt or equity security moves precipitously relative to* *the general market due to a major event, e.g., on a take-over bid or some other shock event; such events would also include the risk of default.*

We emphasize, it is a regulatory requirement that a *broadly based index* (BBI) serves as a reference unit, which we will naturally incorporate in our approach by using a BBI as benchmark, denoted by $S^{(\underline{\pi})}$.

The differentiation between different forms of risk allows a bank to tailor regulatory capital, that is *capital cushions* or *reserves* that correspond to the inherent risks of a given portfolio. Assume that an internal model provides for a portfolio $S^{(\pi)}$, the VaR number $\mathrm{VaR}_{10}(S_i^{(\pi)}, 99\%)$ at time t_i for general market risk. Furthermore, suppose that a prescription is given to calculate separately the specific risk with the associated VaR figure denoted by $\mathrm{VaR}_{10}^{\mathcal{S}}(S_i^{(\pi)}, 99\%)$. To determine the regulatory capital C_i^R at time t_i for the portfolio $S_i^{(\pi)}$ the following formula has to be applied:

$$
\begin{aligned}
C_i^R &= \max\left\{\mathrm{VaR}_{10}\left(S_i^{(\pi)}, 99\%\right) + m \cdot \mathrm{VaR}_{10}^{\mathcal{S}}\left(S_i^{(\pi)}, 99\%\right),\right.\\
&\qquad \left. M \cdot \frac{1}{60}\sum_{\ell=0}^{59} \mathrm{VaR}_{10}\left(S_{i-\ell}^{(\pi)}, 99\%\right) + m \cdot \frac{1}{60}\sum_{l=0}^{59} \mathrm{VaR}_{10}^{\mathcal{S}}\left(S_{i-\ell}^{(\pi)}, 99\%\right)\right\}.
\end{aligned}
$$

Internal models that cover *idiosyncratic risk*, which is specific risk but not event risk, are called *surcharge models*. In that case the variable m equals 1. For those models that cover specific risk including event risk, m is set to zero. M denotes a safety multiplier which is usually set to 3. We remark, that event risk, when it is covered by general market risk, does *not* require particular regulatory capital.

4 Beta Factor Model

It is common practice, see RiskMetrics (1996) and Basle (1996a), to regress the return $R_{i+1}^{(j)}$ of the jth primary security account at time t_{i+1} on the, so called, jth *beta factor* $\beta^{(j)}$ to separate the impact of *general* and *specific market risk*. To apply a *beta factor model* for equities, a system of *linear regression equations* is used, where

$$
R_{i+1}^{(j)} = R_{i+1}^{(0)} + \beta^{(j)} R_{i+1}^{(\underline{\pi})} + \varepsilon_{i+1}^{(j)} \tag{4.1}
$$

for $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, d\}$. Recall that $R_{i+1}^{(0)}$ is the return of the savings account and $R_{i+1}^{(\underline{\pi})}$ is that of the BBI. Hence, $\beta^{(0)} = 0$ and $\varepsilon_{i+1}^{(0)} = 0$ by definition. Furthermore, in a beta factor model the Gaussian random variable $\varepsilon_{i+1}^{(j)}$ for the jth idiosyncratic noise and $R_{i+1}^{(\underline{\pi})}$ are assumed to be such that

$$E_i\left(\varepsilon_{i+1}^{(j)}\right) = 0; \qquad E_i\left(\varepsilon_{i+1}^{(j)}\, R_{i+1}^{(\pi)}\right) = 0;$$

$$E_i\left(\left(\varepsilon_{i+1}^{(j)}\right)^2\right) = \left(\sigma_{\varepsilon,i}^{(j)}\right)^2; \quad E_i\left(\left(R_{i+1}^{(\pi)}\right)^2\right) = \left(\sigma_i^{(\pi)}\right)^2 + \left(E_i\left(R_{i+1}^{(\pi)}\right)\right)^2;$$

$$E_i\left(\varepsilon_{i+1}^{(j)}\, \varepsilon_{i+1}^{(\ell)}\right) = 0; \quad E_i\left(R_{i+1}^{(j)}\, R_{i+1}^{(\pi)}\right) = \beta^{(j)} E_i\left(\left(R_{i+1}^{(\pi)}\right)^2\right) + R_{i+1}^{(0)}\, E_i\left(R_{i+1}^{(\pi)}\right)$$

for $i \in \{0, 1, \ldots, n-1\}$ and $j, \ell \in \{0, 1, \ldots, d\}$ with $\ell \neq j$. Under these assumptions, $\sigma_i^{(0)} = \sigma_{\varepsilon,i}^{(0)} = 0$ by definition. Note that the return $R_{i+1}^{(j)}$ of the jth primary security account in (4.1) depends linearly on the return $R_{i+1}^{(\pi)}$ of the BBI $S^{(\pi)}$. The j-th idiosyncratic noise $\varepsilon_{i+1}^{(j)}$, is neither correlated to the market return $R_{i+1}^{(\pi)}$ nor to the other idiosyncratic noise terms $\varepsilon_{i+1}^{(\ell)}$ for $\ell \neq j$. As a result of these assumptions, the conditional variance of the return of the jth primary security account equals

$$\begin{aligned} \left(\sigma_i^{(j)}\right)^2 &= E_i\left(\left(R_{i+1}^{(j)} - E_i\left(R_{i+1}^{(j)}\right)\right)^2\right) \\ &= (\beta^{(j)})^2 \left(\sigma_i^{(\pi)}\right)^2 + \left(\sigma_{\varepsilon,i}^{(j)}\right)^2 \end{aligned} \tag{4.2}$$

with

$$\left(\sigma_i^{(\pi)}\right)^2 = E_i\left(R_{i+1}^{(\pi)} - E_i\left(R_{i+1}^{(\pi)}\right)\right)^2$$

for $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, d\}$. In this setup the returns and also their conditional variances, see (4.1) and (4.2), are linearly related. The first and second term in (4.2) express the *general* and *specific market risk*, respectively. The quantity $\beta^{(j)}\, R_{i+1}^{(\pi)}$ in (4.1) is the so-called beta-equivalent of the return $R_{i+1}^{(j)}$. Note, the beta factor model, which is a linear regression model with constant beta factor, has a purely statistical motivation.

