MODELING THE CURRENCY FORWARD RISK PREMIUM: THEORY AND EVIDENCE

Ramaprasad Bhar[*], Carl Chiarella and Toan M. Pham[**]

[*]School of Finance and Economics
University of Technology, Sydney
PO Box 123, Broadway, NSW 2007
AUSTRALIA
Fax. +61 2 9514 7711

[**]School of Banking and Finance
The University of New South Wales
Sydney 2052
AUSTRALIA

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[*] Corresponding author: Email:ramaprasad.bhar@uts.edu.au

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Abstract:

There is a huge literature on the existence of risk premia in the foreign exchange market and its influence in explaining the divergence between the forward exchange rate and the subsequently realised spot exchange rate. In this paper, we seek to model directly the risk premium as a mean-reverting diffusion process. This is done by making use of the spot-forward price relationship and assuming a geometric Brownian process for the spot exchange rate. We are able to obtain a stochastic differential equation system for the spot exchange rate, the forward exchange rate and the risk premium which we estimate using Kalman filtering techniques. The model is then applied to the French Franc/USD and Japanese Yen/USD exchange rates from 1 January 1990 to 31 December 1998. For both currencies our main findings show (i) the persistence of substantial positive time variation in the forward risk premium and its alternating regimes; and (ii) the presence of a term structure of the forward risk premia.

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1. Introduction

This paper focuses on a topical and important area of finance theory and practice that analyses risk premia in the foreign exchange market. The notion that the forward exchange rate might be the optimal predictor of the future spot exchange rate has been investigated by a number of researchers. This notion developed as a corollary to the efficient market hypothesis. For market participants it is, therefore, an important issue to monitor whether the forward exchange rate is an unbiased forecast of the future spot exchange rate. This unbiasedness hypothesis has been the subject of several research papers (Engel (1996)).

The typical starting point in these analyses is to consider the following regression of the change in the log of the spot exchange rate on the forward premium:

$$s_{t+k} - s_t = \alpha + \beta\left(f_{t,k} - s_t\right) + u_{t+k} \tag{1}$$

where s_t is the log of the spot price (S) of foreign currency at time t, $f_{t,k}$ is the log of the k-period forward price (F) at time t, and u is the regression error term. The null hypothesis generally tested is that $\alpha = 0, \beta = 1$ and the error term has a conditional mean zero. Thus, under the null hypothesis the log of the forward rate is an unbiased predictor of the log of the future spot exchange rate.

Several papers over the years have examined the regression (1) with various improvements in econometric techniques employed and the overall results may be described as mixed. For example, Wu and Zhang (1997) employ a non-parametric test and not only reject the unbiasedness hypothesis but also conclude that the forward premium either contains no information or wrong information about the future currency depreciation. On the other hand, Bakshi and Naka (1997) derive an error correction model under the assumption that the spot and the forward rates are cointegrated and conclude using the generalised method of moments that the unbiasedness hypothesis cannot be rejected. Phillips and McFarland (1997) develop a

robust test and reject the unbiasedness hypothesis but conclude that the forward rate has an important role as a predictor of the future spot rate.

It has been suggested that the unbiasedness hypothesis may be failing empirical tests due to the existence of a foreign exchange risk premium. This has led to a great deal of research on the modelling of the risk premia in the forward exchange rate market. However, models of risk premia have been unsuccessful in explaining the magnitude of the failure of unbiasedness (Engel (1996), page 124). Under rational expectations,

$$s_{t+k} - s_t = E_t(s_{t+k}) - s_t + \varepsilon_{t+k} \tag{2}$$

where E_t is the mathematical expectation conditional on information at t and ε_{t+k} is uncorrelated with information at time t. We define the term $rp_t \equiv f_{t,k} - E_t(s_{t+k})$ as the foreign exchange risk premium. Under risk-neutrality the market participants would behave in such a way that $f_{t,k}$ equals $E_t(s_{t+k})$ and the expected profit from forward market speculation would be zero.

This definition of risk premium is based on the rational expectations of the market participants. Even then, the measures of rp_t may suffer from small sample biases. If rp_t could be related to underlying economic variables then its theoretical foundation would be firmly based upon economic theory. Several articles (see for example a survey in Stulz (1994)) discuss the models of foreign exchange risk premium based on optimising behaviour of international investors. However, alongside such theoretical developments pure time series studies of rp_t assume a renewed importance. In particular, they are useful in describing the behaviour of $f_{t,k} - E_t(s_{t+k})$, which models of foreign exchange risk premium that assume rational expectations need to be able to explain. Examples of such studies include Backus et al (1993) and Bekaert (1994).

Some researchers, Wolff (1987, 2000) and Chung (1993), have modelled this risk premium as an unobserved component in state space form and estimated it using the Kalman filter. The advantage of this signal extraction approach is that the researcher

can empirically characterise the temporal behaviour of the premium using only data on spot and forward exchange rates. This avoids the problem associated with specifying a functional form of the underlying economic determinants of risk premium and other strong assumptions of the regression based approach. At the same time signal extraction methods do not offer much insight into the relationship between the risk premium and other economic variables.

Wolff (1987) suggests a state space formulation where the risk premium and the unexpected rate of exchange rate depreciation are assumed uncorrelated. Cheung (1993) follows a framework similar to Wolff and treats the unobserved risk premium as a low order ARMA process. In addition, the innovations in rp_t are allowed to be correlated with $s_t - E_{t-k}(s_t)$, the error from previous period's forecast. Using monthly data Cheung (1993) finds that the filtered estimate of rp_t exhibit a great deal of persistence, high variability and negative correlation with $s_t - E_{t-k}(s_t)$. Canova (1991), Canova, and Ito (1991) also find high volatility in $f_{t,k} - E_t(s_{t+k})$. Canova and Marrinan (1993) agree with these findings and further document high serial correlation and volatility clustering in the time series of rp_t. One other common feature of these studies is that the estimate of rp_t switches sign during the sample periods investigated. For a given exchange rate, eg. USD/DEM, this would imply that there are periods when U.S. dollar assets are considered much safer than DEM asset and there are times when the reverse is the case.

An approach to test the hypothesis that the risk premium is a linear function of the conditional variances and covariances as suggested by standard asset pricing theory is based on a multivariate GARCH framework. Baillie and Bollerslev (1990) consider a GARCH in the mean model using weekly data under the assumption of risk neutrality and rational expectations. Tests of their model fail to find support for this theory. They conclude a possible violation of forward market efficiency and this could be due to inefficient information processing by market participants or the fact that other theoretical models are required to deal with the time varying risk premium.

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The risk premia models discussed above may be termed as partial equilibrium in nature since the stochastic process of asset returns is given. Dumas (1993) points out that a full general equilibrium model will relate this process to underlying exogenous economic variables. A good starting point in this respect is the Lucas (1982) model. Bekaert (1994) discusses some of the reasons why general equilibrium models cannot adequately explain the behaviour of the risk premium.

