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**Scenario analysis with recursive utility: dynamic consumption plans for  
charitable endowments**

**Stephen Satchell<sup>a,#</sup>, Susan Thorp<sup>b</sup>**

<sup>a</sup>Faculty of Economics, University of Cambridge, Cambridge, UK

<sup>b</sup>School of Finance and Economics, University of Technology, Sydney, Australia

**Abstract**

We determine optimal consumption paths under a series of returns scenarios for charitable endowments with distinct tastes over investment risk and inter-temporal substitution. Charities typically prefer smooth consumption paths but are investment-risk tolerant. Using a recursive, Kreps-Porteus utility function, we model the optimal disbursement from an infinitely-lived charitable trust, then, allowing a general form for the returns density, we apply stochastic dominance relations to estimate income/substitution effects whereby a change in future returns influences the current consumption rate. The elasticity of intertemporal substitution rather than risk aversion is key: optimal consumption rises or falls as the elasticity diverges from one.

*JEL Classification:* G00; D81; D91

*Keywords:* Recursive utility; Stochastic dominance; Inter-temporal choice

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<sup>#</sup>Faculty of Economics, University of Cambridge, Cambridge, CB3 9DD, UK  
Tel: +44-(0) 1223 335229; Fax: +44(0) 1223 335299  
Email address: [stephen.satchell@econ.cam.ac.uk](mailto:stephen.satchell@econ.cam.ac.uk)

## 1. Introduction

The problem we wish to address is how to determine the optimal consumption rate for a charity<sup>1</sup>. Endowed institutions, foundations and charitable trusts in the UK include universities, schools, research institutions, and grant-making charities. The top 500 charitable trusts in the UK have assets in excess of £33 billion, and investment income of more than £4 billion per annum with a similar amount being raised by donation, and US endowments and foundations control more than \$1300 billion in assets (Brown, Garlappi and Tiu 2007). While there are a few studies of the US university endowment sector there is otherwise surprisingly little quantitative research published in this area<sup>2</sup>.

The more general problem of an entity making spending and investment plans over a finite or infinite horizon, subject to uncertainty, has generated a huge literature. Models usually comprise time-additive von Neumann-Morgenstern utility and uncertainty generated by log-normal diffusions, with explicit solutions for consumption and asset portfolios possible in a limited number of cases.<sup>3</sup> Merton's seminal model (Merton 1969) analyses an infinitely-lived entity with a constant relative risk aversion utility function. In the case where all asset returns are log-normally distributed and some regularity conditions on the rate of discounting of future utility are satisfied, the optimal rate of consumption is constant, and optimal wealth is log-normal and bounded below. All calculations are done continuously

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<sup>1</sup> We shall use the terms foundations, endowments and charities interchangeably.

<sup>2</sup> Discussion of optimal spending and investment plans for university endowments originates with Tobin (1974), Litvack, Malkiel and Quandt (1974), Nichols (1974), but also features in Merton (1990) and more recently in Dybvig (1999), Woglom (2003) and Merton (2003). Empirical studies of the structure and investment performance of endowments include Brown (1999), Lerner, Schoar and Wong (2005), Brown, Garlappi and Tiu (2007) and Dimmock (2007).

<sup>3</sup> This literature mainly originates with Merton (1969, 1971). Campbell and Viceira (2002) is a standard work in the area. In chapter 6, they consider the case where the investor has labour income which could proxy for future endowment income or donations to a charity. See also Korn and Korn (2001).

rather than discretely. Although the key features of the Merton solution, a constant drawdown and strictly positive wealth, are interesting, and the solution is relatively easy to compute, it only partially addresses the problem at hand here.

First, a continuous time framework is hardly an advantage in describing the decision-making of a charitable institution, whose trustees typically meet every six months. And since many charitable endowments are set up to provide funding perpetually, but work with particular clients or projects within quite short time-frames, we define the problem in terms of choosing annual spending rates over an infinite horizon. Secondly, joint log-normality seems excessively restrictive, given the asset classes invested in by endowments, which include hedge funds and private equity (see Brown 1999, Wellcome 2005, Lerner, Schoar and Wong 2005, Dimmock 2007 and Brown et al. 2007). Without making specific distributional assumptions, we derive the optimal spending plan for a charity earning risky investment returns, and go on to examine the way optimal drawdown responds when the density of investment returns is transformed. This is of particular importance to trustees because it allows them to carry out scenario analysis. By shifting probability mass from the lower to the upper tail of the returns density, and by working with mean-preserving spreads, we can examine analytically, and estimate numerically, the trade-off between income and substitution effects and the ensuing changes to disbursement rates. Consequently our analysis is more robust to the real-world peculiarities of financial data than existing models.

Thirdly, charitable trusts and endowments invest large amounts of wealth in risky investment portfolios with volatile returns, but ‘consume’ by making disbursements to beneficiaries who value smooth funding streams. Models of the

drawdown of charities and endowments which apply the usual time-additive, von Neumann-Morgenstern expected utility functions limit the scope of analysis by restricting tastes over inter-temporal substitution and aversion to risk. In particular, the class of constant relative risk aversion (CRRA) utility functions constrains relative risk aversion to be the inverse of the elasticity of inter-temporal substitution, so that agents who have low risk aversion must also be willing to transfer consumption through time. However for charitable trusts, risk aversion and aversion to intertemporal substitution appear conceptually and practically distinct: charities tolerate considerable uncertainty over returns while aiming for fairly smooth payments to beneficiaries over time.