5 Benchmark Framework

In the following, we consider an alternative to the beta factor model, which allows us to separate general and specific market risk in a canonical way. This separation is achieved in a natural setting using a BBI $S^{(\pi)}$ as reference unit or *benchmark*. We introduce the jth *benchmarked growth ratio*

$$\hat{H}_{i+1}^{(j)} = \frac{H_{i+1}^{(j)}}{H_{i+1}^{(\pi)}} \tag{5.1}$$

for $i \in \{0,1,\dots,n-1\}$ and $j \in \{0,1,\dots,d\}$, see (2.1) and (2.6). Equation (5.1) allows us to express the growth ratio of the jth primary security account as the product

$$H^{(j)}_{i+1} = H^{(\pi)}_{i+1}\, \hat{H}^{(j)}_{i+1} \tag{5.2}$$

for $i \in \{0,1,\dots,n-1\}$ and $j \in \{0,1,\dots,d\}$. This product provides a *multiplicative decomposition* of the jth growth ratio. The first factor $H^{(\pi)}_{i+1}$ is related to general market risk, whereas the second factor $\hat{H}^{(j)}_{i+1}$ is naturally tied to the specific market risk of the jth primary security account at time t_{i+1}. In this benchmark framework we denote the logarithm of the j-th benchmarked growth ratio (5.1), by

$$\hat{L}^{(j)}_{i+1} = \log\left(\hat{H}^{(j)}_{i+1}\right) \tag{5.3}$$

and obtain the conditional expectation

$$E_i\left(\hat{L}^{(j)}_{i+1}\right) \approx \hat{\eta}^{(j)}_i\, \Delta \tag{5.4}$$

with $\hat{\eta}^{(j)}_i = \eta^{(j)}_i - \eta^{(\pi)}_i$ for $i \in \{0,1,\dots,n-1\}$ and $j \in \{0,1,\dots,d\}$.

BBIs, for instance, the S&P100, S&P500, S&P1000 and the MSCI world index behave all in a very similar manner. In Platen (2003) it has been shown under general assumptions that BBIs approximate the GOP. Thus the GOP is naturally approximated when modeling general market risk by a BBI.

6 Semiparametric Benchmark Models

Let us introduce a general class of *semiparametric benchmark models*, where we assume that the jth *centralized log-return* admits the structure

$$X^{(j)}_{i+1} = \hat{L}^{(j)}_{i+1} - E_i\left(\hat{L}^{(j)}_{i+1}\right) = -\sum_{k=1}^{d} \sigma^{j,k}_i \sqrt{\Delta}\, Z^{(k)}_{i+1} \tag{6.1}$$

for $j \in \{0,1,\dots,d\}$ and $i \in \{0,1,\dots,n-1\}$. Note that $X^{(0)}_{i+1}$ is the centralized log-return of the benchmarked savings account. Here, $\sigma^{j,k}_i$ is called the jth *volatility* at time t_i with respect to the kth *source of uncertainty* $Z^{(k)}_{i+1}$. We choose $Z^{(1)}_{i+1},\dots,Z^{(d)}_{i+1}$ as random variables with zero conditional mean

$$E_i\left(Z^{(k)}_{i+1}\right) = 0, \tag{6.2}$$

unit conditional variance

$$E_i\left(\left(Z^{(k)}_{i+1}\right)^2\right) = 1 \tag{6.3}$$

and such that

$$E_i\left(Z_{i+1}^{(k)}\, Z_{i+1}^{(\ell)}\right) = 0 \tag{6.4}$$

for $\ell \neq k$ with $i \in \{0, 1, \dots, n-1\}$ and $k, \ell \in \{1, 2, \dots, d\}$. We assume for technical reasons that an absolute conditional moment of order slightly greater than two exists for the vector of uncertainty $Z_{i+1} = (Z_{i+1}^{(1)}, \dots, Z_{i+1}^{(d)})^\top$. Note that in contrast to the beta factor model, we are not restricted to the use of Gaussian random variables. Nor do we assume the independence of market returns and benchmarked individual returns.

