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The private value of public pensions.

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Abstract

Individual retirement savings accounts are replacing or supplementing public basic pensions. However at decumulation, replacing the public pension with an equivalent private sector income stream may be costly. We value the Australian basic pension by calculating the wealth needed to generate an equivalent payment stream using commercial annuities or phased withdrawals, but still accounting for investment and longevity risks. At age 65, a retiree needs an accumulation of about 8.5 years' earnings to match the public pension in real value and insurance features. Increasing management fees by 1% raises required wealth by about one year's earnings. Delaying retirement by 5 years lowers required wealth by about one half year's earnings. Phased withdrawals have money's worth ratios close to 0.5 suggesting that private replacement costs are high.

JEL codes: H55; J14; G11 *Subject category:* IE25 *Insurance branch:* IB80; IB81.

Key words: Social security; Longevity risk; Phased withdrawal; Stochastic present value

1 Introduction

Governments across the world are reviewing retirement saving systems in the light of increasing public pension liabilities. In particular, the role of basic redistributive pensions that comprise the 'first pillar' of retirement savings systems is being reassessed. Basic pensions are designed to ensure that the elderly reach a minimum level of welfare, and are typically payments of between 20% and 40% of average earnings, targeted towards the more needy via income and assets testing (Whitehouse 2007). In countries with high rates of population aging, first pillar pensions are consuming an increasing part of public funds, thus motivating changes to pension indexation systems, eligibility ages and means-tests. In addition, governments are creating incentives for individuals to fund their own retirements through personal earnings-related (second pillar) savings schemes. Policy aims to encourage personal saving, reduce dependence on first pillar provisions and relieve the strain on public funds.

However one implication of replacing public pensions with individual accounts is that personal savers need to decumulate using commercially-provided income streams such as phased withdrawals and/or annuities. Consequently, they may face higher costs and greater risks than under the public pension,

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depending on the prices and insurance features of products offered by the financial services sector. If higher accumulations into individual accounts interact with means tests so as to disqualify the elderly from access to the public basic pension, and commercial provision is expensive or inadequately insured, the result may be significant welfare loss. On the other hand, if commercial products offer more flexibility or better services at a reasonable cost, retirees may be better off than when receiving the government payment.

Here we estimate the wealth that a self-funded retiree needs in order to generate an income stream equivalent to the Australian basic pension in terms of value and insurance protection, but using decumulation products now offered by the financial services sector. Applying a new analytical approximation to the retirement income problem (Huang et al. 2004, Milevsky and Robinson 2005), we calculate the purchase price and money's worth [1] of income streams from private providers that mimic the basic pension.

This analysis serves three purposes. First, we value the basic pension stream in terms of its replacement cost; secondly we gauge whether an average person could feasibly accumulate enough wealth over their working lives to replicate the basic pension payment, and thirdly we measure the difference between the cost of public and private provision at the margin, using the money's worth metric.

Earlier studies of phased withdrawal in retirement (Huang et al. 2004, Milevsky and Robinson 2000) show that for an infinite horizon and lognormally distributed investment returns, the stochastic present value of a desired spending

[1] The money's worth is the ratio of the expected present value of an annuity stream to its purchase price.

plan is reciprocal gamma distributed. Consequently we can evaluate the *ex ante* feasibility of a spending and investment plan by comparing the stochastic present value of the plan with retirement wealth. This is the probability of 'retirement ruin' or the likelihood of running out of money before the end of life.

In the more general case, when time horizons are finite and stochastic, we approximate the probability of running out of resources before the end of life by matching the first two moments of the stochastic present value of the retirement spending plan with the first two moments of the reciprocal gamma distribution. The result is an analytical approximation to the probability of retirement ruin, for random investment returns and uncertain lifetimes. We fix the probability of ruin at an arbitrarily low level to approximate the small regulatory risk of the public pension, then use this moment-matching method to back out the minimum retirement accumulation needed to replace a pension using equivalent private sector income streams.

In addition we compute the money's worth of the best phased withdrawal strategies by calculating the ratio of the simple annuity value of the public pension stream to the total retirement wealth needed for self-insurance (the effective purchase price of self-annuitization). We find that pension eligibility creates a substantial transfer of public wealth to the retiree, but that shifting responsibility for the pension stream from the public to the private sector means higher personal costs.

We estimate the retiree needs close to $380,000 at age 65 to ensure the lifelong $14,000 p.a. real income now paid by the Australian basic pension. This amount of savings is almost seven years of average earnings and more than four times the average retirement savings account balance of current 60-65 year olds. Even if eligibility age is delayed to 70 years, the amount of wealth needed to fund an equivalent real payment is close to \$355,000 or nearly 6.4 years of average earnings. Significantly less is needed if the retiree pools longevity risk by purchasing commercial life annuities, but voluntary annuitization is very rare among Australian retirees.[2] Recipients of the Australian Age Pension enjoy an option over price and wage increases: the pension is adjusted with inflation, but also cannot fall below 25% of Male Total Average Weekly Earnings (MTAWE). When we incorporate an historical rate of wages growth into the pension path, the accumulation required to match it increases further: an additional \$90,000 or close to 8.5 times average annual earnings in total. In addition, we show that for each 1% increase in management fees or administrative loadings, required wealth rises by around \$60,000, or one year's earnings.

The most efficient pension-matching investment strategy is either a 'balanced' or 'growth' portfolio with around 50-70% allocated to equities or property securities. The best portfolio allocation may vary by age, gender and risk tolerance, but more aggressive and more conservative investments generally require higher initial wealth to be secure and sustainable. However the money's worth (ratio of expected discounted value of the payment stream to the wealth needed for the phased withdrawal) of the best strategies are generally below

[2] Data on allocation of Australian retirement savings are sparse, but survey evidence suggests that 12% of retirement savings are used to purchase income stream products (ABS 2006a), and of that, less that 0.2% go to life annuities (Plan for Life 2006). Studies from other countries also note low levels of volutary annuitization (e.g. James and Song 2001).

0.5, declines as the retiree ages, and is lower for men than for women.

Overall, if current rates of individual retirement accumulation are any guide, it is unlikely that many Australian retirees will save sufficient wealth to self-insure a pension-equivalent income stream against longevity and investment risk, using products now available in the retirement incomes market.

Section 2 of the paper sets out the main features of the basic and targeted pensions in the US, UK and Australia. We describe the method for calculating the stochastic present value of a spending plan in Section 3, and Section 4 outlines parameter choices. In Section 5 we compute the wealth required by men and women of retirement age to construct a secure pension payment, and the money's worth of various self-annuitization strategies. Section 6 concludes.

2 Basic pensions

Unlike the US and the UK, Australia does not have a public, earnings-linked pension system. This feature, along with the fact that the majority of Australian retirees rely on both public first-pillar and private second-pillar provisions for retirement income, creates a natural experiment in the interaction between targeted government retirement support and mandatory savings into individual retirement accounts.

