A Benchmark Framework for Risk Management

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Abstract. The paper describes a general framework for contingent claim valuation for finance, insurance and general risk management. It considers security prices and portfolios with finite expected returns, where the growth optimal portfolio is taken as numeraire or benchmark. Benchmarked nonnegative wealth processes are shown to be supermartingales. Fair benchmarked values are conditional expectations of future benchmarked prices under the real world probability measure. Standard risk neutral and actuarial pricing formulas are obtained as special cases of fair pricing. The proposed benchmark framework covers the infinite time horizon and does not require the existence of an equivalent risk neutral pricing measure.

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1 Introduction

New challenges are arising from the need to integrate the modeling of risk in the fields of finance, insurance and other areas of risk management. Also a consistent and more general modeling framework is needed to jointly accommodate the widely used risk neutral and actuarial pricing methodologies. This framework should provide enough flexibility for realistic modeling but, of course, exclude arbitrage.

In the literature one can find various notions of arbitrage, see, for instance, Ross (1976), Harrison & Kreps (1979), Harrison & Pliska (1981), Kreps (1981), Duffie & Huang (1986), Dalang, Morton & Willinger (1990), Lakner (1993), Fritelli & Lakner (1994), Delbaen & Schachermayer (1994, 1998), Karatzas & Shreve (1998), Yan (1998), Jouini, Kallal & Napp (2001), Shiryaev & Cherny (2002), Goll & Kallsen (2003) and Davis (2003). Usually, the word arbitrage means that one cannot generate strictly positive wealth from zero initial capital. In most of the above mentioned cases the exclusion of arbitrage is linked to the existence of an *equivalent risk neutral measure*. In a general semimartingale setting this is formulated in Delbaen & Schachermayer (1994, 1998) as the *Fundamental Theorem of Asset Pricing*.

On the other hand, the *growth optimal portfolio* (GOP) has been studied by several authors, including Kelly (1956), Long (1990), Korn & Schäl (1999), Becherer (2001), Korn (2001), Bühlmann & Platen (2002), Goll & Kallsen (2003) and Platen (2002, 2004). The GOP is the portfolio that maximizes long term expected growth. Furthermore, when used as numeraire under standard risk neutral assumptions, it makes prices that are expressed in units of the GOP martingales under the real world probability measure.

In Platen (2002) and Heath & Platen (2002a, 2002b, 2002c, 2003) realistic asset price models are studied for which no equivalent risk neutral measure exists. To cover these and a wide range of other models this paper assumes weaker conditions for the proposed modeling framework than those typically considered in the literature. It generalizes the results in Platen (2002, 2004) by allowing for a wider range of semimartingale models. In particular, models can be handled where the candidate risk neutral measure is not equivalent to the real world probability measure and/or the corresponding Radon-Nikodym derivative is not a martingale.

The paper presents in Section 2 a general semimartingale benchmark modeling framework. Section 3 lists important properties of benchmarked portfolios. In Section 4 it is shown that the GOP is the best performing portfolio. Contingent claim pricing is considered in Section 5. Finally, Section 6 provides some examples of benchmark models.

2 Semimartingale Benchmark Framework

2.1 Primary Securities

Let us consider a frictionless financial *market model* over the time period $[0, \infty)$. It is assumed that $d+1$ nonnegative *primary security accounts* exist, $d \in \{1, 2, \ldots\}$. Primary securities may generate dividends or other investment income for the owner of the respective security. Let us denote by $S^{(j)}(t)$ the value at time $t \in [0, \infty)$ of the jth *primary security account*, $j \in \{0, 1, \ldots, d\}$. The account $S^{(j)}$ consists of units of the jth primary security with all income reinvested. In the case when the jth primary security account consists of one share starting at the time zero, then $S^{(j)}(t)$ denotes the cum-dividend value of this share at time t including all accumulated dividends. We assume that the units of primary securities are infinitely divisible such that continuous trading is possible.

The primary security account vector process $S = \{S(t) = (S^{(0)}(t), S^{(1)}(t), \ldots, S^{(d)}(t))^\top, t \in [0, \infty)\}$ is assumed to form a semimartingale, which is right continuous with left hand limits defined on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, satisfying the usual conditions, see Protter (1990). The information structure of the market is described by the right continuous filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0,\infty)}$ with $\mathcal{A}_t$ representing the information available at time t and $\mathcal{A}_0$ the σ-algebra consisting of all null sets and their complements. The filtration $\underline{\mathcal{A}}$ is defined as the augmentation under P of the natural filtration $\mathcal{A}^S$, generated by the primary security account vector process S.

In general, not all primary security accounts are available for hedging. However, a unique prescription in the form of a semimartingale vector stochastic differential equation, which characterizes the evolution of the vector $S(t)$ of primary security accounts over the time $t \in [0, \infty)$, is assumed to be given for $S(t)$, see Jacod (1979), Yan (1998) and Shiryaev & Cherny (2002).

2.2 Expected Return

In addition to the given primary security accounts, we consider *wealth processes* $S^{(\delta)} = \{S^{(\delta)}(t), t \in [0, \infty)\}$, also called *portfolios*, which are formed as linear combinations of these accounts, where

$$S^{(\delta)}(t) = \delta^\top(t)\, S(t) \tag{2.1}$$

We denote by $L(S)$ the space of $\Re^{d+1}$-valued, predictable *strategies* $\delta = \{\delta(t) = (\delta^{(0)}(t), \ldots, \delta^{(d)}(t))^\top, t \in [0, \infty)\}$ for which the corresponding gains from trade, that is, the Itô vector integral $\int_0^t \delta^\top(s)\, dS(s)$ exists for all $t \in [0, \infty)$, see Jacod (1979). Here $\delta^{(j)}(t)$ denotes the number of units of the jth primary security account $S^{(j)}(t)$ that are held at time t in the wealth process $S^{(\delta)}(t)$.

A *wealth process* $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0,\infty)\}$ with corresponding *strategy* $\delta \in L(S)$ is called *self-financing* if

$$S^{(\delta)}(t) = S^{(\delta)}(0) + \int_0^t \delta^\top(s)\, dS(s) \tag{2.2}$$

for all $t \in [0,\infty)$. In the following we will only deal with self-financing wealth processes and strategies and therefore from now on omit the word "self-financing". Any process is at any time either nonnegative or negative and can be separated into nonnegative and negative components. A negative wealth process can be interpreted as nonnegative though generated via a short position. Therefore, we concentrate on the analysis of the set of all *nonnegative wealth processes* that are $\underline{\mathcal{A}}$-adapted, right-continuous and have left hand limits. Let us denote by $\mathcal{V}$ ($\mathcal{V}^+$) the set of all nonnegative (strictly positive) wealth processes.

Each market participant holds a nonnegative total wealth process. If the total wealth process becomes zero or negative, then the investor must declare bankruptcy. This means the legally established principle of *limited liability* forces any realistic financial modeling to incorporate the fact that when nonnegative wealth processes reach the level zero they must remain at the level zero. If this were not the case, there would be obvious *arbitrage*. We make this more precise in the following definition.

Definition 2.1 *We say, a nonnegative wealth process* $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0,\infty)\} \in \mathcal{V}$ *permits arbitrage if*

$$P\left(S^{(\delta)}(\tau) = 0\right) = 1 \tag{2.3}$$

and

$$P\left(S^{(\delta)}(\sigma) > 0 \,\middle|\, \mathcal{A}_\tau\right) > 0 \tag{2.4}$$

for any stopping times $\tau \in [0,\infty)$ *and* $\sigma \in [\tau,\infty)$. *A given market model* excludes arbitrage *if no nonnegative wealth process* $S^{(\delta)} \in \mathcal{V}$ *of the above kind exists.*

As can be seen from the above definition, arbitrage will arise if there exists a nonnegative portfolio process, which generates from zero initial capital strictly positive wealth with strictly positive probability. Consequently, by exploiting such arbitrage opportunities, one is able to systematically generate unlimited wealth from nothing.

Let us prepare the formulation of natural assumptions that will exclude arbitrage. An important quantity, of interest to investors, is the expected return of a wealth process. For any portfolio $S^{(\delta)} \in \mathcal{V}$ and pair of stopping times $\tau \in [0,\infty)$ and $\sigma \in (\tau,\infty)$ we call the conditional expectation

$$R^\delta_{\tau,\sigma} = E\left(\frac{S^{(\delta)}(\sigma)}{S^{(\delta)}(\tau)} \,\middle|\, \mathcal{A}_\tau\right) - 1 \tag{2.5}$$

the *expected return of* $S^{(\delta)}$ over the period $[\sigma, \tau]$. Here we set $\frac{0}{0} = 1$. We consider returns also in other denominations. Thus, similarly as above, for any portfolio $S^{(\delta)} \in \mathcal{V}$, strictly positive portfolio $S^{(\underline{\delta})} \in \mathcal{V}^+$ and pair of stopping times $\tau \in [0, \infty)$ and $\sigma \in (\tau, \infty)$ we call

$$R^{\underline{\delta},\delta}_{\tau,\sigma} = E\left(\frac{S^{(\delta)}(\sigma)}{S^{(\underline{\delta})}(\sigma)} \frac{S^{(\underline{\delta})}(\tau)}{S^{(\delta)}(\tau)} \,\middle|\, \mathcal{A}_\tau \right) - 1 \tag{2.6}$$

the expected return of $\frac{S^{(\delta)}}{S^{(\underline{\delta})}}$ over the period $[\sigma, \tau]$. The possibility of unlimited expected returns is economically not realistic and therefore excluded by the following assumption.

Assumption 2.2 *There exists a strictly positive reference portfolio* $S^{(\delta_*)} \in \mathcal{V}^+$ *and for each pair of stopping times* $\tau \in [0, \infty)$ *and* $\sigma \in (\tau, \infty)$ *an* $\mathcal{A}_\tau$*-measurable, nonnegative, integrable random variable* $K^{(\delta_*)}_\tau \in [0, \infty)$ *such that for all nonnegative wealth processes* $S^{(\delta)} \in \mathcal{V}$ *the expected return of* $\frac{S^{(\delta)}}{S^{(\delta_*)}}$ *satisfies the inequality*

$$R^{\delta_*,\delta}_{\tau,\sigma} \leq K^{(\delta_*)}_\tau. \tag{2.7}$$

According to Assumption 2.2 all returns of nonnegative portfolios in the denomination of the strictly positive reference portfolio $S^{(\delta_*)}$ are assumed to be *integrable*, which is a natural condition.

Condition (2.7) in Assumption 2.2 is, for instance, violated if two portfolios exist that have different drifts but the same martingale component. In such circumstances one can form a nonnegative portfolio that starts with zero initial value and rises to any desired expected future value in finite time. Such a possibility indicates the existence of an arbitrage opportunity, which by Assumption 2.2 is excluded.