We obtain from (6.1) - (6.4) the second order normalized conditional moments

$$c_i^{j,\ell} = \frac{1}{\Delta}\, E_i\left(X_{i+1}^{(j)}\, X_{i+1}^{(\ell)}\right) = \sum_{k=1}^{d} \sigma_i^{j,k}\, \sigma_i^{\ell,k} \tag{6.5}$$

for $i \in \{0, 1, \dots, n-1\}$ and $j, \ell \in \{1, 2, \dots, d\}$. Relation (6.5) allows us to introduce the *conditional covariance matrix*

$$\Sigma_i = [c_i^{j,\ell}]_{j,\ell=1}^{d} = D_i\, D_i^\top \tag{6.6}$$

with *volatility matrix*

$$D_i = [\sigma_i^{j,\ell}]_{j,l=1}^{d}, \tag{6.7}$$

$i \in \{0, 1, \dots, n-1\}$. The volatility matrix D_i can be interpreted as the Cholesky decomposition of Σ_i. If the volatility matrix D_i is invertible, then, by (6.1), the vector $Z_{i+1} = (Z_{i+1}^{(1)}, \dots, Z_{i+1}^{(d)})^\top$ of the sources of uncertainty can be explicitly expressed in the form

$$Z_{i+1} = -\frac{1}{\sqrt{\Delta}}\, D_i^{-1}\, X_{i+1},$$

$i \in \{0, 1, \dots, n-1\}$. By equations (5.1), (5.3) and (6.1) with $j = 0$ it can be seen that

$$\begin{aligned} X_{i+1}^{(0)} &= \hat{L}_{i+1}^{(0)} - E_i\left(\hat{L}_{i+1}^{(0)}\right) \\ &= \log\left(H_{i+1}^{(0)}\right) - E_i\left(\log\left(H_{i+1}^{(0)}\right)\right) - \log\left(H_{i+1}^{(\pi)}\right) + E_i\left(\log\left(H_{i+1}^{(\pi)}\right)\right). \end{aligned}$$

Since the growth ratio $H_{i+1}^{(0)}$ of the savings account is known at time t_i the first two terms in the above formula offset each other. Thus, we obtain the log-return of the BBI in the approximate form

$$\log\left(H_{i+1}^{(\pi)}\right) = -X_{i+1}^{(0)} + E_i\left(\log\left(H_{i+1}^{(\pi)}\right)\right) \approx -X_{i+1}^{(0)} \tag{6.8}$$

for $i \in \{0, 1, \dots, n-1\}$, because $E_i(\log(H^{(\pi)}_{i+1})) \approx \eta^{(\pi)}_i \Delta$ is of higher order than $\sqrt{\Delta}$. This means, the uncertainty of the log-return of the BBI is approximately the negative of that of the benchmarked savings account. Since the return $R^{(j)}_{i+1}$ of the jth primary security account is small we obtain from relation (2.1) and the expansion of the logarithm that

$$R^{(j)}_{i+1} = H^{(j)}_{i+1} - 1 \approx \log\left(H^{(j)}_{i+1}\right), \tag{6.9}$$

for $i \in \{0, 1, \dots, n-1\}$ and $j \in \{0, 1, \dots, d\}$. Now, relation (5.2) yields together with (6.8), (5.3) and (6.1) by neglecting higher order terms that

$$R^{(j)}_{i+1} \approx \log\left(H^{(j)}_{i+1}\right) = \log\left(H^{(\pi)}_{i+1}\right) + \log\left(\hat{H}^{(j)}_{i+1}\right) \approx -X^{(0)}_{i+1} + X^{(j)}_{i+1}. \tag{6.10}$$

The above described semiparametric benchmark model is based on the multiplicative relationship (5.2) between growth ratios. As is evident from (6.8) and (6.1), the volatility $\sigma^{0,k}_i$ provides a measure for general market risk, that is the exposure of the BBI towards the k-th source of uncertainty $Z^{(k)}_{i+1}$. Similarly, by (5.3) and (6.1) the volatility $\sigma^{j,k}_i$ quantifies its specific market risk, that means the exposure of the j-th benchmarked primary security account towards the k-th source of uncertainty, $i \in \{0, 1, \dots, n-1\}$, $j \in \{0, 1, \dots, d\}$ and $k \in \{1, 2, \dots, d\}$.

Summarizing (6.10) and (6.1) provides the following representation of the stochastic component of the return of the jth primary security account

$$R^{(j)}_{i+1} \approx \sum_{k=1}^{d} \left(\sigma^{0,k}_i - \sigma^{j,k}_i\right) \sqrt{\Delta}\, Z^{(k)}_{i+1} \tag{6.11}$$

for $i \in \{0, 1, \dots, n-1\}$ and $j \in \{0, 1, \dots, d\}$. Similarly, using (6.8), (6.9) and (6.1), we obtain for the stochastic part of the return of the BBI the approximate expression

$$R^{(\pi)}_{i+1} = H^{(\pi)}_{i+1} - 1 \approx \log\left(H^{(\pi)}_{i+1}\right) \approx -X^{(0)}_{i+1} = \sum_{k=1}^{d} \sigma^{0,k}_i \sqrt{\Delta}\, Z^{(k)}_{i+1} \tag{6.12}$$

for $i \in \{0, 1, \dots, n-1\}$. By (6.12), (6.11), (6.2), (6.3) and (6.4) we get

$$E_i\left(\left(R^{(\pi)}_{i+1}\right)^2\right) \approx \Delta \sum_{k=1}^{d} \left(\sigma^{0,k}_i\right)^2, \tag{6.13}$$

$$E_i\left(\left(R^{(j)}_{i+1}\right)^2\right) \approx \Delta \sum_{k=1}^{d} \left(\sigma^{0,k}_i - \sigma^{j,k}_i\right)^2 \tag{6.14}$$

and

$$E_i\left(R_{i+1}^{(j)} R_{i+1}^{(\underline{\pi})}\right) \approx \Delta \sum_{k=1}^{d} \left(\sigma_i^{0,k} - \sigma_i^{j,k}\right) \sigma_i^{0,k} \tag{6.15}$$