In the US, the Old Age, Survivors and Disability Insurance (OASDI) system provides payments after retirement that are not fixed at a flat rate but will vary with lifetime earnings, subject to a minimum level of participation in the workforce. The redistributive purpose of the OASDI program is achieved by offering proportionately higher old age insurance benefits to workers who have long work histories in low-paid jobs or who have short work histories, compared with benefits to higher earners. Payments under OASDI are not means-tested and accrue to workers as a legal entitlement. Projections of income and cost rates for OASDI indicate that the system will experience a substantial short fall in coming decades (McGill et al. 2005). For those over 65 who fail to qualify for the OASDI benefit, means-tested Supplemental Security Income (SSI) is available to ensure minimum income. SSI is not based on work history and is paid out of general revenue at a rate of US $623 per month. Apart from some exemptions (such as an individual's home and vehicle, and some minimal income concessions), the SSI payment is strictly means-tested across assets and income.

By contrast, recent pension reform in the UK has made first pillar provision there more universal. The Basic State Pension (BSP) of £87.30 (US $177) per week is available at retirement to those who have a sufficient work history. In 2005, about 80% of men but only about 30% of women of eligible age received the full BSP. However recent reforms have reduced the number of years required for eligibility from 44 to 30, with concessions to carers who have an incomplete work history. Forecasts from the Department for Work and Pensions (2006a) indicate that 90% of both men and women are expected to receive the BSP by 2025. For those who still do not qualify for the BSP, or whose pension payment is low, a safety net is provided by the means-tested Pension Credit, which ensures income of £119 (US $241) per week. Projections show that the cost of providing the BSP will rise by 3.4% of Gross Domestic Product (GDP) over 2008 when reforms (including broadening the BSP and linking payments to earnings rather than prices) combine with the effects of population aging.

Both the US and UK regulators will increase pension eligibility ages to 67 and 68 respectively in coming decades.

First pillar provision in Australia falls between the US and UK systems. In Australia, a large majority of current retirees relies on the basic pension, the 'Age Pension', for income, but the payment is means-tested, not universal and not dependent on work history. Recent survey data show that nearly 70% of couple households and nearly 80% of single-person households over the age of 64 depend on the Age Pension (or the war veteran's equivalent) as their primary source of income (Australian Bureau of Statistics 2006b). As the population ages in the next four decades, targeted first pillar pension payments, which now represent nearly 2.5% of GDP are projected to increase to almost 4.4% of GDP (Commonwealth of Australia 2007).

Elderly Australians will continue to rely on first-tier provision despite the introduction in 1992 of mandatory, earnings-related retirement savings under the Australian Superannuation Guarantee. The Superannuation Guarantee compels Australian workers to contribute at least 9% of income to privately-managed and fully-funded personal retirement savings accounts, but 15 years after inception, accumulations into superannuation accounts are still relatively modest, currently averaging less than $100,000[3] at retirement (ASFA 2007) and projected to be less than $150,000 by 2020.[4] As a result, most retirees will continue to depend on first-tier income support and the Age Pension will remain a large and increasing component of fiscal outlays. Nevertheless, it is a stated policy aim of the Australian Government to encourage private saving

[3] This amount is less than twice annual average earnings.

[4] Kelly et al. (2002) projects an average balance of $119709 by 2020 in 1999 dollars, which we scale up by 20% to get an estimate in 2007 dollars.

and to reduce demand for the basic pension (Commonwealth of Australia, 2002).

Population aging has placed considerable pressure on unfunded public provision, and many governments have responded by encouraging saving through personal retirement accounts. In the US, 401(k) coverage is now a substantial component of retirement provision, with 401(k) contributions the most rapidly growing component of private sector pension contributions since 1980. Assets in 401(k) plans are likely to increase greatly in the next 30 years (Poterba, Venti and Wise 2007). In the UK, policy reforms emphasize individual, funded, second pillar provision with some investment choice but with centralized collection and administration (Department for Work and Pensions 2006b). Both the US and the UK plans for private accounts also imply less dependence on earnings-linked public annuities, and more on commercial income streams, whether conventional annuities or phased withdrawals.

2.1 Features of the Australian basic pension

The current Age Pension for a single pensioner who owns their own home is \$525 per fortnight, or \$13,653 p.a. (US \$11,880). Many pensioners are entitled to additional allowances for pharmaceuticals, utilities, telephone, rent assistance and for living in remote areas. Consequently pension eligibility is highly valued by retirees and their advisors: allowances alone mean that qualifying for the pension makes a single retiree at least \$343 p.a. better off, before any basic payment is made. Allowances are adjusted in line with the consumer price index (CPI) once or twice a year, but do not rise in line with the general level of earnings as the base single pension does. In the analysis below we study the case of a single home-owning pensioner whose annual benefit is rounded to $14,000 to reflect the basic payment and the most common allowances.

Some OECD countries have first-pillar pension schemes that are adjusted in line with inflation and others are linked to wages growth. Schemes linked with wages growth ensure that pensioners maintain relativity with wage earners as productivity increases, whereas price-linked schemes shrink coverage to smaller sections of the population as the economy grows. Consequently by de-coupling productivity growth and pension obligations, governments can gradually shrink the size of pension liabilities over time. Whitehouse (2007) cites the example of the UK BSP which was indexed to prices rather than wages in 1981. At the time of the change, the pension represented 23.7% of average earnings, but two decades later was worth less than 16% of average earnings. Re-linking to earnings is a major part of the recent UK reforms.

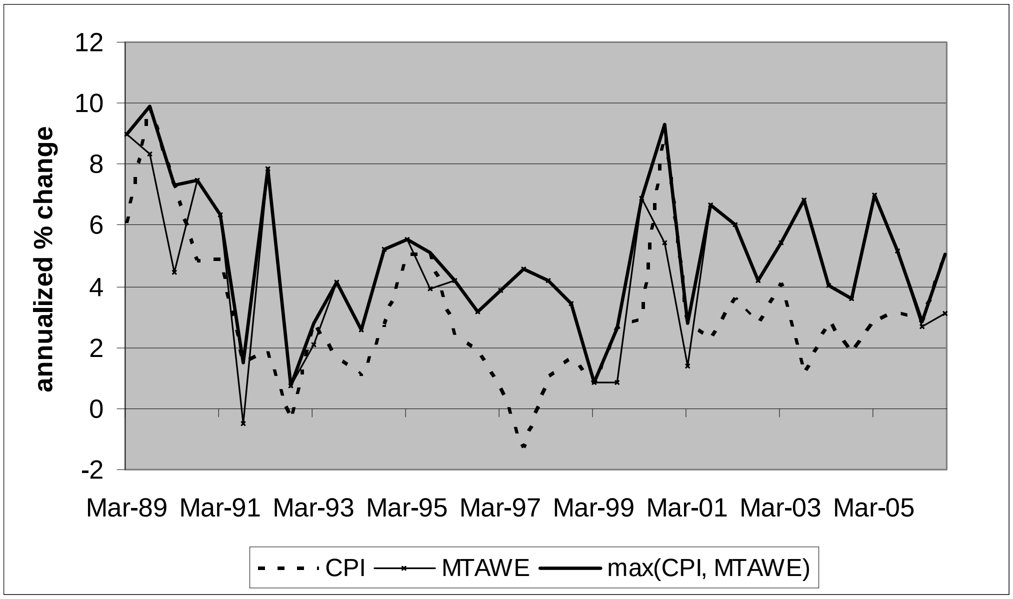
In Australia, the base single pension is recalculated every six months (March and September) to keep up with changes in the CPI and also to ensure that it does not fall below 25% of Male Total Average Weekly Earnings (MTAWE). Pensioners thus hold an option on the general level of wages and prices in the economy so that the relative as well as real value of payments is maintained. The adjustment in the base pension is

$$\frac{P_t}{P_{t-1}} = \max\left[(1+h_t),(1+n_t)\right] \tag{1}$$

where h is the rate of increase in the CPI and n is the rate of increase in MTAWE over the previous six months. This relative-income protection has been very valuable over the past 15 years because earnings growth has exceeded inflation in most years. Figure 1 below graphs annualized 6-monthly paths for inflation and MTAWE, since March 1989.