2.3 Arbitrage

It is important to verify that the above described financial market model does not permit arbitrage. As previously expressed in Definition 2.1, the notion of arbitrage relates to the possibility to generate strictly positive wealth from zero initial capital. We obtained from Assumption 2.2 for nonnegative portfolios an upper bound for their expected returns, when expressed in units of a reference portfolio. Now, we will show that this property is sufficient to exclude arbitrage.

Theorem 2.3 *A financial market model does not permit arbitrage if Assumptions 2.2 is satisfied.*

Proof: Under Assumption 2.2 consider a nonnegative wealth process $S^{(\delta)} \in \mathcal{V}^+$ and a pair of stopping times $\tau \in [0,\infty)$ and $\sigma \in [\tau,\infty)$, where

$$S^{(\delta)}(\sigma) \geq S^{(\delta)}(\tau) = 0 \tag{2.8}$$

almost surely. It follows by (2.6) and (2.7) that

$$E\left(\frac{S^{(\delta)}(\sigma)}{S^{(\delta_*)}(\sigma)} \,\bigg|\, \mathcal{A}_\tau\right) \leq \frac{S^{(\delta)}(\tau)}{S^{(\delta_*)}(\tau)} \left(K_\tau^{\delta_*} + 1\right) = 0 \tag{2.9}$$

almost surely. Obviously, due to relations (2.8) and (2.9) the nonnegative benchmarked value $\hat{S}^{(\delta)}(\sigma)$ cannot be strictly greater than zero with any strictly positive probability. Thus the inequality (2.4) in Definition 2.1 cannot hold, which proves the theorem. □

We remark that in the above financial market model there can be a *free lunch with vanishing risk* in the sense of Delbaen & Schachermayer (1994, 1998). This is equivalent to the absence of an equivalent local martingale measure. Note that the above model does not require any assumption on the existence of an equivalent local martingale measure. However, this does not mean that such a model permits infinite expected returns or infinite expected growth for any nonnegative wealth process, as we will see below. An example of a model without equivalent local martingale measure will be given in Section 6.2. Furthermore, market participants cannot generate in the above model strictly positive wealth from zero initial capital. The existence of an equivalent risk neutral measure seems to be not essential for the construction of a consistent market model, as we will see below. Avoiding this condition provides substantial modeling freedom, as is demonstrated in Platen (2002, 2004) and Heath & Platen (2002a, 2002b, 2002c, 2003) and Breymann, Kelly & Platen (2003). It seems that the condition on the existence of an equivalent local martingale measure is predominantly a mathematical assumption. No economic reason appears to explain why such a property must be imposed on a realistic financial market model.

Some no-arbitrage definitions in the literature permit limited debt, see, for instance, Harrison & Kreps (1979), Harrison & Pliska (1981) and Karatzas & Shreve (1998). For a wide range of models there exists a strictly positive reference portfolio that when used as benchmark or numeraire makes all benchmarked portfolios to local martingales. This applies, for instance, to jump diffusion models. By Fatou's Lemma it can be easily shown that for such models also negative benchmarked portfolios with limited debt form supermartingales and the no-arbitrage property can be extended to include these portfolios.

2.4 Expected Growth

For an investor the *expected growth* of wealth over longer time periods is usually the main quantity of interest. In the following definition we use the convention that $\log(0) = -\infty$ and $\log(\infty) = \infty$. For a pair of stopping times $\tau \in [0,\infty)$, $\sigma \in (\tau,\infty)$ and a given nonnegative wealth process $S^{(\delta)} \in \mathcal{V}$ its *expected growth* $g^{\delta}_{\tau,\sigma}$ over the time interval $[\tau,\sigma]$ is defined as the conditional expectation

$$g^{\delta}_{\tau,\sigma} = E\left(\log\left(\frac{S^{(\delta)}(\sigma)}{S^{(\delta)}(\tau)}\right) \,\bigg|\, \mathcal{A}_\tau\right). \tag{2.10}$$

We can prove the following inequality.

Lemma 2.4 *Under Assumption 2.2 there exists a strictly positive reference portfolio $S^{(\delta_*)} \in \mathcal{V}^+$ such that for all stopping times $\tau \in [0,\infty)$ and $\sigma \in [\tau,\infty)$ any nonnegative wealth process $S^{(\delta)} \in \mathcal{V}$ satisfies the inequality*

$$g^{\delta}_{\tau,\sigma} \leq R^{(\delta_*)}_{\tau,\sigma} + K^{(\delta_*)}_{\tau} \tag{2.11}$$

almost surely.

Proof: Consider a strictly positive reference portfolio $S^{(\delta_*)} \in \mathcal{V}^+$, given by Assumption 2.2, and any nonnegative wealth process $S^{(\delta)} \in \mathcal{V}$ and stopping times $\tau \in [0,\infty)$ and $\sigma \in [\tau,\infty)$. Using the inequality $\log(x) \leq x-1$ for $x \geq 0$ together with (2.5) and (2.10)

$$g^{\delta_*}_{\tau,\sigma} = E\left(\log\left(\frac{S^{(\delta_*)}(\sigma)}{S^{(\delta_*)}(\tau)}\right) \,\bigg|\, \mathcal{A}_\tau\right) \leq R^{(\delta_*)}_{\tau,\sigma} \tag{2.12}$$

and similarly by (2.6) and (2.10) we obtain

$$\begin{aligned} g^{\delta}_{\tau,\sigma} &= E\left(\log\left(\frac{S^{(\delta)}(\sigma)}{S^{(\delta)}(\tau)}\,\frac{S^{(\delta_*)}(\tau)}{S^{(\delta_*)}(\sigma)}\right) \,\bigg|\, \mathcal{A}_\tau\right) + g^{\delta_*}_{\tau,\sigma} \\ &\leq E\left(\frac{S^{(\delta)}(\sigma)}{S^{(\delta)}(\tau)}\,\frac{S^{(\delta_*)}(\tau)}{S^{(\delta_*)}(\sigma)} - 1 \,\bigg|\, \mathcal{A}_\tau\right) + g^{\delta_*}_{\tau,\sigma} \\ &= R^{\delta_*,\delta}_{\tau,\sigma} + g^{\delta_*}_{\tau,\sigma}. \end{aligned} \tag{2.13}$$

Consequently, by (2.13), (2.12) and (2.7) we get

$$g^{\delta}_{\tau,\sigma} \leq R^{\delta_*,\delta}_{\tau,\sigma} + R^{(\delta_*)}_{\tau,\sigma} \leq R^{(\delta_*)}_{\tau,\sigma} + K^{(\delta_*)}_{\tau}.$$

Thus the inequality (2.11) holds almost surely. □

2.5 Growth Optimal Portfolio

The above notion of expected growth allows us to introduce the *growth optimal portfolio* (GOP), which we will use as numeraire or benchmark. The GOP was originally introduced by Kelly (1956). It is the wealth process with maximum expected growth over all finite time intervals. To define this process properly in the given general semimartingale setting we introduce the notion of a *perturbed reference portfolio*.

Definition 2.5 *For a given strictly positive reference portfolio* $S^{(\delta_*)} \in \mathcal{V}^+$*, a nonnegative* perturbing portfolio $S^{(\delta)} \in \mathcal{V}$*, some initial fraction* $\varepsilon \in (0, \frac{1}{2}]$ *and a stopping time* $\tau \in [0,\infty)$ *we define the corresponding* perturbed reference portfolio $S^{(\delta_{\tau,\varepsilon,\delta_*})} = \{S^{(\delta_{\tau,\varepsilon,\delta_*})}(s),\ s \in [\tau,\infty)\}$ *by*

$$S^{(\delta_{\tau,\varepsilon,\delta_*})}(s) = \begin{cases} \varepsilon \left(\frac{S^{(\delta_*)}(\tau)}{S^{(\delta)}(\tau)}\right) S^{(\delta)}(s) + (1-\varepsilon)\, S^{(\delta_*)}(s) & \text{for} \quad S^{(\delta)}(\tau) > 0 \\ S^{(\delta_*)}(s) & \text{for} \quad S^{(\delta)}(\tau) = 0 \end{cases} \tag{2.14}$$

and $s \in [\tau,\infty)$*.*

Note that a perturbed reference portfolio $S^{(\delta_{\tau,\varepsilon,\delta_*})}$ is always strictly positive with value $S^{(\delta_*)}(\tau)$ at initial time τ. Now we define for a given perturbed reference portfolio $S^{(\delta_{\tau,\varepsilon,\delta_*})}$ the derivative of its expected growth with respect to the *initial fraction* $\varepsilon \in (0, \frac{1}{2}]$ of the perturbation.

Definition 2.6 *For a pair of stopping times* $\tau \in [0,\infty)$*,* $\sigma \in [\tau,\infty)$*, a strictly positive reference portfolio* $S^{(\delta_*)} \in \mathcal{V}^+$ *and a nonnegative perturbing portfolio* $S^{(\delta)} \in \mathcal{V}$ *we define the* derivative $Q^{\delta_*,\delta}_{\tau,\sigma}$ of the expected growth $g^{\delta_{\tau,\varepsilon,\delta_*}}_{\tau,\sigma}$ *over the time interval* $[\tau,\sigma]$ *with respect to the initial fraction* ε *as the almost sure limit*

$$Q^{\delta_*,\delta}_{\tau,\sigma} \overset{a.s.}{=} \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left(g^{\delta_{\tau,\varepsilon,\delta_*}}_{\tau,\sigma} - g^{\delta_{\tau,0,\delta_*}}_{\tau,\sigma} \right). \tag{2.15}$$

The derivative $Q^{\delta_*,\delta}_{\tau,\sigma}$ provides information on how a small perturbation of the reference portfolio $S^{(\delta_*)}$ by another wealth process $S^{(\delta)}$ changes the expected growth. If this derivative is zero, then for a small initial fraction $\varepsilon > 0$ there is little change in the expected growth of $S^{(\delta_{\tau,\varepsilon,\delta_*})}$. Note that $Q^{\delta_*,\delta}_{\tau,\sigma}$ is $\mathcal{A}_\tau$-measurable.

The expected growth provides a long term measure for the performance of a portfolio whereas the expected return is more a short term measure for the expected short term increase of wealth. The following lemma establishes an important link between the expected return of the perturbing portfolio and the above derivative of the expected growth. It demonstrates that the expected return is like a derivative of the perturbed expected growth.