Recursive or non-expected utility preferences as proposed by Kreps and Porteus (1978, 1979) allow a partial separation of tastes for risk and inter-temporal consumption. Whereas the von Neumann-Morgenstern agent is interested only in the conditional expectation of all future consumption (the timing of the resolution of uncertain outcomes does not matter), the Kreps-Porteus agent also cares how soon uncertainty over consumption will be resolved. If an entity is highly risk averse but relatively willing to redistribute consumption through time, then they prefer an early resolution of uncertainty, but if an entity is fairly tolerant of risk and, relatively speaking, dislikes transferring consumption through time, then later resolution is better. As Weil (1990) points out, this amounts to a trade-off between the safety and stability of utility, where safety is improved by early resolution of risk and stability by late resolution. Here we adapt Weil's version of Kreps-Porteus preferences to the dynamic consumption problem of charitable trusts, and newly explore the properties of the model under scenario analysis.

We confirm Weil's (1990) result that the optimal constant drawdown for a charity with Kreps-Porteus utility is set by the rule:

$$m = 1 - (\delta \varphi^{\frac{1-\rho}{1-\alpha}})^{\frac{1}{\rho}},$$

where  $m$  is the proportion of wealth spent each year,  $\delta$  is a parameter which is a component of time preference,<sup>4</sup>  $\alpha$  is relative risk aversion,  $1/\rho$  is the elasticity of intertemporal substitution and  $\varphi$  is the expected value of  $\tilde{Z}^{1-\alpha}$ , where  $\tilde{Z}$  is the gross return to the charity's portfolio. We also find the conditions for the convergence of the value function for this problem, a result which to our knowledge has not been derived previously.<sup>5</sup>

Given plausible parameter values and historical estimates of investment returns, optimal drawdown rates might lie between 1% and 3% of wealth per annum in real terms. In practice, some charities may be restricted by regulation to minimum disbursement quotas (rates of spending out of accumulated wealth). The Canada Revenue Agency (2007), for example, currently requires that 3.5% of average value of property owned by a charity but not used directly in activities or administration, be disbursed each year. At a current inflation rate of nearly 2%, this regulation enforces a real drawdown of at least 1.5%. Our analysis suggests that for some preference patterns, such regulations may be a binding constraint which reduces the welfare of

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<sup>4</sup> In a time additive utility model  $\delta$  would simplify to the rate of time preference, but time preference is generally endogenous in non-expected utility settings. See Backus, Routledge and Zin (2004) for a general discussion of recursive preferences.

<sup>5</sup> Under constant relative risk aversion (CRRA) preferences, the drawdown rule simplifies to

$m = 1 - (\delta \varphi)^{\frac{1}{\alpha}}$ . Early analysis of a related problem in growth is due to Phelps (1962) and we do not set out the full derivation of this special case here but a direct proof involving explicit derivation of the value function is available from the authors on request.

the charitable trust.<sup>6</sup> Further, while superficial intuition might predict that spending out of an endowment will be positively related to an optimistic investment outlook and negatively linked to pessimism, we show that this is true only for a sub-set of preferences and the reverse reaction can be optimal. Our estimation of these effects indicates that optimal consumption rates are remarkably sensitive to small changes in beliefs about future returns distributions. The direction of revisions to optimal consumption depends on whether the elasticity of intertemporal substitution is greater than or less than one, not on tastes for risk. We use stochastic dominance arguments to confirm, extend and illustrate analytical results sketched in Weil (1990) and Bhamra and Uppal (2006), which demonstrate the pivotal role of the elasticity of intertemporal substitution rather than risk aversion for consumption paths.

## **2. Literature**

Studies of university endowment behaviour concentrate on finding a drawdown rule that satisfies ‘intergenerational equity’ while preserving capital over the long horizon. (See, for example, Tobin 1974, Litvack, Makiel and Quandt 1974, and Nichols 1974.) Most are not interested in deriving optimal portfolio allocations for endowments.

Tobin’s (1974) main concern is to improve on arbitrary policies which limit consumption from an endowment to either annual cash income (dividends, interest and rent) or to annual cash income plus all capital gains. He sets out a more flexible, far-sighted drawdown rule that distinguishes between changes to the long-run rate of return on assets and temporary revaluation effects, and proposes consuming out of permanent income instead of exploiting temporary revaluation effects caused by changes in the discount rate. However Woglom (2003) shows that Tobin’s definition

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<sup>6</sup> We thank Mr Vincent Taubman of TD Asset Management for advice on this issue.

of intergenerational equity (fixed real consumption through time), implies a zero rate of inter-temporal substitution. For agents with CRRA utility functions this means infinite risk aversion, a hypothesis that is clearly contradicted by endowment investment patterns. Using a deterministic, continuous-time model, Woglom confirms that endowments should consume from recurrent capital gains, but he relaxes the intergenerational equity constraint to allow optimal real consumption to vary over time. Here we manage the fundamental tension between inter-generational equity and efficient wealth management by introducing Kreps-Porteus preferences, hence disentangling tastes for investment risk from tastes for inter-temporal consumption transfers.

University spending and investment was readdressed in later papers by Merton (1990, 2003) who considers optimal consumption and portfolio allocation at the whole university, rather than the endowment, level. When income streams (gifts, bequests etc.) and the costs of university activities covary with investment returns, he argues that university portfolio managers can hedge against future cost changes and adjust to non-tradable income sources by employing replicating strategies.

