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Stephen Satchell and Susan Thorp

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DISCOUNTING AND CONSUMPTION OVER AN UNCERTAIN HORIZON:

DRAW-DOWN PLANS FOR FAMILY TRUSTS.

STEPHEN SATCHELL
Faculty of Economics
University of Cambridge
Cambridge, CB3 9DD United Kingdom
Tel: 44-(0) 1223 335229
Email: Stephen.Satchell@econ.cam.ac.uk

SUSAN THORP
School of Finance and Economics
University of Technology Sydney,
Broadway 2007 NSW Australia
Tel: 61-(0) 2 9514 7784
Email: Susan.Thorp@uts.edu.au

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Abstract

Individuals, endowments and trusts face uncertain lifetimes. When the planning horizon of an entity is stochastic and Pareto distributed, hyperbolic discounting and time-varying consumption rates are optimal. We derive expressions for the optimal rate of consumption (draw-down) from wealth for family trusts facing positive probabilities of extinction at each generation. Using birth statistics for the UK, we compute family extinction probabilities and show that they are well-approximated by a Pareto distribution, hence family trusts will discount hyperbolically. Numerically optimised consumption paths for family trusts with CRRA preferences are decreasing but always higher than for infinitely-lived trusts.

JEL Codes: G0, D9

Keywords: family extinction, hyperbolic discounting, inter-temporal choice

Many problems in economics and finance require planning over long time horizons and in all such problems the discount rate is critical. Researchers from Ramsey (1928) to Stern (2006)[1] have recognised that choosing a discount rate is rarely a disinterested decision, but usually represents some amalgamation of economic 'science' with inter-generational ethics or politics.

While it is analytically convenient for economists to choose non-stochastic boundaries (either fixed or infinite) for multi-period problems, more often than not a problem's inherent uncertainty extends beyond, say, random investment payoffs or uncertain income to the horizon itself. If the planning horizon is stochastic, discounting cannot arbitrarily be fixed at some unobserved level of impatience: it must be treated as a function of the probability density of the horizon. Further, stochastic horizon problems are ubiquitous. The simplest individual consumption problems are subject to uncertainty over the length of life, and the same is true of the majority of plans for firms, financial institutions, governments and societies.

Time-varying discounting has sometimes been used to harmonise observed patterns of behaviour with the predictions of theory, and attributed to weak ethics or a lack of altruism on the part of the decision-maker. For example, hyperbolic discounting, where impatience depends on a person's interest in the well-being of those living in the future, or, indeed, in the well-being of future 'selves', has been proposed to explain anomalous savings behaviour (see Phelps and Pollack, 1968). However we do not have to appeal to weak ethics to justify non-constant discount rates when simple horizon uncertainty may be sufficient.

[1] See also Stern's discussion at http://www.hm-treasury.gov.uk/media/1/8/Technical_annex_to_the_postscript_P1-6.pdf

Here we offer a new explanation for hyperbolic discounting that does not rely on arguments relating to the failure of altruism, requiring only rational uncertainty over the long-term survival of the planning entity. Since, as we show below, the empirical survival function of a multi-generation family has a hazard rate that declines over time, hyperbolic discounting applies naturally to the planning problem of a family trust. As well as analysing the family trust case, we show more generally that the standard constant relative risk aversion (CRRA) consumption planning model under horizon uncertainty implies time-varying draw-downs, and constant draw-down rates are a special case.

We use numerical optimisation methods calibrated to UK birth statistics and a representative investment model to newly estimate optimal spending paths for a family trust. (We make use of the theory of branching processes to calculate the probability of family extinction at each generation.) For a risk-averse foundation or trust expecting real investment returns at 4.75% each year, the ideal real annual spending rate begins around 1.65% of wealth, compared with the infinite horizon optimal rate of 1.54%, and declines slowly as the generations pass, then increases steeply as the family reaches extinction.

1. Literature

There is a well-documented tendency among people and animals to discount near events more than distant events (Loewenstein and Thaler, 1989; Ainslie, 1975). This type of impatience, which decreases as time horizon increases, can be modelled by a hyperbolic function. The hyperbolic discounting function was first used to describe the behaviour of pigeons, but can also explain anomalies in human behaviour, notably in savings patterns

(Laibson, Repetto and Tobacman, 1998), where stated intentions and realized actions are sometimes inconsistent.

In his pioneering work on aggregate savings, Ramsey (1928) asserts that any positive discounting of the future is ‘ethically indefensible and arises merely from the weakness of the imagination’. Ramsey actually relents from this uncompromising view by using a non-zero discount rate in the analysis which follows this statement, but he does rule out the possibility of ‘savings being selfishly consumed by a subsequent generation’. Others are less optimistic about the strength of imagination than Ramsey, allowing that the current generation could be less-than-perfectly altruistic towards future generations. Phelps and Pollack (1968), for example, consider a multi-generation model of consumption and saving: consumption in period t is discounted by $b\upsilon^t$ where υ is the rate of time preference and $b,\ (0 < b < 1)$ represents the current generation’s altruism. The closer b is to one, the more concerned is the current generation about the welfare of future generations. They recognise that if succeeding generations have these same quasi-hyperbolic preferences but cannot control the savings behaviour of their descendants, the outcome is a Nash-equilibrium where saving is lower than the Pareto-optimal level. The current generation rationally consume faster than the Pareto-optimal rate in an effort to limit over-spending by their children and grandchildren.

Similar outcomes can occur when a buffer-stock consumer plays an intra-personal game with future ‘selves’ (Harris and Laibson, 2001). The ‘current self’, a hyperbolic discounter, expects ‘future selves’ to over-consume relative to the current self’s preferences. Harris and Laibson show that the effective rate of impatience in this case depends on future scarcity, is stochastic, and endogenous to the model. Laibson, Repetto and Tobacman argue that this type of discounting can explain savings behaviour that seems inconsistent with a standard exponentially discounted model.

On the other hand, the prior question of why decision makers might use hyperbolic discounting remains open. Two recent studies have given alternative explanations for decreasing impatience over long horizons, both related to uncertainty about future payoffs. Sozou (1998) looks at a payoff of fixed size, v_0, t periods in the future, which has current value $v(t) = v_0 \exp(-\lambda t)$. The instantaneous hazard rate is λ and the probability of receiving the payoff in period t is determined by the survival function $\bar{F}(t) = \exp(-\lambda t)$. However if the consumer does not know the true underlying value of the (constant) hazard rate, but holds a prior belief that λ is exponentially distributed, then he or she will compute a hyperbolic discount function by Bayesian updating.

Dasgupta and Maskin (2005) argue that while both a declining hazard rate and Sozou's analysis can produce hyperbolic discounting behaviour in the sense of decreasing impatience, these cannot explain preference reversals, where a consumer switches from one course of action to another simply because of the passage of time, or time-inconsistent behaviour. Dasgupta and Maskin's own explanation for hyperbolic discounting rests on uncertainty over when, rather than whether, a payoff will be realized. They give the example of a blackbird waiting for fruit to ripen before eating, subject to uncertainty about ripening time and over the 'survival' of the fruit (payoff), which may be ambushed by a flock of impatient crows. Uncertainty over when the payoff will be realized (ripening) rather than just whether it will be realized (crow ambush), can result in a preference reversal. In addition, if the blackbird needs to learn about ripening times, we can observe time-inconsistency.

