A Discrete Time Benchmark Approach for Finance and Insurance

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Abstract. This paper proposes an integrated approach to discrete time modelling in finance and insurance. This approach is based on the existence of a specific benchmark portfolio, known as the growth optimal portfolio. When used as numeraire, this portfolio ensures that all benchmarked price processes are supermartingales. A fair price is characterized in terms of the type of maximum that the growth rate of the perturbed growth optimal portfolio attains. In general, arbitrage amounts arise due to the supermartingale property of benchmarked traded prices. No measure transformation is needed for the fair pricing of insurance policies and derivatives.

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1 Introduction

There exists a stream of literature that exploits the concept of a *growth optimal portfolio* (GOP), originally developed by Kelly (1956) and later extended and discussed, for instance, in Long (1990), Artzner (1997), Bajeux-Besnainou & Portait (1997), Karatzas & Shreve (1998), Kramkov & Schachermayer (1999) and Goll & Kallsen (2002). Under certain assumptions the GOP coincides with the *numeraire portfolio*, which makes prices, when expressed in units of this particular portfolio, into martingales under the given probability measure. In Kramkov & Schachermayer (1999) and Platen (2001) it was demonstrated that prices when benchmarked by the GOP can become supermartingales without assuming the existence of an equivalent martingale measure. The notion of a numeraire portfolio was recently extended by Becherer (2001), taking into account benchmarked prices that are supermartingales when an equivalent local martingale measure exists. In standard cases with an equivalent martingale measure the numeraire portfolio has been shown to coincide with the inverse of the deflator or state price density, see Constatinides (1992), Duffie (1996) or Rogers (1997). Furthermore, in Bühlmann (1992, 1995) and Bühlmann, Delbaen, Embrechts & Shiryaev (1998) the deflator has been suggested for the modelling of financial and insurance markets. Similarly, in Platen (2001) a financial market has been constructed by characterization of the GOP as benchmark portfolio.

Within this paper we follow a discrete time benchmark approach, where we characterize key features of a financial and insurance market via the GOP without assuming the existence of any equivalent local martingale measure as in most of the above mentioned literature. In particular, fair prices of derivatives and insurance policies are obtained in a consistent manner via conditional expectations with respect to the real world probability measure. This provides a basis for the equivalence principle that is widely used by actuaries. An example of a discrete time market, suitable for finance and insurance applications, will illustrate some key features of the benchmark approach.

2 Discrete Time Market

Let us consider a discrete time market on a given probability space $(\Omega, \mathcal{A}, P)$. Asset prices are assumed to change their values only at the given discrete times

$$0 \leq t_0 < t_1 < \ldots < t_n < \infty$$

for $n \in \{0, 1, \ldots\}$. The information structure in this market is described by the filtration $\underline{\mathcal{A}} = (\mathcal{A}_{t_i})_{i \in \{0,1,\ldots,n\}}$ with $\mathcal{A}_{t_0}$ being the σ-algebra consisting of all null sets and their complements. In this paper we consider $d+1$ primary assets, $d \in \{1, 2, \ldots\}$, which generate interest, dividend, coupon or other payments as income or loss, incurred from holding the respective asset. We denote by $S_i^{(j)}$

the nonnegative value at time t_i of a primary security account. This account holds only units of the jth asset in addition to the earnings from holding the jth asset. That is, all income is reinvested into this account. Thus the jth primary security expresses the time value of the jth primary asset. The 0th primary security account is the domestic savings account. For simplicity, we assume at time t_0, that the jth primary security account consists of one unit of the jth primary asset. According to the above description, the domestic savings account $S^{(0)}$ is then a roll-over short term bond account, where the interest payments are reinvested at each time step. If the jth primary asset is a share, then $S_i^{(j)}$ is the value at time t_i of such shares including accrued dividends. The quantity $S_i^{(j)}$ then represents the jth cum-dividend share price at time t_i. We assume that

$$S_0^{(j)} > 0 \tag{2.1}$$

a.s. for all $j \in \{0, 1, \ldots, d\}$.

Now, we introduce the *growth ratio* $h_{i+1}^{(j)}$ of the jth primary security account at time t_{i+1} in the form

$$h_{i+1}^{(j)} = \begin{cases} \frac{S_{i+1}^{(j)}}{S_i^{(j)}} & \text{for} \quad S_i^{(j)} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

for $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, d\}$. Note that the *return* of $S^{(j)}$ at time t_{i+1} equals $h_{i+1}^{(j)} - 1$. We assume that $h_{i+1}^{(j)}$ is $\mathcal{A}_{t_{i+1}}$-measurable and a.s. finite. The growth rate of the primary security account $S^{(0)}$ for the domestic currency shall be strictly positive, that is

$$h_{i+1}^{(0)} > 0 \tag{2.3}$$

a.s. for all $i \in \{0, 1, \ldots, n-1\}$ with $S_0^{(0)} = 1$. We can then express the price of the jth primary security account at time t_i, say the jth cum-dividend share price $S_i^{(j)}$, in the form

$$S_i^{(j)} = S_0^{(j)} \prod_{\ell=1}^{i} h_\ell^{(j)} \tag{2.4}$$

for $i \in \{0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, d\}$. Note that due to (2.4) and (2.3) we have

$$S_i^{(0)} > 0 \tag{2.5}$$

for all $i \in \{0, 1, \ldots, n\}$.