for $i \in \{0, 1, \ldots, n-1\}$. We can define the following approximate ratio of covariances of returns as *generalized jth beta factor* $\beta_i^{(j)}$ at time t_i, where

$$\beta_i^{(j)} = \frac{E_i\left(R_{i+1}^{(j)} R_{i+1}^{(\underline{\pi})}\right)}{E_i\left(\left(R_{i+1}^{(\underline{\pi})}\right)^2\right)} \approx \frac{\sum_{k=1}^{d} \left(\sigma_i^{0,k} - \sigma_i^{j,k}\right) \sigma_i^{0,k}}{\sum_{k=1}^{d} \left(\sigma_i^{0,k}\right)^2} = 1 - \frac{\sum_{k=1}^{d} \sigma_i^{0,k} \sigma_i^{j,k}}{\sum_{k=1}^{d} \left(\sigma_i^{0,k}\right)^2}$$

for $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, d\}$. The above relation provides the equivalent of the common beta factor in a benchmark framework. Note that we get for the domestic savings account the beta factor $\beta_i^{(0)} = 0$, as is to be expected. Moreover, for the generalized beta factor $\beta_i^{(\pi)}$ of a portfolio $S^{(\pi)}$ with

$$\beta_i^{(\pi)} = \frac{E_i\left(R_{i+1}^{(\pi)} R_{i+1}^{(\underline{\pi})}\right)}{E_i\left(\left(R_{i+1}^{(\underline{\pi})}\right)^2\right)}$$

one can show by similar arguments as above that

$$\beta_i^{(\pi)} = 1 - \frac{\sum_{k=1}^{d} \sigma_i^{0,k} \sum_{j=0}^{d} \pi_i^{(j)} \sigma_i^{j,k}}{\sum_{k=1}^{d} \left(\sigma_i^{0,k}\right)^2}. \tag{6.16}$$

Note that the generalized beta factor of the semiparametric benchmark model is time dependent and matches the well-known return relationship of the *Capital Asset Pricing Model*, see Merton (1973). Finally, for the GOP $S^{(\underline{\pi})}$ it can be shown, see Platen (2002), that

$$\sum_{j=0}^{d} \underline{\pi}_i^{(j)} \sigma_i^{j,k} \approx 0 \tag{6.17}$$

for $i \in \{0, 1, \ldots, n-1\}$ and $k \in \{1, 2, \ldots, d\}$. Therefore, using (6.16) and (6.17) the generalized beta factor $\beta_i^{(\underline{\pi})}$ of the GOP, and thus a BBI, is approximately one.

7 Generalized Hyperbolic Benchmark Models

One can now specify appropriate families of distributions for the sources of uncertainty $Z_i^{(1)}, \ldots, Z_i^{(d)}$. Let us choose the log-return distribution from the well established class of generalized hyperbolic distributions, see Barndorff-Nielsen (1978). We assume, for simplicity, that the centralized log-returns of benchmarked share prices have a *symmetric generalized hyperbolic distribution.*

The main feature that we explore in the following concerns the shape of the tails of the probability density f_X of centralized log-returns X. The *symmetric generalized hyperbolic density* is of the form

$$\begin{aligned} f_X(x) &= \frac{1}{\delta\sqrt{\Delta}\,K_\lambda(\bar{\alpha})}\sqrt{\frac{\bar{\alpha}}{2\,\pi}}\left(1+\frac{(x-\eta\,\Delta)^2}{\delta^2\,\Delta}\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} \\ &\quad \cdot K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1+\frac{(x-\eta\,\Delta)^2}{\delta^2\,\Delta}}\right) \end{aligned} \tag{7.1}$$

for $x \in \Re$, where $\eta, \lambda \in \Re$, $\delta \geq 0$ and $\bar{\alpha} = \alpha\,\delta$ with $\alpha \neq 0$ if $\lambda \geq 0$ and $\delta \neq 0$ if $\lambda \geq 0$. Here $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ.

The symmetric generalized hyperbolic distribution is a four parameter family of distributions. The two shape parameters are λ and $\bar{\alpha}$, defined so that they are invariant under scale transformation as described below. For $\bar{\alpha} = 0$ and $\lambda \in [-2, 0]$ we have infinite kurtosis. The parameter η in (7.1) is a location parameter, where the log-return X has mean $m_X = \eta\,\Delta$. We define the parameter c as the unique scale parameter such that the variance of X is $v_X = c^2\Delta$ and

$$c^2 = \begin{cases} \frac{2\,\lambda}{\alpha^2} & \text{if } \delta = 0, \text{ i.e., } \lambda > 0,\ \bar{\alpha} = 0 \\ \frac{\delta^2\,K_{\lambda+1}(\bar{\alpha})}{\bar{\alpha}\,K_\lambda(\bar{\alpha})} & \text{otherwise.} \end{cases}$$

It can be shown that for $\lambda \to \pm\infty$ or $\bar{\alpha} \to \infty$ the symmetric generalized hyperbolic distribution asymptotically approaches the Gaussian distribution. Thus the log-returns of the lognormal model, see Black & Scholes (1973), appear as limiting cases of the above class of distributions.