The preceding analysis suggests that the empirical evidence on the role of the forward rate as a predictor of the future spot rate is mixed, furthermore there seems to be an important influence exerted by the risk premium or even a term structure of risk premium. If the size of the risk premium is unknown and it is time varying then the forward rate will be a poor forecaster of the future spot rate. It is in this context that we attempt an alternative characterisation of the risk premium. We do this by seeking to exploit the information about the risk premium implied in the no arbitrage relationship between spot and forward exchange rates. We use Kalman filtering techniques to extract this information. The theoretical background of our approach is reviewed in Section 2, while a description of the model is given in Section 3. This is followed by the presentation of the Kalman filter estimation procedure in Section 4 and the analysis of the empirical results in Section 5. Finally Section 6 concludes the paper.

2. A New Framework for the Dynamics of Risk Premia

Although there is no unanimity of opinion, the preceding discussion points out the existence of risk premia and its influence in explaining the divergence between the forward exchange rate and subsequently realised spot exchange rate. In most cases the tests rejecting the simple efficiency hypothesis argument are based on asymptotic inferences. Even if the researchers use large data sets, to avoid data correlation problems with overlapped samples, the effective sample size becomes much smaller. For example, when spot exchange rates and one-month forward rates are used in the tests the effective sample frequency becomes monthly. It thus seems to us that a large amount of information in the intervening period are either missed or not utilised effectively.

We propose to adopt a somewhat different approach in this modelling exercise. We start with the usual assumptions in the Black-Scholes option-pricing framework and let the spot exchange rate follow a geometric diffusion process. The standard arbitrage argument is then applied to relate the forward exchange rate to the spot exchange rate through the contract period, and the related interest rates in the two countries. By application of Ito's lemma, we then express the dynamics of the forward price as another stochastic differential equation.

It is clear that in this situation the asset underlying the forward contract is a traded security. Therefore, as discussed in Hull (1997, chapter 13) in order to price the forward contracts the investors may be considered risk-neutral under the equivalent (risk neutral) measure. In operational terms this implies that under the historical measure the expected return part of the underlying security may be replaced by another term involving the risk-free rate, the market price of risk and the volatility of the security. The market price of risk, however, is not observed in the market and has to be inferred from other observable quantities. Hull (1997, page 296) explains why an estimate of market price of risk is usually not needed to price derivative securities when the underlying asset is a traded security.

In our case, however, since we are not pricing the forward contracts as such we incorporate the market price of risk and treat this an unobserved state variable in the system dynamics under the historical measure. Once we express the dynamics of the market price of risk through a suitable stochastic differential equation, we then have a partially observed system involving three variables, the spot exchange rate, the forward exchange rate and the market price of risk. This system can be cast into a state space form and estimated with the help of the Kalman filter after appropriate discretisation. The advantage of this approach is that we get the filtered estimates of the market price of risk, which can be used to form estimates of the risk premium. It should be noted that we are modelling the dynamics of the market price of risk through the discretisation period (e.g. trading day). Thus, there is no need to match the dates for the spot exchange rates with those of the forward exchange rates. This approach therefore has the benefit that we are able to utilise all of the information generated through the trading dates, which is normally not possible in regression-based approaches.

For a suitable representation of the dynamics of the market price of risk, we note the findings in Wolff (1987) and Cheung (1993). Both these authors report empirical support for a low order ARMA process for the risk premium in a number of exchange rates against the U.S. dollar. While Wolff (1987, p. 396) recognises that the economic content of the risk premium may be based upon equilibrium models of international asset pricing, it is not explicitly modeled. In this regard our approach compliments Wolff (1987) and Cheung (1993) by providing an explicit modeling of the risk premium with respect to the market price of risk under the no arbitrage condition. We select a mean reverting form of the stochastic differential equation representing the market price of risk. By suitable change of variable and discretisation, this mean reverting form can also be represented as an AR (1) process. The parameters of the stochastic differential equation representing the market price of risk are to be estimated from the data as well. In the next section, we discuss the details of these modelling issues.

3. The Proposed Model

The proposed model is a result of three main assumptions. Firstly, under the historical probability measure Q (as opposed to the risk neutral probability measure used in derivative security pricing) the spot exchange rate follows the geometric Brownian process

$$dS = \mu S dt + \sigma_S S dW(t) \tag{3a}$$

where dS is the increment of the spot exchange rate, μ is the expected return from the spot rate, σ_S is the volatility of this return and $dW(t)$ is the increment of a Wiener process under the probability measure Q. The second main assumption is that in efficient markets derivative instruments are priced in accordance with the principle of no riskless arbitrage. An expression of this principle is that all derivatives on foreign exchange such as forwards and options are priced such that their expected risk adjusted excess return is constant across all instruments I and foreign exchange itself and equal to the factor λ, the market price of risk[1]:

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[1] See Ross et al (1998, pp. 387-388) for an exposition of the fundamental result that in an active, competitive market the market price of risk must be the same for all the assets in the market.

$$\frac{\mu_I - (r - r_f)}{\sigma_I} = \lambda \qquad (3b)$$

where the subscript I refers to the derivative instrument while r and r_f are the riskfree interest rates in the domestic country and the foreign country respectively.

Considering the foreign exchange itself we can write (3b) as

$$\mu \quad = \quad r - r_f + \lambda\sigma_S$$

(3c)

which highlights the interpretation of λ as the additional return required by investors holding foreign currency for a unit increase in volatility σ_S.

Substituting (3c) into (3a) allows us to re-express the dynamics of the exchange rate S as

$$dS = (r - r_f + \lambda\sigma)Sdt + \sigma SdW(t) \qquad (3d)$$

We stress that the dynamics are still under the historical measure Q. The principle of no riskless arbitrage also allows us to obtain between the forward exchange rate F(t, T) and the spot exchange rate S(t) the relationship

$$F(t,T) = S(t)e^{(r - r_f)(T - t)} \qquad (3e)$$

Using this, equation (3d) for the dynamics of S and Ito's lemma we are able to obtain the dynamics of F under the historical measure Q.

