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EXPLICIT BOUNDS FOR APPROXIMATION RATES FOR BOUNDARY CROSSING PROBABILITIES FOR THE WIENER PROCESS

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Abstract

We give explicit upper bounds for convergence rates when approximating (both one- and two-sided general curvilinear) boundary crossing probabilities for the Wiener process by similar probabilities for close boundaries (of simpler form for which computing the probability is feasible). In particular, we generalize and improve results obtained by Pötzelberger and Wang [13] for the case when approximating boundaries are piecewise linear. Applications to barrier option pricing are discussed as well.

Keywords: Wiener process, boundary crossing probabilities, barrier options

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Secondary 65C50; 91B70; 60G40

1. Introduction and main results

Computing the probability $P(g_-, g_+)$ for the standard Wiener process $\{W_t\}$ to stay within a corridor between two given boundaries $g_-(t) < g_+(t)$ during a specified time interval $[0, T]$ is crucial in many important applications including sequential statistical analysis and pricing financial barrier options. In fact, basing on the Donsker–Prokhorov invariance principle, such a probability is often used as an approximation to a similar

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boundary crossing probability for a random walk (or even a more general process). Computing the probability $P(g_-, g_+)$ in non-trivial cases is, however, a rather tedious task by itself, that in its turn, requires using some approximation methodology as well.

A standard approach to this problem is to approximate the given (general curvilinear) boundaries $g_\pm$ with some other boundaries $f_\pm$ of a form enabling one to relatively easily compute the probability $P(f_-, f_+)$, a popular choice for $f_\pm$ being piecewise linear boundaries (for which a combination of the total probability formula, Markov property and known explicit formulae for linear boundary crossing probabilities for the Brownian bridge process immediately gives the desired probability $P(f_-, f_+)$ as a finite-dimensional Gaussian integral of a product-form integrand [19, 11, 13]; see also our Remark 5 below). To justify the use of $P(f_-, f_+)$ instead of $P(g_-, g_+)$, one must, of course, give an upper bound for the difference between the two values.

As a recent advance in this direction, we mention here a paper by Pötzelberger and Wang [13] (see also further references to be found in that paper). The authors, under the assumptions that the boundaries $g_\pm$ are twice continuously differentiable with $g''_\pm(0) \neq 0$ and $g''_\pm(t) = 0$ at most at finitely many points $t \in (0, T]$, proposed a special rule for choosing a sequence of "optimal partitions" $t_0^{(n)} = 0 < t_1^{(n)} < \cdots < t_n^{(n)} = T$ of $[0, T]$ (generally speaking, depending on the boundaries) with the following property: if $g_\pm^{(n)}$ are piecewise linear boundaries with nodes at $(t_i^{(n)}, g_\pm(t_i^{(n)}))$, $i = 0, 1, \ldots, n$, then for

$$\Delta_n := |P(g_-, g_+) - P(g_-^{(n)}, g_+^{(n)})|$$

one has the asymptotic bound

$$\limsup_{n\to\infty} n^2 \Delta_n \le A, \tag{1}$$

where the constant A depends on both the shape of the boundaries $g_\pm$ and the rule used to form the partitions $\{t_i^{(n)}\}_{0\le i\le n}$(through a couple of integrals that could actually be computed—at least, numerically).

One could observe that the above conditions on the boundaries (in particular, on $g''_{\pm}$ being non-zero) appear to be irrelevant (and are just due to the method employed in [13]). In this paper, we show that this is the case indeed, and that a much nicer than (1) explicit estimate holds for Δ_n under more general assumptions on the boundaries and when one simply uses uniform (or any other fine enough) partitions of $[0, T]$ (see Corollary 1 below).

This finding is based on a general simple result that admits a short self-contained proof, and of which a precise formulation is as follows. Let $g_{\pm}(t)$ be two functions on $[0, T]$, such that $g_-(0) < 0 < g_+(0)$. Denote by

$$P(g_-, g_+) := \boldsymbol{P}(g_-(t) < W_t < g_+(t),\, t \in [0, T])$$

the probability that the trajectory of the standard Wiener process $\{W_t\}_{t\geq 0}$ will stay between the boundaries $g_{\pm}$ during the whole time interval $[0, T]$. If $g_-(t) \geq g_+(t)$ at some $t \in [0, T]$, we simply get $P(g_-, g_+) = 0$. In the case of one-sided (upper) boundary, we will be just using the notation $P(-\infty, g_+)$.