Figure 1

Annualized six-month changes in inflation and earnings, 1989-2006



The average annualized increase in the CPI over this period was 3.1% compared with 4.5% for MTAWE and 4.9% for the maximum of both the CPI and MTAWE.

Means-testing of the Age Pension creates other option-like features over the wealth of the retired. The means tests begin to reduce the pension at fixed levels of income and/or wealth and the pension declines linearly to zero as income and/or wealth increases. Means test boundaries are reviewed in line with changes in the CPI. Since the means tests may interact with each other, the pensioner is entitled to the least payment from either test, or zero. The assets test begins to reduce the pension when wealth (excluding the pensioner's home) reaches \$161,500, reducing the pension payment by \$39 dollars per thousand increase in wealth, reaching zero at wealth of around \$511,600. For income receipts, pension payments begin to reduce when the individual receives \$128 per fortnight or \$3,328 p.a., and reach zero when income is \$1,455 per fortnight or \$37,837 p.a.

This implicit option payoffs in a higher pension as wealth falls below the means-test boundary, so an optimizing retiree will trade off the marginal advantages of pension eligibility against the costs of lower wealth/income. The taper encourages higher rates of consumption early in retirement. For the remaining analysis, however, we do not study the pension taper since the majority of retirees receive the full pension.

The crucial point is that higher personal retirement savings will reduce the public basic pension. Retirees with significant savings thus have less access to this public annuity, and must go to the private sector for decumulation services. As in many developed countries (James and Song 2001) voluntary purchase of term annuities in Australia is very low, and voluntary purchase of life annuities is even less. The majority of Australian retirees either take their second pillar payouts as discretionary lump-sums, or as phased withdrawal plans. Consequently, welfare depends on how efficiently a retiree can replace the pension payment with a phased withdrawal.

In the next section we outline a method for calculating the stochastic present value of a pension-equivalent income stream when the payment is drawn from a phased withdrawal product, as might be purchased with the proceeds of an individual retirement savings account. Comparing the stochastic present value of a phased withdrawal with the expected discounted value of the Age Pension gives the money's worth of self-annuitization. We can also use stochastic present value to estimate the individual savings needed to reproduce the basic pension payments.

3 Stochastic present value of retirement wealth

To make a valuation in terms of phased withdrawal products (which have both investment and longevity risk), we need a method that accounts for the likelihood of failure or ruin under a self-annuitization scheme. [5]

First consider the problem of a retiree who plans to consume one dollar each year from an initial retirement wealth $W_0 = w$. The retiree invests wealth in a portfolio returning a continuously compounded risk-free rate of return, μ. Wealth invested this way has a dynamic path given by the ordinary differential equation

$$dW_t = (\mu W_t - 1)\, dt, W_0 = w, W_t \geq 0, \tag{2}$$

which has a solution

$$W_t = (w - \frac{1}{\mu})e^{\mu t} + \frac{1}{\mu}. \tag{3}$$

We are interested in finding the time t^* at which the investor's wealth is used up, so that

$$W_t = \begin{cases} (w - \frac{1}{\mu})e^{\mu t} + \frac{1}{\mu}, & t < t^* \\ 0, & t \geq t^* \end{cases}.$$

Solving for t^*,

$$t^* = \frac{1}{\mu} \ln \left[\frac{1}{1 - \mu w} \right]. \tag{4}$$

Thus a retiree invested in a risk-free portfolio knows if and when her wealth will expire. For large enough combinations of investment return and initial wealth ($\mu w \geq 1$) she will never reach zero wealth while consuming only one dollar per year.

To model ruin when reforms are stochastic, we consider a portfolio of risky

[5] In this section we follow Milevsky (2006, chapter 9 and appendix to chapter 9) and Huang et. al (2004).

assets following a geometric Brownian motion with known drift and diffusion,

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{5}$$

where B_t is a standard Wiener process. The solution to (5) is

$$S_t = e^{\left(\mu - 1/2\sigma^2\right)t + \sigma B_t}, S_0 = 1. \tag{6}$$

If the retiree keeps consuming at a continuous rate of one dollar per year, the wealth process is

$$dW_t = dS_t - 1dt = \left(\mu W_t - 1\right)dt + \sigma W_t dB_t, W_0 = w, \tag{7}$$

and the solution to this stochastic differential equation is

$$W_t = e^{\left(\mu - 1/2\sigma^2\right)t + \sigma B_t}\left[w - \int_0^t e^{-\left(\mu - 1/2\sigma^2\right)t - \sigma B_t} dt\right], W_0 = w, \tag{8}$$

or equivalently,

$$W_t = S_t \left[w - \int_0^t S_t^{-1} dt\right], W_0 = w.^{6} \tag{9}$$

The draw-down process (8) can become negative if the drift μW_t is small, precipitating ruin.

We evaluate retirement consumption plans by determining the probability of exhausting wealth before the end of life. The probability of reaching 'ruin' $\left(\tilde{t}^*\right)$ is

$$\phi\left(w\right) \equiv \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 | W_0 = w], \tag{10}$$

[6] For some intuition on $\int_0^t S_t^{-1} dt$, consider the discrete time analogue. A \$1 draw-down discounted at a stochastic rate has a present value $SPV = \sum_{i=1}^{t} \prod_{j=1}^{i} \left(1 + \tilde{R}_j\right)^{-1}$. We compare this present value with initial wealth to determine the probability of net wealth reaching zero.

or the likelihood that the lowest value of the stochastic process (8) goes to zero before the retiree reaches the end of life at terminal date, T.[7]

Since the portfolio return S_t is bounded away from zero, retirement wealth can go to zero only if the stochastic present value of the spending plan approaches initial wealth,

$$Z_T \equiv \int_0^T e^{-\left(\mu - 1/2\sigma^2\right)t - \sigma B_t} dt \to w. \tag{11}$$

As t increases, Z_t increases monotonically, so if W_t does becomes negative, it cannot recover (even very high returns cannot increase a zero wealth). As a result, the probability of ruin before a pre-determined time T is

$$\phi\left(w\right) = \Pr[w \leq \int_0^T e^{-\left(\mu - 1/2\sigma^2\right)t - \sigma B_t} dt] = 1 - \Pr[Z_T < w], \tag{12}$$

or the likelihood that the stochastic present value of the spending plan exceeds initial wealth.

If the time horizon is infinite, Huang et al. (2004) prove that the ruin probability has a closed form analytic solution. But here we look at the case of a limited lifetime, $T < \infty$, and more specifically at the case of an uncertain and finite length of life, where $T_x < \infty$ is a random variable following a known mortality law.