We will now describe four particularly important symmetric generalized hyperbolic distributions that coincide with the log-return distributions of important asset price models that have been suggested by different authors:

The *Student t distribution* was originally suggested by Praetz (1972) as a suitable distribution for asset returns. For the Student t distribution one sets $\bar{\alpha} = 0$ and $\lambda < 0$. The parameter $\nu = -2\lambda$ is called the *degrees of freedom.* We have finite variance for $\nu > 2$ and finite kurtosis $\kappa_r = 3\,\frac{\nu-2}{\nu-4}$ for $\nu > 4$. The Student t distribution is a three parameter distribution, where smaller degrees of freedom ν imply heavier tails.

Barndorff-Nielsen (1995) proposed a model, where the log-returns generate a *normal-inverse Gaussian* mixture distribution. Here the shape parameter λ is fixed at the level $\lambda = -0.5$. This distribution is also a three parameter distribution. A smaller shape parameter $\bar{\alpha}$ implies larger tail heaviness with kurtosis $\kappa_r = 3\,(1+\bar{\alpha}^{-1})$.

Eberlein & Keller (1995) suggested asset price models, where log-returns follow a ***hyperbolic distribution*** with shape parameter $\lambda = 1$. It is also a three parameter distribution. Its kurtosis

$$\kappa_r = \frac{3\,K_1(\bar{\alpha})\,K_3(\bar{\alpha})}{K_2(\bar{\alpha})^2}$$

depends on the shape parameter $\bar{\alpha}$, which can be shown to reach its maximum value of $\kappa_r = 6$ for $\bar{\alpha} = 0$.

Madan & Seneta (1990) proposed a class of models that result in log-returns which follow a ***variance gamma distribution***. This distribution arises if one sets $\bar{\alpha} = 0$ and $\lambda > 0$. A smaller λ implies heavier tails for this three parameter distribution with kurtosis $\kappa_r = 3\,(1+\lambda^{-1})$.

8 Testing Benchmark Models

In this section we identify a distribution that best fits the log-returns of BBIs and benchmarked stock prices. We note that for several of the aforementioned distributions the kurtosis can be infinite. Therefore, a statistical method that relies on higher order moments should not be used. The maximum likelihood approach avoids this problem. We recall, for fixed $j \in \{1, 2, \dots, d\}$ that the log-returns $X_i^{(j)}$, $i \in \{0, 1, \dots, n-1\}$, are independent and identically distributed. We define the ***likelihood ratio*** in the standard form $\Lambda = \frac{\mathcal{L}_{\mathrm{m}}}{\mathcal{L}_{\mathrm{gm}}}$, where $\mathcal{L}_{\mathrm{m}}$ represents the maximized likelihood function of a particular three parameter distribution, for instance, the Student t distribution. With respect to this distribution the maximum likelihood estimate for the parameters is computed. On the other hand, $\mathcal{L}_{\mathrm{gm}}$ denotes the maximized likelihood function for the four parameter symmetric generalized hyperbolic distribution.

As $n \to \infty$ the ***test statistic*** $L_n = -2 \ln \Lambda$ is asymptotically chisquare distributed, see Rao (1973), with degrees of freedom one. Asymptotically, we then have the relation

$$P\left(L_n < \chi^2_{1-\alpha,1}\right) \approx F_{\chi^2(1)}\left(\chi^2_{1-\alpha,1}\right) = 1 - \alpha, \tag{8.1}$$

as $n \to \infty$, where $F_{\chi^2(1)}$ denotes the chisquare distribution with one degree of freedom and $\chi^2_{1-\alpha,1}$ its $100(1-\alpha)\%$ quantile.

We can now check, say, for the standard 99% significance level whether or not the test statistic L_n is in the 1% quantile of the chisquare distribution with one degree of freedom. This means, if the relation

$$L_n < \chi^2_{0.01,1} \approx 0.0002 \tag{8.2}$$

is not satisfied, then we reject on a 99% significance level the hypothesis that the suggested distribution is the true underlying log-return distribution.

We study benchmarked US stock price log-returns on the basis of daily benchmarked share price data, provided by Thomson Financial for the eleven year period from 1987 to 1997, using the S&P500 as BBI. The total number of observed daily log-returns for each benchmarked stock is about 2500, which provides reliable asymptotic results for the maximum likelihood ratio test. In Table 1 we show the test statistics L_n for twenty leading shares of the US market. The stock codes correspond to those commonly used. The smallest L_n value identifies the best fit and is displayed in bold type.

Stock	Student t	Inverse Gaussian	Hyperbolic	Variance Gamma
GE	**0.0000**	1.4334	2.2260	3.8726
KO	**0.0000**	14.2692	21.9366	28.6322
XON	**0.0000**	16.0676	23.2020	31.0282
INTC	**0.0000**	15.8856	32.1382	41.9714
MRK	1.6614	**0.8386**	4.8766	10.5624
RD	**0.0000**	6.3572	10.7048	16.2154
IBM	**0.1730**	14.1344	51.5752	64.7730
MO	**0.0000**	18.2024	53.7858	62.0374
PG	**0.0000**	20.7162	32.5862	39.2176
PFE	**0.0170**	4.5236	9.2526	14.2854
BMY	**0.0000**	12.9818	24.0392	30.6054
T	**0.0000**	11.6994	25.4518	34.2240
WMT	**0.0000**	6.7546	14.4522	21.7498
JNJ	1.2272	**1.1620**	5.8662	12.0374
LLY	**0.0000**	6.8344	12.7520	17.8614
DD	5.4536	1.5960	0.5434	**0.0000**
DIS	**0.0000**	16.9802	31.0480	41.4124
HWP	**0.0000**	19.8886	38.8200	50.4892
PEP	**0.0000**	14.7938	25.9036	35.8268
MOB	**0.0398**	1.9694	5.0566	8.7238

Table 1: The L_n-values for log-returns of benchmarked US stocks.