Lemma 2.7 *Under Assumption 2.2 there exists a strictly positive reference portfolio* $S^{(\delta_*)} \in \mathcal{V}^+$ *such that for any nonnegative wealth process* $S^{(\delta)} \in \mathcal{V}$ *the identity*

$$Q^{\delta_*,\delta}_{\tau,\sigma} = R^{\delta_*,\delta}_{\tau,\sigma} \tag{2.16}$$

holds for all pairs of stopping times $\tau \in [0,\infty)$ *and* $\sigma \in (\tau,\infty)$*.*

Proof: For $\varepsilon \in (0, \frac{1}{2}]$, $S^{(\delta)} \in \mathcal{V}$, $S^{(\delta_*)} \in \mathcal{V}^+$, $\tau \in [0,\infty)$ and $\sigma \in (\tau,\infty)$ we introduce the notations

$$h^{(\delta)}_{\tau,\sigma} = \frac{S^{(\delta)}(\sigma)}{S^{(\delta)}(\tau)}, \tag{2.17}$$

and

$$V_{\tau,\sigma}^{\delta_*,\delta}(\varepsilon) = \frac{1}{\varepsilon}\,\log\left(\frac{\varepsilon\, h_{\tau,\sigma}^{(\delta)} + (1-\varepsilon)\, h_{\tau,\sigma}^{(\delta_*)}}{h_{\tau,\sigma}^{(\delta_*)}}\right). \tag{2.18}$$

Applying the inequality $\log(x) \leq x - 1$ for $x \geq 0$ provides the upper bound

$$\begin{aligned} V_{\tau,\sigma}^{\delta_*,\delta}(\varepsilon) &\leq \frac{1}{\varepsilon}\left(\frac{\varepsilon\left(h_{\tau,\sigma}^{(\delta)} - h_{\tau,\sigma}^{(\delta_*)}\right) + h_{\tau,\sigma}^{(\delta_*)}}{h_{\tau,\sigma}^{(\delta_*)}} - 1\right) \\ &= \frac{h_{\tau,\sigma}^{(\delta)}}{h_{\tau,\sigma}^{(\delta_*)}} - 1. \end{aligned} \tag{2.19}$$

Similarly, it can be seen from (2.18) that

$$\begin{aligned} V_{\tau,\sigma}^{\delta_*,\delta}(\varepsilon) &= -\frac{1}{\varepsilon}\,\log\left(\frac{h_{\tau,\sigma}^{(\delta_*)}}{\varepsilon\, h_{\tau,\sigma}^{(\delta)} + (1-\varepsilon)\, h_{\tau,\sigma}^{(\delta_*)}}\right) \\ &\geq -\frac{1}{\varepsilon}\left(\frac{h_{\tau,\sigma}^{(\delta_*)}}{\varepsilon\, h_{\tau,\sigma}^{(\delta)} + (1-\varepsilon)\, h_{\tau,\sigma}^{(\delta_*)}} - 1\right) \\ &= -\frac{h_{\tau,\sigma}^{(\delta_*)} - h_{\tau,\sigma}^{(\delta)}}{\varepsilon\, h_{\tau,\sigma}^{(\delta)} + (1-\varepsilon)\, h_{\tau,\sigma}^{(\delta_*)}}. \end{aligned}$$

Hence since $\varepsilon \leq \frac{1}{2}$, it is straightforward to show that

$$V_{\tau,\sigma}^{\delta_*,\delta}(\varepsilon) \geq -\frac{1}{1-\varepsilon} \geq -2. \tag{2.20}$$

By (2.19), (2.20), the Dominated Convergence Theorem and L'Hospital's rule one obtains from (2.15) and (2.18) for given stopping times $\tau \in [0,\infty)$ and $\sigma \in (\tau,\infty)$ that

$$\begin{aligned} Q_{\tau,\sigma}^{\delta_*,\delta} &= \lim_{\varepsilon\to 0+} E\left(V_{\tau,\sigma}^{\delta_*,\delta}(\varepsilon)\,\Big|\,\mathcal{A}_\tau\right) \\ &= E\left(\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\,\log\left(\varepsilon\left(\frac{h_{\tau,\sigma}^{(\delta)}}{h_{\tau,\sigma}^{(\delta_*)}} - 1\right) + 1\right)\,\Bigg|\,\mathcal{A}_\tau\right) \\ &= E\left(\frac{h_{\tau,\sigma}^{(\delta)}}{h_{\tau,\sigma}^{(\delta_*)}} - 1\,\Bigg|\,\mathcal{A}_\tau\right), \end{aligned}$$

which by (2.7) provides (2.16). $\square$

Note that by Definition 2.6 any strictly positive portfolio $S^{(\delta_*)} \in \mathcal{V}^+$ satisfies under Assumption 2.2 the relation

$$Q_{\tau,\sigma}^{\delta_*,\delta_*} = 0 \tag{2.21}$$

for all stopping times $\tau \in [0,\infty)$ and $\sigma \in [\tau,\infty)$. In general, for a strictly positive reference portfolio $S^{(\delta_*)} \in \mathcal{V}^+$ and a nonnegative portfolio $S^{(\delta)} \in \mathcal{V}$ the derivative $Q^{\delta_*,\delta}_{\tau,\sigma}$ can be positive or negative. Now, let us characterize a GOP as a portfolio that achieves maximum expected growth over any given time interval. The derivative of the expected growth, introduced in Definition 2.6, leads us to the following definition.

Definition 2.8 *A strictly positive reference portfolio $S^{(\delta_*)} \in \mathcal{V}^+$ is called a GOP if for all stopping times $\tau \in [0,\infty)$ and $\sigma \in (\tau,\infty)$ and each nonnegative perturbing wealth process $S^{(\delta)} \in \mathcal{V}$ the derivative of its perturbed expected growth over the time interval $[\tau,\sigma]$ is less than or equal to zero, that is*

$$Q^{\delta_*,\delta}_{\tau,\sigma} \leq Q^{\delta_*,\delta_*}_{\tau,\sigma} = 0. \tag{2.22}$$

Lemma 2.7 and Definition 2.8 allow us then to obtain the following important result.

Corollary 2.9 *Under Assumption 2.2 the strictly positive wealth process $S^{(\delta_*)} \in \mathcal{V}^+$ in this assumption is a GOP if and only if all nonnegative wealth processes $S^{(\delta)} \in \mathcal{V}$, when expressed in units of $S^{(\delta_*)}$, are $(\underline{\mathcal{A}}, P)$-supermartingales.*

This statement expresses a direct relationship between choosing the best performing portfolio as benchmark and obtaining supermartingales as benchmarked price processes. In Bühlmann & Platen (2002) a similar result is derived for the special case of discrete time market models, see also Platen (2003). By assuming additionally that an equivalent local martingale measure exists, Kramkov & Schachermayer (1999), Becherer (2001) and Goll & Kallsen (2003) have obtained analogous supermartingale properties. Corollary 2.9 demonstrates that the existence of an equivalent local martingale measure is *not* necessary for the above type of supermartingale property to hold. It is now straightforward to prove the following statement.

Corollary 2.10 *Under Assumption 2.2 the reference portfolio $S^{(\delta_*)}$ equals a unique GOP if and only if the constant $K^{(\delta_*)}_\tau$ in this assumption can be almost surely set to zero for all stopping times $\tau \in [0,\infty)$.*

Proof: It is clear from Corollary 2.9 that if a unique GOP $S^{(\delta_*)}$ exists, then one can choose in Assumption 2.2 the GOP as reference portfolio and thus set the random variable $K^{(\delta_*)}_\tau$ for all $t \in [0,T]$ to zero. On the other hand, if one has a strictly positive reference portfolio $S^{(\delta_*)} \in \mathcal{V}^+$ with $K^{(\delta_*)}_\tau = 0$ for all $t \in [0,T]$, then all nonnegative portfolios expressed in units of $S^{(\delta_*)}$ are supermartingales. Thus by Corollary 2.9 the portfolio $S^{(\delta_*)}$ is then a GOP. Regarding its uniqueness let us suppose for the moment that there exist two different GOPs. By Corollary

2.9, both GOPs, when expressed in units of the other one, must be supermartingales. This can only be true if both are identical, which proves the uniqueness in Corollary 2.10. $\square$

Note that the GOP is unique as a value process but may be formed by different portfolio strategies. Whether the bound $K_\tau^{(\delta_*)}$ in Assumption 2.2 can be set to zero for a strictly positive reference portfolio $S^{(\delta_*)}$ for all stopping times $\tau \in [0, \infty)$ can best be verified when a particular class of market models is given. Goll & Kallsen (2003) study the dynamics of certain types of semimartingale models in relation to the corresponding GOP. For these types of general models it is straightforward to verify the following assumption. It ensures by Corollary 2.9 the existence and uniqueness of a GOP.

Assumption 2.11 *There exists a reference portfolio $S^{(\delta_*)}$ such that Assumption 2.2 is satisfied with the bound*

$$K_\tau^{(\delta_*)} = 0 \tag{2.23}$$

for all stopping times $\tau \in [0, \infty)$.

Having established by Assumptions 2.2 and 2.11 the existence and uniqueness of the GOP we choose $S^{(\delta_*)}$ as *numeraire* or *benchmark* and call the above modeling framework a *benchmark model*. A portfolio process $S^{(\delta)} = \{S^{(\delta)}(t),\, t \in [0, \infty)\}$, when expressed in units of the GOP $S^{(\delta_*)}$, is then called a *benchmarked* portfolio process and is denoted by $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t),\, t \in [0, \infty)\}$ with

$$\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)} \tag{2.24}$$

for all $t \in [0, \infty)$. According to Corollaries 2.9 and 2.10 we obtain the following result directly.

Corollary 2.12 *Under Assumptions 2.2 and 2.11 all benchmarked, nonnegative portfolio processes $\hat{S}^{(\delta)}$ are $(\underline{\mathcal{A}}, P)$-supermartingales.*

Once a candidate GOP process has been identified in a semimartingale market model, all that remains open is the verification that all benchmarked, nonnegative wealth processes are supermartingales. If this can be shown, then by Corollaries 2.9 and 2.10 the candidate GOP is indeed the unique GOP. To perform such a check is usually much simpler than solving the related complex constrained optimization problem of maximizing the expected growth, where the GOP must remain strictly positive.

3 Properties of Benchmarked Wealth Processes

Under the standard risk neutral modeling setup, see Karatzas & Shreve (1998), it is quite delicate to consider the limit over time of securities when the time horizon tends to infinity. Such limit is, for instance, relevant for long term investments and perpetual derivatives. For the standard Black-Scholes model one already encounters problems, see Karatzas & Shreve (1998), when aiming to extend the time horizon to infinity. Here the equivalence between the risk neutral and real world probability measure breaks down as will be discussed at the end of Section 6.1. Under the benchmark approach we avoid these kind of problems.

Based on the supermartingale property, for any given nonnegative benchmarked portfolio $\hat{S}^{(\delta)}$ various well-known results from martingale theory, see Elliott (1982), Ikeda & Watanabe (1989), Protter (1990) and Jacod & Shiryaev (2003), can be applied to a benchmark model. In the following we list several such properties.