[7] A person's tolerance for the probability of ruin is related to risk preferences: we could think of retirement utility as some general function where the level of (constant real) consumption is a positive argument and the probability of ruin is a negative argument. Further, Milevsky and Robinson (2005) propose that asking a retiree a straightforward question about willingness to tolerate possible ruin may be as good a guide to risk preferences as hypothetical surveys commonly used by financial advisors to assess risk tolerance.

For a random lifetime, the probability of ruin is:

$$Z_{T_x} \equiv \int_0^{T_x} e^{-\left(\mu-1/2\sigma^2\right)t-\sigma B_t} dt \tag{13}$$

$$\phi\left(w\right) = 1 - \Pr[Z_{T_x} < w]. \tag{14}$$

However, since the density function of Z_{T_x} is unknown, we need an approximation method to compute the probability of ruin when the length of life is uncertain.

Huang et al. (2004) outline an approximation based on a moment matching approach. Using the law of iterated expectations, the first moment of the random variable Z_{T_x} is

$$\begin{aligned} E\left(Z_{T_x}\right) &= E[E[Z_t|T_x = t]] \\ &= E\left[\left[E\int_0^t e^{-\left(\mu-1/2\sigma^2\right)s-\sigma B_s} ds\right]|T_x = t\right] \\ &= E\left[\int_0^t e^{-\left(\mu-1/2\sigma^2\right)s} E\left(e^{-\sigma B_s}\right) ds|T_x = t\right] \\ &= E\left[\int_0^t e^{-\left(\mu-1/2\sigma^2\right)s+\frac{1}{2}\sigma^2 s} ds|T_x = t\right] \\ &= \int_0^\infty e^{-\left(\mu-\sigma^2\right)t} {}_tp_x dt \end{aligned} \tag{15}$$

where ${}_tp_x$ is the conditional probability of an individual surviving t more years, having reached age x.

Given an instantaneous force of mortality (hazard rate) $\lambda(t)$, the conditional probability of survival, ${}_tp_x$, can be expressed as

$${}_tp_x = \exp\left[-\int_x^{x+t} \lambda\left(s\right) ds\right] \tag{16}$$

or

$$\begin{aligned} {}_tp_x &= \exp\left[-\int_x^{x+t} \frac{1}{b} \exp\left(\frac{u-m}{b}\right) du\right] \\ &= \exp\left[b\lambda_x\left(1 - e^{\frac{t}{b}}\right)\right], \end{aligned} \tag{17}$$

where under the Gompertz law

$$\lambda(x) = \frac{1}{b} \exp\left(\frac{x-m}{b}\right). \tag{18}$$

The first moment integral in (15) thus evaluates to

$$M_G^{(1)} = E\left[Z_{T_x}\right] = A(\xi|m,b,x), \xi = \left(\mu - \sigma^2\right) \tag{19}$$
$$A(\xi|m,b,x) \equiv b \exp\left\{\exp\left[\frac{x-m}{b}\right] + (x-m)\xi\right\} \Gamma\left(-b\xi, \exp\left[\frac{x-m}{b}\right]\right)$$

where $\Gamma(u,v) = \int_v^\infty e^{-t} t^{(u-1)} dt$ is the incomplete Gamma function. Similarly, the second moment is

$$M_G^{(2)} = E\left[Z_{T_x}^2\right] = \left(\frac{2}{\mu - 2\sigma^2}\right)\left[A\left(\mu - \sigma^2|m,b,x\right) - A\left(2\mu - 3\sigma^2|m,b,x\right)\right]. \tag{20}$$

Having identified the first two moments of the true but unknown density function of Z_{T_x}, the issue is to what known density function can they be approximated so that ruin probabilities can be evaluated analytically? The limiting distribution for Z_∞ $(T \to \infty)$ is a reciprocal Gamma distribution,

$$\Pr[Z < z] \equiv \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_0^t y^{-(\alpha+1)} e^{(-1/y\beta)} dy$$

which has first and second moments of

$$M^{(1)} = \frac{1}{\beta(\alpha-1)}, \quad M^{(2)} = \frac{1}{\beta^2(\alpha-1)(\alpha-2)} \tag{21}$$

so that

$$\alpha = \frac{2M^{(2)} - M^{(1)}M^{(1)}}{M^{(2)} - M^{(1)}M^{(1)}}, \quad \beta = \frac{M^{(2)} - M^{(1)}M^{(1)}}{M^{(2)}M^{(1)}}. \tag{22}$$

Given this limiting result, Huang et al. (2004) propose approximating the distribution of Z_{T_x} using the moments derived above in equations (19) - (20), substituted into (22), and numerically evaluated as a reciprocal Gamma random variable.

The value we are primarily interested in is the probability that the stochastic present value of a consumption and investment plan exceeds initial wealth, $W_0 = w$, $\phi(w) = 1 - \Pr[Z_{T_x} < w] = \Pr[Z_{T_x} > w]$. Furthermore, the probability that a reciprocal Gamma random variable is greater than a particular value is equal to the probability that a Gamma random variable is less than the inverse of that value, or,

$$\phi(w) = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 | W_0 = w] \cong \mathbf{G}\left(\frac{1}{w} | \, \alpha, \beta\right), \tag{23}$$

where the right hand side is the probability that a random variable with a Gamma distribution defined by α and β is less than $\frac{1}{w}$. (However since we evaluate the Gamma distribution over a negative parameter, we need to rescale using the method described in Appendix A.)

The basic pension is like a very safe phased withdrawal plan - one with a ruin probability of less than, say, 1%. (There is always a small, but non-zero, probability that regulators will remove or reduce the pension payment, so we think of this regulatory risk as the probability that the pensioner reaches ruin under public provision.) If we estimate the inflation and earnings adjusted drift and diffusion of the investment plan selected, and model the mortality of a typical retiree, we can fix the probability of ruin at 1%, and infer the size of initial wealth $W_0 = w$ using (23). We conclude that this required wealth $W_0 = w | \phi(w) = 0.01$ is the stochastic present value of a phased withdrawal from the selected investment strategy that substantially replicates the Age Pension.

4 Parameter selection

Reproducing the Age Pension payment stream using the stochastic present value method requires three parameters - the drift and diffusion terms for the portfolio process μ and σ, that is the return and volatility of the portfolio selected by the retiree, and the instantaneous force of mortality, λ, a function of the Gompertz scale and mode parameters b and m.

4.1 Portfolio return and volatility

Consistent with our aim of establishing how much wealth a privately funded retiree would need to generate a consumption stream equal to the Age Pension in value and certainty, we confine ourselves to the portfolios typically offered to Australian superannuants in the retirement incomes market. Most phased withdrawal products amount to holding an account in one or more of these investment portfolios with minimum draw-down rates fixed by regulation (Bateman and Thorp 2007).