We note that for the majority of log-returns of benchmarked stock prices the Student t distribution gives the smallest L_n value and thus yields the best fit in the class of symmetric generalized hyperbolic distributions considered.

We observe that fifteen of the twenty L_n values in Table 1 appear to be smaller than the $\chi^2_{0.01,1} = 0.0002$ quantile. For fourteen of these the hypothesis that the Student t distribution is the true underlying distribution cannot be rejected at a 99% significance level.

A similar study has been performed for the Australian, German, Japanese and UK market using the corresponding market index as BBI. Detailed results for the other four markets can be obtained from the authors. For thirty one of the one hundred stocks considered, the Student t distribution could not be rejected at a 99% significance level as the true underlying distribution. In all five markets the Student t distribution clearly provides the best fit for most log-returns of the benchmarked stocks.

To illustrate the results for all five markets we plot in Figure 1 the estimated $(\bar{\alpha}, \lambda)$-parameter values for the one hundred examined benchmarked stocks. These estimates characterize the specific shape of the tails for the estimated symmetric generalized hyperbolic densities. The positive part of the λ-axis in Figure 1 parameterizes the variance gamma distribution. We note that only one of the one hundred stocks generated an $(\bar{\alpha}, \lambda)$-estimate on the positive λ-axis. The hyperbolic distribution is represented by pairs of shape parameters $(\bar{\alpha}, \lambda)$ with $\lambda = 1$. There are about three to four points in Figure 1 that are located in the neighborhood of the horizontal line $\lambda = 1$. The points $(\bar{\alpha}, \lambda)$ on the horizontal line $\lambda = -\frac{1}{2}$ correspond to the normal-inverse Gaussian distribution. We note that several stocks generate points in the area close to this line. Note that for small $\bar{\alpha}$ the normal-inverse Gaussian distribution coincides asymptotically with the Student t distribution and most $\bar{\alpha}$ parameter estimates are less than one. It is obvious that most of the one hundred $(\bar{\alpha}, \lambda)$-estimates are concentrated close to the negative λ-axis. This is the area that is characteristic for distributions that are similar to the Student t distribution. It appears that the $(\bar{\alpha}, \lambda)$-estimates support a Student t distribution with a parameter value for λ of approximately -2, which corresponds to four degrees of freedom, see Section 7. The average estimated degrees of freedom for the Student t distribution obtained from all benchmarked stocks was $\hat{\nu} = 4.377$.

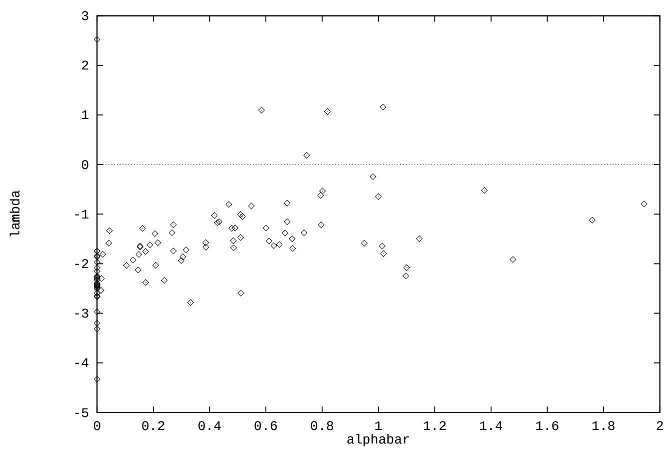

Figure 1: $(\bar{\alpha}, \lambda)$-plot for log-returns of benchmarked stocks.

In a study similar to that described above it has been shown in Hurst & Platen

(1997) that the log-returns of BBIs of the equity markets of most developed economies support the Student t distribution, again with degrees of freedom close to four. As an alternative to the above mentioned studies, a series of papers, see, for instance, Dacorogna et al. (2001), has shown that the tails of asset log-returns follow approximately a *power law*, where the estimates of the so-called tail index are typically in the range of three to five. This is consistent with our findings for benchmarked stock log-returns. Furthermore, we remark that the Student t benchmark model can be interpreted as a discrete time analog of the, so-called, minimal market model, see Platen (2001, 2002), which models the dynamics of a discounted GOP that exhibits minimum variance in its drift. These lead to Student t distributed log-returns of the GOP with theoretically predicted $\nu = 4$ degrees of freedom.

9 VaR Analysis

As outlined in the introduction and in Section 3, the modeling of event risk in internal models is of increasing importance in VaR analysis. The above class of semiparametric benchmark models is able to encompass event risk by choosing an adequate *leptokurtic* distribution for the sources of uncertainty. Incorporating event risk completes VaR modeling for measuring market risk by taking into account all regulatory subcategories of market risk, that is general, idiosyncratic and event risk.

As an alternative to the above approach, Gibson (2001) used a mixing distribution to specify a five parameter model that puts sufficient mass into the tails of log-return distributions by introducing different regimes for the means of certain mixed distributions. Duffie & Pan (2001) apply *jump-diffusions* for modeling idiosyncratic and event risk in the context of VaR, which relies on various parameters. Though the class of jump-diffusions is intuitively appealing, the parameters are difficult to estimate. Huschens & Kim (1999) studied a VaR model that uses Student t distributed returns but not in a benchmark setting.