Here we derive the general result that rational agents facing uncertainty over a long-term planning horizon often exhibit time-varying rates of impatience that are derived from the probability of 'survival'. Constant discounting is a special case arising when horizon length follows a known exponential distribution (section 2 below), but agents will exhibit hyperbolic discounting when survival is Pareto distributed. Pareto distributed survival implies declining hazard rates and consequently decreasing impatience over more distant events. In section 3 we estimate the survival function for a representative UK family: we find that the Pareto distribution is a good fit to current fertility data, that extinction is certain at observed birth rates, and that the mean survival of a UK family is about six generations. Numerical estimates of optimal spending paths for a family trust (section 4) using the estimates hyperbolic discounting function are shallow U-shaped curves, always above the infinite-horizon spending rate.

2. Discounting under horizon uncertainty

Our problem is to generalise the model of optimal draw-down for an infinitely-lived entity facing uncertain investment returns to include the case where the survival of the entity is uncertain.[2] For a family trust, consumption stands for payments to current family beneficiaries, funded from an investment portfolio. The family faces two sources of uncertainty: stochastic returns to the trust fund, and random survival of the family.[3] We

[2] For a general discussion of this standard problem see Ingersoll (1987) and Korn and Korn (2001).
[3] In some respects this analysis follows Dasgupta and Maskin (2005).

do not specify the distribution of investment returns, except that they are assumed independent and identically distributed (i.i.d). The trust is extinguished when the family ceases, so we treat residual trust funds as having no utility value when the family is not alive to enjoy them.

In most common law jurisdictions a family trust deed is invalid if it attempts to tie up wealth for the benefit of generations not yet in existence. The common law 'rule against perpetuities' (Burke 1976), or codified law relating to the same issue, usually requires the interest (assets) in the trust to be vested (passed to beneficiaries) within 80 or 90 years. However some US jurisdictions are allowing large dynastic trusts to avoid the restrictions of the rule and exist for much longer before vesting. In our analysis we assume either that the trust is exempt from the 'rule against perpetuities', or equivalently that the trust continues under a 'rolling' deed which the family voluntarily recreates at each generation.[4]

Let T be the random time the family survives. We treat the survival time as a continuous random variable and denote as *pdf(t)* the probability density of T, the extinction density with distribution function $F(t)$, and $\bar{F}(t)$ its complementary distribution function. It follows that $\frac{\partial \bar{F}(t)}{\partial t} = -pdf(t)$ and $\bar{F}(0) = 1,\ \bar{F}(\infty) = 0$ so that the family survives almost surely in period zero but eventual extinction is inevitable. We discuss the inevitability of extinction further below.

The trust aims to maximise expected utility for as long as the family survives, where utility is derived from consumption (the distributions of funds) out of stochastic

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[4] We thank Mr Vincent Taubman of TD Asset Management for drawing our attention to some of the legal constraints on trust deeds and trustees.

wealth. Let the utility of consumption be $\bar{U}(C(t)) = U(C(t))h(t)$ where *h(t)* is some positive discount function expressing general impatience; possibly *h(t) = 1*, and for now we assume *h(0) = 1*. If we write $\underset{C}{E}$ to mean expectation over consumption, the value of expected utility conditioning on survival until time *t* is:

$$L(t) = \underset{C}{E}\left[\int_0^t \bar{U}\left(C(s)\right)ds \mid T = t\right] \tag{1}$$

and $L = \int_0^\infty L(t)\, pdf(t)dt,$ the unconditional value.

Now, integrating by parts,

$$\begin{aligned} L &= \int_0^\infty \underset{C}{E}\left[\int_0^t \bar{U}(C(s))ds\right] pdf(t)dt \\ &= -\left[\underset{C}{E}\int_0^t \bar{U}(C(s))ds\bar{F}(t)\right]_0^\infty + \underset{C}{E}\left[\int_0^\infty \bar{F}(t)\bar{U}(C(t))dt\right], \end{aligned} \tag{2}$$

using Leibnitz's rule, simplifying, and noting the first expression is zero when $\underset{C}{E}\left[\int_0^0 \bar{U}(C(s))ds\right] = 0$, we arrive at

$$L = \underset{C}{E}\left[\int_0^\infty \bar{F}\left(t\right)\bar{U}(C(t))dt\right]. \tag{3}$$

If survival is exponentially distributed with a constant hazard rate, $\bar{F}(t) = \exp(-\lambda t)$ and if general impatience is constant so that $h(t) = \exp(-pt)$, we recover Blanchard's (1985) result,

$$L = \underset{C}{E}\left\{\int_0^\infty \exp\left[-(p+\lambda)t\right]U(C(t))dt\right\}. \tag{4}$$

In other words, uncertainty over family survival simply increases impatience by a constant hazard rate, raising consumption permanently above the optimal rate for an infinitely-lived dynasty.

However this analysis can also deal with hyperbolic discounting and quasi-hyperbolic discounting.[5] Define hyperbolic discounting as a discount function

$$D(t) = (1+\beta t)^{-\gamma/\beta}; \quad \beta > 0,\ \gamma > 0, \tag{5}$$

and quasi-hyperbolic discounting by

$$D(t) = b\upsilon^t;\ 1 > b > 0,\ 0 < \upsilon < 1. \tag{6}$$

If we arbitrarily set general impatience at zero so that *h(t) = 1*, then the survival function is

$$\bar{F}(t) = (1+\beta t)^{-\gamma/\beta}\ , \tag{7}$$

and

$$pdf\,(t) = \gamma(1+\beta t)^{-(\gamma/\beta+1)} \tag{8}$$

is the density of family extinction, whilst for quasi-hyperbolic discounting the survival function is

$$\bar{F}(t) = b\upsilon^t\ ,$$

and the density of family extinction is

$$\begin{aligned} pdf\,(t) &= -\ln(\upsilon) b\upsilon^t = \ln\left(\frac{1}{\upsilon}\right) b\upsilon^t \\ &= \ln\left(\left(\frac{1}{\upsilon}\right)^b\right)\upsilon^t. \end{aligned} \tag{9}$$

[5] We note that Harris and Laibson (2001) actually include $\bar{U}(C_0)$ as a component of *L*. They call this the ‘Utility Boost’, a term we set to zero.

The first density (8) can be thought of as Pareto, where the family survives almost surely in the initial period $\bar{F}(0)=1$, but is extinct in the limit $\bar{F}(\infty)=0$. The second (9) is not normalised in that $\bar{F}(0)=b,$ and $\bar{F}(\infty)=0.$ If $\upsilon=\exp(-\lambda), \lambda>0$, and $\bar{F}(t)=b\exp(-\lambda t)$, this is similar to (4) and corresponds to the case where a proportion of families, *(1-b)*, die initially at *t = 0*; so $\bar{F}(0)=b$. However, we could also have the case of a degenerate extinction probability in that $\bar{F}(t)=(1-\rho)+\rho\bar{F}^{*}(t)$ where $\bar{F}^{*}(t)$ has the usual property ensuring extinction in the limit, $\bar{F}^{*}(\infty)=0$. This brings about no important changes but allows us to incorporate $(1-\rho)$, a finite probability that the family will last forever.