In the given discrete time market it is possible to form self-financing portfolios containing the above primary security accounts. For the characterization of a self-financing portfolio at time t_i it is sufficient to describe the *proportion* $\pi_i^{(j)} \in$

$(-\infty, \infty)$ of its value that at this time is invested in the jth primary security account, $j \in \{0, 1, \dots, d\}$. Obviously, the proportions add to one, that is

$$\sum_{j=0}^{d} \pi_i^{(j)} = 1 \tag{2.6}$$

for all $i \in \{0, 1, \dots, d\}$. The vector process $\pi = \{\pi_i = (\pi_i^{(0)}, \pi_i^{(1)}, \dots, \pi_i^{(d)}),\ i \in \{0, 1, \dots, n\}\}$ denotes the corresponding *process of proportions*. We assume that π_i is $\mathcal{A}_{t_i}$-measurable, which means that all proportions at a given time do not depend on any future events. The value of the corresponding self-financing portfolio at time t_i is denoted by $S_i^{(\pi)}$ and we write $S^{(\pi)} = \{S_i^{(\pi)},\ i \in \{0, 1, \dots, n\}\}$. We obtain the *growth ratio* $h_\ell^{(\pi)}$ of this portfolio process at time t_ℓ in the form

$$h_\ell^{(\pi)} = \sum_{j=0}^{d} \pi_{\ell-1}^{(j)}\, h_\ell^{(j)} \tag{2.7}$$

for $\ell \in \{1, 2, \dots, n\}$, where its value at time t_i satisfies the expression

$$S_i^{(\pi)} = S_0^{(\pi)} \prod_{\ell=1}^{i} h_\ell^{(\pi)} \tag{2.8}$$

for $i \in \{0, 1, \dots, n\}$.

3 Discrete Time Market of Finite Growth

In the given discrete time market let us denote by $\mathcal{V}$ the set of all self-financing, strictly positive portfolio processes $S^{(\pi)}$ This means, for a portfolio process $S^{(\pi)} \in \mathcal{V}$ it holds $h_{i+1}^{(\pi)} \in (0, \infty)$ a.s. for all $i \in \{0, 1, \dots, n-1\}$. Due to (2.3) - (2.5), $S^{(0)}(t)$ is always strictly positive. Consequently, $\mathcal{V}$ is not empty.

We define for a given portfolio process $S^{(\pi)} \in \mathcal{V}$ with corresponding process of proportions π its *growth rate* $g_i^{(\pi)}$ at time t_i by the following conditional expectation

$$g_i^{(\pi)} = E\left(\log(h_{i+1}^{(\pi)}) \,\middle|\, \mathcal{A}_{t_i}\right) \tag{3.1}$$

for all $i \in \{0, 1, \dots, n-1\}$. This allows us to introduce the *optimal growth rate* $\underline{g}_i$ at time t_i as the supremum

$$\underline{g}_i = \sup_{S^{(\pi)} \in \mathcal{V}} g_i^{(\pi)} \tag{3.2}$$

for all $i \in \{0, 1, \dots, n-1\}$.

If the optimal growth rate could reach any arbitrarily large value at some time, then some self-financing portfolio would have unlimited growth with strictly positive probability. This can be interpreted as some form of arbitrage. We exclude such arbitrage opportunity by introducing the following notions.

Definition 3.1 *We say that a discrete time market is of* finite growth *if*

$$\max_{i\in\{0,1,\dots,n-1\}} \underline{g}_i < \infty, \tag{3.3}$$

a.s.

Definition 3.2 *If there exists in a discrete time market of finite growth a portfolio* $S^{(\underline{\pi})} \in \mathcal{V}$ *with*

$$S_0^{(\underline{\pi})} = 1, \tag{3.4}$$

a corresponding process of proportions $\underline{\pi}$ *such that*

$$g_i^{(\underline{\pi})} = \underline{g}_i \tag{3.5}$$

and

$$E\left(\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_i}\right) < \infty \tag{3.6}$$

for all $i \in \{0,1,\dots,n-1\}$ *and* $S^{(\pi)} \in \mathcal{V}$*, then we call* $S^{(\underline{\pi})}$ growth optimal *and the market* integrable.

From the viewpoint of an investor, a *growth optimal portfolio*, GOP, can be interpreted as a best benchmark portfolio because there is no other strictly positive, self-financing portfolio that can outperform its growth rate. In what follows we call prices, which are expressed in units of a GOP, *benchmarked* prices and their growth ratios *benchmarked* growth ratios. The condition (3.6) guarantees the integrability of benchmarked growth ratios.

We can now formulate the following result.

Theorem 3.3 *In an integrable market a portfolio process* $S^{(\underline{\pi})} \in \mathcal{V}$ *is growth optimal if and only if all portfolios* $S^{(\pi)} \in \mathcal{V}$*, when expressed in units of this portfolio, are* $(\underline{\mathcal{A}}, P)$*-supermartingales, that is*

$$E\left(\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_i}\right) \le 1 \tag{3.7}$$

for all $i \in \{0,1,\dots,n-1\}$.

The proof of this theorem is given in Appendix A. Under the assumption of the existence of an equivalent local martingale measure, Becherer (2001) proved a similar result for semimartingales.

Let us consider two portfolios that are both growth optimal in an invertible market. According to Theorem 3.3 the first portfolio, when expressed in units of the second, must be a supermartingale. Additionally, by the same argument the second, expressed in units of the first, must be also a supermartingale. This can only be true if both processes are identical, which yields the following result.

Corollary 3.4 *In an integrable market the GOP is unique.*

Note that the stated uniqueness of the GOP does not imply that its proportions $\underline{\pi}$ have to be unique.

4 Fair Market

The benchmarked price $\hat{S}_i^{(\pi)}$ at time t_i of a self-financing portfolio $S^{(\pi)}$ is defined by the relation

$$\hat{S}_i^{(\pi)} = \frac{S_i^{(\pi)}}{S_i^{(\underline{\pi})}} \tag{4.1}$$

for all $i \in \{0, 1, \ldots, n\}$.

By Theorem 3.3, in an integrable market the benchmarked price of a strictly positive, self-financing portfolio $S^{(\pi)} \in \mathcal{V}$ is a supermartingale and it follows

$$\hat{S}_i^{(\pi)} \geq E\left(\hat{S}_k^{(\pi)} \,\middle|\, \mathcal{A}_{t_i}\right) \tag{4.2}$$

for all $k \in \{0, 1, \ldots, n\}$ and $i \in \{0, 1, \ldots, k\}$.