In the following version of a Student t benchmark model we specify in a parsimonious way the sources of uncertainty, where we exploit the fact that symmetric generalized hyperbolic distributions admit a representation of a mixture of normal distributions. This means, if one chooses the variance of a conditionally Gaussian distribution as the inverse of a Gamma distributed random variable, then the resulting distribution is a Student t distribution. To generate the Student t distributed sources of uncertainty $Z_{i+1}^{(0)}, \ldots, Z_{i+1}^{(d)}$ appropriately, we set

$$Z_{i+1}^{(k)} = \sqrt{\tau_{i+1}}\, Y_{i+1}^{(k)}, \tag{9.1}$$

where τ_{i+1} denotes the *market activity* with

$$\tau_{i+1} = \left(1 - \frac{2}{\nu}\right) \left(\frac{1}{\nu} \sum_{\ell=1}^{\nu} \left(\psi_{i+1}^{(\ell)}\right)^2\right)^{-1} \tag{9.2}$$

for $i \in \{0, 1, \ldots, n-1\}$ and $k \in \{1, 2, \ldots, d\}$ with $\nu \in \{3, 4, \ldots\}$. Additionally to the independent standard Gaussian distributed random variables $Y_{i+1}^{(k)}$ that appear in (9.1) we employ further independent standard Gaussian random variables $\psi_{i+1}^{(\ell)}$. Hence, the random variables τ_{i+1} are chisquare distributed with ν degrees of freedom. Consequently, the random variables $Z_{i+1}^{(k)}$ are Student t distributed with unit variance and ν degrees of freedom. The market activity τ_{i+1} can be interpreted as a daily increment of the random intrinsic time of the market. Note, the market activity converges to 1 as the degrees of freedom ν tend to infinity, which yields the lognormal benchmark model.

In addition to the typical parameters of the lognormal benchmark model we have used here only the extra parameter ν, which is sufficient to characterize the leptokurtosis of the Student t distribution. As shown previously, the typical parameter choice for ν is about 4. Smaller degrees of freedom generate log-returns with more extreme movements.

An important feature of the resulting multivariate Student t distribution is its copula. It realistically captures the dependence of extreme asset price movements as shown in Embrechts et al. (2002) and related work. The Student t-copula, which is characterizing the joint occurrence of extreme moves in log-returns of several benchmarked stocks, is in our model automatically captured.

To calibrate the above Student t benchmark model one needs to estimate the volatilities $\sigma^{j,k}$, typically obtained from a standard calibration using (6.5) and a Cholesky decomposition to derive $\sigma^{j,k}$. The above model then represents a simple extension of the lognormal model, where one needs only to add an estimate for the degrees of freedom ν. It is reasonable to set $\nu = 4$, as we will see from Table 2 and is theoretically predicted in Platen (2002).

The equations (9.1) and (9.2) give a prescription that can also be used for simulation purposes. They involve the Cholesky decomposition D_i of the covariance matrix Σ_i, see (6.7). The Student t benchmark model provides accurate results if the VaR calculation is based on extensive Monte-Carlo simulations. However, sufficiently accurate Monte-Carlo simulations are extremely time consuming. To circumvent this problem, we propose the following highly efficient method for VaR calculations:

In practice, equity portfolios, are typically dominated by large linear portfolios. Note that the joint distribution of the random vector $X_{i+1} = (X_{i+1}^{(1)}, \ldots, X_{i+1}^{(d)})^\top$ is a multivariate Student t distribution with ν degrees of freedom, where

$$X_{i+1} = \sqrt{\tau_{i+1}}\, D_i\, Y_{i+1}.$$

Since Y_{i+1} is Gaussian and $\frac{1}{\tau_{i+1}}$ is independent chisquare distributed, the resulting multivariate Student t distribution for X_{i+1} belongs to the class of *elliptical distributions*. For linear portfolios the calculation of VaR numbers is for this class of distributions analytically tractable. More precisely, a theorem in Fang et al. (1990) yields the representation

$$a^\top X_{i+1} = |a^\top D_i|\,\zeta_{i+1} \tag{9.3}$$

for any given weight vector a, where $|\cdot|$ is the Euclidean norm, $a \in \Re^d$, $D_i^\top D_i = \Sigma_i$ and ζ_{i+1} denotes a Student t distributed scalar random variable with ν degrees of freedom. The representation (9.3) significantly simplifies the VaR calculation for linear portfolios with an extremely large number of constituents. Furthermore, we mention that if a portfolio with many constituents is not linear but represents a *diversified portfolio* in the sense described by Platen (2003), then it approximates the GOP and thus also our benchmark, the BBI. This asymptotics yields accurate VaR numbers and saves computational time when compared to standard Monte-Carlo simulation.

Since the multivariate Student t distribution is an elliptical distribution, it is shown by Embrechts et al. (2002), that VaR is in this case a *coherent risk measure*, see Artzner et al. (1997). This fact is highly important for the consistent use of VaR as a risk measure for the internal capital allocation to particular business lines.

In order to calculate VaR for short term horizons we apply, for simplicity, the so-called square root time rule. This approximation is in line with the regulatory recommendations of the Basle Committee. From (9.3) we obtain then the following formula for the VaR number of a given portfolio $S^{(\pi)}$ at time t_i.

$$\mathrm{VaR}_h(S_i^{(\pi)}, \alpha) \approx V_i\,\sqrt{a^\top \Sigma_i\, a}\,\sqrt{h\,\Delta}\,\tilde{t}_\alpha(\nu). \tag{9.4}$$

Here V_i denotes the market value of the portfolio at time t_i, $\sqrt{a^\top \Sigma_i\, a}$ characterizes the volatility of the portfolio, Δ the time step size for a trading day, h the number of trading days and $\tilde{t}_\alpha(\nu)$ the α-quantile of the standardized Student t distribution with ν degrees of freedom.