We now apply this analysis by estimating the density function of family survival for a typical UK family using current fertility data.

3. Estimating family extinction

The probability that a family eventually reaches extinction along the female or male line generally will depend on the average number of daughters (sons) born to women (men) in the family. The expected number of children of one gender or the other born to any individual mother or father can be written as

$$\begin{aligned} n &= a_0 + 2a_2 + 3a_3 + ..., \\ \sum a_i &= 1, \end{aligned} \tag{10}$$

where a_i is the probability that a parent has exactly *i* children of their own gender. It is possible to show that, under some simplifying assumptions, the critical rule for eventual family extinction is that $n<1$ (Steffensen, 1995 and Christensen, 1995). So in the case where the expected number of, say, daughters born to mothers is greater than one, the probability of family survival down the female line, $(1-\rho)$, is non-zero in the limit. In the case where $n<1$, $\bar{F}(\infty)=0$ and the family will eventually become extinct.

Theories of family extinction and the related literature on branching processes are associated with Sir Francis Galton, who posed the problem of the survival of aristocratic surnames in 1873 (see Harris, 1963 and Kendall, 1966.) Despite the fact that this is a standard problem in population studies, it appears that few empirical estimates of family extinction are available (Albertsen, 1995). But the question remains interesting since changing fertility patterns in the 20th century mean that extinction probabilities are likely to have increased in western countries. In the UK for example, the number of children of either sex being born to each woman (total fertility rate) was around 1.79 in 2005, which is below the long-term replacement rate required to maintain a stable population, and much less than the peak total fertility rate of 2.95 occurring in 1960.

Earlier studies are rare, but do give estimates of extinction probabilities that are less than one in the limit, reflecting higher historical fertility rates. For example, Lotka (1931) published an estimate of 0.8797 for the probability of male line extinction for the US white population of 1920, and Keyfitz (1968) calculated the likelihood of female line extinction at 0.8206 using 1960-61 US data, along with similar calculations for Hungary, Israel, Mexico and Japan. Hull (1998) reconsidered Lotka’s calculations in the context of a population with two sexes and concluded that, under some restrictions over the availability of partners to the males of concern, the extinction probability lies in the range (0.856, 0.992], not greatly different from Lotka’s original estimate.

Our estimation follows the method of Keyfitz: beginning with the official UK statistics on the distribution of women over the age of 45 by number of live births, we adjust this probability to the number of daughters and then compute the likelihood that the female line becomes extinct in the limit, along with the probability that the family line becomes extinct at any particular generation. We cannot be sure that the same distribution of birth probabilities applies to the male line, since paternity data collected in the UK are incomplete and there is no comparable table of birth probabilities for men, but it is plausible that a dynasty which passes its wealth through sons rather than daughters might face similar survival probabilities. In addition, we work on the assumption that the group of families who create inherited trusts have fertility patterns the same as the population average. There might be reasons to assume both higher (better health prospects), or lower (more educated women with later first births, Rendall et al., 2005) birth rates.

In 1930, two Danish mathematicians, Steffensen and Christensen, separately and simultaneously solved Galton's problem, proving that the probability that any family line reaches extinction at generation *g* can be computed by the recursion

$$x_g = a_0 + a_1 x_{g-1} + a_2 x_{g-1}^2 + a_3 x_{g-1}^3 + ... \qquad (11)$$

where x_{g-1} is the probability of extinction at or before generation *g-1,* and a_i is as defined for equation (10). In the limit, this probability approaches one when $n<1$. If, at the first generation, we set the probability of family extinction $x_1 = a_0$, where a_0 is the probability that the first female in the family (the establisher of the trust) has no daughters, and we assume that the probability that each generation has exactly zero, one, two or more daughters is the same for this particular family as for the a certain representative cohort of mothers, we can estimate a series for x_g using UK national cohort data on births. We also assume that each subsequent generation has the same constant and known fertility distribution.

Table 1 sets out the estimated probability that a woman born in 1960 in England or Wales has a specified number of children. By assuming that girls and boys are equally likely to be born live,[6] we also derive the corresponding probability that a woman gives birth to the specified number of girls, where the probability of R=r girls among n children[7] is

$$P(R=r)=C_r^n\left(0.5\right)^n . \qquad (12)$$

The values in the lowest row of Table 1 are estimates of the probability that a particular family has exactly zero, one, two, three or four or more daughters, that is, a_i in equations (10) and (11). By substituting these values into (10) and checking whether $n<1$, we can infer the overall likelihood of family extinction along the female line: the expected number of daughters to a woman born in 1960 is 0.945<1, which satisfies the condition for eventual family extinction. Further, by substituting these values into (11), setting the initial probability of extinction at $x_1=a_0=0.38$, and then generating $\left\{x_g\right\}_{g=2}^{\infty}$ recursively as $x_g=a_0+a_1x_{g-1}+a_2x_{g-1}^2+a_3x_{g-1}^3+a_4x_{g-1}^4$, we can compute the likelihood that the representative family becomes extinct at any particular future generation.

[INSERT TABLE 1 HERE]

The generational survival probability

[6] In fact, boys are slightly more likely to be born than girls, but also suffer higher average mortality for most of life.

[7] We assume that the probability that a woman has exactly 4 children is 0.1 to make this estimate, i.e., that all families are 4 children or less.

$$s_g = 1 - x_g\,,\;\; x_g = a_0 + a_1 x_{g-1} + a_2 x_{g-1}^2 + a_3 x_{g-1}^3 + a_4 x_{g-1}^4 \qquad (13)$$

derived using the probabilities a_i in Table 1, begins at 1 initially, decreases steeply over the first few generations and converges slowly towards zero, as we can see from the second column in Table 2. The generation *g* hazard rate, λ_g, that is the risk of extinction at the current generation conditioning on the family having survived so far, is set out in column four.

[INSERT TABLE 2 HERE]

Since the estimated hazard rate is declining with time, we expect that family trustees with rational uncertainty over survival will discount future consumption with decreasing impatience as the time horizon lengthens.

In section 2 above, we proposed that the planning horizon, here limited by family survival, might be exponentially distributed, so that $\bar{F}(t) = \exp(-\lambda t)$ or Pareto distributed so that $\bar{F}(t) = (1+\beta t)^{-\gamma/\beta}$. The recursively computed survival function in Table 2 represents a discrete analogue to the continuous cumulative survival distribution $\bar{F}(t)$. By fitting both an exponential and a hyperbolic curve to the discrete survival function, we can estimate values for the constant exponential hazard rate λ, and the parameters of the hyperbolic function, γ and β.