By Lip_K we will denote the class of Lipschitz functions on $[0, T]$ with the constant $K \in (0, \infty)$: $g \in \mathrm{Lip}_K$ iff

$$|g(t + h) - g(t)| \leq Kh, \quad 0 \leq t < t + h \leq T,$$

and by $\|\cdot\|$ the uniform norm of a (bounded) function on $[0, T]$: $\|g\| = \sup_{0\leq t\leq T} |g(t)|$.

Theorem 1. *If $g_{\pm} \in \mathrm{Lip}_K$ and $\|g_{\pm} - f_{\pm}\| \leq \varepsilon$ for some functions $f_{\pm}$ on $[0, T]$, then*

$$|P(-\infty, g_+) - P(-\infty, f_+)| \leq (2.5K + 2T^{-1/2})\varepsilon \tag{2}$$

and

$$|P(g_-, g_+) - P(f_-, f_+)| \leq (5K + 4T^{-1/2})\varepsilon. \tag{3}$$

The same bounds will also hold for the differences

$$P(-\infty, g_+; B) - P(-\infty, f_+; B) \quad \text{and} \quad P(g_-, g_+; B) - P(f_-, f_+; B), \tag{4}$$

where, for a Borel set B,

$$P(g_-, g_+; B) := \boldsymbol{P}(g_-(t) < W_t < g_+(t),\, t \in [0,T]; W_T \in B).$$

Remark 1. The result (or a weaker form thereof) might actually be already known. It was observed in Borovkov [4] that, with a right-hand side of the form $C(K+T^{-1/2})\varepsilon$, where C is *some* absolute constant, the above inequalities can be derived from relation (2.22) in Nagaev [10], estimates in Sahkanenko [16] and the Donsker–Prokhorov invariance principle.

Remark 2. It is clear from the proof of Theorem 1 that a somewhat more precise than (3) bound holds in the two-sided boundary case: assuming that $g_\pm \in \mathrm{Lip}_{K_\pm}$, one can replace $5K$ on the right-hand side of that bound with $2.5(K_- + K_+)$. Observe also that, under additional assumptions about the monotonicity of the boundaries $g_\pm$, the values for the constants in bounds (2) and (3) can be made somewhat smaller (see Lemma 1 below).

Next we will formulate our improvement of (1) which is a simple consequence of Theorem 1 based on the fact that, for smooth enough functions, the approximation rate by piecewise linear functions will be a quadratic function of the partition rank. More precisely, the following result holds.

Corollary 1. *Let $g_\pm$ be continuously differentiable on $[0,T]$, $K = \max\{\|g_-'\|, \|g_+'\|\}$, and let $g_\pm'$ be absolutely continuous satisfying $|g_\pm''| \le \gamma < \infty$ a.e. If $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0,T]$ of rank $\delta = \max_{0<i\le n} |t_i - t_{i-1}|$, and $f_\pm$ are piecewise linear with nodes at the points $(t_i, g_\pm(t_i))$, then*

$$|P(-\infty, g_+) - P(-\infty, f_+)| \le (0.313K + 0.25T^{-1/2})\gamma\delta^2. \qquad (5)$$

and

$$|P(g_-, g_+) - P(f_-, f_+)| \le (0.625K + 0.5T^{-1/2})\gamma\delta^2 \qquad (6)$$

In particular, if the partition is uniform: $t_i = iT/n$, $0 \le i \le n$, and $g_\pm^{(n)}$ denote the respective piecewise linear approximations to $g_\pm$, then $\delta = T/n$

and hence, instead of the asymptotic bound (1)*, we obtain the following inequality:*

$$\Delta_n \le Dn^{-2}, \qquad D = (0.625K + 0.5T^{-1/2})\gamma T^2. \tag{7}$$

The same bounds will hold for the differences (4)*.*

Remark 3. From the proof of the corollary it is obvious that its assumptions can be somewhat relaxed: we can only assume that the boundaries $g_\pm$ are piecewise continuously differentiable (with the derivatives satisfying the stated conditions). The inequalities (6)–(5) remain valid as long as all the points at which any of $g_\pm$ is not differentiable belong to the partition.