One can interpret the last observed benchmarked price of a self-financing, strictly positive portfolio as its benchmarked traded price. In the sense of relation (4.2) it turns out that this benchmarked traded price is never below the best forecast of its future benchmarked values.

Definition 4.1 *If in an integrable market the benchmarked price $\hat{S}_i^{(\pi)}$ at time t_i of a self-financing portfolio process $S^{(\pi)}$ equals the best forecast of its future benchmarked values, that is*

$$\hat{S}_i^{(\pi)} = E\left(\hat{S}_{i+1}^{(\pi)} \,\middle|\, \mathcal{A}_{t_i}\right) \tag{4.3}$$

for all $i \in \{0, 1, \ldots, n-1\}$, then we call $S^{(\pi)}$ a fair *price process.*

Equation (4.3) means that $\hat{S}^{(\pi)}$ is an $(\underline{\mathcal{A}}, P)$-martingale. Note, for a fair price process $S^{(\pi)}$ we have always

$$\hat{S}_i^{(\pi)} = E\left(\hat{S}_n^{(\pi)} \,\middle|\, \mathcal{A}_{t_i}\right) \tag{4.4}$$

for all $i \in \{0, 1, \ldots, n\}$, which means that we can interpret $\hat{S}^{(\pi)}$ as the best forecast process of its final value. Furthermore, equivalently to (4.3) we have for a fair price process $S^{(\pi)}$ that

$$E\left(\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_i}\right) = 1 \tag{4.5}$$

for all $i \in \{0, 1, \ldots, n-1\}$. If all self-financing portfolio processes $S^{(\pi)}$ in an integrable market are fair, then we call it a *fair market*. Since in a fair market all benchmarked, self-financing portfolio processes are martingales one can show that such a market is arbitrage free in the sense of Harrison & Kreps (1979) and Harrison & Pliska (1981) and an equivalent risk neutral measure exists.

To see whether a portfolio in an integrable market is fair it is essential to identify the type of maximum that is obtained by the growth rate of the GOP if it would be perturbed by this portfolio.

For $\theta \in (0, \frac{1}{2})$ and $S^{(\pi)} \in \mathcal{V}$ we construct a *perturbed* GOP $V^{\theta,\pi} \in \mathcal{V}$ with growth ratio

$$h_{i+1}^{\theta,\pi} = \frac{V_{i+1}^{\theta,\pi}}{V_i^{\theta,\pi}} = \theta\, h_{i+1}^{(\pi)} + (1-\theta)\, h_{i+1}^{(\underline{\pi})} \tag{4.6}$$

at time t_{i+1} and corresponding growth rate

$$g_i^{\theta,\pi} = E\left(\log\left(h_{i+1}^{\theta,\pi}\right) \,\middle|\, \mathcal{A}_{t_i}\right) \tag{4.7}$$

at time t_i for $i \in \{0, 1, \ldots, n-1\}$. This allows us to define the *derivative of the growth rate* of the perturbed GOP in the direction of the portfolio $S^{(\pi)} \in \mathcal{V}$ at time t_i as the limit

$$\left.\frac{\partial g_i^{\theta,\pi}}{\partial \theta}\right|_{\theta=0} = \lim_{\theta \to 0+} \frac{1}{\theta}\left(g_i^{\theta,\pi} - g_i^{(\underline{\pi})}\right) \tag{4.8}$$

for $i \in \{0, 1, \ldots, n-1\}$.

Now we can formulate the following identity.

Theorem 4.2 *In an integrable market for a portfolio $S^{(\pi)} \in \mathcal{V}$ and $i \in \{0, 1, \ldots, n-1\}$ the following equality holds*

$$\left.\frac{\partial g_i^{\theta,\pi}}{\partial \theta}\right|_{\theta=0} = E\left(\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_i}\right) - 1. \tag{4.9}$$

The proof of this theorem follows directly from the proof of Theorem 3.3 that will be given in Appendix A.

From Theorem 4.2 and (4.5) we obtain immediately the following characterization of a fair portfolio.

Corollary 4.3 *In an integrable market a portfolio $S^{(\pi)} \in \mathcal{V}$ is fair if and only if for all $i \in \{0, 1, \ldots, n-1\}$ it holds*

$$\left.\frac{\partial g_i^{\theta,\pi}}{\partial \theta}\right|_{\theta=0} = 0. \tag{4.10}$$

Intuitively, Corollary 4.3 expresses the fact that a portfolio is fair if the maximum that the growth rate of the perturbed GOP attains, is a genuine maximum that satisfies the usual first order condition in the direction of the portfolio. It must not be a maximum that arises, for instance, at the boundary of $\mathcal{V}$ because of the constraint that the GOP has to remain strictly positive.

5 Fair Contingent Claim Pricing

We consider a *contingent claim* H_i, which is an $\mathcal{A}_{t_i}$-measurable, possibly negative payoff, expressed in units of the domestic currency, that has to be paid at a maturity date t_i, $i \in \{0, 1, \ldots, n\}$. Note that the claim H_i is not only contingent on the information provided by the observed primary assets $S_\ell^{(j)}$ up until time t_i, $j \in \{0, 1, \ldots, d\}$, $\ell \in \{0, 1, \ldots, i\}$, but as well on additional information contained in $\mathcal{A}_{t_i}$ as, for instance, the occurrence of defaults or insured events.

If we define the *fair price* $U_k^{(H_i)}$ at time t_k for the contingent claim H_i by the relation

$$U_k^{(H_i)} = S_k^{(\underline{\pi})} \, E\left(\frac{H_i}{S_i^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_k} \right), \tag{5.1}$$

for $k \in \{0, 1, \ldots, i\}$, then we obtain in a natural way a consistent system of prices. All fair prices of instruments including primary securities, derivative products and contingent claims have then a corresponding *benchmarked price* of the type

$$\hat{U}_k^{(H_i)} = \frac{U_k^{(H_i)}}{S_k^{(\underline{\pi})}} \tag{5.2}$$

for all $k \in \{0, 1, \ldots, i\}$, $i \in \{0, 1, \ldots, n\}$. Obviously, the process $\hat{U}^{(H_i)} = \{\hat{U}_k^{(H_i)},\, k \in \{0, 1, \ldots, i\}\}$ forms an $(\underline{\mathcal{A}}, P)$-martingale. The argument can be easily extended to sums of contingent claims with $\underline{\mathcal{A}}$-measurable maturity dates.