To find the best fitting continuous distribution function, we calculate 100 generations of discrete survival probabilities, and space each generation 45 years apart.[8] We then fit the curve

[8] Average age at first birth for this birth cohort is 27.8, and higher (over 30) for more educated women (see Rendall et al. 2005), but we complete a generation after all children are born, and the end of the fertile period is assumed to be age 45.

$$\hat{y}_{\exp,i} = e^{-\hat{\lambda}_{\exp} 45i} = e^{-\hat{\lambda}_{\exp} t}, \tag{14}$$

where $\hat{\lambda}$ is the estimated hazard rate which minimises the sum of squared errors

$$\min_{\lambda} f(\lambda) = \sum_{i=0}^{100} (y_i - \hat{y}_i)^2 = \sum_{i=0}^{100} (y_i - e^{-\lambda_{\exp} 45i})^2 . \tag{15}$$

The fitted exponential curve is shown in Figure 1 below. Here, $\hat{\lambda}$ is 0.0063, which is analogous to a constant discrete-time subjective discount factor, of 0.994 per year.[9] In other words, under these assumptions, an expectation of current average rates of family extinction creates mild impatience. However the graph shows that the fit of the function is poor, with the exponential approximation under-predicting and then over-predicting discrete recursive survival probabilities. The sum of scaled squared errors, a guide to the accuracy of the exponential approximation, is $sse_{\exp} = \sum_{i=0}^{100} \frac{(y_i - \hat{y}_{\exp,i})^2}{\hat{y}_{\exp,i}} = 3,071,600$. The mean time to extinction under the estimated exponential distribution is 158.7 years, or 3.53 generations of 45 years.

[INSERT FIGURE 1 HERE]

A hyperbolic function is a better approximation to the family survival function. Using the same recursively computed discrete survival probabilities, we fit the Pareto function

$$\hat{y}_{hyp,i} = (1 + \hat{\beta}(45i))^{-\hat{\gamma}/\hat{\beta}} \;, \hat{\beta} \geq 0, \; \hat{\gamma} > 0 \,, \tag{16}$$

where the parameters γ and β minimise the sum of squared errors,

[9] The exponential approximation is fitted by the Matlab function FMINSEARCH which computes a numerical non-linear optimisation by a simplex method.

$$\min_{\gamma,\beta} f(\gamma,\beta) = \sum_{i=0}^{100}(y_i - \hat{y}_i)^2 = \sum_{i=0}^{100}(y_i - (1+\beta(45i))^{-\gamma/\beta})^2 \tag{17}$$

Figure 2 shows the fitted curve when the estimated survival function is $\bar{F}(t) = (1+\hat{\beta}t)^{-\hat{\gamma}/\hat{\beta}}$; and the estimated parameter values are $\hat{\beta} = 0.0076$ and $\hat{\gamma}$=0.0111. In this case the sum of scaled squared errors is $sse_{hyp} = \sum_{i=0}^{100} \frac{(y_i - \hat{y}_{hyp,i})^2}{\hat{y}_{hyp,i}} = 0.2471$, much lower than the exponential curve. The mean of the extinction function is

$$E(t) = \int_0^\infty t\gamma(1+\beta t)^{-(\frac{\gamma}{\beta}+1)} dt = \left[\frac{1+t\gamma}{(\beta-\gamma)} \frac{1}{(1+\beta t)^{\frac{\gamma}{\beta}}} \right]_0^\infty = \frac{1}{(\gamma-\beta)}. \tag{18}$$

The limit of the integral as $t \to \infty$ can be derived by L'Hopital's rule since

$$\lim_{t\to\infty} \frac{1+t\gamma}{(\beta-\gamma)(1+\beta t)^{\frac{\gamma}{\beta}}} = \lim_{t\to\infty} \frac{1}{(\beta-\gamma)(1+\beta t)^{\frac{\gamma}{\beta}-1}},$$

which goes to zero when $\frac{\gamma}{\beta} > 1$.

For the estimated parameter values $\hat{\beta} = 0.0076, \hat{\gamma} = 0.0111$, the expected value of the distribution, or the mean survival of the typical UK family from this cohort is 285.7 years, or 6.3 generations of 45 years. Hence the hyperbolic distribution predicts a much slower mean extinction time than the exponential distribution.

[INSERT FIGURE 2 HERE]

Had we sufficient data, we could make a statistical comparison between the rival exponential and hyperbolic functions, but that would also entail dealing with some complex issues of testing.[10] Statistically, the exponential distribution is nested inside the

[10] We intend to test this hypothesis in later work.

hyperbolic distribution, being the special case where $\beta = 0$. This restriction corresponds to a boundary value for the parameter space of beta values for the hyperbolic distribution. Further, the distribution of a test statistic based upon likelihood ratio principles is a weighted sum of chi squared variables with the weights depending upon nuisance parameters. For the moment, we shall content ourselves with noting that the hyperbolic seems a much better fit, both on visual grounds, and in terms of sum of squared errors, and continue to work with the assumption that the survival probabilities are known with certainty.

Using the estimated parameters, the formula for the hyperbolic hazard rate is

$$\hat{\lambda}_{hyp} = -\frac{\bar{F}'(t)}{\bar{F}(t)} = \frac{\hat{\gamma}}{1+\hat{\beta}t} \qquad (19)$$

whereas the exponential hazard rate is the constant $\lambda_{\exp}$. We compare the constant exponential hazard with the hyperbolic hazard in Figure 3 below. The 225 years along the horizontal axis corresponds to five 45-year generations. Over that time the hyperbolic hazard rate declines from around 0.011 to close to 0.004, against the constant exponential approximation of 0.0063.

[INSERT FIGURE 3 HERE]

Having derived an approximate survival density for a family, we can now apply the analysis of section 3 to the trust planning problem.

4. Family trust draw-down with hyperbolic discounting

Here we present estimates of the impact of uncertain survival on the optimal draw-down rate of a family trust. The continuous time hyperbolic function we fitted to recursive family survival probabilities acts a time-varying discount rate for the optimal draw-down of the family trust. In other words, the rate at which the family becomes extinct is the rate at which future consumption will be discounted.

Consider the discrete-time approximation to the utility maximisation problem set out in equation (3). The family trust plans to maximise expected utility over consumption C_t (payments to beneficiaries or disbursements to worthy causes), by choosing each period a draw-down from uncertain wealth $m_t W_t$, where the gross returns to the trust's investment portfolio are denoted $\tilde{Z}_t$. We represent the probability of family survival at time *t* by the time-varying parameter, $\delta_t \left(= (1+\beta t)^{-\gamma/\beta}\right)$, where *t* now takes integer values for years. This parameter can be interpreted as a discrete-time analogue to the continuous cumulative survival density $\bar{F}(t)$ and represents the discount factor at time *t*.