Corollary 3.1 *Under Assumptions 2.2 and 2.11 a nonnegative, benchmarked portfolio value $\hat{S}^{(\delta)}(t)$ converges almost surely to an integrable random variable $\hat{S}^{(\delta)}(\infty)$ as t tends to infinity, that is*

$$\hat{S}^{(\delta)}(\infty) \overset{a.s.}{=} \lim_{t\to\infty} \hat{S}^{(\delta)}(t), \tag{3.1}$$

where

$$E\left(\hat{S}^{(\delta)}(\infty)\right) < \infty. \tag{3.2}$$

This means, benchmarked nonnegative portfolios are well defined and integrable at the infinite time horizon. If we denote the information set at $t = \infty$ by $\mathcal{A}_\infty = \bigvee_{t\in[0,\infty)} \mathcal{A}_t$, then for a stopping time $\tau \in [0, \infty)$, $A \in \mathcal{A}_\tau$ and $s > \tau$ a.s. one has by the supermartingale property of $\hat{S}^{(\delta)}$ that

$$\int_A \hat{S}^{(\delta)}(\tau,\omega)\, dP(\omega) \geq \int_A \hat{S}^{(\delta)}(s,\omega)\, dP(\omega). \tag{3.3}$$

Letting s tend to infinity we get

$$\hat{S}^{(\delta)}(\tau) \geq E\left(\hat{S}^{(\delta)}(\infty) \,\middle|\, \mathcal{A}_\tau\right) \tag{3.4}$$

a.s. and $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t),\, t \in [0,\infty]\}$ is then an $(\underline{\mathcal{A}}, P)$-supermartingale on $[0, \infty]$.

Similarly, we have the Optional Sampling Theorem for supermartingales on $[0, \infty]$, which shows that observed benchmarked nonnegative prices at any stopping time are at least equal to their expected values at any future stopping time.

Corollary 3.2 *Under Assumptions 2.2 and 2.11 for a benchmarked nonnegative portfolio* $\hat{S}^{(\delta)}$ *and two stopping times* σ, $\tau \in [0, \infty]$ *such that* $\sigma \leq \tau$ *a.s. the benchmarked portfolio values* $\hat{S}^{(\delta)}(\sigma)$ *and* $\hat{S}^{(\delta)}(\tau)$ *are integrable and*

$$\hat{S}^{(\delta)}(\sigma) \geq E\left(\hat{S}^{(\delta)}(\tau) \,\middle|\, \mathcal{A}_\sigma\right) \tag{3.5}$$

a.s.

The following definition of what constitutes a *fair* value is shown to be natural and rather useful in derivative pricing.

Definition 3.3 *A wealth process* $V = \{V(t),\, t \in [0, \infty)\}$ *is called* fair *if its benchmarked value* $\hat{V}(t) = \frac{V(t)}{S^{(\delta_*)}(t)}$ *forms an* $(\underline{\mathcal{A}}, P)$*-martingale* $\hat{V} = \{\hat{V}(t),\, t \in [0, \infty)\}$.

From Lemma 2.7 we obtain the following interesting property of expected returns of benchmarked fair portfolios.

Corollary 3.4 *Under Assumptions 2.2 and 2.11 for a nonnegative portfolio process* $S^{(\delta)}$ *the expected return* $R^{\delta_*,\delta}_{\tau,\sigma}$ *of its benchmarked value is zero for all* $\tau \in [0, \infty)$ *and* $\sigma \in [\tau, \infty]$ *if and only if the portfolio process* $S^{(\delta)}$ *is fair.*

This shows that the benchmarked nonnegative wealth processes that achieve the highest expected returns are martingales. Additionally, it follows by Corollary 3.4 and Lemma 2.7 that the fair, strictly positive wealth processes provide the highest expected growth. This indicates that the GOP is in several ways the best performing portfolio and fair portfolios play a preferred role for investors.

Let us apply further standard results from martingale theory.

Corollary 3.5 *Under Assumptions 2.2 and 2.11 a uniformly integrable, benchmarked, fair portfolio process* $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t),\, t \in [0, \infty]\}$ *is an* $(\underline{\mathcal{A}}, P)$*-martingale on* $[0, \infty]$*, where*

$$\hat{S}^{(\delta)}(\infty) = \lim_{t \to \infty} \hat{S}^{(\delta)}(t) \tag{3.6}$$

almost surely, and for all stopping times σ, $\tau \in [0, \infty]$ *such that* $\sigma \leq \tau$ *a.s. one has*

$$\hat{S}^{(\delta)}(\sigma) = E\left(\hat{S}^{(\delta)}(\tau) \,\middle|\, \mathcal{A}_\sigma\right) \tag{3.7}$$

a.s. with integrable $\hat{S}^{(\delta)}(\sigma)$ *and* $\hat{S}^{(\delta)}(\tau)$.

This means, the actual benchmarked value of a fair wealth process is at any time the best forecast of its future benchmarked values, including the one at the infinite time horizon, provided the benchmarked wealth process is uniformly integrable. Note that it holds also an Optional Sampling Theorem, similar to Corollary 3.2, for uniformly integrable, benchmarked fair wealth processes.

For a nonnegative portfolio process $S^{(\delta)} \in \mathcal{V}$ let

$$T_{S^{(\delta)}} = \inf\{t : S^{(\delta)}(t) = 0\} \tag{3.8}$$

denote its *default time*, which is the time when it first reaches the level zero. For the total wealth process of a market participant this is the time when bankruptcy must be declared. The following result shows that any nonnegative portfolio that reaches the level zero remains zero afterwards.

Corollary 3.6 *Under Assumptions 2.2 and 2.11 a nonnegative portfolio process $S^{(\delta)} \in \mathcal{V}$ has almost surely the value*

$$S^{(\delta)}(t) = 0 \tag{3.9}$$

for all $t \in [T_{S^{(\delta)}}, \infty)$.

This is a fundamental property of nonnegative portfolios. It provides a mathematical basis for the legal principle of limited liability. Note that if in Corollary 3.6 the benchmarked portfolio $\hat{S}^{(\delta)}$ is uniformly integrable, then it follows that $S^{(\delta)}(t) = 0$ for all $t \in [T_{S^{(\delta)}}, \infty]$, which includes the infinite time horizon.

Finally, we describe the Doob-Meyer decomposition, which provides a fundamental unique characterization of the structure of benchmarked nonnegative wealth processes.

Corollary 3.7 *Under Assumptions 2.2 and 2.11 a nonnegative wealth process $S^{(\delta)} \in \mathcal{V}$ with value $S^{(\delta)}(t)$ at time t has a unique decomposition of the form*

$$S^{(\delta)}(t) = (M^{(\delta)}(t) + A^{(\delta)}(t))\, S^{(\delta_*)}(t) \tag{3.10}$$

for $t \in [0, \infty)$. Here $A^{(\delta)} = \{A^{(\delta)}(t),\, t \in [0, \infty)\}$ is a nonincreasing, predictable process with $A^{(\delta)}(0) = 0$ a.s. and $M^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-local martingale.

By equation (3.10) we obtained a unique decomposition of each nonnegative portfolio. Its benchmarked value splits into the sum of a local martingale and a nonincreasing predictable process. By splitting a general portfolio into its nonnegative and negative components and applying the above Doob-Meyer decomposition to each of these components one obtains a unique decomposition for all portfolios.

4 The GOP as Best Performing Portfolio

The GOP can be considered to be the best performing portfolio in different ways. In the following, we would like to substantiate this property by describing some general mathematical results.

4.1 Expected Growth

By relation (2.13) it follows that for any nonnegative wealth process $S^{(\delta)} \in \mathcal{V}$ that the difference between the expected growths $g^{\delta}_{\tau,\sigma}$ and $g^{\delta_*}_{\tau,\sigma}$ over the time interval $[\tau, \sigma]$ is bounded by the corresponding expected return $R^{\delta_*,\delta}_{\tau,\sigma}$ for $\hat{S}^{(\delta)}$. Furthermore, from Corollary 2.9 we know that the expected return $R^{\delta_*,\delta}_{\tau,\sigma}$ for $\hat{S}^{(\delta)}$ is not greater than zero. This leads directly to the following result.

Corollary 4.1 *Under Assumptions 2.2 and 2.11 for any nonnegative wealth process $S^{(\delta)} \in \mathcal{V}$ its expected growth over the time interval $[\tau, \sigma]$ with stopping times $\tau \in [0, \infty)$ and $\sigma \in (\tau, \infty)$ satisfies the inequality*

$$g^{\delta}_{\tau,\sigma} \leq g^{\delta_*}_{\tau,\sigma} \tag{4.1}$$

almost surely.

Thus, over any finite period the expected growth of the GOP is never smaller than that of any other nonnegative wealth process. This is one characterization which shows that the GOP outperforms all other portfolios.

4.2 Systematic Outperformance

For an investor it is of interest to know whether or not it is possible to *systematically outperform* the GOP by constructing another portfolio with better performance over any time period. To make this mathematically precise we introduce the following definition.

Definition 4.2 *A strictly positive wealth process $S^{(\delta)} \in \mathcal{V}^+$ is said to* systematically outperform *another strictly positive wealth process $S^{(\bar{\delta})} \in \mathcal{V}^+$ if for some stopping times $\tau \in [0, \infty)$ and $\sigma \in [\tau, \infty)$ with*

$$S^{(\delta)}(\tau) = S^{(\bar{\delta})}(\tau) \tag{4.2}$$

and

$$S^{(\delta)}(\sigma) \geq S^{(\bar{\delta})}(\sigma) \tag{4.3}$$

almost surely, then

$$P\left(S^{(\delta)}(\sigma) > S^{(\bar{\delta})}(\sigma) \,\middle|\, \mathcal{A}_\tau\right) > 0. \tag{4.4}$$

According to the above definition, if a nonnegative wealth process systematically outperforms the GOP, then it can generate, with strictly positive probability, over some period certain wealth that is strictly greater than what can be achieved by the GOP. We can prove the following theorem.

Theorem 4.3 *Under Assumptions 2.2 and 2.11 any nonnegative wealth process cannot systematically outperform the GOP.*

Proof: Consider a benchmarked, nonnegative wealth process $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t),\, t \in [0,\infty)\}$, where we have at a stopping time $\tau \in [0,\infty)$ the benchmarked value

$$\hat{S}^{(\delta)}(\tau) = 1 \tag{4.5}$$

almost surely and for a later stopping time $\sigma \in [\tau,\infty)$ the inequality

$$\hat{S}^{(\delta)}(\sigma) \geq 1 \tag{4.6}$$

almost surely. Then it follows by the supermartingale property (3.4) of $\hat{S}^{(\delta)}$, the Optional Sampling Theorem and the property (4.5) that

$$0 \geq E\left(\hat{S}^{(\delta)}(\sigma) - \hat{S}^{(\delta)}(\tau) \,\middle|\, \mathcal{A}_\tau\right) = E\left(\hat{S}^{(\delta)}(\sigma) - 1 \,\middle|\, \mathcal{A}_\tau\right). \tag{4.7}$$

Obviously, due to (4.7) and (4.6), the benchmarked value $\hat{S}^{(\delta)}(\sigma)$ cannot be strictly greater than $\hat{S}^{(\delta)}(\tau) = 1$ with any strictly positive conditional probability, which means that

$$P\left(\hat{S}^{(\delta)}(\sigma) - 1 > 0 \,\middle|\, \mathcal{A}_\tau\right) = 0. \tag{4.8}$$

Therefore, it follows by (4.7) and (2.24) that

$$P\left(S^{(\delta)}(\sigma) > S^{(\delta_*)}(\sigma) \,\middle|\, \mathcal{A}_\tau\right) = 0 \tag{4.9}$$

almost surely. This proves by Definition 4.2 the theorem. □

Similarly, by exploiting the martingale property of benchmarked fair portfolios one can prove the following result.