Dybvig (1995, 1999) views the inter-generational equity question differently, proposing that most endowments will want short-run spending certainty while maintaining long-run viability. In Dybvig's set-up the endowment maximises CRRA utility over current spending subject to non-negative wealth, and non-negative, non-decreasing spending, hence utility is still time-additive and separable conditioning on consumption never falling, and tastes for risk and inter-temporal substitution are linked. By moving a varying proportion of funds into the risk-free asset as the level of spending increases, the institution creates a riskless perpetuity matched to the current



minimum spending level while maintaining exposure to risky returns, and the resulting strategy is a dynamic generalisation of constant proportion portfolio insurance.<sup>7</sup>

While Dybvig's proposal is an interesting financial engineering plan for endowments, his model takes a more rigid approach to preferences than seems necessary for charities in general, and so we build our analysis in a framework which allows more flexible inter-temporal consumption and investment plans.

### **3. Defining the problem**

The standard problem for inter-temporal utility maximisation is to find the optimal functional form for consumption and the set of asset-allocations that maximise the expected multi-period utility of wealth functional through time. Indeed, in many cases, foundations state in their charter that they are endowed to provide some sort of support in perpetuity, but the needs of their clients, at least in a research context, may be very short-lived indeed. One large UK foundation, for example, offers funding to charity projects over a two to three year horizon subject to an option for either the charity or the beneficiary to withdraw at six months notice. Here we assume that charities are infinitely lived but make annual consumption plans, making calculations easier and rendering any finite-horizon bequest function irrelevant.

#### *3.1. Recursive utility*

Following Weil (1990) and Bhamra and Uppal (2006), we find the closed form solution for the optimal consumption path of an infinitely-lived entity which

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<sup>7</sup> Constant proportion portfolio insurance is the optimal investment strategy of an investor or endowment protecting a fixed minimum level of consumption, a result implicit in Merton (1971) and explicit in Kingston (1989).

maximises a discrete-time recursive utility function. The aggregator function for utility has two arguments, the first represents the value of current consumption and the second represents expected future utility over uncertain future consumption:

$$L_t = U[C_t, E_t L_{t+1}] \quad -$$

$$\equiv \frac{\{(1-\delta)C_t^{1-\rho} + \delta[1 + (1-\delta)(1-\alpha)E_t L_{t+1}]^{(1-\rho)/(1-\alpha)}\}^{(1-\alpha)/(1-\rho)} - 1}{(1-\delta)(1-\alpha)} \quad (1)$$

where  $\delta \in (0,1)$ ,  $\alpha > 0$ , and  $\rho > 0$  and where  $C_t$  is consumption in the form of payments to worthy causes and costs.<sup>8</sup>

The aggregator function represents time preference in (1), so that the derivative of  $U(.,.)$  with respect to expected future utility can be viewed as a subjective discount factor. If  $U(.,.)$  is convex with respect to expected future utility, the agent prefers early resolution of uncertainty, or safety over stability. If  $U(.,.)$  is concave with respect to its second argument, then the agent prefers a stable certainty equivalent path of future consumption. As Weil points out,  $\delta$  is the subjective discount factor in the case of certainty and in the linear constant relative risk aversion (CRRA) case where  $\alpha = \rho$ .

Consider an agent who faces two lotteries offering consumption over three periods which differ only by the timing of the resolution of the lottery.

[INSERT FIGURE 1 HERE]

If we assume that the agent discounts risk according to  $C^{1-\alpha}$ , that each branch of the consumption path is equally likely, and we fix  $\alpha = 2$ , the certainty equivalent

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<sup>8</sup> We would like to thank Professor James Sefton for persuading us of the applicability of this approach.

of expected future utility at time  $t_1$  is 1.67 for lottery A and either 5 or 1 for lottery B depending on the branch selected by the lottery. Similarly, the conditional variance of lottery A at time  $t_1$  is non-zero ( $=4$ ), but the conditional variance of lottery B is zero. If we stood at time  $t_0$  and computed the certainty equivalent of A and B, they would both be 8.21 and indistinguishable in terms of von Neumann-Morgenstern preferences. However, under Kreps-Porteus preferences, lottery A presents a more risky conditional path, but with less variation in certainty equivalent consumption over time, whereas B is a less risky conditional path with larger swings in certainty equivalent consumption.

It is straightforward to show that the convexity or concavity of  $U(.,.)$  depends on the relative sizes of  $\alpha$  and  $\rho$ , being convex when  $\alpha > \rho$  and concave when  $\alpha < \rho$ . Convexity implies more rapidly increasing patience and concavity more slowly increasing patience as expected future utility rises. Agents who are more risk tolerant and value smoothness ( $\alpha < \rho$ ) prefer late resolution (lottery A), and agents who dislike risk but tolerate larger swings in certainty equivalent utility ( $\alpha > \rho$ ) prefer early resolution (Lottery B).

Another way to view the parameters of the model is to recognise that the coefficient of relative risk aversion for timeless gambles is  $\alpha$  and the constant elasticity of inter-temporal substitution for deterministic consumption paths is  $1/\rho$ . If either parameter approaches unity, then preferences become logarithmic in that dimension, so that we get logarithmic risk preferences when  $\alpha \rightarrow 1$  and logarithmic inter-temporal substitution preferences when  $\rho \rightarrow 1$ . Under the special case where  $\alpha = \rho$ , the utility function represents the preferences of an individual with constant

relative risk-aversion (CRRA) and for whom the inverse of the risk aversion parameter is the elasticity of inter-temporal substitution.