We label our five artificial portfolios as High Growth, Growth, Balanced, Conservative and Capital Stable, where each is a combination of two or more asset classes from Australian shares, international shares, Australian property securities, Australian fixed interest and cash. We make no claim that these constructed investments are optimal, since a wider variety of assets and possibly more efficient weighting schemes are available. However a quick survey of providers (such as Vanguard and Colonial First State) will show that our choices are typical. Portfolio weights are set out in Table 1. The portfolios decline in exposure to growth assets, from 90% allocation to shares and property in the High Growth fund, 70% in Growth, 50% in Balanced, 30% in the Conservative fund and Capital Stable entirely invested in cash and fixed-interest securities. [8]

Table 1

Portfolio weights

Asset Class	Portfolio				
	High Growth	Growth	Balanced	Conservative	Capital Stable
Australian Equities	50%	37%	26%	16%	0%
International Equities	30%	23%	17%	10%	0%
Property Securities	10%	10%	7%	4%	0%
Fixed Interest	10%	28%	28%	28%	30%
Cash	0%	2%	22%	42%	70%

To estimate real returns and volatility for each of these portfolios we collect monthly time series of returns indices for each asset class over the 16-year period, 30 December 1989 – 30 June 2006. For each asset class, we compute a monthly periodic return and apply portfolio weights (Table 1) to get a portfolio return, so that $(1 + i_{P,t}) = \sum_{j=1}^{n} \omega_j (1 + i_{j,t})$ where $(1 + i_{P,t})$ is the gross nominal monthly portfolio return over month t, ω_j is the proportion allocated to asset class j and $(1 + i_{j,t})$ is the nominal monthly gross return to asset index j. To translate this to a real return, we derive a monthly percentage change in the quarterly Consumer Price Index by linear interpolation h_t (or for MTAWE, n_t) and compute the monthly log-change in the real portfolio return as

$$r_{P,t} = \ln S_t - \ln S_{t-1} = \ln (1 + i_{P,t}) - \ln (1 + h_t) \qquad (24)$$

[8] The average Australian retiree chooses to hold a relatively high proportion of growth assets in their portfolio - more than 50% in property and equities - according to survey data (see Thorp et al. 2007).

or if we deflate by the greater of inflation and earnings growth

$$r_{P,t} = \ln S_t - \ln S_{t-1} = \ln (1 + i_{P,t}) - \ln (\max [(1 + h_t), (1 + n_t)]) . \qquad (25)$$

The annualized expected value and volatility of this process are:

$$\begin{aligned}
\mu &= 12 \frac{1}{T} \sum_{t=1}^{T} (r_{P,t}) + \frac{1}{2}\sigma^2 \\
\sigma &= s\sqrt{12}, \\
s &= \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (r_{P,t})^2 - \frac{1}{T(T-1)} \left[\sum_{t=1}^{T} (r_{P,t}) \right]^2},
\end{aligned}$$

where T is the number of observations.

Table 2 shows the nominal and real returns and volatilities for each of the portfolios. The average inflation rate is 2.8% and the average earnings-augmented deflation is 4.4%. Historical returns are tempered by poor international equity results in the later part of the sample, but, coming as they do out of 16 years of economic expansion in Australia, and coinciding with strong domestic equity and property performance, may be overly optimistic as a proxy for future returns.

We also deduct an indicative management fee from the real returns, such as are charged by providers of phased withdrawal products offering similar investments (see, for example, AMP (2007) for accounts of value 100-499K). Management fees for retail investors in retirement income products are in the range of 1-3% of account value, and are high by international standards, although a wide variety of fee structures are on offer. We note that such high management expenses in the accumulation stage were a motivation for the UK Government's plan to centralize the collection and administration of individual retirement savings accounts (Department for Work and Pensions 2006b).

Table 2

Portfolio summary statistics

	Nominal Return	CPI-adj Return	less fees	CPI/AWE adj Return	less fees	Std. Dev.
High Growth	10.4%	7.6%	5.7%	6.0%	4.1%	9.9%
Growth	10.0%	7.2%	5.4%	5.6%	3.8%	7.9%
Balanced	9.2%	6.4%	4.7%	4.8%	3.1%	5.8%
Conservative	8.5%	5.7%	4.0%	4.1%	2.4%	3.8%
Capital Stable	7.5%	4.7%	3.1%	3.1%	1.5%	1.5%

Note: Portfolio weights are in Table 1. Australian equities are the Australia-DS Market index, International equities are the AC WORLD INDEX ex AUSTRALIA translated into Australian dollars at the end-month AUD/USD exchange rate, fixed income is the UBS Composite All Maturities index for Australia, property is the S&P/ASX 300 Property index and cash is the UBS AU Bank Bills All Maturities index, all from Datastream. The total return price index (RI) of the relevant asset class index is used for calculations of the periodic monthly returns. As a measure of inflation (earnings), we use a linear interpolation of quarterly annualized growth in the CPI (MTAWE), translated into a monthly log change. The CPI data are from the Reserve Bank of Australia database and MTAWE is from the Australian Bureau of Statistics Publication 6302.0. Indicative management fees are taken from the AMP Allocated Pension Product Disclosure Statement (AMP 2007) for managed fund investments of similar risk exposure and account size $100-$499k.

4.2 Force of mortality

Over the past hundred years, mortality rates in Australia have been declining rapidly. By the publication of the most recent (2000-2002) Life Tables (Commonwealth of Australia 2004), mortality rates were 40-45% lower than in the mid-1960s. Improved life expectancy implies longer retirements and raises the value of a guaranteed income stream such as the Age Pension. Even so, uncertainty over the length of life is a crucial factor in life-cycle planning and one of the advantages of the stochastic present value model is that it incorporates this risk via approximations to the survival density.

4.2.1 Gompertz force of mortality

We estimate parameters b and m using non-linear least squares as $\log(p_x) = \exp\left(\frac{x-m}{b}\right)\left(1 - \exp\frac{1}{b}\right)$, taking discrete mortality data, p_x, from the Australian Life Tables 2000-2002. The conditional survival probability p_x we use in estimation is adjusted by the 25 year improvement factors as described in the Life Tables. Model fit worsens if the sample includes the thin mortality data at extreme old age, so the sample runs from ages 50 to 90. Table 3 reports estimation results for males and females.

Table 3

Estimated Gompertz parameters.

Estimated equation: $\log(p_x) = \exp\left(\frac{x-m}{b}\right)\left(1 - \exp\frac{1}{b}\right)$		
Sample: 50-90 years		
	Males	
	$\hat{m}$	$\hat{b}$
Coefficient	86.91	8.95
p-value	(0.000)	(0.000)
R^2	0.998	
	Females	
	$\hat{m}$	$\hat{b}$
Coefficient	90.70	7.60
p-value	(0.000)	(0.000)
R^2	0.999	

Note: Estimated coefficients and fit statistics for the non-linear least-squares estimation of the Gompertz equation, where p_x, is the improved probability of surviving one more year having reached age x, and b and m are the scale and mode parameters of the distribution. Data are the discrete survival probabilities for males (females) aged 50 to 90 years from the 2000-2002 Australian Life Tables, improved by the 25-year improvement factors.

Having estimated a range of parameter values to reflect investment choices and current mortality for Australians eligible for the Age Pension we can input these to equation (23) and infer a probability of retirement ruin. Alternatively we can fix the likelihood of ruin, the drift and diffusion, and infer the initial wealth needed to fund a payment scheme. This is the amount of private savings a worker needs to accumulate to safely guarantee that income stream.