Corollary 4.4 *Under Assumptions 2.2 and 2.11 any fair portfolio cannot be systematically outperformed by the GOP.*

This is an interesting statement. It shows that in terms of expected returns all fair, benchmarked portfolios are equally attractive to an investor. However, as we will see below, in the long term only one of these fair portfolios is almost surely outgrowing the others.

4.3 Long Term Growth Rate

Let us now mention also a pathwise result. We define the *long term growth rate* $\tilde{g}_\delta$ for a strictly positive wealth process $S^{(\delta)}$ as the almost sure limit

$$\tilde{g}^{\delta}_{\infty} \overset{\text{a.s.}}{=} \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{S^{(\delta)}(t)}{S^{(\delta)}(0)}\right). \tag{4.10}$$

Note that $\tilde{g}^{\delta}_{\infty}$ is random and there is no expectation taken in (4.10). The following result shows that the GOP has the largest long term growth rate. Thus, over a sufficiently long period it is almost surely pathwise superior to all other nonnegative portfolios.

Corollary 4.5 *Under Assumptions 2.2 and 2.11 for all strictly positive portfolio processes $S^{(\delta)}$ the GOP has the maximum long term growth rate, that is*

$$\tilde{g}^{\delta_*}_{\infty} \overset{a.s.}{=} \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(0)}\right) \geq \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{S^{(\delta)}(t)}{S^{(\delta)}(0)}\right) \overset{a.s.}{=} \tilde{g}^{\delta}_{\infty} \tag{4.11}$$

almost surely.

Proof:

This proof uses analogous arguments as applied for a similar result in Karatzas & Shreve (1998). Consider a nonnegative portfolio $S^{(\delta)}$ with

$$S^{(\delta)}(0) = S^{(\delta_*)}(0). \tag{4.12}$$

By Corollary 2.12 the nonnegative benchmarked portfolio $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale. As a supermartingale on $[0,\infty]$, see (3.4), the benchmarked portfolio $\hat{S}^{(\delta)} = \{\hat{S}^{(\delta)}(t),\, t \in [0,\infty]\}$ satisfies by (4.12) the well-known inequality

$$\exp\{\varepsilon\, k\}\, P\left(\sup_{k\leq t\leq\infty} \hat{S}^{(\delta)}(t) > \exp\{\varepsilon\, k\} \,\big|\, \mathcal{A}_0\right) \leq E\left(\hat{S}^{(\delta)}(k) \,\big|\, \mathcal{A}_0\right) \leq \hat{S}^{(\delta)}(0) = 1 \tag{4.13}$$

for all $k \in \{1, 2, \ldots\}$ and $\varepsilon \in (0,1)$, see Elliott (1982). Let us fix $\varepsilon \in (0,1)$, then

$$\sum_{k=1}^{\infty} P\left(\sup_{k\leq t\leq\infty} \log\left(\hat{S}^{(\delta)}(t)\right) > \varepsilon\, k \,|\, \mathcal{A}_0\right) \leq \sum_{k=1}^{\infty} \exp\{-\varepsilon\, k\} < \infty. \tag{4.14}$$

The Lemma of Borel-Cantelli implies the existence of a random variable $\bar{K}_\varepsilon$ such that

$$\log\left(\hat{S}^{(\delta)}(t)\right) \leq \varepsilon\, k \leq \varepsilon\, t$$

for all $k \geq \bar{K}_\varepsilon$ and $t \geq k$ almost surely. Thus, one has almost surely

$$\sup_{t\geq k} \frac{1}{t} \log\left(\hat{S}^{(\delta)}(t)\right) \leq \varepsilon$$

for all $k \geq \bar{K}_\varepsilon$ and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{S^{(\delta)}(t)}{S^{(\delta)}(0)}\right) \leq \limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(0)}\right) + \varepsilon \tag{4.15}$$

almost surely. Noting that relation (4.15) holds for all $\varepsilon \in (0,1)$ the inequality (4.11) follows by (4.10). $\square$

The above results indicate that actively managed funds cannot systematically outperform the GOP, unless they have access to non-public information, invest in securities that are not publicly traded or detect arbitrage in the market. Therefore, the best that a long term investor in a no-arbitrage world can do, based on these probabilistic considerations, is to simply invest in some approximation of the GOP. Of course, investors with a short time horizon may not invest fully in the GOP. For them it is optimal to form according to the mutual fund theorem, see Platen (2002), in a portfolio process that holds some wealth invested in a risk free asset but keeps the remaining wealth in the GOP.

5 Fair Valuation

5.1 Contingent Claim Pricing

It is of fundamental importance to formulate a consistent pricing concept that is both economically reasonable and computationally tractable. The previously established notion of a fair price, given in Definition 3.3, satisfies this criterion.

From Definition 3.3 and Corollary 3.4 it follows that over any time interval $[t, s]$ for a fair, nonnegative portfolio $S^{(\delta)}$ the derivative of the expected growth $Q_{t,s}^{\delta_*,\delta}$ is zero. Thus, a small perturbation of the GOP by a fair portfolio does not alter the expected growth of the perturbed GOP greatly. This constitutes an important robustness property, which applies to all fair investments that are added in small quantities to the GOP. Fair pricing has some similarity to utility indifference pricing, see Davis (2003), however no utility function is here involved.

From Definition 3.3 and Corollary 3.5 it follows by the martingale property of a uniformly integrable, benchmarked, fair value process $\hat{V} = \{\hat{V}(t),\, t \in [0, \infty)\}$ that

$$E\left(\hat{V}(\sigma)\,\big|\,\mathcal{A}_\tau\right) = \hat{V}(\tau) \tag{5.1}$$

a.s. for all stopping times $\tau \in [0, \infty]$ and $\sigma \in [\tau, \infty]$. The benchmarked fair price can therefore be interpreted as the best forecast of all of its future benchmarked values including that at infinity.

Definition 5.1 *A* contingent claim H_τ *is defined as an* $\mathcal{A}_\tau$*-measurable payoff at a stopping time* $\tau \in [0, \infty]$ *with*

$$E\left(\frac{|H_\tau|}{S^{(\delta_*)}(\tau)}\right) < \infty. \tag{5.2}$$

From Definitions 3.3 and 5.1 the following *fair pricing formula* is obtained.

Corollary 5.2 *For a contingent claim H_τ the fair price $U_{H_\tau}(t)$ at time $t \in [0,\tau]$ is given by the* fair pricing formula

$$U_{H_\tau}(t) = S^{(\delta_*)}(t)\, \hat{U}_{H_\tau}(t), \tag{5.3}$$

where the corresponding fair, benchmarked contingent claim price process $\hat{U}_{H_\tau} = \{\hat{U}_{H_\tau}(t),\ t \in [0,\tau]\}$ has at time $t \in [0,\tau]$ the value

$$\hat{U}_{H_\tau}(t) = \frac{U_{H_\tau}(t)}{S^{(\delta_*)}(t)} = E\left(\frac{H_\tau}{S^{(\delta_*)}(\tau)} \,\middle|\, \mathcal{A}_t \right). \tag{5.4}$$

For a given contingent claim H_τ the corresponding benchmarked, fair price process $\hat{U}_{H_\tau} = \{\hat{U}_{H_\tau}(t),\ t \in [0,\tau]\}$ is a martingale, see (5.4) and (5.1).

For hedging it is important to identify for a given contingent claim a wealth process that perfectly replicates this claim.

Definition 5.3 *For a contingent claim H_τ we call a wealth process $S^{(\delta)} = \{S^{(\delta)}(t),\ t \in [0,\tau]\}$* replicating *if*

$$\hat{S}^{(\delta)}(\tau) = \frac{H_\tau}{S^{(\delta_*)}(\tau)} \tag{5.5}$$

almost surely.

This means, a replicating wealth process equals at the maturity date τ the payoff of the corresponding contingent claim. If in the given benchmark model each contingent claim can be replicated, then we call the model *complete*. Otherwise, it is called *incomplete*. Due to the supermartingale property of benchmarked, nonnegative portfolio processes we can prove that at any given time a fair, replicating price process equals the *minimal value* of all replicating portfolio processes.

Corollary 5.4 *Under Assumptions 2.2 and 2.11, if for a nonnegative contingent claim H_τ its fair price process U_{H_τ} equals a replicating portfolio process $S^{(\delta_{H_\tau})} \in \mathcal{V}$, then any other nonnegative replicating portfolio process $S^{(\bar{\delta})} \in \mathcal{V}$ is less than or equal to the corresponding fair price process, that is*

$$U_{H_\tau}(t) = S^{(\delta_{H_\tau})}(t) \le S^{(\bar{\delta})}(t) \tag{5.6}$$

almost surely for all $t \in [0,\tau]$.

Proof: Note that the replicating, benchmarked, nonnegative, fair price process $\hat{U}_{H_\tau} = \{\hat{U}_{H_\tau}(t),\ t \in [0,\tau]\}$ given in (5.4) is an $(\underline{\mathcal{A}}, P)$-martingale. By Corollary 2.12 all nonnegative benchmarked wealth processes are supermartingales. Consequently, if the nonnegative contingent claim H_τ can be perfectly replicated by a nonnegative wealth process $S^{(\bar{\delta})}$ at maturity τ, then we have

$$\hat{U}_{H_\tau}(\tau) = \hat{S}^{(\delta_{H_\tau})}(\tau) = \hat{S}^{(\bar{\delta})}(\tau). \tag{5.7}$$

At any earlier time $t \in [0, \tau]$ the value $\hat{S}^{(\bar{\delta})}(t)$ of the supermartingale $\hat{S}^{(\bar{\delta})}$ is greater than or equal to $\hat{U}_{H_\tau}(t)$, which proves by (2.24) the inequality (5.6). $\square$

The fair price is as the minimal replicating price economically the rational price when there are several replicating portfolios. The corresponding hedging strategy, also for the case of incomplete markets, will be discussed elsewhere.