Assuming that utility is time-separable and additive, the trust's problem is to maximise expected utility from consumption, L_0,

$$L_0 = E_0\left(\sum_{t=0}^{\infty} \delta_t U(C_t)\right), \quad \text{where } 0 < \delta_t < 1, \tag{19}$$

$$C_t = m_t W_t \tag{20}$$

$$W_{t+1} = (1 - m_t) W_t \tilde{Z}_t . \tag{21}$$

We can rewrite (21), the difference equation in wealth as,

$$W_t = W_0 \prod_{i=0}^{t-1} (1 - m_i) \tilde{Z}_i . \tag{22}$$

We define $\tilde{V}_{t-1} = \prod_{i=0}^{t-1} \tilde{Z}_i$, where $\tilde{V}_{t-1}$ is the accumulated value of one unit of wealth invested at time 0 and held until time *t*; it is random and assumed non-negative. If

$\tilde{Z}_i$ are also non-negative and independent and identically distributed (i.i.d.), then $\left(\tilde{V}_{t-1}\right)^{\theta}$ has a constant mean and $\left[E\left(\tilde{V}_{t-1}{}^{\theta}\right)\right]^{\frac{1}{t}}$ is constant for all t, if the mean exists. Thus equation (22) can be written as:

$$W_t = W_0 \tilde{V}_{t-1} \prod_{i=0}^{t-1} \left(1 - m_i\right), \tag{23}$$

and expected utility as,

$$L_0 = E_0\left[\sum_{t=0}^{\infty} \delta_t U\left(m_t W_0 \tilde{V}_{t-1} \prod_{i=0}^{t-1} (1 - m_i)\right)\right]. \tag{24}$$

The first-order condition for optimal draw-down at time t is therefore:

$$\frac{\partial L_0}{\partial m_t} = E_0\left[\begin{array}{l} \delta_t U'\left(m_t \prod_{i=0}^{t-1} (1 - m_i) W_0 \tilde{V}_{t-1}\right) \cdot W_0 \tilde{V}_{t-1} \prod_{i=0}^{t-1} (1 - m_i) - \\ \sum_{j=1}^{\infty} \delta_{t+j} U'\left(m_{t+j} \prod_{i=0}^{t+j-1} (1 - m_i) W_0 \tilde{V}_{t+j-1}\right) \cdot m_{t+j} W_0 \tilde{V}_{t+j-1} \prod_{i=0, i \neq t}^{t+j-1} (1 - m_i) \end{array}\right] = 0. \tag{25}$$

Explicit solutions for the draw-down rate depend on the form of the utility function. For log utility, $U(C_t) = \ln(C_t)$,

$$U'(C_t) = C_t^{-1} = \left[m_t W_0 \tilde{V}_{t-1} \prod_{i=0}^{t-1} (1 - m_i)\right]^{-1}. \tag{26}$$

After substituting (26) into (25) and simplifying we get

$$\frac{\partial L_0}{\partial m_t} = \frac{\delta_t}{m_t} - \frac{\sum_{j=1}^{\infty} \delta_{t+j}}{\left(1 - m_t\right)} = 0,$$

so that,

$$m_t = \frac{\delta_t}{\delta_t + \sum_{j=1}^{\infty} \delta_{t+j}} = \frac{\delta_t}{\sum_{j=0}^{\infty} \delta_{t+j}} . \qquad (27)$$

Equation (27) will be constant for a constant δ, so that draw-down depends entirely on the constant rate of time preference and

$$\bar{m} = 1 - \delta . \qquad (28)$$

Using the condition that $\frac{\partial L_0}{\partial m_t} = \frac{\partial L_0}{\partial m_{t+1}} = 0$ at the optimum we can write the change in the draw-down path as

$$\frac{m_{t+1} - m_t}{m_t} = \frac{\delta_{t+1}}{\delta_t} \left(\frac{\sum_{j=0}^{\infty} \delta_{t+j}}{\sum_{j=1}^{\infty} \delta_{t+j}} \right) - 1, \qquad (29)$$

As the term in brackets in (29) approaches one, the proportional change in the optimal draw-down rate varies with the discrete-time hazard rate, $\lambda = \frac{\delta_{t+1} - \delta_t}{\delta_t}$. For the hyperbolic (Pareto) survival function, this hazard rate is declining over time, so the proportional change in the draw-down also declines.

For constant relative risk aversion (CRRA) utility where $U(C_t) = \frac{C_t^{1-\alpha}}{1-\alpha}, U'(C_t) = C_t^{-\alpha}$, and α is the coefficient of relative risk aversion, an analogous result obtains. Defining the risk-adjusted expected return to wealth as $\varphi = E(\tilde{Z}^{1-\alpha})$ and $\varphi^t = E(\tilde{V}_{t-1}^{1-\alpha})$, the optimal draw-down at time *t* when the discount rate is constant is

$$m = 1 - (\delta\varphi)^{\frac{1}{\alpha}} . \qquad (30)$$

However if the discount rate varies with time, then combining the utility function with (25) gives

$$m_t = \left\{ \left[\frac{1}{\delta_t} (\delta_{t+1} m_{t+1}^{1-\alpha} \varphi + \sum_{j=2}^{\infty} \delta_{t+j} m_{t+j}^{1-\alpha} \varphi^j \prod_{i=t+1}^{t+j-1} (1-m_i)^{1-\alpha}) \right]^{\frac{1}{\alpha}} + 1 \right\}^{-1} \tag{31}$$

and the change in the drawdown rate will be

$$\begin{aligned} \frac{m_{t+1}}{m_t} &= \frac{\left[\frac{1}{\delta_t} (\delta_{t+1} m_{t+1}^{1-\alpha} \varphi + \sum_{j=2}^{\infty} \delta_{t+j} m_{t+j}^{1-\alpha} \varphi^j \prod_{i=t+1}^{t+j-1} (1-m_i)^{1-\alpha}) \right]^{\frac{1}{\alpha}} + 1}{\left[\frac{1}{\delta_{t+1}} (\delta_{t+2} m_{t+2}^{1-\alpha} \varphi + \sum_{j=3}^{\infty} \delta_{t+j} m_{t+j}^{1-\alpha} \prod_{i=t+2}^{t+j-1} (1-m_i)^{1-\alpha} \varphi^{j-1}) \right]^{\frac{1}{\alpha}} + 1} \\ &= \frac{\left(\frac{1}{\delta_t} \varphi \right)^{\frac{1}{\alpha}} \left[(\delta_{t+1} m_{t+1}^{1-\alpha} + \sum_{j=2}^{\infty} \delta_{t+j} m_{t+j}^{1-\alpha} \prod_{i=t+1}^{t+j-1} (1-m_i)^{1-\alpha} \varphi^{j-1}) \right]^{\frac{1}{\alpha}} + 1}{\left(\frac{1}{\delta_{t+1}} \right)^{\frac{1}{\alpha}} \left[(\delta_{t+2} m_{t+2}^{1-\alpha} \varphi + \sum_{j=3}^{\infty} \delta_{t+j} m_{t+j}^{1-\alpha} \prod_{i=t+2}^{t+j-1} (1-m_i)^{1-\alpha} \varphi^{j-1}) \right]^{\frac{1}{\alpha}} + 1} \\ &= \frac{\left(\delta_{t+1} \varphi \right)^{\frac{1}{\alpha}} \left[(\delta_{t+1} m_{t+1}^{1-\alpha} + \delta_{t+2} m_{t+2}^{1-\alpha} \varphi [(1-m_{t+1})^{1-\alpha} - 1] + \Upsilon \right]^{\frac{1}{\alpha}} + 1}{\delta_t^{\frac{1}{\alpha}} \left[\Upsilon \right]^{\frac{1}{\alpha}} + 1} \end{aligned} \tag{32}$$