Remark 4. If the assumption $g_+, f_+ \in \mathrm{Lip}_K$ fails, but the functions are absolutely continuous with square integrable derivatives, and $g_+(0) - f_+(0) = g_+(T) - f_+(T)$, then the following bound holds:

$$|P(-\infty, g_+; B) - P(-\infty, f_+; B)| \le \boldsymbol{P}(W_T \in B)\left[\frac{1}{2\pi}\int_0^T (g_+'(s) - f_+'(s))^2\, ds\right]^{1/2}.$$

The proof of this result and also similar bounds for the two-sided boundary crossing probabilities in the case $B = (M, \infty)$ was given in [11].

Remark 5. It appears that Wang and Pötzelberger [19] were the first to combine the total probability formula, the Markov property of the Wiener process and a known explicit formula for a (one-sided) linear boundary crossing probability for the Brownian bridge process to show that the one-sided boundary crossing probability $P(-\infty, g^{(n)})$ for a piecewise linear function $g^{(n)}$ can be represented as an n-fold Gaussian integral. Novikov et al. [11] gave in their Theorem 1 a more general formula for the two-sided boundary crossing probabilities with arbitrary (measurable) boundaries $g_\pm$ that is equivalent to the following representation: for any Borel set B,

$$P(g_-, g_+;\, B) = \boldsymbol{E}\Big[\mathbf{1}_{\{W_T \in B\}} \prod_{i=0}^{n-1} p_i(g_-, g_+|\, W_{t_i}, W_{t_{i+1}})\Big] \tag{8}$$

with $\mathbf{1}_A$ being the indicator function of the event A and

$$p_i(g_-, g_+|x_i, x_{i+1}) := \boldsymbol{P}\big(g_-(s) < W_s < g_+(s),\ s \in [t_i, t_{i+1}]\,|\, W_{t_i} = x_i, W_{t_{i+1}} = x_{i+1}\big)$$

(with $B = \boldsymbol{R}$, see also Theorem 2 in [13]).

In the special case when $g_-(s) = -\infty$, $s \in [t_i, t_{i+1}]$, and $g_+(s)$ is a linear boundary on this interval, the last probability has the following simple form used in [19]:

$$p_i(-\infty, g_+ |\, x_i, x_{i+1}) = 1 - \exp\left\{ -\frac{2(g_+(t_i) - x_i)(g_+(t_{i+1}) - x_{i+1})}{t_{i+1} - t_i} \right\} \quad (9)$$

(this is a well-known expression for the linear boundary crossing probability by the Brownian bridge process, see e.g. p.63 in [2]). In the case of two-sided linear boundaries $g_\pm$, the probability $p_i(g_-, g_+ | x, y)$ is given by a rapidly convergent infinite series of exponential functions (for details and numerical examples see e.g. [11] or [13]). In both cases (of one-sided or two-sided piecewise linear boundaries), the complexity of the numerical computation of the n-fold Gaussian integral on the right-hand side of (8) appears to be acceptable due to a relatively simple form of the functions $p_i(g_-, g_+ |\, x_i, x_{i+1})$.

Nardo et al. [9] found another parametric family of one-sided boundaries (that could be called "generalized Daniels' boundaries", cf. [5]), for which the probability $p_-(-\infty, g_+|, x_i, x_{i+1})$ also has a relatively simple form: if, for $t \in [t_i, t_{i+1}]$,

$$y(t) := (d_1 - d_2)(t - t_i) + d_2 < u(t) := (d_1^* - d_2^*)(t - t_i) + d_2^*$$

with $d_1, d_2, d_1^*, d_2^* \in \mathbf{R}$ and, on that time interval,

$$g_+(t) = x_i + \frac{(t - t_i)(x_{i+1} - x_i)}{t_{i+1} - t_i} + u(t) - \frac{(t - t_i)\ln(D(t)/2)}{2(u(t_i) - y(t_i))} \quad (10)$$

with

$$D(t) = C_1 + \sqrt{C_1^2 + 4C_2^2 \exp\left\{ -4\frac{(u(t) - y(t))(u(t_i) - y(t_i))}{t - t_i} \right\}} > 0,$$

where $C_j > 0$, then

$$p_i(-\infty, g_+ |\, x_i, x_{i+1}) = 1 - C_1 \exp\left\{ -\frac{2d_1^*(u(t_i) - y(t_i))}{t_{i+1} - t_i} \right\} - C_2 \exp\left\{ -\frac{4(2d_1^* - d_1)(u(t_i) - y(t_i))}{t_{i+1} - t_i} \right\} \quad (11)$$

(our notation is slightly different from that in [9]).