The third assumption is that the market price of risk of foreign exchange risk follows a mean-reverting stochastic differential equation. In Appendix A we show how these three assumptions allow us to express the spot exchange rate, the forward exchange rate and the market price of risk as the stochastic dynamical system

$$dS = (r - r_f + \lambda\sigma_S)Sdt + \sigma_S SdW(t), \qquad (4a)$$

$$d\lambda = \kappa(\bar{\lambda} - \lambda)dt + \sigma_\lambda dW(t) \qquad (4b)$$

$$d\Phi(t, x_i) = (r - r_f + \lambda\sigma)\Phi(t, x_i)dt + \sigma_\Phi \Phi(t, x_i)dW(t) \qquad (4c)$$

$$\pi(t, x) = -0.5\sigma^2 + \sigma_S\left[\left(\lambda(t) - \bar{\lambda}\right)\left(\frac{1 - e^{-\kappa x}}{\kappa}\right) + \bar{\lambda}x\right] \qquad (4d)$$

where:

S	= spot exchange rate process
$W(t)$	= standard Wiener process under the historical probability measure
λ	= market price of risk
$\Phi(t, x_i)$	= forward price $\Phi(t, x_i) \equiv F(t, \quad t + x_i)$ with maturity x_i ahead
$\pi(t, x)$	= the risk premium for the x- period ahead spot rate
r	= domestic risk free interest rate
r_f	= risk-free interest rate in the country of the foreign currency
$\bar{\lambda}$	= long-run equilibrium of market price of risk
κ	= the speed of adjustment of the process for λ to its long run equilibrium

Equations (4a), (4b) and (4c) are the stochastic processes describing the behavior of the spot exchange rate, the market price of risk and the forward price respectively. Equation (4d) is the risk premium expressed in term of the variables embedded in Equations (4a), (4b) and (4c). As discussed in the introduction, assuming rational expectations previous studies attributed the difference between the forward rate and the subsequently realised spot exchange rate to a risk premium and some noise term (eg Wolff (1987), Cheung (1993)). We are, however, able to characterise how the market price of risk enters the expectation formation and thus determine the risk premium. In fact, the noise terms identified in Wolff (1987) and Cheung (1993) can now be explained in terms of an integral with respect to the Wiener increments [see equations (A13) and (B4) in Appendices A and B respectively].

We would now like to compare the time variation of risk premia for different maturities of forward contracts obtained from equation (4d) for different exchange rates. As a result we will be able to examine the term structure of forward risk premia present in the quoted forward exchange rates. To carry out these procedures we will require estimates of the parameters describing the stochastic process for λ given by equation (4b). In the next section, we briefly describe the state space formulation of

the system and estimation of these parameters while a detailed technical exposition is contained in Appendix C.

4. State Space Framework

Broadly speaking our empirical procedure involves the discretisation of the continous sytem dynamics given by the equations (3a) through (3c). A number of discretisation schemes for stochastic differential equations are discussed in Kloeden and Platen (1992) and we choose to work with the Euler-Maruyama scheme. The next step is to express the discretised system in state space form.

As it stands, the equations (4a) and (4c) suggest that the diffusion terms are dependent on the state variables themselves and are thus stochastic in nature. By a simple transformation of variables using the natural logarithm and application of Ito's lemma we can transform these to equations with constant diffusion terms. Using these transformations and after discretisation of the equations (4a) through (4c) we obtain for the time interval between k-1 and k:

$$s_{k+1} = s_k + (r - r_f - 0.5\sigma_S^2)\Delta t + \lambda_k \sigma_S \Delta t + \sigma_S \sqrt{\Delta t}\, \tilde{\xi}_k \tag{5}$$

$$\phi_k = \phi_{k-1} + (r - r_f - 0.5\sigma_S^2)\Delta t + \lambda_{k-1} \sigma_S \Delta t + \sigma_S \sqrt{\Delta t}\tilde{\xi}_k \tag{6}$$

$$\lambda_{k+1} = (1 - \kappa\Delta t)\lambda_k + \kappa\bar{\lambda}\Delta t + \sigma_\lambda \sqrt{\Delta t}\tilde{\xi}_k \tag{7}$$

where $s = lnS,\ \phi = ln\Phi$ and $\tilde{\xi}_k \sim N(0,1)$

The equations (5) – (7) describe the dynamics of the partially observed system and in the state space framework it is generally referred to as the state transition equation.

Once the system is specified in state space form a recursive algorithm such as the Kalman filter (see e.g. Harvey (1990)) can be applied to obtain the optimal estimate of the state vector at time k based upon all the information available at that time. In this sense the Kalman filter is forward looking. However, more efficient estimates of the state vector and its error covariance matrix can be obtained if after the initial estimation all the information up to the final observation is utilised in a sort of second

pass process. The smoothing algorithm provides such a procedure (see Harvey (1990) for details). In most applications in finance and economics, such as ours, all the observations are already available. Therefore, the smoothing algorithm can be easily applied and has been incorporated in the study of this paper.

Among other things, the Kalman filter provides exact finite sample forecasts. These forecasts are used to form prediction errors at each time step which in turn are used to form the log likelihood function from which maximum likelihood estimates of the parameters of the system are obtained. A further by product of this estimation procedure is the set of filtered estimates of the unobserved market price of risk λ at each time step, which is then used to form the filtered estimates of the risk premium in equation (4c).

5. Data and Empirical Results:

We apply the methodology outlined in Section 4 to two exchange rates and three different maturities for the forward rates. Specifically, we use JPY/USD and FF/USD exchangerates and forward exchange rates of 1-month, 2-months and 3-months maturities to investigate the variation of the risk premia over a nine year period from January 1, 1990 to December 31, 1998. The exchange rate data reflects the daily 4PM London quotation obtained from Datastream™ and the interest rate data are the daily closing 3-month Treasury bill yield for the period from 1 January 1990 to 31 December 1998. Thus, the inputs to the Kalman filtering estimation consists of the returns on the spot rate, forward rate for 1, 3, 6- month maturities, the bill rates in the home country and the foreign country (r, r_f) and the volatility of the returns on the spot rate, σ_S, being fixed at sample values[2]. The outputs are the parameter estimates $\overline{\lambda}$, σ_λ, κ and the filtered estimates of λ {λ_0, λ_1, …, λ_T} (where T is the number of observations) which are then used to estimate the risk premium in equation (4d).
The results are shown in Tables 1, 2, 3 and graphed in Fig. 1, Fig. 2 and Fig. 3 from which several observations and interpretations can be made.

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[2] In order to improve the stability of the estimation process the volatility of the return on the spot exchange rate (see equation (4a)) is fixed at the values calculated as standard deviation from the

With respect to the statistical significance of the parameter estimates (see Table 1), all estimates are statistically significant apart from the equilibrium market price of risk of the French Franc, $\overline{\lambda}$. The French Franc appears to have adjusted more slowly than the Japanese Yen, which is also consistent with its lower level of volatility (see κ and σ in Table 1). This may be attributed to the fact that since 1979 France has been part of the European Monetary System whose purpose is to foster currency stability in Europe while the Japanese Yen has been operating in a relatively free floating environment. From a different point of view the relatively large fluctuations of the French Franc risk premium (see Fig. 1 and Fig. 3) in the 1991-1993 period reflect the currency turmoil in Europe which culminated in the currency crisis of 1992. The catalyst for this volatile period was the deliberate attempt of the Bundesbank to tighten monetary policy by raising interest rates to combat inflation (caused by the expansionary policies to shore up the economy of East Germany) and to attract foreign capital to finance the resulting budget deficits.

We also notice high values of σ_λ for all three forward exchange rates (see Table 1) and this is consistent with the finding of Canova and Ito (1991) who reported high volatility in $f_{t,k} - E_t(s_{t+k})$. Furthermore, the diagnostic statistics to determine the adequacy of the estimates of the model market price of risk, λ (see Table 2) indicate that the residuals are white noise.