where $\Upsilon = (\delta_{t+2} m_{t+2}^{1-\alpha} \varphi + \sum_{j=3}^{\infty} \delta_{t+j} m_{t+j}^{1-\alpha} \prod_{i=t+2}^{t+j-1} (1-m_i)^{1-\alpha} \varphi^{j-1})$. Taking logs,

$$\begin{aligned} \ln(m_{t+1} / m_t) \approx &\frac{1}{\alpha} \ln(\frac{\delta_{t+1}}{\delta_t}) + \frac{1}{\alpha} \ln(\varphi) \\ &+ \left[\frac{1}{\alpha} \ln(\delta_{t+1} m_{t+1}{}^{1-\alpha} + \delta_{t+2} m_{t+2}{}^{1-\alpha} \varphi [(1-m_{t+1}{}^{1-\alpha}) - 1] + \Upsilon) - \frac{1}{\alpha} \ln(\Upsilon) \right]. \end{aligned} \tag{33}$$

So the change in the optimal rate of draw-down will be approximately proportional to

$\left(\frac{\delta_{t+1}\varphi}{\delta_t}\right)^{\frac{1}{\alpha}}$ a function of the decreasing survival probability δ_t, increasing with expected returns and shrinking as risk aversion rises.

Hence the hyperbolic discounting resulting from uncertain family survival results in subtle but important differences in optimal spending plans when compared with the constant draw-downs under an infinite horizon.

Since we cannot find analytical solutions to the spending problem, we compute a numerical optimisation calibrated to historical data. Consider a family trust whose investment return is 4.75% p.a. in real terms.[11] This figure is close the 15 year historical average for a typical UK trust with a well-diversified portfolio. Figure 5 below sets out a numerical estimate of the first 500 years of the optimal draw-down of a family trust whose survival is modelled by the Pareto distribution estimated in Figure 2, where $\delta_t = \bar{F}(t) = (1+\beta t)^{-\gamma/\beta}; \quad \beta \geq 0, \gamma > 0$ and the estimated parameter values are $\hat{\beta} = 0.0076$ and $\hat{\gamma}$=0.0111. The first panel is for a trust with relative risk aversion at 5 and the second panel for a trust with relative risk aversion at 2. The solid line shows the optimal spending rate for the Pareto (hyperbolic) path, while the dotted line is the infinite horizon path where general impatience is set to zero, and the dashed line is the optimal path using the exponential approximation to family survival set out in Figure 1.

Declining hazard rates create a decreasing shape in the hyperbolic curves, but the certainty of eventual extinction ensures that both the hyperbolic and exponential draw-down rates are higher than that for an infinitely lived-trust. An ideal spending plan at

[11] For estimates of investment returns to UK charitable trusts see Satchell and Thorp (2007).

lower risk aversion ($\alpha = 2$) begins close to 1.65% p.a., and drops toward 1.5%. The more risk averse trust ($\alpha = 5$) consumes more slowly, but the optimal path still shows a declining rate in the first few generations. Generally speaking, a lower level of risk aversion means a much steeper decrease in spending earlier in the life of the trust, indicating a stronger preference for current consumption.

[INSERT FIGURE 5 HERE]

5. Conclusion

Recent studies (Sozou, 1998 and Dasgupta and Maskin, 2005) have shown that decreasing impatience can be a rational response to uncertainty over whether or when a future payoff might occur. Uncertainty over horizon is a very common problem for long-term investors, and family trusts are just one example of the many bodies that must consider stochastic 'survival'. Indeed we all have to plan for uncertain lifetimes. By contrast with family survival, which we have modelled using a Pareto distribution with hyperbolically declining hazard rates, individual mortality (at least later in life) is better fitted by the increasing hazards typical of a Gompertz function. An increasing hazard rate suggests rationally increasing impatience, perhaps motivating the elderly aunt who says 'but I'll be dead by then' as a reason for not thinking further than next Christmas.

In the standard inter-temporal consumption model with i.i.d. returns, time-varying hazards mean time-varying optimal draw-down rates, a result that goes against the customary advice to trusts and endowments to spend at a constant rate. Our results have interesting implications for family foundation trustees. Estimates for UK families along the female line, assuming that current fertility patterns stay constant into the future, signal eventual extinction for a typical family. Faced with the resulting hyperbolic survival function, it seems hard for a trustee behaving in the interest of the multi-generational family to justify a policy of constant consumption. The ideal plan spends more rapidly in the near future and steadily but more slowly as the trust ages, moving to a rapid increase in the rate of spending as extinction approaches.

Our results could be applied to more general survival problems, including the survival of financial institutions such as banks, mutual funds or hedge funds, or to more general macroeconomic questions such as the estimation of a social discount rate, questions which we plan to look into in future work.

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Table 1

Estimated distribution of women of child-bearing age by number of children

Population proportion of women born 1960 having number of children (live births) by age 45				
0	1	2	3	4 or more
0.18	0.13	0.38	0.20	0.10
Estimated probability of women born 1960 having number of daughters (live births) by age 45				
0	1	2	3	4
$a_0 = 0.380$	$a_1 = 0.355$	$a_2 = 0.208$	$a_3 = 0.050$	$a_4 = 0.007$

Source: Table 10.5, Birth Statistics, Series FM1 no.34, Office for National Statistics, London, UK.

Note: We infer the probability of daughters by assuming that girls and boys are equally likely, so if the probability that a woman from this cohort has 2 children is 0.38, the probability that one will be a girl is 0.5x0.38 and that 2 will be girls is 0.25x0.38 etc. We use these as estimates of the probabilities a_i in equation (10) that a typical family has exactly zero, one, two, three or four or more daughters.