In finance, the pricing formula (5.1) is typically used for projecting future cashflows into present values, that is for $t_k \leq t_i$. Formally, one can use (5.1) also for assessing the present value for cashflows that occurred in the past, that is for $t_k > t_i$. Then we obtain

$$U_k^{(H_i)} = \frac{H_i}{S_i^{(\underline{\pi})}} S_k^{(\underline{\pi})} \tag{5.3}$$

for $i \in \{0, 1, \ldots, k\}$, $k \in \{0, 1, \ldots, n\}$. In (5.3) we express the benchmarked accumulated value of the payment H_i at time t_i in terms of time t_k value. This interpretation is important for insurance accounting as we will discuss below.

For the pricing of an insurance policy the actuarial task is the valuation of a sequence of cashflows $X_0, X_1, \ldots, X_n$, which are paid at the times $t_0, t_1, \ldots, t_n$, respectively. After each payment, its value is invested by the insurance company in a strictly positive, self-financing portfolio, characterized by a process of proportions π. The benchmarked fair price $\hat{Q}_0$ at time t_0 for the above sequence of cashflows is according to (5.2) given by the expression

$$\hat{Q}_0 = E\left(\sum_{k=0}^{m} \frac{X_k}{S_k^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_0} \right). \tag{5.4}$$

The benchmarked fair price $\hat{Q}_i$ at time t_i for $i \in \{0, 1, \ldots, n-1\}$ of this sequence of cashflows becomes

$$\hat{Q}_i = \hat{C}_i + \hat{R}_i \tag{5.5}$$

for $i \in \{0, 1, \ldots, n\}$. Here we choose an arbitrary process of proportions π, representing the investment portfolio of the insurance company, to obtain

$$\hat{C}_i = \frac{1}{S_i^{(\underline{\pi})}} \sum_{k=0}^{i} X_k \prod_{\ell=k}^{i-1} h_{\ell+1}^{(\pi)}, \tag{5.6}$$

which then expresses the benchmarked fair value of the already *accumulated payments*. Furthermore,

$$\hat{R}_i = E\left(\sum_{k=i+1}^{n} \frac{X_k}{S_k^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_i} \right) \tag{5.7}$$

is the benchmarked fair price at time t_i for the remaining payments, which is called the *prospective reserve*. It is easy to check that the process $\hat{Q} = \{\hat{Q}_i,\ i \in \{0, 1, \ldots, n\}\}$ forms an $(\underline{\mathcal{A}}, P)$-martingale for all choices of π.

When expressed in units of the domestic currency, we have at time t_i for the above sequence of cashflows the fair value

$$Q_i = S_i^{(\underline{\pi})} \hat{Q}_i \tag{5.8}$$

for all $i \in \{0, 1, \ldots, n\}$.

The above result is important, for instance, for the fair pricing of life insurance policies. Each insurance carrier can choose its own process of proportions π to invest the payments that arise. However, the GOP, that is needed to value the prospective reserve, must be the same for all insurance companies in the same market. Above we clarified the role of the GOP for pricing the prospective reserve. We point out that the above analysis says nothing about the performance and riskiness of different investment strategies that the insurance carrier can choose. The growth rate for the investment portfolio becomes optimal, if the proportions of the GOP are used. If the insurance company aims to maximize the growth rate of its wealth, then the pricing of an insurance policy and the optimization of the investment portfolio both involve the GOP.

Note that we did not use any measure transformation to obtain a fair price system. In the case of a fair market with continuous security prices one can equivalently derive the resulting fair prices by the use of the minimal equivalent martingale measure $\tilde{P}$, which is related to local risk minimization, as described in Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) or Hofmann, Platen & Schweizer (1992). The corresponding Radon-Nikodym derivative process for the minimal equivalent martingale measure is then $\frac{d\tilde{P}}{dP} = \frac{S^{(0)}}{S^{(\underline{\pi})}}$, which in this case is an $(\underline{\mathcal{A}}, P)$-martingale.

6 Arbitrage Amount

Typically, arbitrage free markets have been studied in a risk neutral setting in the literature. In our terminology these are fair markets. As we will see, it appears to be realistic to allow some form of arbitrage in a given market. In an integrable market a particular form of arbitrage becomes visible in the positive difference between a traded benchmarked price of a portfolio $S^{(\pi)} \in \mathcal{V}$ and its expected benchmarked future price, see (4.2). We call this difference the *benchmarked arbitrage amount*

$$\hat{A}_i^{(\pi)} = \hat{S}_i^{(\pi)} - E\left(\hat{S}_{i+1}^{(\pi)} \,\middle|\, \mathcal{A}_{t_i}\right) \tag{6.1}$$

at time t_i, $i \in \{0, 1, \ldots, n-1\}$. As has been shown for continuous time in Heath & Platen (2002), one can model and quantify arbitrage amounts, see also our example in Section 8.

In a developed market it is reasonable to expect that arbitrage amounts are typically small. However, strictly positive arbitrage amounts may exist.

In an integrable market one could, in principle, interpret any nonnegative contingent claim as primary asset. Its benchmarked traded price forms the corresponding benchmarked primary security account, which is a supermartingale. If the corresponding contingent claim prices are additionally chosen to be fair, that means they have zero arbitrage amounts, then they are equivalent to those obtained in (5.1). In general, however, demand and supply primarily determine the security price evolution and extreme demand or supply can lead to some strictly positive arbitrage amounts. Under the above described benchmark framework, prices are allowed to be different from fair prices. Note, in an integrable market the minimal possible price for a nonnegative contingent claim is always given by the fair price. This is a consequence of the supermartingale property of benchmarked prices.