Overall there is evidence supporting correct model identification with 5 out of the 6 parameter estimates being statistically significant (see Table 1) while the behavior of the risk premium estimates of the French Franc (see Fig. 1) reflects the currency crisis in Europe in the early 1990's.

Both currencies exhibit substantial maturity variation in their respective risk premium (see Table 3) and also sign switching between positive and negative (see Fig. 1, Fig. 2 and Fig. 3). Thus, this finding convincingly rejects constancy of the mean of the risk premium of both currencies particularly for the French Franc. While our modeling approach is different, this result of positive serial correlation and alternating regimes

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sample. These annualised values are 0.0146 and 0.0194 for the French Franc and Japanese yen respectively.

is consistent with previous evidence [Engel (1996), Wolff (1987, 2000), Nijman et al (1993)]. Furthermore, a 'whiteness' test (see Table 3) is also performed to investigate the behavior of the demeaned risk premia and the statistics (Table 3) indicate the fluctuations of the risk premia of both currencies around their respective means are non-white, thus further reinforcing the persistence of the positive correlation of the risk premia. This feature of the behavior of the forward risk premium is now catalogued as one of the new facts in finance (see Cochrane (1999)). Lastly the negative risk premia (see Table 3) for both currencies are feasible in the *ex-ante* sense and consistent with recent research [(see Boudoukh, Richardson and Smith (1993), Ostdiek (1998)] while their changing values across maturities clearly indicate a term structure of risk premia.

On balance our empirical results reaffirm the presence of the time varying property of forward risk premia while our model provides an integrated framework where the risk premium is tied to the market price of risk in the context of rational expectations and no riskless arbitrage.

6. Conclusions:

In this paper we have presented a new approach to analyse the risk premium in forward exchange rates. This involves exploiting the no arbitrage relationship that links the spot exchange rate and the forward exchange rate through the market price of risk under the historical probability measure. By directly modelling the market price of risk as a mean reverting process we are able to show how the market price of risk enters into expectation formation for a future spot exchange rate.

This methodology allows us to quantify the risk premium associated with a particular forward exchange rate in terms of the parameters of the process describing the market price of risk. We also demonstrate how these parameters can be estimated in a state space framework by application of the Kalman filter. This procedure, in turn, generates the filtered and the smoothed estimates the unobserved market price of risk.

We apply the procedure developed in the paper to French Franc/USD and JPY/USD spot exchange rates and 1-month, 2-months and 3-months forward exchange rates.

For both currencies the analysis of the results shows (i) the persistence of substantial time variation in the forward risk premium on the positive side and its alternating regimes; and (ii) the presence of a term structure of the forward risk premia.

Table 1
Estimated Parameters of the Market Price of Risk

	κ	$\bar{\lambda}$	σ_{λ}
French Franc	25.22	-1.555	172.53
	(5.81)	(-1.13)	(8.48)
Japanese yen	49.99	-0.372	325.5
	(12.61)	(-12.28)	(12.88)

Numbers in parentheses are t-statistics computed from standard errors obtained using the heteroscedasticity consistent covariance matrix at the point of convergence. The annualised volatility of the spot exchange rate process is set to the sample values and these are 0.0146 and 0.0194 for French Franc and Japanese Yen respectively.

Table 2
Diagnostic Tests of the Estimated Model Market Price of Risk (λ)

	1-Month Forward	2-Month Forward	3-Month Forward
French Franc			
Q(10)	0.757	0.998	0.876
$Q^2(10)$	0.999	0.999	0.999
Japanese Yen			
Q(10)	0.141	0.539	0.602
$Q^2(10)$	0.939	0.999	0.999

The entries in the table are p-values. Q(10) measures the Ljung-Box statistics (order 10) for serial correlation in the respective residual series. $Q^2(10)$ is similar to Q(10) but computed with the squared residual. The asymptotic distribution of both these statistics are Chi-squared with degrees of freedom 10.

Table 3
Descriptive Statistics of the Estimated Risk Premia (π)

	1-Month Forward	2-Month Forward	3-Month Forward
French Franc			
Mean	-0.0013	-0.0031	-0.0050
Std. Dev.	0.0109	0.0122	0.0124
Q(10)	0.000	0.000	0.000
Japanese Yen			
Mean	-0.0010	-0.0015	-0.0020
Std. Dev.	0.0126	0.0128	0.0128
Q(10)	0.033	0.033	0.033

Descriptive statistics of the risk premia for the three different forward exchange rates computed from the parameters estimates in Table 1 and the smoothed estimates of the market price of risk.

Figure 1
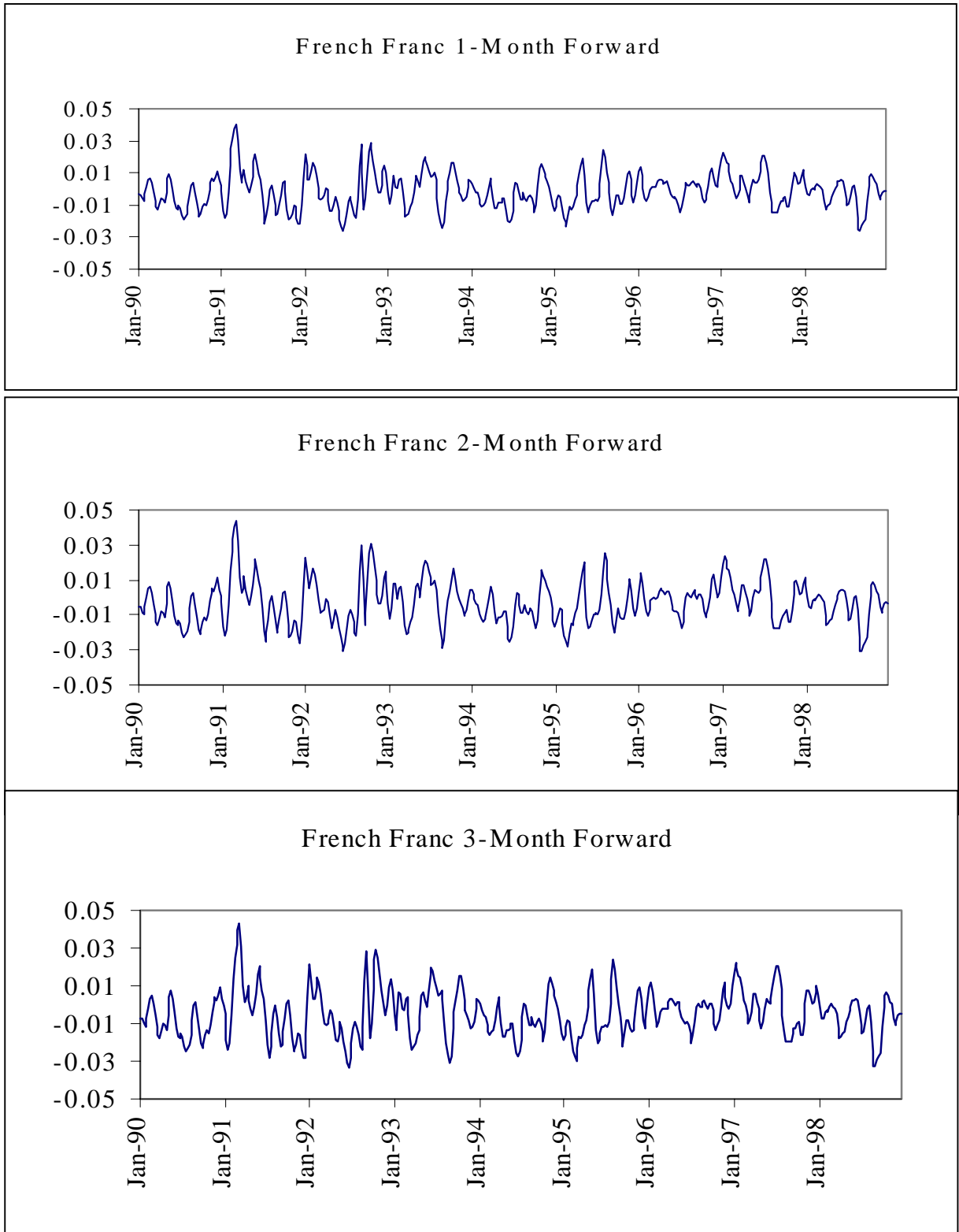Estimated Risk Premia in French Franc Forward Exchange Rates