5 Valuing the basic pension

The most obvious way to value the basic pension is as a standard annuity. We begin with this computation and then use the method of Section 3 to revalue the pension using the stochastic present value of some typical commercial decumulation plans. Finally we compute the money's worth of phased withdrawals and show effects on welfare of delaying retirement and of varying management fees.

5.1 Annuity value of the Age Pension

The Age Pension payment is an indexed immediate life annuity. $V_x(A)$ is the expected discounted value of the annuity,

$$V_x(A) = \sum_{t=1}^{T-x} {}_t\bar{p}_x \frac{A}{(1+\bar{r})^t} \tag{26}$$

where A is the pension payment, here assumed to be \$14000 p.a., $\bar{r}$ is the real rate of interest on long duration government debt, ${}_t\bar{p}_x$ is the (discrete) probability that an individual of age x survives another t years, and T is the oldest old age in the Life Tables. For simplicity and for consistency with the return estimates set out in Section 4, we assume a flat term structure, and set $\bar{r}$ as the geometric mean of the annualized monthly yield on a 5 year Treasury Bond (Reserve Bank of Australia series) divided by either the geometric mean inflation or the geometric mean of the maximum of the increase in the CPI or MTAWE, over the sample 30 December 1989 – 30 June 2006. We compute ${}_t\bar{p}_x$ using the improved probabilities for males/females from the 2002 Australian Life Tables. Table 4 sets out these annuity values by gender and age.

Table 4

Expected discounted value of the basic Age Pension by age and gender (annuity factor in italics).

Age	CPI Indexed		CPI/MTAWE Indexed	
	F	M	F	M
65	\$197908	\$178240	\$236870	\$209825
	14.136	*12.731*	*16.919*	*14.987*
70	\$172373	\$143571	\$201365	\$165142
	12.312	*10.255*	*14.383*	*11.796*
75	\$144881	\$112938	\$165247	\$127076
	10.349	*8.067*	*11.803*	*9.077*
80	\$116337	\$82796	\$129708	\$91317
	8.310	*5.914*	*9.265*	*6.523*

We use these valuations to calculate the money's worth of a commercial annuity or phased withdrawal.

5.2 Risk of ruin

The stochastic present value model can be used to predict the sustainability of a self-funded retiree's investment and spending plan without using simulation experiments. For example, Table 5 shows the probability that an individual with a fixed consumption plan will run out of money before the end of life. In this example our investor reaches age 65 and retires with a net \$1,000,000 and then decides on a fixed real spending plan of between \$20,000 and \$100,000 each year. The lowest retirement ruin probabilities for each draw-down rate is marked with an asterisk. Which is the least-risk investment strategy depends on required levels of drawdown: for 4-6% of initial retirement wealth, the growth portfolio is least likely to be exhausted, whereas at higher expenditure

(8% of initial wealth) the high growth portfolio is safer.[9]

Table 5

Probability of retirement ruin for female (male) age 65, initial wealth $1 million.

Probability of retirement ruin (%), $w = \$1$ million										
	Real spending rate, $000 p.a.									
	20		40		60		80		100	
	F	M	F	M	F	M	F	M	F	M
High Growth	0.03	0.04	2.6	2.1	18.6	13.3	47.4*	34.8	73.7*	58.6*
Growth	0.008	0.02	1.8*	1.6*	17.0*	12.1*	48.2	34.6*	76.4	59.9
Balanced	0.004*	0.01*	1.8	1.7	19.5	13.6	54.9	38.7	83.0	65.4
Conservative	0.004	0.02	2.2	2.1	24.0	16.3	63.1	44.3	88.7	71.4
Capital Stable	0.01	0.03	3.9	3.3	33.6	22.1	74.0	53.1	93.9	78.7

If preferences are measurable in terms of ruin probability, then a retiree could use this table to decide on an investment and spending plan by trading off an increase in ruin probability against an increase in spending. However we note that in some respects the stochastic present value method is at odds with conventional utility theory. A constant real level of consumption is not an optimal strategy for, say, a power utility maximizer - the best plan is a constant rate of draw-down (leaving aside survival uncertainty). As Brown (2000) points out, a constant level of consumption implies infinite risk aversion for an individual with power utility preferences over consumption, manifesting in complete unwillingness to transfer consumption across time. And further, the utility maximizing consumer with conventional preferences will never allow wealth to fall to zero because at zero wealth the marginal utility of consumption is infinite.

[9] The Association of Superannuation Funds of Australia (ASFA) estimate that a single, home-owning retiree in 2006 needed around $18,200 p.a. for a modest lifestyle and $35,400 p.a. to maintain a comfortable lifestyle.

While theoretically sub-optimal for conventional preferences, a constant real consumption stream is exactly what is offered under the Age Pension and similar basic pension schemes around the world, so the stochastic present value method is a reasonable valuation approach.

5.3 Wealth value of the Age Pension

Given the Australian Government's historical support for the program, we could argue that the likelihood of running out of money when receiving the Age Pension approaches zero, but in the analysis below we accept some regulation risk. In other words we assume that the Age Pensioner still faces a very small but non-zero, probability of ruin.

How much retirement wealth would an individual self-funded retiree need to generate a constant real income of $14,000 p.a. and what would be the least-risk approach to creating that income stream? Table 6 sets out the amount of initial wealth needed to support the pension payment for men and women of average improved mortality who invest in standard managed funds. We compute this wealth amount for age 65 and report it as a multiple of average annual earnings. In February 2007, full-time adult ordinary time earnings were $1,072 per week or $55,728 p.a. seasonally adjusted (ABS release 6302.0).

Table 6

Wealth required at age 65 to produce inflation adjusted $14,000 p.a. real income.

Initial wealth premium for $14000 income as multiple of average earnings								
	Female 65 yrs				Male 65 yrs			
Portfolio	Ruin probability				Ruin probability			
	0.5%	1%	3%	5%	0.5%	1%	3%	5%
High Growth	8.24	7.39	6.13	5.41	8.19	7.24	5.86	5.27
Growth	7.54	6.84	5.79*	5.31*	7.63	6.80*	5.60*	5.07*
Balanced	7.43*	6.79*	5.82	5.38	7.62*	6.84	5.68	5.16
Conservative	7.59	6.97	6.01	5.41	7.85	7.06	5.88	5.36
Capital Stable	8.24	7.56	6.52	6.04	8.54	7.67	6.38	5.80
CPI Indexed life annuity	6.06				5.90			

The required wealth level at retirement for a 65 year-old female ranges from as much as 8.24 times average annual earnings for very high or low risk investment portfolios at 0.5% probability of failure, to 5.31 times earnings for the growth portfolio with a 5% probability of failure. At our benchmark 1% failure probability, the most efficient investment allocation is to a balanced fund, needing 6.79 times average earnings or accumulated wealth of $378,581. This amount is 8.6 times more than current average superannuation balances for females 60-65 years (approximately $44,000: ASFA 2007). More conservative and more risky investment strategies need more savings to generate the real income stream with the desired level of security. By contrast, commercial insurance firms currently offer CPI-indexed single life annuities paying $14,000 p.a. at a premium of $337,455, or 6.06 times average annual earnings. [10]

For males, wealth requirements are similar to females at the 1% probability of failure. The slightly riskier growth portfolios, with 70% exposure to equity

[10] Average purchase price to generate $14000 p.a. using CPI indexed single life annuities without gurantee for 65 year old male and female, Table F, DeXX&R Retirement Incomes League Tables, Quarterly Statistics ending December 2006.

and property assets, are most efficient for generating the real income stream. A 65 year old male needs 6.8 times average earnings ($379,194) a sum about three times as large as estimates of the current average male accumulation of approximately $130,000 (ASFA 2007). The cost of a single-life, CPI-indexed annuity for a 65 year old male is $328,844 or 5.9 times average earnings, which again is less costly than the phased withdrawals.