7 Unit Linked Insurance Contracts

In the insurance context we look again at the cash flows $X_0, X_1, \ldots, X_n$ but assume a specific form for these random variables. Intuitively, they stand now for unit linked claims and premiums. Hence they can be of either sign. The cashflow at time t_i is of the form

$$X_i = D_i \, S_i^{(\pi)} \tag{7.1}$$

for $i \in \{1, 2, \ldots, n\}$. The payments are linked to some self-financing, strictly positive reference portfolio $S^{(\pi)} \in \mathcal{V}$ with given proportions π. The insurance contract specifies the reference portfolio $S^{(\pi)}$ and the random variables D_i, which are contingent on the occurrence of insured events during the period $(t_{i-1}, t_i]$, for instance, death, disablement or accidents.

The standard actuarial technique treats such contracts by using the reference portfolio process $S^{(\pi)}$ as numeraire and then deals with the unit linked random variables $D_0, D_1, \ldots, D_n$ at interest rate zero. It is reasonable to assume that these random variables are $\underline{\mathcal{A}}$-adapted and independent of the reference portfolio process $S^{(\pi)}$.

As in Section 5, let us now look at the value $W_i^{(\pi)}$ of the payment stream at time t_i. It is determined by the accumulated payments $C_i^{(\pi)}$ and the liability or prospective reserve r_i. Let us follow the standard actuarial methodology, assuming that the insurer invests all accumulated payments in the reference portfolio $S^{(\pi)}$. Then we obtain for $W_i^{(\pi)}$, when expressed in units of the domestic currency, the expression

$$W_i^{(\pi)} = C_i^{(\pi)} + r_i \tag{7.2}$$

with accumulated payments

$$C_i^{(\pi)} = S_i^{(\pi)} \sum_{k=1}^{i} D_k \tag{7.3}$$

and the liability or prospective reserve

$$r_i = S_i^{(\pi)}\, E\left(\sum_{k=i+1}^{n} D_k \,\Big|\, \mathcal{A}_{t_i}\right) \tag{7.4}$$

for $i \in \{0, 1, \ldots, n\}$.

In an integrable market the benchmarked value $\hat{W}_i^{(\pi)} = \frac{W_i^{(\pi)}}{S_i^{(\underline{\pi})}}$ at time t_i for the cashflows of this unit linked insurance contract is then by (7.2) of the form

$$\hat{W}_i^{(\pi)} = \frac{C_i^{(\pi)} + r_i}{S_i^{(\underline{\pi})}} \tag{7.5}$$

for $i \in \{0, 1, \ldots, n\}$. On the other hand, the benchmarked fair value $\hat{Q}_i^{(\pi)}$ at time t_i of the cashflows of this contract is according to (5.4) - (5.8) given by the expression

$$\hat{Q}_i^{(\pi)} = \frac{C_i^{(\pi)} + R_i}{S_i^{(\underline{\pi})}} \tag{7.6}$$

with benchmarked fair prospective reserve

$$R_i = S_i^{(\underline{\pi})}\, E\left(\sum_{k=i+1}^{n} \frac{D_k\, S_k^{(\pi)}}{S_k^{(\underline{\pi})}} \,\Bigg|\, \mathcal{A}_{t_i}\right) \tag{7.7}$$

for $i \in \{0, 1, \ldots, n\}$.

It is important to note that under quite natural conditions one can prove that the benchmarked fair prospective reserve is less or equal the actuarial prospective reserve. The proof of this inequality relies on the supermartingale property of $\frac{S_k^{(\pi)}}{S_k^{(\underline{\pi})}}$, $k \in \{0, 1, \ldots, n\}$ shown in Theorem 3.3 and the inequalities

$$E\left(\sum_{k=i+1}^{n} \frac{D_k\, S_k^{(\pi)}}{S_k^{(\underline{\pi})}} \,\Bigg|\, \mathcal{A}_{t_{n-1}}\right) \leq E\left(\sum_{k=i+1}^{n} \frac{D_k\, S_{k\wedge(n-1)}^{(\pi)}}{S_{k\wedge(n-1)}^{(\underline{\pi})}} \,\Bigg|\, \mathcal{A}_{t_{n-1}}\right)$$

if $E(D_n \,|\, \mathcal{A}_{t_{n-1}}) \geq 0$,

$$E\left(\sum_{k=i+1}^{n} \frac{D_k\, S_{k\wedge(n-1)}^{(\pi)}}{S_{k\wedge(n-1)}^{(\underline{\pi})}} \,\Bigg|\, \mathcal{A}_{t_{n-2}}\right) \leq E\left(\sum_{k=i+1}^{n} \frac{D_k\, S_{k\wedge(n-2)}^{(\pi)}}{S_{k\wedge(n-2)}^{(\underline{\pi})}} \,\Bigg|\, \mathcal{A}_{t_{n-2}}\right)$$

if $E(D_n + D_{n-1} \,|\, \mathcal{A}_{t_{n-2}}) \geq 0$,

$$\vdots$$

$$E\left(\sum_{k=i+1}^{n} \frac{D_k\, S^{(\pi)}_{k\wedge(i+1)}}{S^{(\underline{\pi})}_{k\wedge(i+1)}} \,\middle|\, \mathcal{A}_{t_i}\right) \;\leq\; E\left(\sum_{k=i+1}^{n} \frac{D_k\, S^{(\pi)}_{k\wedge i}}{S^{(\underline{\pi})}_{k\wedge i}} \,\middle|\, \mathcal{A}_{t_i}\right)$$
$$= \frac{r_i}{S^{(\underline{\pi})}_i}$$

if $E(D_n + D_{n-1} + \ldots + D_{i+1} \,|\, \mathcal{A}_{t_i}) \geq 0$, for $i \in \{0, 1, \ldots, n-1\}$. Taking conditional expectation with respect to $\mathcal{A}_{t_i}$, the inequalities above become a chain, whose first member equals $\frac{R_i}{S^{(\underline{\pi})}_i}$, and the last member becomes $\frac{r_i}{S^{(\underline{\pi})}_i}$. We formulate the above result as a lemma.