If $C_1 = 1$ and $C_2 = 0$, then the function $g_+(t)$ in (10) is linear, and putting $d_2^* = g_+(t_i) - x_i$, $d_1^* - d_2^* = g_+(t_{i+1}) - x_{i+1}$, we get from (11) the well-known result (9). Generally, the parametric curve (10) depends on six parameters, so we can use it as a second-order spline with the boundary conditions

$$g_+^{(n)}(t_i) = g_+(t_i), \quad \frac{d}{dt} g_+^{(n)}(t_i) = \frac{d}{dt} g_+(t_i), \quad t_i = \frac{iT}{n}, \quad i = 0, 1, ..., n,$$

It is well-known (see e.g. Chapter 1 in [17]) that the approximation rate of four times continuously differentiable functions $g_\pm(t)$ by the second-order spline functions $g_\pm^{(n)}(t)$ on uniform partitions is

$$\varepsilon := \| g_\pm - g_\pm^{(n)} \| = O(n^{-4}) \quad \text{as } n \to \infty.$$

Therefore by Theorem 1

$$|P(g_-, g_+; B) - P(g_-^{(n)}, g_+^{(n)}; B)| = O(n^{-4}).$$

Of course, to meet the additional boundary conditions for such an approximation, one has to solve a system of nonlinear equations, and therefore the computational complexity of this approach could be higher compared to the piecewise linear approximation.

In conclusion, note that in the literature, there exist several quite different approaches to computing numerical approximations to $P(g_-, g_+)$ and $P(-\infty, g)$, which have different computation complexity (see e.g. [6, 7, 8, 15, 9] and references therein). Knowing not only the order, but also the *form* of the approximation error allows one to further improve the approximation rate by using the so-called Richardson extrapolation that is based on the idea of extrapolating computed results to much bigger values of n (see e.g. [1]).

In Section 3 we will discuss an application of Theorem 1 to estimating the accuracy of approximations for double-barrier option prices.

2. Proofs

Due to the self-similarity property of the Wiener process, we can assume without loss of generality that $T = 1$ (the general case bounds will follow then by the standard scaling argument).

We will begin the proof of Theorem 1 with the obvious relation

$$P(g_- + \varepsilon, g_+ - \varepsilon) \le P(f_-, f_+) \le P(g_- - \varepsilon, g_+ + \varepsilon). \tag{12}$$

Note that

$$\begin{aligned} 0 &\le P(g_- - \varepsilon, g_+ + \varepsilon) - P(g_-, g_+) \\ &= [P(g_- - \varepsilon, g_+ + \varepsilon) - P(g_-, g_+ + \varepsilon)] + [P(g_-, g_+ + \varepsilon) - P(g_-, g_+)], \end{aligned}$$

where both terms on the right-hand side can be dealt with in the same way. So it suffices to consider the second one only, and for that term we clearly have

$$\begin{aligned} P(g_-, g_+ + \varepsilon) &- P(g_-, g_+) \\ &= \boldsymbol{P}\Big(0 \le \sup_{0\le t\le 1}(W_t - g_+(t)) < \varepsilon,\ \inf_{0\le t\le 1}(W_t - g_-(t)) > 0\Big) \\ &\le \boldsymbol{P}\Big(0 \le \sup_{0\le t\le 1}(W_t - g_+(t)) < \varepsilon\Big) = P(-\infty, g_+ + \varepsilon) - P(-\infty, g_+) \\ &=: \Delta_\varepsilon(g_+) \le \Delta_\varepsilon := \sup_{f\in \mathrm{Lip}_K} \Delta_\varepsilon(f). \end{aligned} \tag{13}$$

As the same argument applies to the first term as well, we get

$$0 \le P(g_- - \varepsilon, g_+ + \varepsilon) - P(g_-, g_+) \le 2\Delta_\varepsilon.$$