ν	∞	10	5	4	3	2
φ	1	1.06	1.11	1.12	1.14	1.16

Table 2: Event factor φ in dependence on degrees of freedom ν.

Obviously, the product (9.4) generalizes the well-known short hand formula, used in RiskMetrics, to calculate VaR by including the *event factor*

$$\varphi = \frac{\tilde{t}_\alpha(\nu)}{q_\alpha}, \tag{9.5}$$

that is

$$\mathrm{VaR}_h(S_i^{(\pi)}, \alpha) \approx V_i \sqrt{a^\top \Sigma_i a} \sqrt{h \Delta}\, q_a\, \varphi. \tag{9.6}$$

Here q_α is the α-quantile of the standard Gaussian distribution. Consequently, the event factor φ adjusts the standard VaR formula to a level that captures approximately event risk when one uses the Student t benchmark model as internal model. According to the quantiles of the Gaussian and Student t distribution one obtains by (9.4) the event factors shown in Table 2. Even for rather small degrees of freedom, say $\nu \approx 2$, the additional regulatory capital will not surpass 16%. To confirm the appropriate choice of the model and its calibration one needs to perform *stress tests*, see Basle (1996a). Along these lines, Gibson (2001) performed an extensive study using representative portfolios for US institutions, which identified an event factor of about $\varphi = 1.12$. This is exactly the event factor that matches in Table 2 the degrees of freedom $\nu = 4$, which again supports the model proposed in Platen (2002) and also our empirical findings.

References

Alexander, C. (1996). *Handbook of Risk Management and Analysis.* Wiley, Chichester.

Artzner, P., F. Delbaen, J. M. Eber, & D. Heath (1997). Thinking coherently. *Risk* **10**, 68–71.

Barndorff-Nielsen, O. (1978). Hyperbolic distributions and distributions on hyperbolae. *Scand. J. Statist.* **5**, 151–157.

Barndorff-Nielsen, O. (1995). Normal\\Inverse Gaussian processes and the modelling of stock returns. Technical report, University of Aarhus. 300.

Basle (1996a). *Amendment to the Capital Accord to Incorporate Market Risks.* Basle Committee on Banking and Supervision, Basle, Switzerland.

Basle (1996b). *Modifications to the Market Risk Amendment.* Basle Committee on Banking and Supervision, Basle, Switzerland.

Black, F. & M. Scholes (1973). The pricing of options and corporate liabilities. *J. Political Economy* **81**, 637–659.

Dacorogna, M., U. A. Müller, O. V. Pictet, & C. G. De Vries (2001). Extremal forex returns in extremely large data sets. *Extremes* **4**(2), 105–127.

Duffie, D. & J. Pan (1997). An overview of Value at Risk. *J. of Derivatives* **4**(3), 7–49.

Duffie, D. & J. Pan (2001). Analytic Value at Risk with jumps and credit risk. *Finance Stoch.* **5**, 155–180.

Eberlein, E. & U. Keller (1995). Hyperbolic distributions in finance. *Bernoulli* **1**, 281–299.

Embrechts, P., A. McNeal, & D. Straumann (2002). Correlation and dependencies in risk management: Properties and pitfalls. In *Risk Management: Value at Risk and Beyond*, pp. 176–223. Cambridge Univ. Press.

Fang, K. T., S. Kotz, & K. W. Ng (1990). *Symmetric Multivariate and Related Distributions*. Chapman Hall, London.

Gibson, M. S. (2001). Incorporating event risk into Value at Risk. Discussion Paper Federal Reserve Bord, Washington (http:\\www.gloriamundi.org).

Hurst, S. R. & E. Platen (1997). The marginal distributions of returns and volatility. In Y. Dodge (Ed.), *L_1-Statistical Procedures and Related Topics*, Volume 31 of *IMS Lecture Notes - Monograph Series*, pp. 301–314. Institute of Mathematical Statistics Hayward, California.

Huschens, S. & Y. Kim (1999). Measuring risk in Value-at-Risk based on Student's t-distribution. In W. Gaul and H. Locarek-Junge (Eds.), *Proceedings of the GfKl-Conference in Dresden 1998*, pp. 453–459. Berlin, Springer.

Jorion, P. (2000). *Value at Risk: The New Benchmark for Controlling Market Risk* (2nd ed.). Irwin, Chicago.

Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

Madan, D. B. & E. Seneta (1990). The variance gamma (V.G.) model for share market returns. *J. Business* **63**, 511–524.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica* **41**, 867–888.

Platen, E. (2001). A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.

Platen, E. (2002). Arbitrage in continuous complete markets. *Adv. in Appl. Probab.* **34**(3), 540 – 558.

Platen, E. (2003). Diversified portfolios in a benchmark framework. Technical report, University of Technology, Sydney. QFRG Research Paper 87.

Praetz, P. D. (1972). The distribution of share price changes. *J. Business* **45**, 49–55.

Rao, C. R. (1973). *Linear Statistical Inference and Its Applications* (2nd ed.). Wiley, New York.

RiskMetrics (1996). *Technical Document* (4th ed.).