Lemma 7.1 *If*

$$E\left(\sum_{k=m+1}^{n} D_k \,\middle|\, \mathcal{A}_{t_m}\right) \geq 0 \tag{7.8}$$

for all $m \in \{0, 1, \ldots, n-1\}$*, then*

$$R_i \leq r_i \tag{7.9}$$

for all $i \in \{0, 1, \ldots, m-1\}$.

As by (7.4)

$$r_m = S^{(\pi)}_m\, E\left(\sum_{k=m+1}^{n} D_k \,\middle|\, \mathcal{A}_{t_m}\right),$$

the condition (7.8) of the lemma means that the insurance contract defines a cashflow whose actuarial prospective reserve never becomes negative. This is usually observed as practical rule, since products that allow for negative reserves have many defects. For instance, they permit antiselection.

From (7.5) and (7.6) we immediately have under condition (7.8)

$$\hat{Q}^{(\pi)}_i \leq \hat{W}^{(\pi)}_i$$

for $i \in \{0, 1, \ldots, n\}$. However, we observe that neither $\hat{Q}^{(\pi)} = \{\hat{Q}^{(\pi)}_i,\, i \in \{0, 1, \ldots, n\}\}$ nor $\hat{W}^{(\pi)} = \{\hat{W}^{(\pi)}_i,\, i \in \{0, 1, \ldots, n\}\}$ is in general a supermartingale, even under condition (7.8).

Reverting to property (7.9) we observe that there is, in general, a nonnegative difference

$$A_i = r_i - R_i \geq 0$$

between the actuarial and the fair prospective reserve. This arbitrage amount is a consequence of the classical actuarial price calculation leading to the prospective reserve r_i in (7.4). Of course, the actuarial and the fair prospective reserve coincide if one uses the GOP as reference portfolio.

8 A Lognormal Example

To illustrate key features of the discrete time benchmark approach, let us discuss a simple example of a market with two primary assets, which is a discrete time version of the Black-Scholes model, see also Becherer (2001). Other examples, which also demonstrate a supermartingale property for benchmarked prices, can be found in Kramkov & Schachermayer (1999) and Heath & Platen (2002). The two primary assets are the domestic currency, which is assumed to pay no interest, and a stock that pays also no dividends. The primary security account at time t_i for the domestic currency is then simply the constant $S_i^{(0)} = 1$ of one unit of the domestic currency for $i \in \{0, 1, \ldots, n\}$. The stock price $S_i^{(1)}$ at time t_i is given by the product

$$S_i^{(1)} = S_0^{(1)} \prod_{\ell=1}^{i} h_\ell^{(1)}, \tag{8.1}$$

for $i \in \{0, 1, \ldots, n\}$, where we assume the growth ratio at time t_ℓ to be an independent, lognormal distributed random variable, that is

$$Y_\ell = \log\left(h_\ell^{(1)}\right) \sim \mathcal{N}(\mu\,\Delta, \sigma^2\,\Delta) \tag{8.2}$$

with mean $\mu\Delta$, variance $\sigma^2\Delta > 0$ and some given parameter $\Delta > 0$. The GOP proportions $\underline{\pi}_i^{(0)}$ and $\underline{\pi}_i^{(1)}$ must be such that

$$\underline{\pi}_i^{(0)} = 1 - \underline{\pi}_i^{(1)} \tag{8.3}$$

for all $i \in \{0, 1, \ldots, n\}$. Furthermore, since the GOP has to be always strictly positive we must have

$$\underline{\pi}_i^{(1)} \in [0, 1] \tag{8.4}$$

for all $i \in \{0, 1, \ldots, n\}$. Obviously, the set $\mathcal{V}$ of strictly positive, self-financing portfolios $S^{(\pi)}$ is here characterized by those processes of proportions π for which $\pi_i^{(1)} \in [0, 1]$ for all $i \in \{0, 1, \ldots, n\}$. The growth rate at time t_i for a portfolio $S^{(\pi)} \in \mathcal{V}$ is then given by the expression

$$g_i^{(\pi)} = E\left(\log\left(1 + \pi_i^{(1)}\left(\exp(Y_{i+1}) - 1\right)\right) \,\Big|\, \mathcal{A}_{t_i}\right) \tag{8.5}$$

for all $i \in \{0, 1, \ldots, n-1\}$. Its first derivative with respect to $\pi_i^{(1)}$ is

$$\frac{\partial g_i^{(\pi)}}{\partial \pi_i^{(1)}} = E\left(\frac{\exp(Y_{i+1}) - 1}{1 + \pi_i^{(1)}\left(\exp(Y_{i+1}) - 1\right)} \,\Bigg|\, \mathcal{A}_{t_i}\right) \tag{8.6}$$

and its second derivative has the form

$$\frac{\partial^2 g_i^{(\pi)}}{\partial\left(\pi_i^{(1)}\right)^2} = -E\left(\frac{\left(\exp(Y_{i+1}) - 1\right)^2}{\left(1 + \pi_i^{(1)}\left(\exp(Y_{i+1}) - 1\right)\right)^2} \,\Bigg|\, \mathcal{A}_{t_i}\right) \tag{8.7}$$

for $i \in \{0, 1, \dots, n-1\}$. We note that the second derivative is always negative, which indicates that the growth rate has at most one maximum for some proportion $\pi_i^{(1)} \in [0, 1]$. We also observe that the first derivative of the growth rate for $\pi_i^{(1)} = 0$ has the value

$$\left.\frac{\partial g_i^{(\pi)}}{\partial \pi_i^{(1)}}\right|_{\pi_i^{(1)}=0} = \exp\left(\Delta\left(\mu + \frac{\sigma^2}{2}\right)\right) - 1 \tag{8.8}$$

and is for $\pi_i^{(1)} = 1$ given by the expression

$$\left.\frac{\partial g_i^{(\pi)}}{\partial \pi_i^{(1)}}\right|_{\pi_i^{(1)}=1} = 1 - \exp\left(\Delta\left(-\mu + \frac{\sigma^2}{2}\right)\right) \tag{8.9}$$

for $i \in \{0, 1, \dots, n-1\}$.