Similarly,

$$0 \le P(g_-, g_+) - P(g_- + \varepsilon, g_+ - \varepsilon) \le 2\Delta_\varepsilon,$$

and together with (12) these inequalities imply that

$$|P(g_-, g_+) - P(f_-, f_+)| \le 2\Delta_\varepsilon.$$

Basically the same argument shows that

$$|P(g_-, g_+; B) - P(f_-, f_+; B)| \le 2\Delta_\varepsilon$$

as well. In the case of one-sided boundaries, even a simpler argument gives

$$|P(-\infty, g_+) - P(-\infty, f_+)| \le \Delta_\varepsilon.$$

The desired bounds (2)–(3) will now follow from the next assertion.

Lemma 1. *Let g be a function on $[0,1]$, $g(0) \ge 0$, such that for some $K^\pm \in [0,\infty)$*

$$-K^- h \le g(t+h) - g(t) \le K^+ h, \quad 0 \le t < t+h \le 1. \tag{14}$$

Then

$$\Delta_\varepsilon(g) \le (2K^+ + 0.5K^- + 2)\varepsilon, \quad \varepsilon > 0. \tag{15}$$

Proof. Put $\tau := \inf\{t > 0 : W_t > g(t)\}$ and observe that

$$\begin{aligned} \Delta_\varepsilon(g) &= \boldsymbol{P}\Big(0 \le \sup_{0\le t\le 1} (W_t - g(t)) < \varepsilon\Big) \\ &= \int_0^1 \boldsymbol{P}(\tau \in dt)\, \boldsymbol{P}\Big(\sup_{t\le s\le 1} (W_s - g(s)) < \varepsilon |\, W_t = g(t)\Big) \\ &\le \int_0^1 \boldsymbol{P}(\tau \in dt)\, \boldsymbol{P}\Big(\sup_{0\le s\le 1-t} (W_s - K^+ s) < \varepsilon\Big). \end{aligned} \tag{16}$$

The last probability is known in explicit form (see e.g. formula 1.1.4 on p.197 in [2]): denoting by Φ the standard normal distribution function, we get

$$\begin{aligned} &\boldsymbol{P}\Big(\sup_{0\le s\le 1-t} (W_s - K^+ s) < \varepsilon\Big) \\ &\qquad = \Phi\Big(K^+\sqrt{1-t} + \frac{\varepsilon}{\sqrt{1-t}}\Big) - e^{-2K^+\varepsilon}\Phi\Big(K^+\sqrt{1-t} - \frac{\varepsilon}{\sqrt{1-t}}\Big) \\ &\le \Phi\Big(K^+\sqrt{1-t} + \frac{\varepsilon}{\sqrt{1-t}}\Big) - \Phi\Big(K^+\sqrt{1-t} - \frac{\varepsilon}{\sqrt{1-t}}\Big) + (1 - e^{-2K^+\varepsilon}). \end{aligned}$$

Using the obvious inequalities $\Phi'(x) \le 1/\sqrt{2\pi}$ and $1 - e^{-2K^+\varepsilon} \le 2K^+\varepsilon$, we get from the above representation that that probability on the left-hand side does not exceed $\sqrt{2/\pi(1-t)}\varepsilon + 2K^+\varepsilon$.

Now we need to deal with $\boldsymbol{P}(\tau \in dt)$. For any fixed $t \in [0,1]$, introduce the boundary

$$g_t(s) := g(t) + K^-(t-s), \quad 0 \le s \le 1,$$

and put

$$\tau_t := \inf\{s > 0 : W_s > g_t(s)\} = \inf\{s > 0 : W_s + K^- s > g(t) + K^- t\}.$$

Obviously, due to our assumption about g,

$$\boldsymbol{P}(\tau \in (t, t+h)) \le \boldsymbol{P}(\tau_t \in (t, t+h)), \quad 0 \le t < t+h \le 1, \tag{17}$$

and hence it just remains to bound the right-hand side of this inequality.