As discussed in Section 2, Age Pension payments are adjusted to be no less than 25% of MTAWE. In Table 7 we value this connection with earnings growth by computing the wealth needed at retirement to generate an income stream that maintains real value and parity with earnings. We do this by 'deflating' nominal returns by the maximum of monthly changes in prices and earnings over the sample. Since earnings have outpaced inflation historically, larger accumulations are needed to match the growth in wages.

Table 7

Wealth required at age 65 to produce the earnings and inflation adjusted equivalent to $14000 p.a.

Initial wealth premium for $14000 income as multiple of average earnings								
	Female 65 yrs				Male 65 yrs			
Portfolio	Ruin probability				Ruin probability			
	0.5%	1%	3%	5%	0.5%	1%	3%	5%
High Growth	10.48	9.30	7.57	6.82	10.26	8.96	6.64	6.34
Growth	9.60	8.62	7.16*	6.52*	9.57	8.44*	6.82*	6.12*
Balanced	9.46*	8.55*	7.20	6.59	9.55*	8.47	6.90	6.22
Conservative	9.74	8.84	7.49	6.88	9.91	8.81	7.21	6.51
Capital Stable	10.83	9.82	8.30	7.62	11.02	9.77	7.95	7.17
5% indexed life annuity	7.56				6.61			

At our benchmark 1% probability of failure, a 65 year old female needs 8.55 times average annual earnings ($476,694) in retirement savings to generate the earnings- and inflation-adjusted pension payment. This wealth is 26% more than required wealth if the pension tracks inflation only, and we conclude that the link to earnings is very valuable to pensioners. Males require 8.44 times average earnings at age 65, a 24% increase over the amount needed to match inflation increases only. The closest commercial single life annuity to the earnings-linked Age Pension payment is a single life indexed to rise at 5% p.a. A 65 year old female would pay $421,559 or 7.56 times average earnings for a 5% indexed annuity paying $14,000 in the first year, whereas the premium for a male is currently $368,227, or 6.6 times average annual earnings, again below the cost of self-insurance via phased withdrawals.

5.4 Money's worth

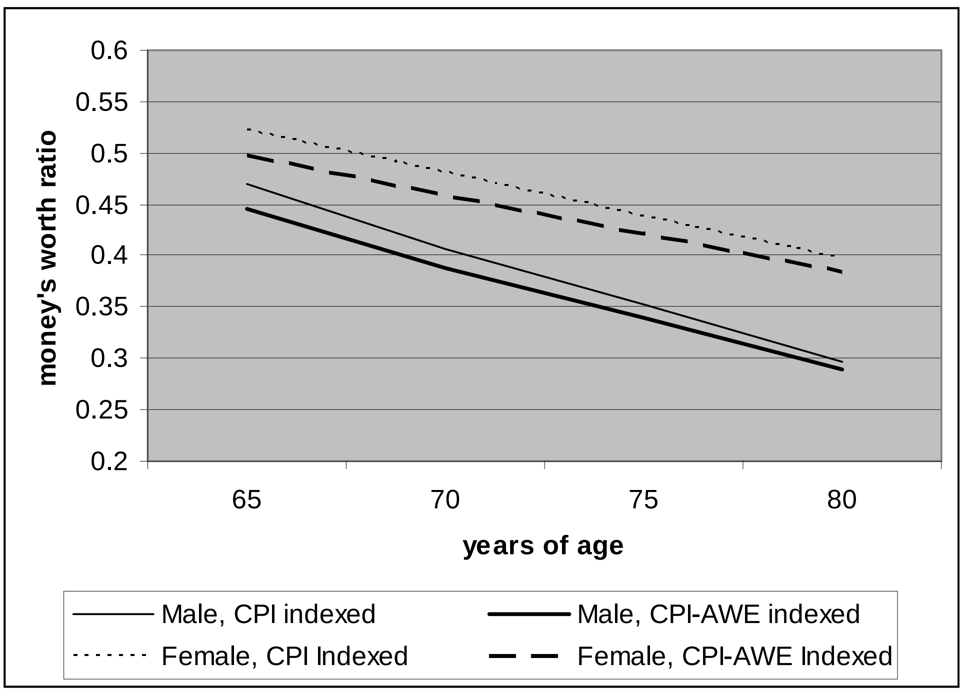
If we choose the least-cost investment strategy and allow the time of retirement to vary, we can compare the money's worth of phased withdrawal plans over a range of ages. The money's worth is the ratio of the expected net present value of the annuity stream to the purchase price (Mitchell et al. 1999), or in our case, the ratio of the expected net present value of the pension payment to required initial wealth of the phased withdrawal, allowing for very improbable plan failure.

Studies of the money's worth of immediate nominal single life annuities across a range of countries find that commercial offerings represent reasonable value for consumers (James and Song 2001). For the US and Australia, Mitchell and McCarthy (2002) report money's worth ratios above 0.8 and 0.9 respectively. Similarly, Cannon and Tonks (2004) put the money's worth of UK annuities above 0.9. Compared with these, the value of the phased withdrawals we report here is very low.

In Figure 2 below, we graph the money's worth of the least-cost phased withdrawal as the age and gender of the pensioner varies. For women this is either the balanced or growth portfolio, and for men, the growth portfolio. Expected discounted values of the pension stream at each age are from Table 4 above.