### 3.2. Wealth

The amount of money available to the charity for investment,  $I_t$ , is given by,

$$I_t = W_t + D_t - C_t, \quad (2)$$

where  $W_t$  is the wealth at time  $t$  and  $D_t$  is the income from donations or bequests. If  $I_t$  is invested in  $n$  assets, buying  $N_{i,t}$  shares in the  $i^{\text{th}}$  asset at a price  $P_{i,t}$ , then,

$$I_t = \sum_{i=1}^n N_{i,t} P_{i,t}. \quad (3)$$

If one defines the return to the  $i^{\text{th}}$  asset as the random variable

$$\tilde{z}_{i,t} = \tilde{P}_{i,t+1} / P_{i,t}, \quad (4)$$

then it is possible to write an expression for the stochastic wealth of the charity at time  $t+1$ ,

$$W_{t+1} = I_t \sum_{i=1}^n w_{i,t} \tilde{z}_{i,t}. \quad (5)$$

where  $w_{i,t} = N_{i,t} P_{i,t} / I_t$ , represents the relative weights of the assets.

The charity consumes at the constant proportional rate  $C_t = mW_t \quad \forall t$ , by spending on administration and providing funding to beneficiaries. Setting aside

questions of portfolio allocation, and assuming that no donation income is received, the budget constraint is

$$W_{t+1} = (W_t - C_t)\tilde{Z}_t \quad (6)$$

where  $\tilde{Z}_t$  is the random growth in investments from  $t$  to  $t+1$ . If  $C_t = mW_t$ ,

$$W_{t+1} = (1-m)W_t\tilde{Z}_t. \quad (7)$$

This gives us a difference equation,

$$\begin{aligned} W_t &= (1-m)^t W_0 \prod_{i=0}^{t-1} \tilde{Z}_i \\ &= (1-m)^t W_0 \tilde{V}_{t-1} \end{aligned} \quad (8)$$

where  $\tilde{V}_{t-1} = \prod_{i=0}^{t-1} \tilde{Z}_i$ .  $\tilde{V}_{t-1}$  is the accumulated value of one unit of wealth invested at

time 0 and held until time  $t$ ; it is random and assumed non-negative.

**Proposition 1.** If  $\tilde{Z}_i$  is positive i.i.d and  $\tilde{Z}_i^{1-\alpha}$  is a well defined random variable such that  $E(\tilde{Z}_i^{1-\alpha}) = \varphi$  exists for  $0 < \alpha < \infty$ , it follows that  $E_0(V_{t-1}^{1-\alpha}) = \varphi^t$  for all integer  $t > 0$ .

**Proof.** Since  $\tilde{Z}_i$  is iid,  $\tilde{Z}_i^{1-\alpha}$  is iid and

$$\begin{aligned}
E_0(\tilde{V}_{t-1}^{1-\alpha}) &= E\left[\left(\prod_{i=0}^{t-1} \tilde{Z}_i\right)^{1-\alpha}\right] \\
&= E\left[\prod_{i=0}^{t-1} \tilde{Z}_i^{1-\alpha}\right] \\
&= \prod_{i=0}^{t-1} E(\tilde{Z}_i^{1-\alpha}) \\
&= \prod_{i=0}^{t-1} \varphi = \varphi^t.
\end{aligned} \tag{9}$$

### 3.3. Optimal consumption path

Optimising (1) subject to (8)<sup>9</sup>, and assuming that  $\tilde{Z}_i$  is positive i.i.d, the drawdown rate is:

$$m = 1 - \delta^{1/\rho} [\varphi^{1/(1-\alpha)}]^{(1-\rho)/\rho} \tag{10}$$

and the optimised value of (1) above is

$$L(W) = \frac{(\psi W)^{1-\alpha} - 1}{(1-\delta)(1-\alpha)}, \tag{11}$$

for  $\psi = [(1-\delta)m^{-\rho}]^{1/(1-\rho)}$ . In the special case of CRRA utility, when  $\alpha = \rho$ , the

optimal drawdown simplifies to  $m = 1 - (\delta\varphi)^{\frac{1}{\alpha}}$ . In the case of logarithmic risk preferences when  $\alpha \rightarrow 1$  the rule is  $m = 1 - \delta^{1/\rho}$  and in the case of logarithmic inter-temporal substitution preferences when  $\rho \rightarrow 1$ ,  $m = 1 - \delta$  for all values of  $\alpha$ . Thus in either logarithmic case, the optimal drawdown  $m = 1 - \delta$  or  $m = 1 - \delta^{1/\rho}$  is independent of our assumption about  $\sum_{t=1}^{\infty} E_0 \tilde{V}_{t-1}$ . This result is simple but not terribly useful, as it

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<sup>9</sup> Appendix A sets out an explicit derivation of this result originally reported in Weil (1990) but not explicitly derived there. Bhamra and Uppal (2006) derive the related result for a finite horizon.

depends primarily on the unknown discount factor,  $\delta$ . The optimal drawdown is independent of asset allocation although the amount of wealth drawn down is not. However, as  $\delta$  increases  $m$  decreases; this means the charity values future utility more and the value of current consumption decreases.