5.2 Risk Neutral Pricing

Now, let us show that fair pricing generalizes the established standard risk neutral pricing methodology. For this purpose we consider the Radon-Nikodym derivative process $\Lambda = \{\Lambda(t),\, t \in [0, T]\}$ with $T < \infty$ for the candidate risk neutral pricing measure $\tilde{P}$ with

$$\frac{d\tilde{P}}{dP} = \Lambda(T) \tag{5.8}$$

with

$$\Lambda(t) = \frac{B(t)\, S^{(\delta_*)}(0)}{S^{(\delta_*)}(t)\, B(0)} \tag{5.9}$$

for $t \in [0, T]$. Here $B(t)$ denotes the riskless asset at time t, which is usually called the *savings account*. By using (5.9) one can rewrite for a contingent claim H_τ the fair pricing formula (5.3) for the fair value process U_{H_τ} in the form

$$\begin{aligned} U_{H_\tau}(t) &= E\left(\frac{S^{(\delta_*)}(t)}{S^{(\delta_*)}(\tau)}\, H_\tau \,\middle|\, \mathcal{A}_t \right) \\ &= E\left(\frac{\Lambda(\tau)}{\Lambda(t)}\, \frac{B(t)}{B(\tau)}\, H_\tau \,\middle|\, \mathcal{A}_t \right) \end{aligned} \tag{5.10}$$

for $t \in [0, \tau]$. In the special case when an equivalent risk neutral martingale measure $\tilde{P}$ exists, we obtain from (5.10) with (5.9) the *risk neutral pricing formula*

$$U_{H_\tau}(t) = \tilde{E}\left(\frac{B(t)}{B(\tau)}\, H_\tau \,\middle|\, \mathcal{A}_t \right) \tag{5.11}$$

for all $t \in [0, \tau]$, provided that the assumptions for the Girsanov Theorem are satisfied. Here $\tilde{E}$ denotes expectation under the equivalent risk neutral martingale measure $\tilde{P}$. This confirms that standard risk neutral pricing is a particular case of fair pricing if an equivalent risk neutral martingale measure $\tilde{P}$ exists.

In general, in a benchmark model the candidate risk neutral measure $\tilde{P}$ may not be equivalent to the real world probability measure P. Also the Radon-Nikodym derivative process Λ may not be an $(\underline{\mathcal{A}}, P)$-martingale. An example is given in Section 6.2 where the risk neutral pricing formula (5.11) breaks down. Examples of benchmark models that reflect stylized empirical facts and where standard risk neutral pricing does not apply, are described in Platen (2001, 2002), Heath & Platen (2002a, 2002b, 2002c, 2003) and Breymann, Kelly & Platen (2003).

5.3 Actuarial Pricing

Using Corollary 5.2 we note that the *zero coupon bond* that pays one monetary unit at the fixed maturity date $\tau \in [0, T]$ with $T < \infty$ has at time $t \in [0, \tau]$ the fair price

$$P(t, \tau) = S^{(\delta_*)}(t)\, E\left(\frac{1}{S^{(\delta_*)}(\tau)} \,\middle|\, \mathcal{A}_t \right). \tag{5.12}$$

For a contingent claim H_τ, which is independent from the GOP value $S^{(\delta_*)}(\tau)$ and has a given deterministic maturity date τ, the following widely used *actuarial pricing formula* can be directly obtained from the fair pricing formula (5.3) and relation (5.12).

Corollary 5.5 *For a contingent claim H_τ, which is independent of the GOP value $S^{(\delta_*)}(\tau)$ and has a deterministic maturity date $\tau \in [0, T]$, its fair price $U_{H_\tau}(t)$ satisfies the* actuarial pricing formula

$$U_{H_\tau}(t) = P(t, \tau)\, E\big(H_\tau \,\big|\, \mathcal{A}_t\big) \tag{5.13}$$

for $t \in [0, \tau]$.

To obtain (5.13) one only needs to use the fact that the expectation of the product of independent random variables equals the product of their expectations. Thus, the commonly used *net present value pricing rule* is recovered as a consequence of the fair pricing formula. Note that this applies only for contingent claims that are independent of the GOP, which is typical for a range of valuation problems in the insurance area, see Bühlmann & Platen (2002). It also applies for most real option valuations and weather derivatives, see Platen & West (2003). It should be emphasized again that the discounted conditional expectation of the given payoff in (5.13) is taken with respect to the real world probability measure P. Furthermore, the interest rate in formula (5.13) that is implicitly used to compute (5.12) may be stochastic. This case is typically not considered in most actuarial pricing problems but can be easily handled under (5.13).

6 Examples of Benchmark Models

Let us illustrate the above benchmark framework through some examples. In each of the following benchmark models we set, for simplicity, the interest rate to zero such that

$$B(t) = S^{(0)}(t) = 1 \tag{6.1}$$

denotes the constant savings account for $t \in [0, T]$ with $T < \infty$. Furthermore, we identify a risky primary security account process by $S^{(1)} = \{S^{(1)}(t),\, t \in [0, T]\}$.

6.1 Black-Scholes Model

At first, we study the well-known Black-Scholes model. Here $S^{(1)}(t)$ satisfies the stochastic differential equation (SDE)

$$dS^{(1)}(t) = S^{(1)}(t)\,(a\,dt + \sigma\,dW(t)) \tag{6.2}$$

for $t \in [0,T]$ with $S^{(1)}(0) > 0$. The appreciation rate a and the volatility σ are constants and $W = \{W(t),\, t \in [0,T]\}$ is a standard Wiener process on $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$. A portfolio value

$$S^{(\delta)}(t) = \delta^{(0)}(t)\,S^{(0)}(t) + \delta^{(1)}(t)\,S^{(1)}(t) \tag{6.3}$$

satisfies the SDE

$$dS^{(\delta)}(t) = S^{(\delta)}(t)\,\pi_\delta^1(t)\,\sigma\,(\theta\,dt + dW(t)) \tag{6.4}$$

for $t \in [0,T]$. Here

$$\theta = \frac{a}{\sigma} \tag{6.5}$$

is known as the market price for risk and

$$\pi_\delta^1(t) = \frac{\delta^{(1)}(t)\,S^{(1)}(t)}{S^{(\delta)}(t)} \tag{6.6}$$

is the fraction of the portfolio value $S^{(\delta)}(t)$ that is invested in the risky primary security account $S^{(1)}$. Obviously,

$$\pi_\delta^0(t) = 1 - \pi_\delta^1(t) \tag{6.7}$$

for $t \in [0,T]$.

By the Itô formula we obtain for the logarithm of a strictly positive portfolio value $S^{(\delta)}(t)$ the SDE

$$d\log(S^{(\delta)}(t)) = g_\delta(t)\,dt + \pi_\delta^1(t)\,\sigma\,dW(t)$$

with growth rate

$$g_\delta(t) = \pi_\delta^1(t)\,\sigma\,\theta - \frac{1}{2}\left(\pi_\delta^1(t)\,\sigma\right)^2$$

for $t \in [0,T]$. Thus, by maximizing the growth rate we find the optimal fraction

$$\pi_{\delta_*}^1(t) = \frac{\theta}{\sigma} \tag{6.8}$$

for the GOP $S^{(\delta_*)}(t)$. The GOP satisfies with this fraction the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t)\,\theta\,(\theta\,dt + dW(t)) \tag{6.9}$$

for $t \in [0, T]$ with $S^{(\delta_*)}(0) > 0$.

By the Itô formula it follows from (6.4) and (6.9) that the benchmarked portfolio value $\hat{S}^{(\delta)}(t)$ satisfies the SDE

$$d\hat{S}^{(\delta)}(t) = \hat{S}^{(\delta)}(t)\, \pi^1_\delta(t)\, \sigma\, dW(t) \tag{6.10}$$

for $t \in [0, T]$. Obviously, when $\pi^1_\delta(t)$ is a deterministic function of time, then $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-martingale. Thus, the benchmarked risky primary security account $\hat{S}^{(1)}(t) = \frac{S^{(1)}(t)}{S^{(\delta_*)}(t)}$ and the benchmarked savings account $\hat{S}^{(0)}(t) = \frac{S^{(0)}(t)}{S^{(\delta_*)}(t)}$ are $(\underline{\mathcal{A}}, P)$-martingales. Obviously, the Radon-Nikodym derivative process $\Lambda = \{\Lambda(t) = \frac{\hat{S}^{(0)}(t)}{\hat{S}^{(0)}(0)},\, t \in [0, \infty)\}$ is an $(\underline{\mathcal{A}}, P)$-martingale. For the Black-Scholes model the predictable process $A^{(\delta)}$ in the Doob-Meyer decomposition (3.10) is zero.

Already under the simple Black-Scholes model and risk neutral pricing the infinite time horizon is difficult to analyze. This problem is caused by the fact that the Radon-Nikodym derivative process $\lambda = \{\lambda(t) = S^{(\delta_*)}(t)^{-1},\, t \in [0, \infty)\}$ for the risk neutral measure $\tilde{P}$ is *not* uniformly integrable. Thus, the risk neutral measure $\tilde{P}$ is not necessarily equivalent to P on $\mathcal{A}_\infty$.

6.2 Strict Local Martingale Portfolio

By an appropriate choice of the strategy $\delta = \{\delta(t) = (\delta^0(t), \delta^1(t))^\top,\, t \in [0, T]\}$ under the previously introduced Black-Scholes model one can construct benchmarked portfolios that are strict local martingales.

As an example, let us consider a squared Bessel process $Z = \{Z(t),\, t \in [0, T]\}$ of dimension $\nu > 2$, which satisfies the SDE

$$dZ(t) = \nu\, dt + 2\sqrt{Z(t)}\, dW(t) \tag{6.11}$$

for $t \in [0, T]$ and $Z(0) > 0$. We then note that with the fraction

$$\pi^1_{\bar{\delta}}(t) = \frac{2 - \nu}{\sigma\, Z(t)} \tag{6.12}$$

we get the portfolio $S^{(\bar{\delta})}$ with benchmarked value

$$\hat{S}^{(\bar{\delta})}(t) = Z(t)^{1 - \frac{\nu}{2}} \tag{6.13}$$

for $t \in [0, T]$, when $\hat{S}^{(\bar{\delta})}(0) = Z(0)^{1 - \frac{\nu}{2}}$. As is known from Revuz & Yor (1999), the process $\hat{S}^{(\bar{\delta})}$ is an $(\underline{\mathcal{A}}, P)$-local martingale but not a martingale, despite the fact that $E((\hat{S}^{(\bar{\delta})}(t))^\alpha) < \infty$ for exponents $\alpha < \frac{\nu}{\nu - 2}$ for $t \in [0, T]$. For instance, for $\alpha = 1$ and $\nu = 4$ its expectation equals

$$E\left(\hat{S}^{(\bar{\delta})}(t) \,\middle|\, \mathcal{A}_0\right) = \hat{S}^{(\bar{\delta})}(0) \left(1 - \exp\left\{-\frac{1}{2\, \hat{S}^{(\bar{\delta})}(0)\, t}\right\}\right) < \hat{S}^{(\bar{\delta})}(0) \tag{6.14}$$

for $t \in (0, T]$, see Platen (2002). This shows that $\hat{S}^{(\bar{\delta})}$ cannot be a martingale. In this case the Doob-Meyer decomposition for $\hat{S}^{(\bar{\delta})}(t)$ forms a, so called, potential, which is a nonnegative supermartingale that converges over time to zero.