French Franc 1-Month Forward

French Franc 2-Month Forward

French Franc 3-Month Forward

Figure 2
Estimated Risk Premia in Japanese Yen Forward Exchange Rates

Japanese Yen 1-Month Forward

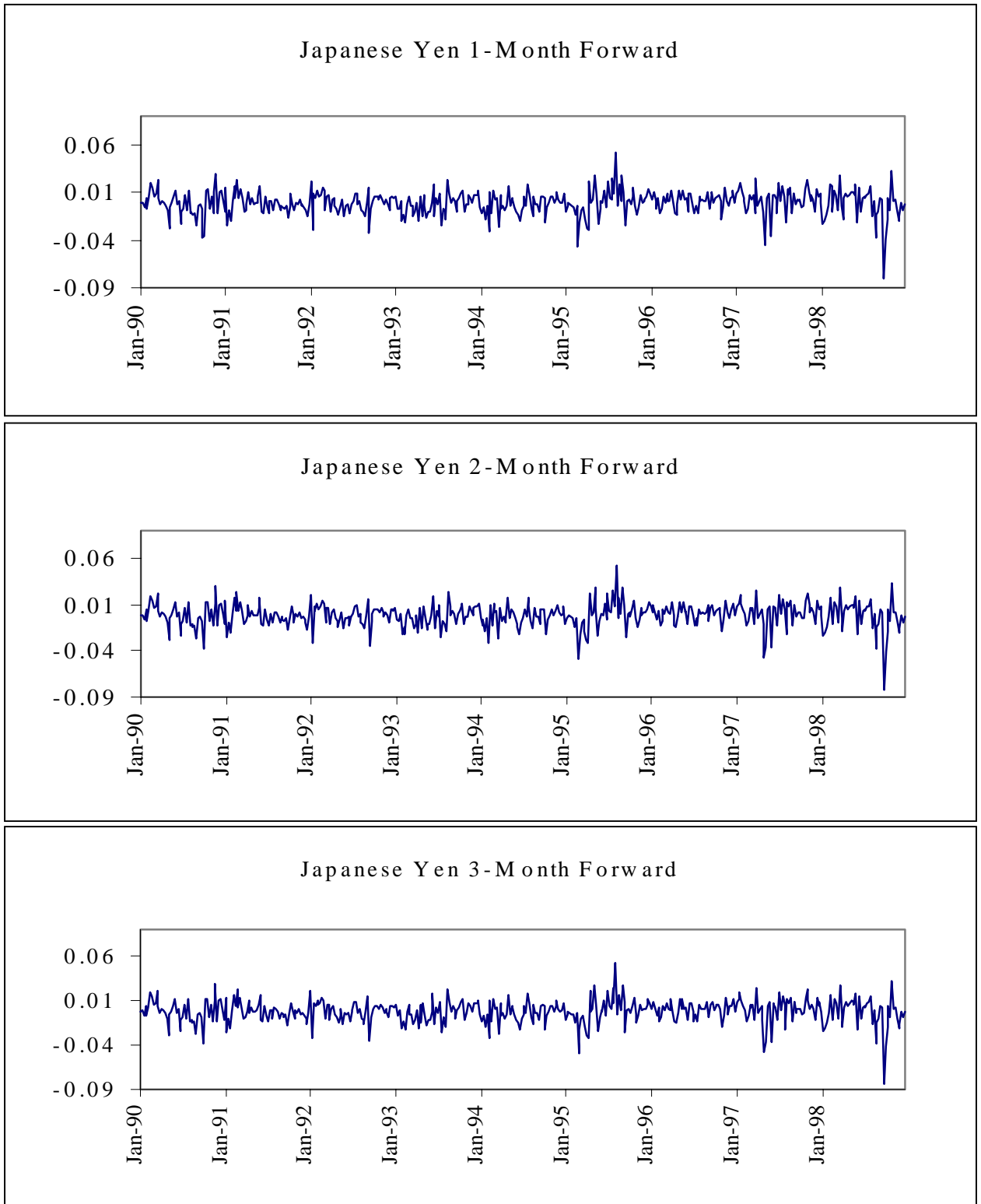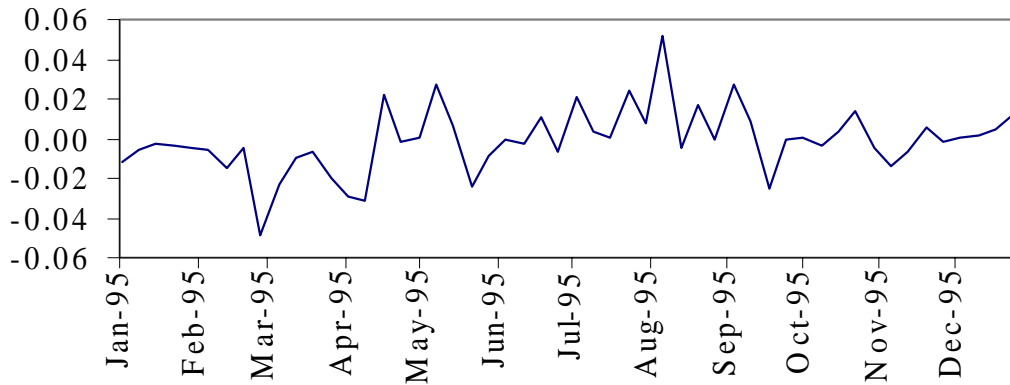Japanese Yen 2-Month Forward