Table 2

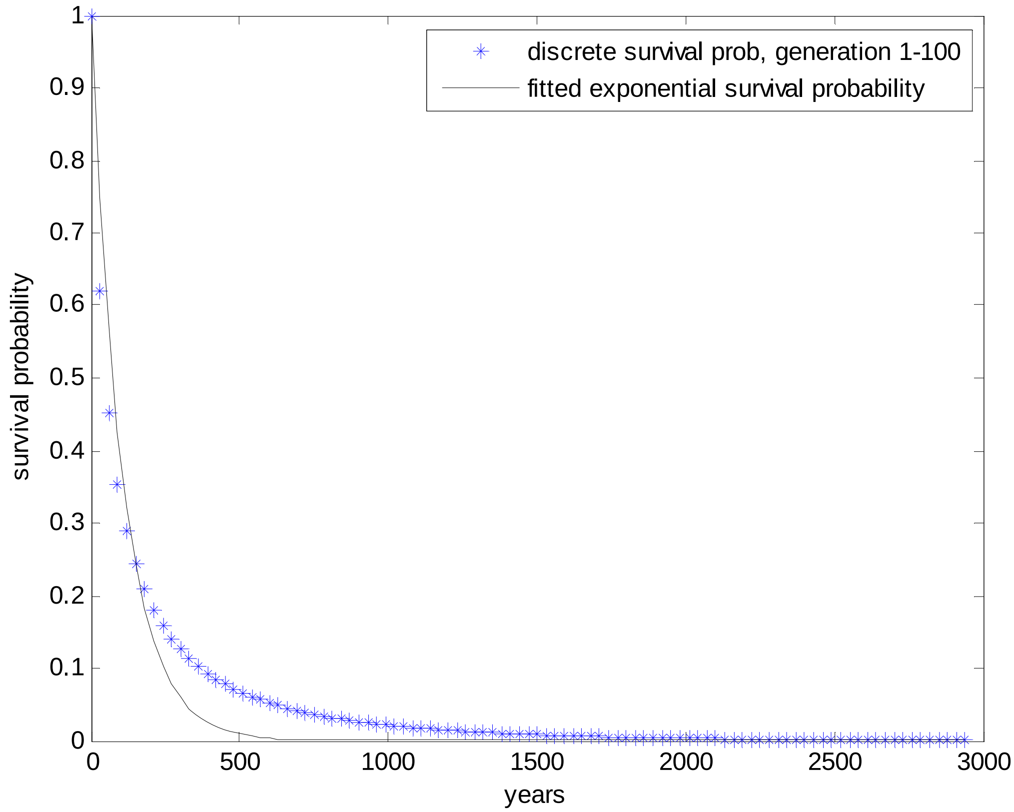
Estimated probability of family survival

Number of Generations	Extinction Probability x_g	Survival Probability $s_g = 1 - x_g$	Hazard Rate λ_g
0	0.000	1.000	
1	0.380	0.620	0.380
2	0.548	0.452	0.271
3	0.646	0.354	0.217
4	0.711	0.289	0.183
5	0.757	0.243	0.160
6	0.792	0.208	0.144
7	0.819	0.181	0.131
8	0.841	0.159	0.121
9	0.859	0.141	0.113
10	0.874	0.126	0.106
11	0.887	0.114	0.100
12	0.897	0.103	0.095
13	0.907	0.093	0.092
14	0.915	0.085	0.088
15	0.922	0.078	0.085
16	0.928	0.072	0.081
17	0.934	0.066	0.080
18	0.939	0.061	0.077
19	0.944	0.056	0.076
20	0.948	0.052	0.073
25	0.964	0.037	0.066
30	0.974	0.026	0.064
35	0.981	0.019	0.059
40	0.986	0.014	0.053
45	0.989	0.011	0.053
50	0.992	0.008	0.047
130	≈ 1.000	≈ 0.000	

Note: Table shows the probability of family extinction and survival down the female line where the probability of extinction at generation *g* is given by $x_g = a_0 + a_1 x_{g-1} + a_2 x_{g-1}^2 + a_3 x_{g-1}^3 + a_4 x_{g-1}^4$ and a_i is the probability that a mother has exactly *i* daughters. Values of a_i are taken from the last row of Table 1, the estimated distribution of daughters to the cohort of mothers born in England and Wales in 1960. The recursion begins with $x_1 = a_0 = 0.38$ and continues with a_i fixed. The hazard rate is the probability of extinction between generation *g-1* and *g*, conditional on having survived to time *g*, which is computed by $\lambda_g = \frac{s_g - s_{g-1}}{s_{g-1}}, s_g = 1 - x_g$.

Fig. 1

Fitted exponential survival function

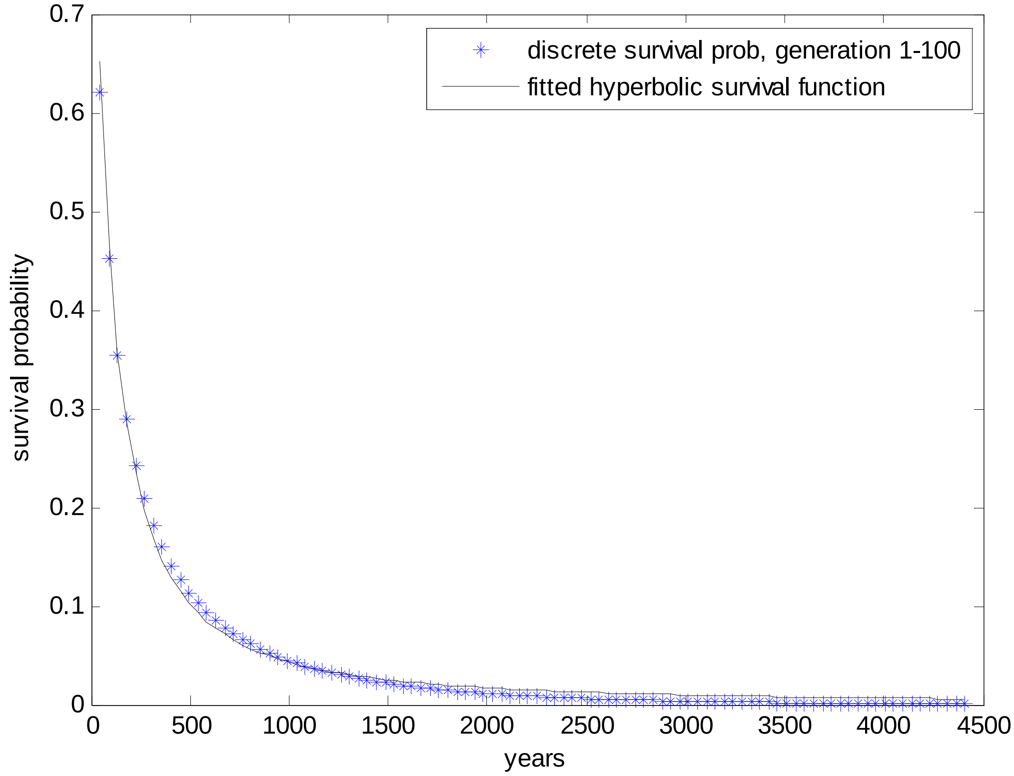


Note: This figure shows a graph of the fitted exponential function $\hat{y}_t = e^{-\hat{\lambda}_{\exp} t}$ where y_i is the probability of family survival at generation *i=t/45* and *t* are years. The generation t/45 survival probabilities are calculated recursively from equation (11) along the female line for the 1960 birth cohort of English and Welsh women, assuming that the likelihood of the birth of 0-4 girls exactly is constant over time and homogeneous across the population. (See Tables 1 and 2 in the text.) Function is fitted by fminsearch in Matlab which uses a simplex method for non-linear optimisation to minimise the sum of squared errors

$$\min_{\lambda} f(\lambda) = \sum_{i=0}^{100} (y_i - \hat{y}_i)^2 = \sum_{i=0}^{100} (y_i - e^{-\lambda_{\exp} 45 i})^2 .$$