The dynamic spending plan in (10) is feasible (satisfying strictly positive wealth and consumption constraints) when the rate of drawdown is positive so that  $\delta\varphi^{(1-\rho)/(1-\alpha)} < 1$ , or for the CRRA case, when  $\delta\varphi < 1$ . Dynamic stability, such that the expected value of optimised utility is bounded at the infinite horizon, is also satisfied by  $\delta\varphi < 1$  in the CRRA case, but the conditions for feasibility and dynamic stability do not always coincide in the non-linear recursive utility case.<sup>10</sup>

**Proposition 2.** Under Newton's generalised binomial theorem (Graham et al. 1994), the aggregator function  $L_t = U[C_t, E_t L_{t+1}]$  in (1) is sum of a convergent infinite series if  $\frac{m}{1-m} < 1$ .

**Proof.** Newton's generalised binomial theorem states that for any  $r \in \mathbb{R}$ , if  $|a| < 1$

then  $\sum_{t \geq 0} \binom{r}{t} a^t$  converges to  $(1+a)^r$ . This results implies

$$(y+x)^r = \sum_{t \geq 0} \binom{r}{t} y^{r-t} x^t = \sum_{t \geq 0} \binom{r}{t} a^t = (1+a)^r,$$

for  $a = x/y$ . Using the aggregator function (1), and substituting the value function

(11),

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<sup>10</sup> Smith (1996) derives the feasibility and transversality condition for a related aggregator function in continuous time, but the model we work with here is different in significant ways and Smith's result does not transfer directly.

$$\begin{aligned}
L_t &= \frac{\left\{ (1-\delta)C_t^{1-\rho} + \delta \left[ E_t \left( [(1-\delta)m^{-\rho}]^{1/(1-\rho)} W_{t+1} \right)^{(1-\alpha)} \right]^{(1-\rho)/(1-\alpha)} \right\}^{(1-\alpha)/(1-\rho)} - 1}{(1-\delta)(1-\alpha)} \\
&= \frac{\left\{ (1-\delta)C_t^{1-\rho} + \delta[(1-\delta)m^{-\rho}] \left[ E_t \left( W_{t+1}^{(1-\alpha)} \right) \right]^{(1-\rho)/(1-\alpha)} \right\}^{(1-\alpha)/(1-\rho)} - 1}{(1-\delta)(1-\alpha)} \\
&= \frac{\left\{ (1-\delta)m^{(1-\rho)}W_t^{(1-\rho)} + \delta(1-\delta)m^{-\rho}(1-m)^{(1-\rho)}W_t^{(1-\rho)}\varphi^{(1-\rho)/(1-\alpha)} \right\}^{(1-\alpha)/(1-\rho)} - 1}{(1-\delta)(1-\alpha)}
\end{aligned} \tag{12}$$

and (12) will be the sum of the generalised binomial expansion above if,

$$a = \frac{x}{y} = \frac{(1-\delta)m^{(1-\rho)}W_t^{(1-\rho)}}{\delta(1-\delta)m^{-\rho}(1-m)^{(1-\rho)}W_t^{(1-\rho)}\varphi^{(1-\rho)/(1-\alpha)}} = \left| \frac{m}{(1-m)} \right| < 1 \tag{13}$$

and  $r = \frac{1-\alpha}{1-\rho}$  is a real number. The generalised binomial expansion in this case is:

$$\sum_{t \geq 0} \binom{r}{t} z^t = \sum_{t \geq 0} \frac{r(r-1)(r-2)\dots(r-t+1)}{t!} \left( \frac{m}{1-m} \right)^t. \tag{14}$$

Each period the summation in (14) grows by a factor  $\frac{r-t+1}{t} \left( \frac{m}{1-m} \right)$ , which in the

limit goes to

$$\lim_{t \rightarrow \infty} \left[ \frac{r-t+1}{t} \left( \frac{m}{1-m} \right) \right] = \lim_{t \rightarrow \infty} \left[ \frac{r}{t} - \frac{t}{t} + \frac{1}{t} \left( \frac{m}{1-m} \right) \right] = - \left( \frac{m}{1-m} \right), \tag{15}$$

confirming that for  $\left( \frac{m}{1-m} \right) < 1$  the sum converges to a finite value.



The convergence condition (13) applies where the discounted value of expected future utility (the second argument in the aggregator function) exceeds the value of current consumption (the first argument in the aggregator function), and amounts to the requirement that the optimal spending rate,  $m$ , be less than the saving rate,  $(1-m)$ . If the reverse is true and the value of current consumption exceeds discounted expected utility, then the rate of spending needs to exceed the rate of saving to achieve dynamic stability. For most of the empirical applications to follow, we need the spending rate to be less than the saving rate. This condition is equivalent to  $m < 1/2$ .

### 3.3 Donation income

We could also generalise the problem to the case where ‘income’ is included, by which we mean donations. Donations  $\tilde{D}_t$  are received during the period time  $t-1$  to  $t$  but invested at the end of the period. (Income received from donations during the period cannot be invested in this discrete time framework until the market opens in integer time.) This means that the wealth equation (2) needs to be adjusted to

$$W_t = W_{t-1}(1-m)\tilde{Z}_{t-1} + \tilde{D}_{t-1}, \quad (16)$$

then

$$W_t = (1-m)^t W_0 \tilde{V}_{t-1} + \sum_{j=0}^{t-1} \tilde{D}_j (\tilde{V}_{t-1} / \tilde{V}_j) (1-m)^{t-1-j}, \quad (17)$$

where  $\tilde{V}_0$  is assumed to equal one.