Japanese Yen 3-Month Forward

Figure 3
Estimated Risk Premia in 3-Month Forward Exchange Rates
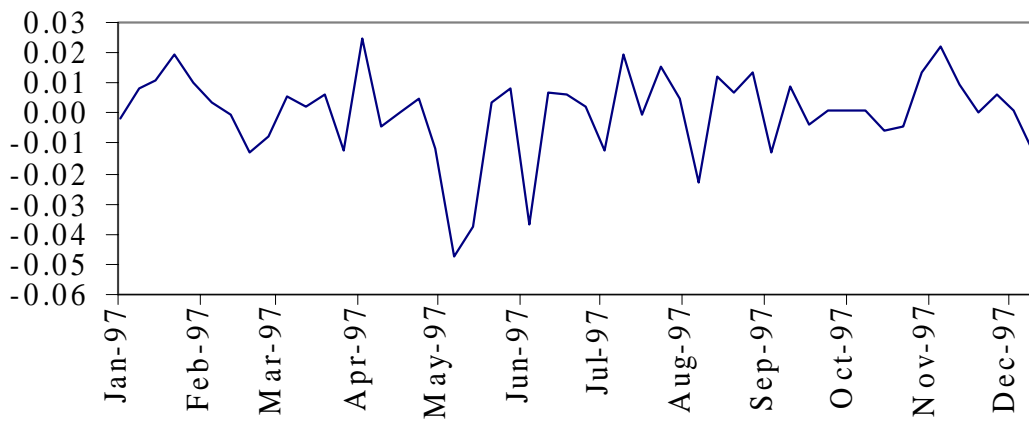### French Franc and Japanese Yen (Selected Periods)

French Franc 3-Month Forward (1992)
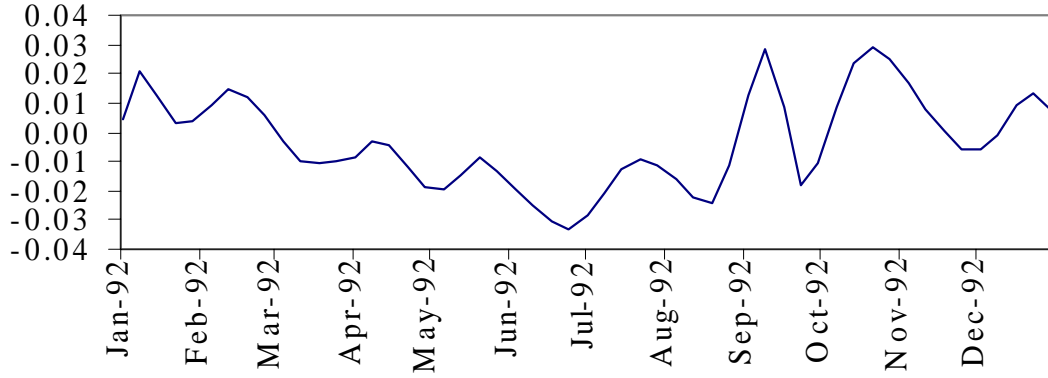
Japanese Yen 3-Month Forward (1995)

Japanese Yen 3-Month Forward (1997)

Appendix A

Derivation of the Stochastic Dynamical System

While this paper essentially adopts the methodology of Bhar and Chiarella (2000), in order to make it self-contained the basic elements of this methodology are summarised in this appendix.

Let the spot exchange rate follow the one-dimensional geometric diffusion process,

$$dS = \mu S dt + \sigma S dW(t) \qquad\qquad (A1)$$

where μ is the expected return from the spot asset, σ is the volatility of this return, both measured per unit of time and dW is the increment of a Wiener process under the historical probability measure Q. Let us define r as the domestic risk-free interest rate and r_f as the counter-part in the foreign currency. Since r_f can be interpreted as a continuous dividend yield, the instantaneous expected return to an investor holding foreign exchange is $(\mu + r_f)$. Thus the relationship between the excess return demanded and the market price of risk (λ) should become

$$(\mu + r_f) - r = \lambda\sigma \text{, or}$$

$$\mu = (r - r_f) + \lambda\sigma . \qquad\qquad (A2)$$

Thus, equation (A1) can be rewritten as

$$dS = (r - r_f + \lambda\sigma)S dt + \sigma S dW(t) \text{, under } Q . \qquad\qquad (A3)$$

Alternatively we may write

$$dS = (r - r_f)S dt + \sigma S d\tilde{W}(t) \text{, under } \tilde{Q} \qquad\qquad (A4)$$

where, $\tilde{W}(t) = W(t) + \int\limits_0^t \lambda(u)du$ and $\tilde{Q}$ is the risk neutral probability measure .

We recall that under the historical measure Q, the process $\tilde{W}(t)$ is not a standard Wiener process since $E[d\tilde{W}(t)] = \lambda dt \neq 0$ in general. However, Girsanov's theorem allows us to obtain the equivalent risk neutral measure $\tilde{Q}$ under which $\tilde{W}(t)$ does become a standard Wiener process. The measures Q and $\tilde{Q}$ are related via the Radon-Nikodym derivative details of which may be found in Kloeden and Platen (1992).

Using standard arbitrage arguments for pricing derivative securities (see for example, Hull (1997), chapter 13), the forward price at time t for a contract maturing at T ($>t$), is

$$F(t,T) = \tilde{E}_t(S_T). \tag{A5}$$

But from equation (A4), by Ito's lemma,

$$d[S(t)e^{-(r-r_f)t}] = \sigma S(t)e^{-(r-r_f)t}d\tilde{W}(t),$$

so that under $\tilde{Q}$, the quantity $S(t)e^{-(r-r_f)t}$ is a martingale from which it follows immediately that

$\tilde{E}_t(S_T) = S_t e^{(r-r_f)(T-t)}$, ie.

$$F(t,T) = S_t e^{(r-r_f)(T-t)}. \tag{A6}$$

If the maturity date of the contract is a constant period, x, ahead then (A6) may be written as,

$$F(t,t+x) = S_t e^{(r-r_f)x}. \tag{A7}$$

Let $\Phi(t,x) \equiv F(t,t+x)$ and $\phi(t,x) \equiv \ln F(t,t+x)$, then from (A3), (A4) and (A7) and by a trivial application of Ito's lemma we obtain the stochastic differential equation for Φ under Q and $\tilde{Q}$. Thus, under $\tilde{Q}$

$$d\Phi(t,x) = (r - r_f)\Phi(t,x)dt + \sigma\Phi(t,x)d\tilde{W}(t) \tag{A8}$$

whilst under Q,

$$d\Phi(t,x) = (r - r_f + \lambda\sigma)\Phi(t,x)dt + \sigma\Phi(t,x)dW(t) \tag{A9}$$

with, $\Phi(0,x) = S_0 e^{(r-r_f)x}$.