1. To show that we have a fair market, we first clarify whether the derivative $\frac{\partial g_i^{(\pi)}}{\partial \pi_i^{(1)}}$ can become zero for $\pi_i^{(1)} \in [0, 1]$. Taking (8.6) - (8.9) into account, this can be only the case for $|\mu| \leq \frac{\sigma^2}{2}$. In this case it is then straightforward to show that the optimal proportion $\underline{\pi}_i^{(1)}$ for the GOP converges to the value

$$\lim_{\Delta \to 0} \underline{\pi}_i^{(1)} = \frac{1}{2} + \frac{\mu}{\sigma^2}$$

   as $\Delta \to 0$ for $i \in \{0, 1, \dots, n-1\}$. Asymptotically for $\Delta \to 0$ we have a fair market for this parameter choice because

$$\lim_{\Delta \to 0} E\left(\left.\frac{h_{i+1}^{(0)}}{h_{i+1}^{(\underline{\pi})}}\,\right|\,\mathcal{A}_{t_i}\right) = 1$$

   and

$$\lim_{\Delta \to 0} E\left(\left.\frac{h_{i+1}^{(1)}}{h_{i+1}^{(\underline{\pi})}}\,\right|\,\mathcal{A}_{t_i}\right) = 1.$$

   Obviously, by Corollary 4.3 and (6.1) we have then for all $S^{(\pi)} \in \mathcal{V}$ and $i \in \{0, 1, \dots, n-1\}$ a vanishing arbitrage amount $\hat{A}_i^{(\pi)} = 0$ and $\left.\frac{\partial g_i^{\theta,\pi}}{\partial \theta}\right|_{\theta=0} = 0$. The Radon-Nikodym derivative process $\hat{S}^{(0)} = \frac{S^{(0)}}{S^{(\underline{\pi})}}$ becomes here for $\Delta \to 0$ a martingale and the standard risk neutral approach can be applied.

2. In the case $\mu < -\frac{\sigma^2}{2}$ we obtain from (8.6) - (8.7) the optimal proportion

$$\underline{\pi}_i^{(1)} = 0$$

for all $i \in \{0, 1, \ldots, n-1\}$. This requires to hold for the GOP all investments in domestic currency. Here we get

$$E\left(\frac{h_{i+1}^{(1)}}{h_{i+1}^{(\underline{\pi})}} \,\Big|\, \mathcal{A}_{t_i}\right) = \exp\left(\Delta\left(\mu + \frac{\sigma^2}{2}\right)\right) < 1,$$

which shows that the benchmarked stock price process $\hat{S}^{(1)} = \frac{S^{(1)}}{S^{(\underline{\pi})}}$ is a strict supermartingale and not a martingale. Obviously, $\hat{S}^{(0)} = \frac{S^{(0)}}{S^{(\underline{\pi})}} = 1$ is a martingale and would be the candidate for the Radon-Nikodym derivative process for an equivalent risk neutral measure. However, since $\hat{S}^{(1)}$ is not a martingale we have not a fair market and the standard risk neutral pricing approach does not apply. For $\pi_i = (\pi_i^{(0)}, \pi_i^{(1)}) = (0, 1)$ we have the arbitrage amount

$$\hat{A}_i^{(\pi)} = 1 - \exp\left(\Delta\left(\mu + \frac{\sigma^2}{2}\right)\right) > 0$$

and

$$\frac{\partial g_i^{\theta,\pi}}{\partial \theta}\Big|_{\theta=0} = \exp\left(\left(\mu + \frac{\sigma^2}{2}\right)\Delta\right) - 1 < 0.$$

3. For $\mu > \frac{\sigma^2}{2}$ it follows by (8.6) - (8.7) the optimal proportion

$$\underline{\pi}_i^{(1)} = 1$$

for $i \in \{0, 1, \ldots, n-1\}$. This means for sufficiently large mean of the logarithm of the growth ratio of the stock one has to hold in the GOP all investments in the stock. In this case we get

$$E\left(\frac{h_{i+1}^{(0)}}{h_{i+1}^{(\underline{\pi})}} \,\Big|\, \mathcal{A}_{t_i}\right) = \exp\left(\Delta\left(-\mu + \frac{\sigma^2}{2}\right)\right) < 1,$$

which says that the benchmarked domestic savings account $\hat{S}^{(0)} = \frac{S^{(0)}}{S^{(\underline{\pi})}}$ is a strict supermartingale and not a martingale. This means that $\hat{S}^{(0)}$, which is the candidate for the Radon-Nikodym derivative process of an equivalent risk neutral measure, is not a martingale. This means, the market is not fair and the standard risk neutral approach does not apply. However, note that $\hat{S}^{(1)} = \frac{S^{(1)}}{S^{(\underline{\pi})}} = 1$ is a martingale. For $\pi_i = (\pi_i^{(0)}, \pi_i^{(1)}) = (1, 0)$ we have then the arbitrage amount

$$\hat{A}_i^{(\pi)} = 1 - \exp\left(\Delta\left(-\mu + \frac{\sigma^2}{2}\right)\right) > 0$$

and

$$\frac{\partial g_i^{\theta,\pi}}{\partial \theta}\Big|_{\theta=0} = \exp\left(-\left(\mu + \frac{\sigma^2}{2}\right)\Delta\right) - 1 < 0.$$

This example demonstrates that benchmarked prices are not always martingales and strictly positive arbitrage amounts may exist. However, these benchmarked prices become martingales if the corresponding first derivatives of the growth rate of the perturbed GOP in the direction of these securities are zero. In all above cases the benchmark approach provides a unique price system, whereas the standard risk neutral approach is not applicable in the last two cases.