It is apparent that no closed-form solution exists for additive income for general distributions. However, using the fact that donation income must be positive, we can use instead a multiplicative addition to wealth, thus we get, defining the cumulative growth in income from donations as a proportion of wealth  $\tilde{Y}_{t-1} = \prod_{i=0}^{t-1} \tilde{y}_i$ ,

$$W_t = W_0(1-m)^t \tilde{V}_{t-1} \tilde{Y}_{t-1}. \quad (18)$$

In this case we can derive a solution exactly as for (10), but the new interpretation of  $\varphi$  is

$$\begin{aligned} E(Z^{1-\alpha}_t \tilde{y}_t^{1-\alpha}) &= E(Z^{1-\alpha}_t)E(\tilde{y}_t^{1-\alpha}) + \text{cov}(Z^{1-\alpha}_t \tilde{y}_t^{1-\alpha}) \\ &= \mu_{Z^{1-\alpha}} \mu_{\tilde{y}^{1-\alpha}} + \sigma_{Y^{1-\alpha} Z^{1-\alpha}}, \end{aligned} \quad (19)$$

where previously  $\varphi$  was  $\mu_{Z^{1-\alpha}}$ .

This now allows us to include donations in our general model where the necessary assumptions are the same as before. We might expect the covariance term to be positive (as in the case studied by Merton 1990) but there may be reasons why it could be negative.<sup>11</sup> An increase in financial market returns may be co-existent with a fall in donations as the population shifts from altruism to greed.

### 3.4 Asset allocation

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<sup>11</sup> Dimmock (2007) offers some analysis of non-investment income to US university endowments. He reports low negative correlation between equity indices and non-endowment income which includes private donations.

Whilst in principle we could solve numerically for optimal asset allocation, we may not wish to do so.<sup>12</sup> This is for two reasons: first, the analysis is complex and dependent on distributional assumptions, or, if based on sample data, dependent on making assumptions that the future will be similar to the past; and secondly, asset allocation is in practice determined by decentralised committees via consultation<sup>13</sup>. Portfolio optimisation tools are theoretically important but their practical application is restricted by significant parameter uncertainty and the complex management structures of institutions.

For any given asset allocation, however determined, we can calculate the impact on the optimal drawdown of varying assumptions about the distribution of future returns, of changes in risk attitudes and changes in portfolio weights. We proceed to this scenario analysis in section 4 below.

### *3.5. Empirical illustration*

To illustrate the explicit solution in (10), we create a representative portfolio for a charitable trust, simulate returns and compute optimal drawdown rates for some feasible parameter ranges. Here we design the portfolio of our artificial entity to approximate the publicly available asset allocation of an independent UK biomedical research-funding charity, the Wellcome Trust (Wellcome 2005).

The Wellcome Trust Annual Report for 2005 states the principal investment objective as ‘total return in inflation-adjusted terms over the long term in order to

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<sup>12</sup> Giovannini and Jorion (1993) test the asset pricing implications of the model for the general non-i.i.d. case. Bhamra and Uppal (2006) set out the implicit portfolio optimality condition, and explicit optimal portfolio weights for simple examples of constant and stochastic investment opportunity sets.

<sup>13</sup> See Brown et al. (2007) for a description of university endowment structure and their decentralised process of investment management. While university endowment boards or investment committees make high level investment policy, day to day decisions are often delegated to groups of fund managers or to sub-committees.

provide for real increases in annual expenditure while preserving at least the Trust's capital base in real terms'. Consistent with the aim of maximising total returns, more than 90% of capital is held in public and private equity, hedge funds and property, with a small proportion in gilts and cash. Our returns data are constructed using an asset allocation similar to the Trust's allocation set out in their 2005 annual report (Figure 2). If we aimed to analyse the Wellcome Trust as an entity we would want an historical return series that reflects changes to investment policy as well as changes to underlying returns over time, but it serves our purpose simply to fix portfolio weights close to the 2005 report levels and pass these back through the historical asset class data.

[INSERT FIGURE 2 HERE]

We calculate monthly real portfolio returns over the period January 1990 to June 2006, (198 observations) deriving individual asset class returns from standard indexes, and deflate using consumer prices and earnings data.<sup>14</sup> It is reasonable to expect that wages are the majority of costs for most beneficiary projects, and deflation using consumer prices alone will overstate the real spending power of the charity, so we treat inflation as 50% consumer-price-driven and 50% purely due to wage increases.<sup>15</sup>

The mean (log) real return to this portfolio is 4.75% annualised with volatility of 13%. Summary statistics in Table 1 show that the data are significantly non-normal: negatively skewed and leptokurtic. However the autocorrelation structure of the de-meaned returns and squared de-meaned returns supports an assumption that

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<sup>14</sup> See Appendix B for data sources and calculations.

<sup>15</sup> We use an arbitrary rule for deflation here, but most research into endowment spending (e.g., Tobin 1974 and Woglom 2003) recognises the importance of an institution-specific deflation procedure, while Merton (1990) goes further and suggests hedging strategies for future cost changes.

real portfolio returns are i.i.d. Ljung-Box Q statistics, not reported here, are insignificant to at least 50 lags for the de-meaned returns. The squared residuals have one significant autocorrelation at lag ten.