We now propose, under Q the historical measure, for the market price of risk, λ, the mean reverting stochastic process

$$d\lambda = \kappa(\lambda - \bar{\lambda})dt + \sigma_\lambda dW \tag{A10}$$

where $\bar{\lambda}$ is the long-term equilibrium market price risk, and κ defines the speed of mean reversion. It should be pointed out here that when discretised the stochastic differential equation (A10) would become a low order ARMA type process of the kind reported in Wolff (1987) and Cheung (1993)[3]. The parameters in equation (A10) may be estimated from the data using the Kalman filter.

Suppose we have n forward prices, $\Phi(t,x_1), \Phi(t,x_2),...\Phi(t,x_n)$, then we have a system of (n+2) stochastic differential equations. These are (under the historical measure Q),

$$dS = (r - r_f + \lambda\sigma)Sdt + \sigma SdW(t), \tag{A11a}$$

$$d\lambda = \kappa(\bar{\lambda} - \lambda)dt + \sigma_\lambda dW(t) \tag{A11b}$$

$$d\Phi(t, x_i) = (r - r_f + \lambda\sigma)\Phi(t, x_i)dt + \sigma\Phi(t, x_i)dW(t) \tag{A11c}$$

where, $S(0) = S_0$, $\lambda(0) = \lambda_0$, $\Phi(0, x_i) = S_0 e^{(r-r_f)x_i}$, $i = 1,2,...n$.

It should be noted that the information contained in equations (A11c) is also contained in the pricing relationships,

$$\Phi(t, x_i) = S_t e^{(r-r_f)x_i} . \tag{A12}$$

To estimate the parameters in the filtering framework, however, we choose to work with the equation (A11c).

From equation (A3), we can write the spot price at time $t + x$ as, using $s(t) = \ln S(t)$, as

$$s(t + x) = s(t) + (r - r_f - 0.5\sigma^2)x + \sigma\int_t^{t+x} \lambda(\tau)d\tau + \sigma\int_t^{t+x} dW(\tau) . \tag{A13}$$

From equation (A13) we can write the expected value $s(t + x)$ as,

$$E_t[s(t + x)] = s_t + \left(r - r_f - 0.5\sigma^2\right)x + \sigma E_t\left[\int_t^{t+x}\lambda(\tau)d\tau\right] . \tag{A14}$$

The calculations outlined in Appendix B (see in particular equation (B5)) allow us to then write,

$$E_t[s(t + x)] = s(t) + (r - r_f - 0.5\sigma^2)x + \sigma\left[\left(\lambda(t) - \bar{\lambda}\right)\left(\frac{1 - e^{-\kappa x}}{\kappa}\right) + \bar{\lambda}x\right] . \tag{A15}$$

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[3] As we have pointed out in Section 2, Wolff and Cheung report an ARMA type process for the risk premium itself. However, we see from equation (4d) that $\pi(t, x)$ and $\lambda(t)$ must follow the same type of

The above equation may also be expressed (via use of equation (A7)) as,

$$E_t\left[s(t + x)\right] = \phi(t, x) - 0.5\sigma^2 + \sigma\left[\left(\lambda(t) - \bar{\lambda}\right)\left(\frac{1 - e^{-\kappa x}}{\kappa}\right) + \bar{\lambda}x\right].$$

(A16)

Let $\pi(t, x) \equiv (E[s(t + x) - \phi(t, x))$ represent the risk premium (under the historical measure Q) for the x period ahead spot rate, then from equation (A16),

$$\pi(t, x) = -0.5\sigma^2 + \sigma\left[\left(\lambda(t) - \bar{\lambda}\right)\left(\frac{1 - e^{-\kappa x}}{\kappa}\right) + \bar{\lambda}x\right].$$

(A17)

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stochastic process.

Appendix B

Evaluation of Forward Expectations

The stochastic differential equation for λ (equation 3b, Section 3) can be expressed as,

$$d\left(e^{\kappa u}\lambda(u)\right) = \kappa\overline{\lambda}e^{\kappa u} + \sigma_{\lambda}e^{\kappa u}dW(u) \tag{B1}$$

Integrating (B1) from t to τ ($> t$),

$$e^{\kappa\tau}\lambda(\tau) - e^{\kappa t}\lambda(t) = \overline{\lambda}\left(e^{\kappa\tau} - e^{\kappa t}\right) + \sigma_{\lambda}\int_{t}^{\tau}e^{\kappa u}dW(u), \tag{B2}$$

from which

$$\lambda(\tau) = e^{-\kappa(\tau-t)}\lambda(t) + \overline{\lambda}\left(1 - e^{-\kappa(\tau-t)}\right) + \sigma_{\lambda}\int_{t}^{\tau}e^{-\kappa(\tau-u)}dW(u). \tag{B3}$$

Now integrating (B3) from t to $t+x$,

$$\int_{t}^{t+x}\lambda(\tau)d\tau = \lambda(t)\int_{t}^{t+x}e^{-\kappa(\tau-t)}d\tau +$$

$$\overline{\lambda}x - \overline{\lambda}\int_{t}^{t+x}e^{-\kappa(\tau-t)}d\tau +$$

$$\sigma_{\lambda}\int_{t}^{t+x}\int_{t}^{\tau}e^{-\kappa(\tau-u)}dW(u)d\tau.$$

The first two integrals in the foregoing equation are readily evaluated. However, in order to proceed the third integral needs to be expressed as a standard stochastic integral, having the $dW(u)$ term in the outer integration. This is achieved by an

application of Fubini's theorem, (see Kloeden and Platen (1992)) which essentially allows us to interchange the order of integration in the obvious way. Thus,

$$\int_{t}^{t+x} \lambda(\tau)d\tau = \left(\lambda(t) - \overline{\lambda}\right)\left(\frac{1-e^{-\kappa x}}{\kappa}\right) + \overline{\lambda}x + \sigma_{\lambda}\int_{t}^{t+x}\left(\int_{u}^{t+x}e^{-\kappa(\tau-u)}d\tau\right)dW(u)$$

$$= \left(\lambda(t) - \overline{\lambda}\right)\left(\frac{1-e^{-\kappa x}}{\kappa}\right) + \overline{\lambda}x + \frac{\sigma_{\lambda}}{\kappa}\int_{t}^{t+x}\left[1 - e^{-\kappa(t+x-u)}\right]dW(u). \tag{B4}$$

Thus, $E_t\left[\int_{t}^{t+x}\lambda(\tau)d\tau\right] = \left(\lambda(t) - \overline{\lambda}\right)\left(\frac{1-e^{-\kappa x}}{\kappa}\right) + \overline{\lambda}x. \tag{B5}$

Appendix C

State Space Framework and the Kalman Filter Updating Equations

For a particular maturity, the dynamics of the spot exchange rate, forward exchange rate and the market price of risk are described by the equations (4a) through (4c) in Section 3 of this paper. The key concept in understanding the state space formulation is the separation of the noise driving the system dynamics and the observational noise. What we observe in practice may not be the system variables directly and these may be masked by measurement noise. Besides, we are dealing with a partially observed system since the market price of risk is not observable.