Since $\{W_s + K^- s\}_{s\ge 0}$ is a continuous processes with stationary independent increments, the distribution of τ_t can be readily found from the Kendall's formula (see e.g. Theorem 1 on p.66 in [3]; see also formula 2.0.2 on p.223 in [2] for our special case): it will have the density

$$v_t(s) := \frac{\boldsymbol{P}(\tau_t \in ds)}{ds} = \frac{g(t) + K^- t}{\sqrt{2\pi s^3}} \exp\{-(g(t) + K^-(t-s))^2/2s\}, \quad s > 0.$$

Now from (17) it follows that τ will also have a density $p(t)$ and

$$p(t) \le v_t(t).$$

Returning to (16) and our bound for the integrand in it, we have, for any $r \in (0, 1)$,

$$\begin{aligned}\Delta_\varepsilon(g) &\le \varepsilon\sqrt{\frac{2}{\pi}}\int_0^1 \frac{p(t)\,dt}{\sqrt{1-t}} + 2K^+\varepsilon\int_0^1 p(t)\,dt \le \varepsilon\sqrt{\frac{2}{\pi}}\left[\int_0^r + \int_r^1\right] + 2K^+\varepsilon \\ &\le \varepsilon\sqrt{\frac{2}{\pi(1-r)}}\int_0^r p(t)\,dt + \varepsilon\sqrt{\frac{2}{\pi}}\int_r^1 \frac{v_t(t)\,dt}{\sqrt{1-t}} + 2K^+\varepsilon. \end{aligned} \tag{18}$$

As $\int_0^r p(t)\,dt < 1$, we just have to estimate the last integral in (18), and for that, we will find the maximum possible value of $v_t(t)$ over all admissible under the lemma's assumptions values $g(t) \ge g(0) - K^- t \ge -K^- t$ (as $g(0) \ge 0$). To this end, compute

$$\sup\bigl\{m(y) := (y + K^- t)e^{-y^2/2t} : y \ge g(0) - K^- t\bigr\}$$

by taking the derivative of the function $m(y)$ with respect to y and equating it to zero, which yields

$$0 = 1 - (y + K^- t)y/t.$$

Solving this for y, we get

$$y_\pm = -\frac{K^- t}{2} \pm \sqrt{t + \frac{K^- t}{2}}.$$

Now noting that the root y_- is inadmissible and that $m''(y_+) < 0$, we see that we found the maximum indeed, so that

$$v_t(t) \le \frac{y_+ + K^- t}{\sqrt{2\pi t^3}} = \frac{1}{\sqrt{2\pi}}\left(\frac{K^-}{2\sqrt{t}} + \sqrt{\frac{1}{t^2} + \frac{(K^-)^2}{4t}}\right) \le \frac{1}{\sqrt{2\pi}}\left(\frac{K^-}{\sqrt{t}} + \frac{1}{t}\right)$$

using $\sqrt{a^2+b^2} \le a + b$, $a, b \ge 0$.

Therefore

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_r^1 \frac{v_t(t)\,dt}{\sqrt{1-t}} &\le \frac{1}{\pi}\left[\int_r^1 \frac{K^- dt}{\sqrt{t(1-t)}} + \int_r^1 \frac{dt}{t\sqrt{1-t}}\right] \\ &= \frac{1}{\pi}\left[K^-(\pi - 2\arcsin\sqrt{r}) + 2\int_{\arcsin\sqrt{r}}^{\pi/2} \frac{du}{\sin u}\right] \\ &\le \frac{K^-}{\pi}(\pi - 2\arcsin\sqrt{r}) + \frac{2(\pi/2 - \arcsin\sqrt{r})}{\pi\sqrt{r}}\Bigg|_{r=1/2} = \frac{K^-}{2} + \frac{1}{\sqrt{2}}. \end{aligned}$$

Substituting this into (18) (with $r = 1/2$) and noting that $\sqrt{4/\pi} + 1/\sqrt{2} < 2$, we readily get inequality (15). This completes the proof of Lemma 1 and therefore that of Theorem 1 as well.

To prove Corollary 1, it suffices to show that, for the maximum deviation of the piecewise linear approximant f from the original boundary function g with the assumed properties, one has

$$\|f - g\| \le \gamma\delta^2/8, \tag{19}$$

as the desired result will then immediately follow from Theorem 1. The proof of bound (19) is elementary and is included only for the exposition completeness' sake.