Fig. 2

Fitted hyperbolic survival function

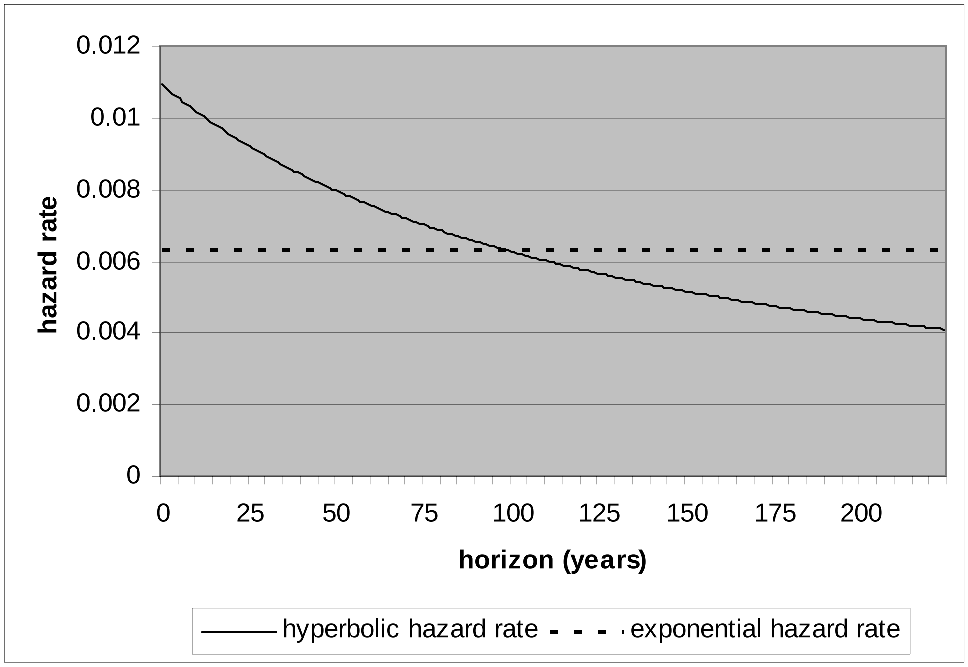


Note: This figure shows a graph of the fitted hyperbolic function $\hat{y}_i = (1+\hat{\beta}(45i))^{-\hat{\gamma}/\hat{\beta}}$. where y_i is the probability of family survival at generation *i=t/45* and *t* are years. The generation *i* survival probabilities are calculated recursively from equation (11) along the female line for the 1960 birth cohort of English and Welsh women, assuming that the likelihood of the birth of 0-4 girls exactly is constant over time and homogeneous across the population. (See Tables 1 and 2 in the text.) Function is fitted by fminsearch in Matlab which uses a simplex method for non-linear optimisation to minimise the sum of squared errors.

$$\min_{\gamma,\beta} f(\gamma,\beta) = \sum_{i=0}^{100}(y_i - \hat{y}_i)^2 = \sum_{i=0}^{100}(y_i - (1+\beta(45i))^{-\gamma/\beta})^2$$

Fig. 3

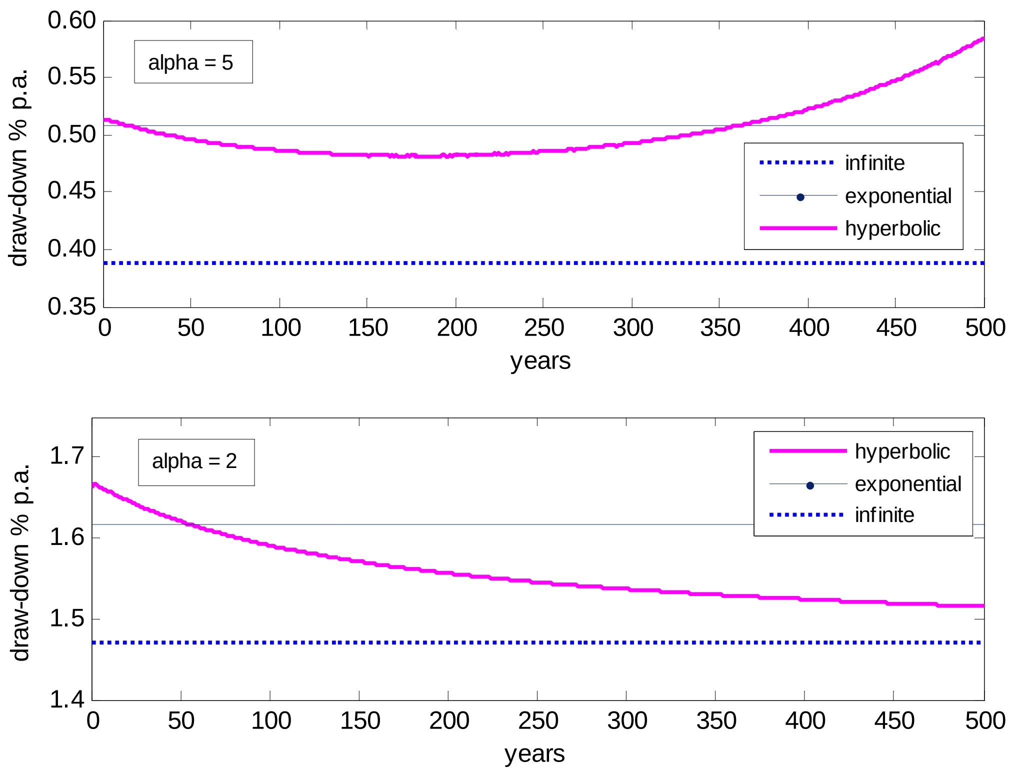
Estimated hazard rate for multi-generational family survival



Note: Figure shows the estimated exponential hazard rate $\hat{\lambda}_{\exp} = 0.0063$ and the estimated hyperbolic hazard rate $\hat{\lambda}_{hyp} = -\frac{\bar{F}'(t)}{\bar{F}(t)} = \frac{\hat{\gamma}}{1+\hat{\beta}t}$ where $\hat{\beta} = 0.0076$ and $\hat{\gamma}$=0.011. See Figures 1 and 2 and the text for estimation details.

Fig. 5

Optimal draw-down with survival uncertainty



Note: Figure shows the estimated optimal spending rates with and without uncertainty over family survival for a trust with power utility preferences $U(C_t) = \dfrac{1}{1-\alpha} C_t^{1-\alpha}$, and investment returns close to 4.75% p.a. Estimates of the risk-scaled investment returns are bootstrapped from historical portfolio returns to a typical investment trust (See Satchell and Thorp 2007). The hyperbolic survival probability of the family is given by the distribution function, $\bar{F}(t) = (1+\beta t)^{-\gamma/\beta}; \quad \beta \geq 0,\ \gamma > 0,\ \ \hat{\beta} = 0.0076$ and $\hat{\gamma}$=0.0111 and the exponential survival probability is $\hat{y}_t = e^{-\hat{\lambda}_{\exp} t}, \hat{\lambda}_{\exp} = 0.0063$. Numerical optimisation is via the fminimax routine in Matlab.