Conclusion

We have shown that the growth optimal portfolio plays a central role in the understanding of key properties of a financial and insurance market. If the first derivatives of its growth rate vanish, then this turns out to be equivalent to having a fair market. It is well known that this portfolio also plays a key role in portfolio optimization, equivalent to the mutual fund. The proposed benchmark approach provides not only an integrated framework to main applications in finance and insurance but fund management, also.

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A Appendix

Proof of Theorem 3.3

1. For $\theta \in (0, \frac{1}{2})$ and $S^{(\pi)} \in \mathcal{V}$ we consider the perturbed GOP $V^{\theta,\pi} \in \mathcal{V}$, that is with growth ratio

$$h^{\theta,\pi}_{i+1} > 0 \tag{A.1}$$

given in (4.6) for $i \in \{0, 1, \ldots, n-1\}$. One can then show, using $\log(x) \leq x - 1$ and (4.6), that

$$G^{\theta,\pi}_{i+1} = \frac{1}{\theta} \log\left(\frac{h^{\theta,\pi}_{i+1}}{h^{(\pi)}_{i+1}}\right) \leq \frac{1}{\theta}\left(\frac{h^{\theta,\pi}_{i+1}}{h^{(\pi)}_{i+1}} - 1\right) = \frac{h^{(\pi)}_{i+1}}{h^{(\pi)}_{i+1}} - 1 \tag{A.2}$$

and

$$G_{i+1}^{\theta,\pi} \geq -\frac{1}{\theta}\left(\frac{h_{i+1}^{(\underline{\pi})}}{h_{i+1}^{\theta,\pi}} - 1\right) = \frac{h_{i+1}^{(\pi)} - h_{i+1}^{(\underline{\pi})}}{h_{i+1}^{\theta,\pi}}. \tag{A.3}$$

We obtain in (A.3) for $h_{i+1}^{(\pi)} - h_{i+1}^{(\underline{\pi})} \geq 0$ because of $h_{i+1}^{\theta,\pi} > 0$ the inequality

$$G_{i+1}^{\theta,\pi} \geq 0 \tag{A.4}$$

and for $h_{i+1}^{(\pi)} - h_{i+1}^{(\underline{\pi})} < 0$ from (A.3) because of $\theta \in (0, \frac{1}{2})$, $h_{i+1}^{(\pi)} > 0$, and $h_{i+1}^{\theta,\pi} > 0$ that

$$G_{i+1}^{\theta,\pi} \geq -\frac{h_{i+1}^{(\underline{\pi})}}{h_{i+1}^{\theta,\pi}} = -\frac{1}{1-\theta+\theta\,\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}}} \geq -\frac{1}{1-\theta} \geq -2. \tag{A.5}$$

Summarizing (A.2) - (A.5) we have for $i \in \{0, 1, \ldots, n-1\}$ and $S^{(\pi)} \in \mathcal{V}$ the upper and lower bounds

$$-2 \leq G_{i+1}^{\theta,\pi} \leq \frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} - 1, \tag{A.6}$$

where by (3.6)

$$E\left(\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}}\right) < \infty \tag{A.7}$$

if $S^{(\underline{\pi})}$ is growth optimal. Then by using (A.6) and (A.7) it follows by the Dominated Convergence Theorem that

$$\begin{aligned}
0 &\geq \lim_{\theta\to 0+} E\left(G_{i+1}^{\theta,\pi} \,\Big|\, \mathcal{A}_{t_i}\right) \\
&= E\left(\lim_{\theta\to 0+} G_{i+1}^{\theta,\pi} \,\Big|\, \mathcal{A}_{t_i}\right) \\
&= E\left(\frac{\partial}{\partial\theta} \log\left(\frac{h_{i+1}^{\theta,\pi}}{h_{i+1}^{(\underline{\pi})}}\right)\Bigg|_{\theta=0} \,\Bigg|\, \mathcal{A}_{t_i}\right) \\
&= E\left(\frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} \,\Bigg|\, \mathcal{A}_{t_i}\right) - 1
\end{aligned} \tag{A.8}$$

for $i \in \{0, 1, \ldots, n-1\}$ and $S^{(\pi)} \in \mathcal{V}$. This shows us that the benchmarked portfolio process $\hat{S}^{(\pi)}$ is then a supermartingale.

2. We prove now the reverse. Since $\log(y) \geq \log(x) + 1 - \frac{x}{y}$ for $\frac{x}{y} > 0$ we have with $x = h_{i+1}^{(\pi)}$ and $y = h_{i+1}^{(\underline{\pi})}$ that

$$E\left(\log(h_{i+1}^{(\underline{\pi})}) \,\middle|\, \mathcal{A}_{t_i}\right) \geq E\left(\log(h_{i+1}^{(\pi)}) \,\middle|\, \mathcal{A}_{t_i}\right) + E\left(1 - \frac{h_{i+1}^{(\pi)}}{h_{i+1}^{(\underline{\pi})}} \,\middle|\, \mathcal{A}_{t_i}\right). \tag{A.9}$$

If for all portfolio processes $S^{(\pi)} \in \mathcal{V}$ the benchmarked process $\hat{S}^{(\pi)}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale, then it follows by (A.9) that

$$E\left(\log(h_{i+1}^{(\underline{\pi})}) \,\middle|\, \mathcal{A}_{t_i}\right) \geq E\left(\log(h_{i+1}^{(\pi)}) \,\middle|\, \mathcal{A}_{t_i}\right) \tag{A.10}$$

for all $i \in \{0, 1, \ldots, n-1\}$. This proofs by (3.2) and (3.5) that the portfolio $S^{(\underline{\pi})}$ is growth optimal. $\square$

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