[INSERT TABLE 1 HERE]

Equation (10) above is the optimal rate of drawdown for an infinitely-lived charity under a fixed asset allocation, given time preference parameter  $\delta$ , consumption smoothing parameter  $\rho$ , and relative risk aversion  $\alpha$ . Another key determinant is the mean of the risk-aversion-scaled portfolio return,  $E(\tilde{Z}_t^{1-\alpha}) = \varphi$ . To estimate  $\varphi$ , we bootstrap the monthly portfolio returns using 120,000 random draws from our historical sample, and sum to get 10,000 annual real (gross) returns. We then use these to compute the sample mean

$$\hat{\varphi} = \frac{1}{10,000} \sum_{i=1}^{10,000} \tilde{Z}_i^{1-\alpha}, \quad (20)$$

for  $\delta = 0.97$  and  $\alpha > 0$ .

The estimated optimal drawdown rate  $\hat{m}$  is shown in Figure 3 for values of the inter-temporal substitution parameter  $\rho$  between 0.2 and 5, and with risk aversion  $\alpha = 2.6$ , an estimated value we infer from the portfolio weights of the Wellcome Trust<sup>16</sup>. The fine grey curves give an approximate 95% error range for the estimate of

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<sup>16</sup> The condition for portfolio optimality for this model gives a vector of moment conditions in the scaled portfolio return and returns to individual assets given a constant rate of consumption  $E\left[m^{(1-\alpha)\rho/(1-\rho)} \tilde{Z}_{t+1}^{-\alpha} (z_{it} - R)\right] = E\left[\tilde{Z}_{t+1}^{-\alpha} (z_{it} - R)\right] = 0$ , where  $R$  is the return to the risk-free asset (see Bhamra and Uppal 2006 equation (17) under i.i.d. returns). We use this system of moment conditions and the portfolio returns data described above to estimate  $\hat{\alpha} = 2.6$  by Generalised Method of Moments assuming that the real risk-free rate is zero. Estimation results are available from the authors on request.

$m$ . Since under feasibility  $0 < \left( \frac{\delta \hat{\varphi}^{(1-\rho)}}{\delta \hat{\varphi}^{(1-\alpha)}} \right)^{1/\rho} < 1$ , we can fit a beta distribution to 1000

bootstrapped estimates of  $\hat{v}_\alpha \equiv \left( \frac{\delta \hat{\varphi}^{(1-\rho)}}{\delta \hat{\varphi}^{(1-\alpha)}} \right)^{1/\rho}$  by maximum likelihood, after filtering out

values that do not meet the feasibility and boundary conditions. From the estimated

beta parameters, we can back out  $1 - \hat{v}_{\alpha,0.025} = 1 - F^{-1}(0.025)$  and

$1 - \hat{v}_{\alpha,0.975} = 1 - F^{-1}(0.975)$  as a guide to the accuracy of  $\hat{m}$ . Consistent with the

solution for logarithmic inter-temporal substitution preferences, the optimal

consumption rate is 3% per annum when  $\rho = 1$ . As tolerance for consumption transfer

through time decreases and rho increases, the disbursements falls from around 4.7%

when  $\rho = 0.2$ , reaching 2.8% when  $\rho = 5$ .<sup>17</sup>

[INSERT FIGURE 3 HERE]

The error range around  $m$  widens rapidly as the elasticity of inter-temporal substitution (EIS) diverges from one in either direction. Figure 4 graphs the estimated beta distributions of the optimal drawdown at three indicative values of the inter-temporal substitution parameter. When the EIS is relatively high at 1.33 ( $\rho = 0.75$ ), the error distribution is more right-skewed and disbursed than when the EIS falls to 0.8 ( $\rho = 1.25$ ) where the distribution is more tightly packed around the 3% logarithmic drawdown. However as EIS moves away from one, falling to 0.2 ( $\rho = 5$ ), the probability distribution becomes more right-skewed again, and uncertainty over

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<sup>17</sup> We choose parameters that are roughly consistent with the empirical estimates of tastes for risk and inter-temporal substitution made by Epstein and Zin (1991) who find that the elasticity of inter-temporal substitution (EIS) is small and always less than one (implying that  $\rho > 1$ ) and that risk preferences are close to one, conditions which together imply a preference for the late resolution of uncertainty. Earlier studies find a low value for the EIS, but for a contrasting view see Gruber (2006).

the optimal spending rate increases. This pattern indicates the increasing importance of the stochastic risk-scaled returns parameter  $\varphi = E(Z^{1-\alpha})$  to optimal consumption paths as the EIS diverges from one, since at  $\rho = 1$ , consumption depends only on the discount parameter  $\delta$ , which is assumed to be known with certainty.

Hence we conclude that a moderately risk averse charity will spend between 5% and 2% of wealth each year, but that the uncertainty surrounding that optimal solution is very large and increasing as the EIS diverges from one.

[INSERT FIGURE 4 HERE]

#### 4. Scenario analysis

Trustees need a way of assessing whether their chosen drawdown rate is robust to changes in beliefs about future returns, an exercise usually called scenario analysis. A natural approach is to set past history as the benchmark and build optimistic or pessimistic outlooks relative to recent experience. In this section we set out a simple procedure to conduct scenario analysis that is not highly dependent on complicated assumptions about distributions of returns.