Clearly,

$$\|f - g\| = \max_{0<i\le n} \max_{t\in[t_{i-1}, t_i]} |f(t) - g(t)|,$$

so we have to show that the bound holds for all the maximum deviations on each of the subintervals $[t_{i-1}, t_i]$, $i = 1, \ldots, n$. Consider the first of them (the same argument will clearly work for all the others as well). Put

$$\xi = \arg \max_{t\in[t_0,t_1]} |f(t) - g(t)|$$

and observe that $g'(\xi) = f'(\xi)$. Putting

$$h(t) = g(\xi) + g'(\xi)(t - \xi), \quad t \in [t_0, t_1].$$

we see that the last observation means that the plots of the linear (at least, on the subinterval $[t_0, t_1]$) functions f and h are parallel to each other: $f(t) - h(t) = \text{const}$, $t \in [t_0, t_1]$.

Next, for $j = 0, 1$, one has

$$\begin{aligned}\max_{t\in[t_0,t_1]} |f(t) - g(t)| &= |f(\xi) - g(\xi)| = |f(\xi) - h(\xi)| = |f(t_j) - h(t_j)| \\ &= |g(t_j) - h(t_j)| = |g(t_j) - g(\xi) - g'(\xi)(t_j - \xi)| \\ &= \left| \int_\xi^{t_j} \left[\int_\xi^u g''(v)\, dv \right] du \right| \le \frac{\gamma}{2}(t_j - \xi)^2.\end{aligned}$$

Now since $\min_{j=0,1}(t_j - \xi)^2 \le ((t_1 - t_0)/2)^2 \le \delta^2/4$, the bound (19) and hence the statement of Corollary 1 are proved.

3. Approximations for time-dependent barrier options prices

In this section we will discuss how the above results can be applied to barrier options pricing.

It is well known that, under the no-arbitrage assumption, the fair price of a (replicable) option (on an underlying asset with a price process $\{S_t\}_{t\ge0}$) with maturity T and payoff f_T is given by $\boldsymbol{E}(f_T/B_T)$, where $\boldsymbol{E}$ denotes the operation of taking expectation with respect to a risk-neutral measure $\boldsymbol{P}$ and $\{B_t\}_{t\ge0}$ is the bank account process (for detail see e.g. [18]). We will consider a call option with strike K_T and time-dependent lower/upper barriers $G_\pm(t)$ such that $G_-(t) < G_+(t)$, $t \le T$. In this case, the payoff function is given by

$$f_T = (S_T - K_T)\mathbf{1}_{\{S_T>K_T;\, G_-(t)<S_t<G_+(t),\, t\in[0,T]\}}.$$

Assume that the bank account process is non-random and has the form

$$B_t = \exp\left\{\int_0^t r(s)\,ds\right\},$$

where $r(t)$ is a positive function of time (the spot interest rate). Under the assumptions of the standard diffusion model, the price of the underlying asset S_t (under risk-neutral measure) has the following representation:

$$S_t = S_0 \exp\left\{\int_0^t [r(s) - \sigma^2/2]\,ds + \sigma W_t\right\}, \tag{20}$$

where σ is the (constant) volatility of the price process.

The following statement gives the representation of the option price in terms of the boundary crossing probabilities.

Proposition 1. *The fair price of the above double-barrier call option is given by*

$$S_0 p_1 - K_T \exp\left\{-\int_0^T r(s)\,ds\right\} p_0,$$

where

$$\begin{aligned}
p_1 &= \boldsymbol{P}\big(f_-(t) < \sigma W_t + \sigma^2 t < f_+(t),\ t \in [0,T]; \sigma W_T + \sigma^2 T > F\big),\\
p_0 &= \boldsymbol{P}\big(f_-(t) < \sigma W_t < f_+(t),\ t \in [0,T];\ \sigma W_T > F\big),\\
F &= \ln(K_T/S_0) + \frac{1}{2}\sigma^2 T - \int_0^T r(s)\,ds,
\end{aligned}$$

and

$$f_\pm(t) = \ln(G_\pm(t)/S_0) + \frac{1}{2}\sigma^2 t - \int_0^t r(s)\,ds, \quad t \in [0,T].$$

One can easily prove this statement using Girsanov's transformation (for details in the case of a one-sided barrier, see e.g. [12]).