The system dynamics given by the equations (4a) through (4c) in the paper (Section 3) are in continuous time and we usually measure in discrete intervals, so we need to discretise the equations for the purposes of implementation and estimation. A number of discretisation schemes for stochastic differential equations are discussed in Kloeden and Platen (1992) and we choose to work with the Euler-Maruyama scheme.

As it stands, the equations (4a) and (4c) suggest that the diffusion terms are dependent on the state variables themselves and are thus stochastic in nature. By a simple transformation of variables using the natural logarithm and application of Ito's lemma we can transform these to equations with constant diffusion terms. Using these transformations and after discretisation of the equations (4a) through (4c) (see Section 3) we obtain for the time interval between *k-1* and *k*:

$$s_{k+1} = s_k + (r - r_f - 0.5\sigma_S^2)\Delta t + \lambda_k \sigma_S \Delta t + \sigma_S \sqrt{\Delta t}\, \widetilde{\xi}_k \tag{C1}$$

$$\phi_k = \phi_{k-1} + (r - r_f - 0.5\sigma_S^2)\Delta t + \lambda_{k-1} \sigma_S \Delta t + \sigma_S \sqrt{\Delta t}\, \widetilde{\xi}_k \tag{C2}$$

$$\lambda_{k+1} = (1 - \kappa\Delta t)\lambda_k + \kappa\bar{\lambda}\Delta t + \sigma_\lambda \sqrt{\Delta t}\, \widetilde{\xi}_k \tag{C3}$$

where $\widetilde{\xi}_k \sim N(0,1)$

The equations (C1) – (C3) describe the dynamics of the partially observed system and in the state space framework it is generally referred to as the state transition equation. In a multivariate situation it is convenient to express these in matrix notation and following Harvey (1990) this turns out as follows:

$$a_k = T_k a_{k-1} + c_k + R_k \eta_k,$$ (C4)

where,

$$a_k = \begin{bmatrix} s_k \\ \phi_k \\ \lambda_k \end{bmatrix}, \ T_k = \begin{bmatrix} 1 & 0 & \sigma\Delta t \\ 0 & 1 & \sigma\Delta t \\ 0 & 0 & 1-\kappa\Delta t \end{bmatrix}, \ c_k = \begin{bmatrix} (r-r_f-0.5\sigma^2)\Delta t \\ (r-r_f-0.5\sigma^2)\Delta t \\ \kappa\bar{\lambda}\Delta t \end{bmatrix}, \ R_k = \begin{bmatrix} \sigma & 0 \\ \sigma & 0 \\ \sigma_{\lambda 1} & \sigma_{\lambda 2} \end{bmatrix}$$

and η_k is a (2×1) vector of noise sources that are serially uncorrelated, with expected values zeros and the covariance matrix,

$$Cov.(\eta_k) = Q_k = \begin{bmatrix} \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}.$$

The observations in our system are related to the state variables in an obvious way as,

$$y_k = Z_k a_k + \varepsilon_k$$ (C5)

where,

$$y_k = \begin{bmatrix} s_k \\ \phi_k \end{bmatrix}, \ Z_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ Cov.(\varepsilon_k) = H_k = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}.$$

As described before the variance in measurement of the observables are represented by h. Another assumption in this set up is that the noise sources in the state and the measurement equations are independent of each other. The state space system requires

specification of the initial state vector. As suggested in Harvey, Ruiz and Shepherd (1994) the first observations can be used to initialise it if non-stationarity is suspected.

Once the system is specified in state space form a recursive algorithm such as the Kalman filter can be applied to obtain the optimal estimate of the state vector at time k based upon all the information available at that time. As the system given by equation (C4) is conditionally Gaussian, by recursively calculating the first two moments of the conditional distribution, the Kalman filter gives the minimum mean square estimates of the state vector. Another advantage of the conditionally Gaussian case is that the likelihood function can be precisely calculated from the prediction error and its covariance. When this likelihood function is maximised with respect to the unknown parameters of the model their estimates and the corresponding standard errors are obtained. We can now write down the main updating equations of the Kalman filter for this system. An intuitive explanation of the operation of the filter can also be found in Bhar and Chiarella (1997).

If $\hat{a}_{k-1}$ is the optimal estimator of the state vector based upon the observation up to and including y_{k-1}, and P_{k-1} is the covariance matrix of the estimation error then the optimal estimator of the state vector at k is given by,

$$\hat{a}_{k|k-1} = T_k \hat{a}_{k-1} + c_k \tag{C6a}$$

and the covariance matrix of the estimation error is,

$$P_{k|k-1} = T_k P_{k-1} T_k' + R_k Q_k R_k'. \tag{C6b}$$

The equations (C6a) and (C6b) are the prediction equations. Once the observation at k becomes available these estimates can be updated as follows:

$$\hat{a}_k = \hat{a}_{k|k-1} + P_{k|k-1} Z_k' F_k^{-1} (y_k - Z_k \hat{a}_{k|k-1}), \tag{C6c}$$

$$P_k = P_{k|k-1} - P_{k|k-1} Z_k' F_k^{-1} Z_k P_{k\_k-1}, \tag{C6d}$$

where,

$$F_k = Z_k P_{k|k-1} Z_k' + H_k. \tag{C6e}$$

Given the starting values, a_0 and P_0, Kalman filter gives the optimal estimator of the state vector, as each new observation becomes available. The prediction error at each step and its covariance matrix can be used to construct the likelihood function, which (without the constant term) for T observations is given by,

$$\log L = -0.5 \sum_{k=1}^{T} \log |F_k| - 0.5 \sum_{k=1}^{T} v_k' F_k^{-1} v_k \tag{C7}$$

where, $v_k = y_k - Z_k \hat{a}_{k|k-1}$.

As we have seen, the filter algorithm provides the optimal estimates of the state vector, $\hat{a}_k$, based upon all the information up to time k. However, in our application we can also take into account of all the information up to T, once the maximum likelihood estimates of the parameters are obtained. This is known as fixed interval smoothing and we will be using these smoothed estimates of the market price of risk to compute the risk premium. The smoothing algorithm consists of a set of recursions starting at the final point and working backwards to the starting point. We summarise these equations below and the details can be found in Harvey (1990, pp. 149-155) as well as Jazwinski (1970, pp. 216-217):

$$a_{k|T} = a_k + P_k^* (a_{k+1|T} - T_{k+1} a_k), \tag{C8a}$$

$$P_{k|T} = P_k + P_k^* (P_{k+1|T} - P_{k+1|k}) P_k^{*'}, \tag{C8b}$$

$$P_k^* = P_k T_{k+1}' P_{k+1|k}^{-1}, \text{ for k = T-1, T-2, ..., 1.} \tag{C8c}$$

These recursions require final values of a_k and P_k for all k and initialisation as

$$a_{T|T} = a_T, P_{T|T} = P_T.$$

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