6.3 Alternative Market Models

In Platen (2001, 2002) a class of alternative market models has been suggested, where the time transformed GOP value $S^{(\delta_*)}(t)$ satisfies an SDE of the form

$$dS^{(\delta_*)}(t) = 4\,dt + 2\sqrt{S^{(\delta_*)}(t)}\,dW(t)$$

for $t \in [0, T]$ with $S^{(\delta_*)}(0) > 0$. Obviously, $S^{(\delta_*)}$ is here a squared Bessel process of dimension four, see (6.11). The benchmarked savings account

$$\hat{S}^{(0)}(t) = \frac{S^{(0)}(t)}{S^{(\delta_*)}(t)} = \left(S^{(\delta_*)}(t)\right)^{-1}$$

is then by (6.1) its inverse and therefore also a strict $(\underline{\mathcal{A}}, P)$-local martingale. As is well known, the Radon-Nikodym derivative process $\Lambda = \{\Lambda(t) = \frac{\hat{S}^{(0)}(t)}{\hat{S}^{(0)}(0)},\, t \in [0, T]\}$ with $\Lambda(T) = \frac{d\tilde{P}}{dP}$ for the corresponding candidate risk neutral measure $\tilde{P}$ is then a strict $(\underline{\mathcal{A}}, P)$-local martingale. Under P the probability of the squared Bessel process $S^{(\delta_*)}$ of dimension four for reaching zero is zero. However, under $\tilde{P}$ the corresponding risk neutral probability is strictly positive since the squared Bessel process $S^{(\delta_*)}$ has under $\tilde{P}$ the dimension zero, see Revuz & Yor (1999). Consequently, in this case the measures P and $\tilde{P}$ are not equivalent. Furthermore, Λ is not a martingale. These properties of the model do not permit the application of the risk neutral pricing methodology. The model also provides a *free lunch with vanishing risk* in the sense of Delbaen & Schachermayer (1994, 1998). The above benchmark framework encompasses the class of alternative market models mentioned above, which provide realistic models for the world stock index in different denominations, as shown in Breymann, Kelly & Platen (2003).

6.4 Jump Diffusion Model

We consider as risky primary security account $S^{(1)}(t)$ a stock that follows a jump diffusion process with SDE

$$dS^{(1)}(t) = S^{(1)}(t-)\,(a\,dt + \sigma\,dW(t) - (dN(t) - \lambda\,dt)), \tag{6.15}$$

for $t \in [0, T]$ with $S^{(1)}(0) > 0$ and $T < \infty$, where the appreciation rate a and the volatility σ are constant. Here W is a standard Wiener process and $N = \{N(t),\, t \in [0, T]\}$ is an independent Poisson process with constant intensity $\lambda > 0$. Obviously, by the SDE (6.15) it follows that when the Poisson process generates the first jump, then the stock price process $S^{(1)}$ defaults to the level zero and remains zero thereafter, see Corollary 3.6.

Additionally to the savings account $B(t) = S^{(0)}(t) = 1$ and the stock $S^{(1)}(t)$ we consider also a defaultable bond $S^{(2)}(t)$ which matures at $T > 0$. The SDE for $S^{(2)}(t)$ let be given in the form

$$dS^{(2)}(t) = S^{(2)}(t-)\,(\lambda\,dt - dN(t)) \tag{6.16}$$

for $t \in [0,T]$ with $S^{(2)}(0) = \exp\{-\lambda\,T\}$. The defaultable bond pays one monetary unit at time T when the stock does not default until T.

A portfolio

$$S^{(\delta)}(t) = \sum_{j=0}^{2} \delta^{(j)}(t)\,S^{(j)}(t)$$

with strategy $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \delta^{(2)}(t))^\top,\, t \in [0,T]\}$ satisfies the SDE

$$\begin{aligned} dS^{(\delta)}(t) &= S^{(d)}(t-)\Big(\pi^1_\delta(t-)\,[\sigma\,(\theta\,dt + dW(t)) - (dN(t) - \lambda\,dt)] \\ &\qquad - \pi^2_\delta(t-)(dN(t) - \lambda\,dt)\Big) \end{aligned}$$

with market price for risk $\theta = \frac{a}{\sigma}$. It therefore has the growth rate

$$\tilde{g}_\delta(t) = \pi^1_\delta(t)\,\sigma\,\theta - \frac{1}{2}\left(\pi^1_\delta(t)\,\sigma\right)^2 + \left(\pi^1_\delta(t) + \pi^2_\delta(t)\right)\lambda + \log\left(1 - \pi^1_\delta(t) - \pi^2_\delta(t)\right)\lambda$$

for $t \in [0,T]$. By maximizing this growth rate the optimal fractions for the GOP $S^{(\delta_*)}$ are obtained as

$$\pi^1_{\delta_*}(t) = \frac{\theta}{\sigma} \quad \text{and} \quad \pi^2_{\delta_*}(t) = -\frac{\theta}{\sigma}$$

for $t \in [0,T]$. Thus the GOP value $S^{(\delta_*)}(t)$ satisfies the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t)\,(\theta\,dt + dW(t)),$$

which is the same as in the above Black-Scholes example. Note that the GOP is here not dependent on the jumps of the stock. In practical terms this means that the corresponding risk is diversifiable and a zero market price for risk has been allocated to the uncertainty that a default arises or not.

Consider now a portfolio $S^{(\delta)}$. Then we get for its benchmarked value $\hat{S}^{(\delta)}(t) = \frac{S^{(\delta)}(t)}{S^{(\delta_*)}(t)}$ by the Itô formula the SDE

$$d\hat{S}^{(\delta)}(t) = \hat{S}^{(d)}(t-)\Big((\pi^1_\delta(t)\,\sigma - \theta)\,dW(t) - \left(\pi^1_\delta(t-) + \pi^2_\delta(t-)\right)(dN(t) - \lambda\,dt)\Big)$$

for $t \in [0,T]$. This SDE is driftless and $\hat{S}^{(\delta)}$ is therefore an $(\underline{\mathcal{A}}, P)$-local martingale. Consequently, for this jump diffusion model we have in the Doob-Meyer decomposition of benchmarked portfolios (3.10) the nonincreasing process $A^{(\delta)}$ being zero, that is $A^{(\delta)}(t) = 0$ for all $t \in [0,T]$.

6.5 A Discrete Time Model

Finally, we consider an example with predictable jump times. Let us assume that the risky primary security account $S^{(1)}(t)$ jumps only at the discrete time points $\tau_i = i\Delta$ for $i \in \{1, 2, \ldots\}$, where we use some small time step size $\Delta > 0$. We denote by

$$i_\tau = \max\{i \in \{0, 1, \ldots\} : \tau_i \le t\}$$

the index of the last discretization time before time t. The risky primary security account price satisfies then the expression

$$S^{(1)}(t) = S^{(1)}(0) \prod_{\ell=1}^{i_\tau} h_\ell \tag{6.17}$$

for $t \in [0, T]$. Here the jump ratio h_ℓ is assumed to be independent and lognormally distributed, where

$$\log(h_\ell) \sim \mathcal{N}(\mu\, \Delta, \sigma^2\, \Delta)$$

is a Gaussian random variable with mean $\mu\Delta$ and variance $\sigma^2\Delta$, $\ell \in \{1, 2, \ldots, i_T\}$.

A strictly positive portfolio $S^{(\delta)}$ needs to have a fraction $\pi^1_\delta(t) \in [0, 1]$ that lies between zero and one for all $t \in [0, T]$. Otherwise, $S^{(\delta)}$ can become negative. Its ith jump ratio is then

$$\frac{S^{(\delta)}(\tau_{i+1})}{S^{(\delta)}(\tau_i)} = \pi^0_\delta(\tau_i) + \pi^1_\delta(\tau_i)\, h_{i+1} = 1 + \pi^1_\delta(\tau_i)\,(h_{i+1} - 1).$$

The expected growth, see (2.10), of a strictly positive portfolio for the ith time step τ_i is

$$\begin{aligned} g^\delta_{\tau_i,\tau_{i+1}} &= E\left(\log\left(\frac{S^{(\delta)}(\tau_{i+1})}{S^{(\delta)}(\tau_i)}\right) \,\Big|\, \mathcal{A}_{\tau_i}\right) \\ &= E\left(\log\left(1 + \pi^1_\delta(\tau_i)\,(h_{i+1} - 1)\right) \,\Big|\, \mathcal{A}_{\tau_i}\right) \end{aligned} \tag{6.18}$$

for all $i \in \{0, 1, \ldots, i_T\}$. Let us now compute the optimal expected growth $g^{\delta_*}_{\tau_i,\tau_{i+1}}$ at time τ_i. The first derivative of $g^\delta_{\tau_i,\tau_{i+1}}$ with respect to $\pi^1_\delta(\tau_i)$ is

$$\frac{\partial g^\delta_{\tau_i,\tau_{i+1}}}{\partial \pi^1_\delta(\tau_i)} = E\left(\frac{h_{i+1} - 1}{1 + \pi^1_\delta(\tau_i)\,(h_{i+1} - 1)} \,\Bigg|\, \mathcal{A}_{\tau_i}\right) \tag{6.19}$$

and the second derivative has the form

$$\frac{\partial^2 g^\delta_{\tau_i,\tau_{i+1}}}{\partial (\pi^1_\delta(\tau_i))^2} = -\,E\left(\frac{(h_{i+1} - 1)^2}{(1 - \pi^1_\delta(\tau_i)\,(h_{i+1} - 1))^2} \,\Bigg|\, \mathcal{A}_{\tau_i}\right) \tag{6.20}$$

for $i \in \{0, 1, \ldots, i_T\}$. We observe that the second derivative is always negative. Therefore, the expected growth has at most one maximum. However, this maximum may arise for a fraction $\pi^1_\delta(\tau_i)$ with a value outside the interval $[0, 1]$. To clarify this possibility we compute with (6.19) the values

$$\frac{\partial g^\delta_{\tau_i,\tau_{i+1}}}{\partial \pi^1_\delta(\tau_i)}\bigg|_{\pi^1_\delta(\tau_i)=0} = E\left(h_{i+1} \,\big|\, \mathcal{A}_{\tau_i}\right) - 1 \tag{6.21}$$

and

$$\frac{\partial g^\delta_{\tau_i,\tau_{i+1}}}{\partial \pi^1_\delta(\tau_i)}\bigg|_{\pi^1_\delta(\tau_i)=1} = 1 - E\left(\frac{1}{h_{i+1}} \,\bigg|\, \mathcal{A}_{\tau_i}\right) \tag{6.22}$$

for $i \in \{0, 1, \ldots, i_T\}$. Due to (6.20) the first derivative $\frac{\partial g^\delta_{\tau_i,\tau_{i+1}}}{\partial \pi^1_\delta(\tau_i)}$ is decreasing for $\pi^1_\delta(\tau_i)$ increasing if

$$E\left((h_{i+1})^p \,\big|\, \mathcal{A}_{\tau_i}\right) \geq 0 \tag{6.23}$$

for both exponents $p = 1$ and $p = -1$. Then the values in (6.21) and (6.22) have opposite signs and there exists some fraction $\pi^1_{\delta_*}(\tau_i) \in [0, 1]$ such that

$$\frac{\partial g^\delta_{\tau_i,\tau_{i+1}}}{\partial \pi^1_\delta(\tau_i)}\bigg|_{\pi^1_\delta(\tau_i)=\pi^1_{\delta_*}(\tau_i)} = 0 \tag{6.24}$$

for $i \in \{0, 1, \ldots, i_T\}$. On the other hand, when condition (6.23) is violated, then the value of the optimal fraction $\pi^1_{\delta_*}(\tau_i)$ is located at one of the endpoints of the interval $[0, 1]$. Note that in this case the derivative (6.19) will, in general, not be zero. This means, in such a case we do not find a genuine maximum for $\pi^1_{\delta_*}(\tau_i)$ within the interval $[0, 1]$. Essentially we have three cases to consider:

1. At first we need to study the situation when the derivative $\frac{\partial g^\delta_{\tau_i,\tau_{i+1}}}{\partial \pi^1_\delta(\tau_i)}$ becomes zero for $\pi^1_{\delta_*}(\tau_i) \in [0, 1]$. This is the case, which provides a genuine maximum in $[0, 1]$. Using the well-known Laplace transform of a Gaussian random variable we get

$$E\left((h_{i+1})^p \,\big|\, \mathcal{A}_{\tau_i}\right) = \exp\left\{\left(p\,\mu + \frac{\sigma^2}{2}\right)\Delta\right\} \tag{6.25}$$

for exponents $p \in \{-1, 1\}$. Therefore, (6.24) only holds for

$$|\mu| \leq \frac{\sigma^2}{2}. \tag{6.26}$$

For this case we can show for $\Delta \to 0$ that the optimal fraction $\pi^1_{\delta_*}(\tau_i)$ approaches asymptotically the value

$$\lim_{\Delta \to 0} \pi^1_{\delta_*}(\tau_i) = \frac{1}{2} + \frac{\mu}{\sigma^2}.$$

With the strategy $\delta(\tau_i) = (\delta^{(0)}(\tau_i), \delta^{(1)}(\tau_i))^\top$ with $\delta^{(0)}(\tau_i) = 1$, $\delta^{(1)}(\tau_i) = 0$ we obtain for the expected return of the benchmarked savings account and also for $\delta(\tau_i) = (\delta^{(0)}(\tau_i), \delta^{(1)}(\tau_i))^\top$ with $\delta^{(0)}(\tau_i) = 0$ and $\delta^{(1)}(\tau_i) = 1$, that is for the expected return of the benchmarked risky primary security account, the almost sure limit

$$\lim_{\Delta \to 0} R^{\delta_*,\delta}_{\tau_i,\tau_{i+1}} \overset{\text{a.s.}}{=} 0$$

for $i \in \{0, 1, \ldots, i_T\}$. In this case, where $|\mu| \le \frac{\sigma^2}{2}$, it follows by Corollary 3.5 for all strictly positive portfolios $S^{(\delta)}$ that these can be interpreted in the limit to be fair.

2. In the case when $\mu < -\frac{\sigma^2}{2}$ the risky primary security account underperforms markedly when compared to the GOP. The optimal fraction is then

$$\pi^1_{\delta_*}(\pi_i) = 0$$

for all $i \in \{0, 1, \ldots, i_T\}$. The GOP consists of the savings account $S^{(0)}$. We obtain with $\delta^{(0)}(\tau_i) = 0$ and $\delta^{(1)}(\tau_i) = 1$ the ith expected return

$$R^{\delta_*,\delta}_{\tau_i,\tau_{i+1}} = \exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)\Delta\right\} - 1 < 0$$

for $i \in \{0, 1, \ldots, i_T\}$. This shows that the benchmarked risky primary security account price process $\hat{S}^{(1)}$ is a strict supermartingale, which is not a local martingale as in the previous continuous time examples. It can be seen that the martingale property is already violated when $\hat{S}^{(1)}$ is immediately stopped after one time step. Furthermore, the Doob-Meyer decomposition (3.10) shows a corresponding process $A^{(\delta)}$ that is strictly decreasing. Clearly, by Definition 3.3 the stock $S^{(1)}$ is here not a fair price process. We can check whether $S^{(0)}$ is possibly fair. This is simple because the GOP $S^{(\delta_*)}$ is equivalent to the savings account $S^{(0)}$. Therefore, the benchmarked savings account is constant and thus a martingale. Consequently, by Definition 3.3 the savings account is in this case a fair price process.

3. It remains to study the case when $\mu > \frac{\sigma^2}{2}$, where the stock is performing extremely well. Obviously, the optimal fraction is then

$$\pi^{(1)}_{\delta_*}(\tau_i) = 1$$

for all $i \in \{0, 1, \ldots, i_T\}$. This means, the GOP is formed when all wealth is invested in the risky primary security account $S^{(1)}$. For $\delta^{(0)}(\tau_i) = 1$ and $\delta^{(1)}(\tau_i) = 0$ the ith expected return of the benchmarked savings account satisfies the expression

$$R^{\delta_*,\delta}_{\tau_i,\tau_{i+1}} = \exp\left\{\left(-\mu + \frac{\sigma^2}{2}\right)\Delta\right\} - 1 < 0,$$

which shows that the benchmarked savings account is a strict supermartingale and not a local martingale. Consequently, $S^{(0)}$ is not fair for $\mu > \frac{\sigma^2}{2}$.

On the other hand, the benchmarked risky primary security account $\hat{S}^{(1)}(t)$ is constant and thus a martingale and fair.

This example demonstrates that there are benchmark models with benchmarked portfolios that are supermartingales but not local martingales. As we have seen, this arises for instance, when there are jumps in the underlying security at predictable stopping times and the corresponding jump ratios vary anywhere between zero and infinity.

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References

Becherer, D. (2001). The numeraire portfolio for unbounded semimartingales. *Finance Stoch.* **5**, 327–341.

Breymann, W., L. Kelly, & E. Platen (2003). An intraday empirical analysis of diversified world stock indices. University of Technology Sydney, (working paper).

Bühlmann, H. & E. Platen (2002). A discrete time benchmark approach for finance and insurance. *ASTIN Bulletin*. (to appear).

Dalang, R. C., A. Morton, & W. Willinger (1990). Equivalent martingale measures and no-arbitrage in stochastic security market models. *Stochastics Stochastics Rep.* **29**(2), 185–201.

Davis, M. H. A. (2003). Valuation, hedging and investing in incomplete financial markets. In *Proceedings ICIAM03 World Congress, Sydney, June 2003.* (to appear).

Delbaen, F. & W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann* **300**, 463–520.

Delbaen, F. & W. Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann* **312**, 215–250.

Duffie, D. & C. F. Huang (1986). Multiperiod security markets with differential information; martingales and resolution times. *J. Math. Economics* **15**, 283–303.

Elliott, R. J. (1982). *Stochastic Calculus and Applications*. Springer.

Fritelli, M. & P. Lakner (1994). Almost sure characterization of martingales. *Stochastics Stochastics Rep.* **49**, 181–190.

Goll, T. & J. Kallsen (2003). A complete explicit solution to the log-optimal portfolio problem. *Adv. in Appl. Probab.* **13**(2), 774–799.

Harrison, J. M. & D. M. Kreps (1979). Martingale and arbitrage in multiperiod securities markets. *J. Economic Theory* **20**, 381–408.

Harrison, J. M. & S. R. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11**(3), 215–260.

Heath, D. & E. Platen (2002a). Consistent pricing and hedging for a modified constant elasticity of variance model. *Quant. Finance.* **2**(6), 459–467.

Heath, D. & E. Platen (2002b). Perfect hedging of index derivatives under a minimal market model. *Int. J. Theor. Appl. Finance* **5**(7), 757–774.

Heath, D. & E. Platen (2002c). Pricing and hedging of index derivatives under an alternative asset price model with endogenous stochastic volatility. In J. Yong (Ed.), *Recent Developments in Mathematical Finance*, pp. 117–126. World Scientific.

Heath, D. & E. Platen (2003). Pricing of index options under a minimal market model with lognormal scaling. *Quant. Finance.* **3**(6), 442–450.

Ikeda, N. & S. Watanabe (1989). *Stochastic Differential Equations and Diffusion Processes* (2nd ed.). North-Holland. (first edition (1981)).

Jacod, J. (1979). *Stochastic calculus and martingale problems*, Volume 714 of *Lecture Notes in Math.* Springer. (in French).

Jacod, J. & A. N. Shiryaev (2003). *Limit Theorems for Stochastic Processes* (2nd ed.). Springer.

Jouini, E., H. Kallal, & C. Napp (2001). Arbitrage and viability in security markets with fixed trading costs. *J. Math. Economics* **35**(2), 197–221.

Karatzas, I. & S. E. Shreve (1998). *Methods of Mathematical Finance*, Volume 39 of *Appl. Math.* Springer.

Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

Korn, R. (2001). Value preserving strategies and a general framework for local approaches to optimal portfolios. *Math. Finance* **10**(2), 227–241.

Korn, R. & M. Schäl (1999). On value preserving and growth-optimal portfolios. *Math. Methods Oper. Res.* **50**(2), 189–218.

Kramkov, D. O. & W. Schachermayer (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9**, 904–950.

Kreps, D. M. (1981). Arbitrage and equilibrium in economics with infinitely many commodities. *J. Math. Economics* **8**, 15–35.

Lakner, P. (1993). Martingale measure for a class of right-continuous process. *Math. Finance* **3**, 43–53.

Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.

Platen, E. (2001). A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.

Platen, E. (2002). Arbitrage in continuous complete markets. *Adv. in Appl. Probab.* **34**(3), 540 – 558.

Platen, E. (2003). A semimartingale benchmark framework for asset pricing. University of Technology Sydney, (working paper).

Platen, E. (2004). A class of complete benchmark models with intensity based jumps. *J. Appl. Probab.* **41**(1). (to appear in March 2004).

Platen, E. & J. West (2003). Fair pricing of weather derivatives. Technical report, University of Technology, Sydney. QFRG Research Paper 106.

Protter, P. (1990). *Stochastic Integration and Differential Equations*. Springer.

Revuz, D. & M. Yor (1999). *Continuous Martingales and Brownian Motion* (3rd ed.). Springer.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *J. Economic Theory* **13**, 341–360.

Shiryaev, A. N. & A. S. Cherny (2002). Vector stochastic integrals and the fundamental theorems of asset pricing. *Proceedings of the Steklov Mathematical Institute Moscow* **237**, 6–49.

Yan, J. A. (1998). A new look at the fundamental theorem of asset pricing. *J. Korean Math. Soc.* **35**(3), 659–673.