To calculate the probabilities p_0 and p_1, one could use several different techniques: a PDE approach [20], integral equations for the case of one-sided barriers [15, 9], and Monte-Carlo simulation [14]. As both probabilities p_0 and p_1 are of the form $P(g_-, g_+; (M, \infty))$, one can also use a numerical approximation based on the integral representation (8) with a proper chosen spline approximations and respective probabilities

$p_i(g_-, g_+ \,|\, x_i, x_{i+1})$. In particular, using piecewise linear (on uniform partitions) approximations for boundaries will yield, by Corollary 1, approximation rate $O(n^{-2})$. Using generalized Daniels' boundaries (see Remark 5 above), the rate of convergence could potentially be improved up to $O(n^{-4})$ (or even to a higher order). Discussing the computational efficiencies of different numerical techniques is beyond the scope of the present paper.

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References

[1] Acton, F. S. (1990). *Numerical methods that work.* Mathematical Association of America, Washington.

[2] Borodin, A.N. and Salminen, P. (2002). *Handbook of Brownian motion—facts and formulae,* 2nd edn. Birkhäuser Verlag, Basel.

[3] Borovkov, A.A. (1976). *Stochastic Processes in Queueing Theory.* Springer, New York.

[4] Borovkov, K.A. (1982). *Limit Theorems for Some Types of Random Walks.* Cand.Sci. Thesis, Steklov Mathem. Institute, Moscow.

[5] Daniels, H. E. (1969). The minimum of a stationary Markov process superimposed on a U-shaped trend. *J. Appl. Probability* **6,** 399–408.

[6] Daniels, H. (1996). Approximating the first crossing-time density for a curved boundary. *Bernoulli*, **2,** 133–143.

[7] Durbin, J. (1992). The First-Passage Density of the Brownian Motion Process to a Curved Boundary. *J. Appl. Prob.*, **29,** 291–304.

[8] Lerche, H. R. (1986). *Boundary Crossing of Brownian Motion.* Lecture Notes in Statistics, v.40, Springer-Verlag, Berlin.

[9] Di Nardo, E., Nobile, A. G., Pirozzi, E. and Ricciardi, L. M. (2001). A computational approach to first-passage-time problems for Gauss-Markov processes. *Adv. Appl. Prob.* **33,** 453–482.

[10] Nagaev, S.V. (1970). The rate of convergence in a certain boundary value problem.I. *Theor. Probab. Appl.* **15,** 179–199; II, *ibid.* 419–444.

[11] Novikov, A. A., Frishling, V. and Kordzakhia, N. (1999). Approximations of Boundary Crossing Probabilities for a Brownian Motion. *J. Appl. Prob.* **36,** 1019–1030.

[12] Novikov, A. A., Frishling, V. and Kordzakhia, N. (2003). Time-Dependent Barrier Options and Boundary Crossing Probabilities. *Georgian Math. J.* **10,** 325–334.

[13] Potzelberger, K. and Wang, L. (2001). Boundary crossing probability for Brownian motion. *J. Appl. Probab.* **38,** 152–164.

[14] Roberts, G.O. and Shortland, C.F. (1997). Pricing Barriers Options with Time-Dependent Coefficients. *Mathem. Finance* **7,** 83–93.

[15] Sacerdote, L. and Tomasseti, F. (1996). On evaluations and asymptotics approximations of the first-passage-time probabilities. *Adv. Appl.Prob.* **28,** 270–284.

[16] Sahanenko, A.I. (1974). On the rate of convergence in a boundary value problem. *Theor. Probab. Appl.* **19,** 416–421.

[17] Shikin, E.V. and Plis, A.I. (1995). *Handbook on splines for the user.* CRC Press, Boca Raton.

[18] Shiryaev, A. N. (1999). *Essentials of Stochastic Finance: Facts, Models, Theory.* World Scientific, Singapore.

[19] Wang, L. and Pötzelberger, K. (1997). Boundary Crossing Probability for Brownian Motion and General Boundaries. *J. Appl. Prob.* **34,** 54–65.

[20] Wilmont, P., Dewynne, J. and Howison, S. (1995). *Option Pricing: Mathematical Models and Computation.* Cambridge Financial Press, Cambridge.