Figure 2

Money's worth of best phased withdrawal strategy at increasing ages

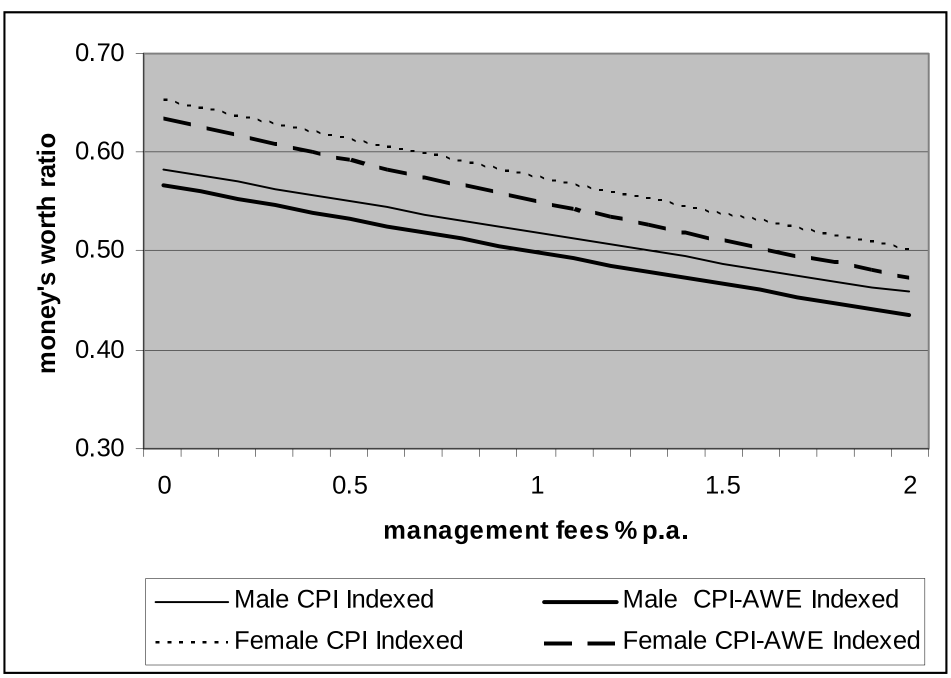


The money's worth of the phased withdrawal strategy decreases with increasing age, and is higher for females than for males. Most phased withdrawals offer a money's worth ratio below 0.5. The highest ratio is 0.52 for a 65 year old female matching an inflation-indexed payment, and the lowest is 0.29 for an 80-year-old male matching a CPI/AWE indexed payment. By comparison, the money's worth of the commercial immediate life annuities for CPI/CPI-AWE indexing are 0.54/0.57 for males, and 0.59/0.56 for females.

Management fees are a major component of the cost of these self-insurance strategies. Figure 3 graphs the impact of varying management fees from 0-2% p.a. in the best phased withdrawal strategy for 65 year old males and females. The money's worth of each strategy declines linearly as fees increase: the ratio for females falls by 0.15 as fees increase from 0-2%, and the ratio for males falls less steeply by 0.12. The impact on wealth required at retirement is substantial. For the CPI-AWE indexed plans the difference in wealth between zero and two per cent fees for females is $126,867 and for males is $112,048. In other words, for each 1% increase in fees, a retiree needs an additional $60,000 in wealth, or more than one year's earnings.

Figure 3

Money's worth of best phased withdrawal strategies under increasing management fees



6 Conclusion

The majority of elderly Australians rely on a targeted public pension, the Age Pension, as their main source of retirement income. Despite the maturing of the mandatory, earnings-linked personal savings scheme (Superannuation Guarantee), individual retirement accumulations are modest and likely to remain so in coming decades. General dependence by the elderly on transfers from public revenue is forecast to continue. However, Australian Government policy aims to alleviate increasing demands on public funds by encouraging more reliance on private retirement savings rather than the basic pension.

Here we ask how much private savings would an individual have to accumulate to reproduce the payments and insurance features of the basic public pension using commercial annuities or phased withdrawals? We compute the retirement wealth that would allow a retiree to enjoy the same benefits as the basic pension using the standard draw-down products of the Australian retirement incomes market. This amount represents the stochastic present value at retirement of the Age Pension payment stream.

Since the exact density function of the stochastic present value of any retirement spending plan is not known when lifetimes are uncertain, we use a moment-matching approximation (Milevsky and Robinson 2000, 2005 and Huang et al. 2004) to value a spending plan equivalent to the pension. Allowing for a very low probability of reaching 'ruin', we back out the initial nest egg needed to replicate a $14,000 pension while accounting for investment and longevity risk. We interpret this initial wealth as the value that a self-insured retiree would attach to full pension eligibility.

We estimate that the implicit public transfer to Age Pensioners is substantial, in the order of $450,000 at age 65 or 8.5 times current average annual earnings. This amount is many times larger than current average personal retirement accumulations. The implicit transfer is generally larger for women because of longer life expectancy, and harder for women to attain by private savings because earnings-related accumulations are commonly smaller than for men. Delaying retirement by five years reduces required wealth by only 5% or less. On the other hand, 25% more wealth is needed to maintain the relative level of the pension with wages, as compared with indexing to consumer prices, and around 6% more wealth is needed for each 1% increase in investment management fees.

The money's worth ratio of phased withdrawal products is very low, generally less than 0.5, so we conclude that switching from public to private provision is expensive, with substantial extra wealth needed to cover management and administrative fees, as well as the costs of self-insuring against investment and longevity risk. On the other hand low levels of voluntary annuitization suggest that retirees are willing to bear risks and costs in exchange for continued ownership of their lump sum, and we have not explicitly valued this feature of phased withdrawals. Finally, despite their marked unpopularity with the retired, commercial life annuity products mimic public pension payment paths more cheaply than drawn-down plans invested in managed funds.

Dramatic increases in retirement savings are needed if lower levels of basic pension reliance are to be realized in Australia. The likelihood of the majority of employees accumulating sufficient in second pillar savings to generate a basic pension equivalent payment seems remote. On the other hand, the high implicit value of pension eligibility creates incentives for retirees to draw down private savings faster in order to access pension benefits.

Appendix A: Rescaling the incomplete gamma function

A numerical complication arises from the fact that the incomplete gamma function which appears in the moments (19) and (20) is difficult to compute in most software packages because $-1 < -\xi b < 0$ and the packages will not return gamma values defined over negative parameters. A rescaling derived from Milevsky (2001) allows the incomplete gamma function to be rewritten over $(-\xi b + 1) > 0$.

The standard probability density function of the gamma distribution is

$$g_a(x) = \frac{e^{-x} x^{(a-1)}}{\Gamma(a)}, \tag{27}$$

where $x, a > 0$, and the cumulative density function is given by:

$$G_a(c) = \Pr[X \leq c] = \int_0^c \frac{e^{-x} x^{(a-1)}}{\Gamma(a)} dx. \tag{28}$$

To evaluate the moments we need values of the incomplete gamma function:

$$\begin{aligned} \Gamma(a, c) &= \int_c^\infty e^{-x} x^{(a-1)} dx && (29) \\ &= \Gamma(a)(1 - G_a(c)). && (30) \end{aligned}$$

where $a = -\xi b$ and $c = \exp\left[\frac{x-m}{b}\right]$. The negative value $-\xi b$ rules out using standard software to retrieve these values. Milevsky (2001) suggests redefining the incomplete gamma function over $-\xi b + 1$, which will be non-negative, and then rescaling to get back to the original problem.

Integrating (29) by parts gives

$$\int e^{-x} x^{(a-1)} dx = e^{-x} \frac{1}{a} x^a + \frac{1}{a} \int e^{-x} x^a dx, \tag{31}$$

and so

$$\Gamma(a, c) = -\frac{c^a e^{-c}}{a} + \frac{1}{a} \Gamma(a+1, c). \tag{32}$$

Using (32) we can rewrite $\Gamma(a, c)$ as:

$$\Gamma(a, c) = \frac{1}{a} \Gamma(a+1)(1 - G_{a+1}(c)) - \frac{c^a e^{-c}}{a}. \tag{33}$$

Equation (33) is easily programmed into standard spreadsheet packages. For the cases where $-\xi b < -1$, we rescale to $-\xi b + 2$ using the recursion:

$$\Gamma(a, c) = \frac{1}{a} \left[\frac{1}{a+1} \Gamma(a+2)(1 - G_{a+2}(c)) - \frac{c^{a+1} e^{-c}}{a+1} \right] - \frac{c^a e^{-c}}{a}.$$

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