The influence of the returns distribution on optimal spending rates for a charity is via the expectation of risk-scaled portfolio returns,  $\varphi = E(\tilde{Z}^{1-\alpha})$ . To gauge the optimal spending response to optimistic and pessimistic investment scenarios, we consider changes in the expected risk-scaled portfolio return  $\varphi$ , where we keep tastes for risk,  $\alpha$ , and inter-temporal substitution,  $\rho$ , fixed but vary distributional

parameters. The change in optimal drawdown as  $\varphi$  varies depends on the relative size of  $\alpha$  and  $\rho$ ,

$$\frac{\partial m}{\partial \varphi} = -\delta^{\frac{1}{\rho}} \frac{(1-\rho)}{(1-\alpha)\rho} \varphi^{\frac{(1-\rho)}{(1-\alpha)\rho}-1}. \quad (21)$$

Since  $\delta, \rho$  and  $\varphi$  are positive, the response of the optimal drawdown to an increase in  $\varphi$  will be positive when  $\rho > 1$  and  $\alpha < 1$ , and when  $\rho < 1$  and  $\alpha > 1$ . If both  $\alpha$  and  $\rho$  are greater than one or less than one, then the response of the optimal drawdown to an increase in  $\varphi$  will be negative. However we need to account for the influence on  $\varphi$  itself of changes in relative risk aversion. It turns out that this can be done using the properties of stochastic dominance.

#### 4.1 First order stochastic dominance

**Proposition 3.** If  $Z_i^\Delta$  first order stochastically dominates (FSD)  $Z_i$ , then  $\varphi$  is increased if  $0 < \alpha < 1$  and decreased if  $\alpha > 1$ .

**Proof.** Note that if  $Z_i^\Delta$  FSD  $Z_i$  then  $F_\Delta(Z) \leq F(Z)$  where  $F_\Delta(Z)$  and  $F(Z)$  are the respective distribution functions. Denote expectations with respect to them by  $E_\Delta(\cdot)$  and  $E(\cdot)$ . First order stochastic dominance implies that  $E_\Delta[G(Z)] \geq E[G(Z)]$  for  $G(\cdot)$ , any increasing function.<sup>18</sup> If  $G(Z) = Z^{1-\alpha}$ ,  $0 < \alpha < 1$ , then  $G(Z)$  is increasing

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<sup>18</sup> See Huang and Litzenberger (1988) for proof.



in  $Z$ , hence under  $F_\Delta(Z)$ ,  $\varphi$  is increased. If  $\alpha > 1$  we have  $G(\cdot)$  a decreasing function in  $Z$  and under  $F_\Delta(Z)$  the reverse happens,  $\varphi$  decreases.<sup>19</sup>

We now set out a method for reshaping the returns distribution to reflect optimistic and pessimistic scenarios for investment. For optimistic outlooks, our aim is to make extremely poor payoffs unlikely relative to the recent past by shifting tail mass from the left to the right tail of the distribution. For an arbitrary positive continuous density,  $pdf(x)$ ; we consider two points  $x_l$  and  $x_u$  and the probabilities

$$P_l = \int_0^{x_l} pdf(x)dx, \quad P_u = \int_{x_u}^{\infty} pdf(x)dx \text{ and } P_{md} = \int_{x_l}^{x_u} pdf(x)dx; \text{ clearly } P_u + P_l + P_{md} = 1.$$

We can construct a new density by the following shift,

$$\begin{aligned} P'_u &= P_u + \Delta \\ P'_l &= P_l - \Delta \\ P'_{md} &= P_{md} \end{aligned} \tag{22}$$

where  $0 < \Delta < \min(P_u, P_l)$ , and

$$\begin{aligned} pdf'(x) &= \frac{P'_u pdf(x)}{P_u} & x_u < x < \infty \\ &= pdf(x) & x_l \leq x \leq x_u \\ &= \frac{P'_l pdf(x)}{P_l} & 0 \leq x \leq x_l \end{aligned} \tag{23}$$

It is easy to check that  $pdf'(x)$  is still a well-defined density although no longer continuous at  $x = x_l$  or  $x = x_u$ . Note that since we assumed a continuous density

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<sup>19</sup> Bhamra and Uppal (2006) derive expressions for the income and substitution effects of changes in the risk-free rate on consumption and portfolio choice in the finite horizon setting.

with zero probability mass at any point, the discontinuities induced by our transformation will not affect the existence of the integrals. Furthermore the above transformation can be called optimistic in that it transfers probability from the lower tail to the upper tail of the density while a pessimistic transformation does the reverse.

**Corollary.** If  $G(x)$  is a positive increasing function then

$$\int_0^{\infty} G(x) pdf'(x) dx > (<) \int_0^{\infty} G(x) pdf(x) dx \quad (24)$$

for  $pdf'(x)$  the result of an optimistic (pessimistic) transformation. An opposite result applies to positive decreasing functions. If we now apply the result for  $\varphi = E(\tilde{Z}_i^{1-\alpha})$ , we see that  $\tilde{Z}_i^{1-\alpha}$  is positive increasing for  $0 < \alpha < 1$  hence  $\varphi_{\Delta} > \varphi$ , and positive decreasing for  $\alpha > 1$  so that  $\varphi_{\Delta} < \varphi$ , where  $\Delta$  is a positive transformation.

We consider now the change in  $m$  under a FSD shift for each of four combinations of  $\alpha$  and  $\rho$ .

$$\text{For } 0 < \rho < 1, 0 < \alpha < 1, \begin{cases} \frac{\partial G}{\partial Z} > 0 \Rightarrow \varphi \text{ increases under } F_{\Delta}(Z) \\ \frac{\partial m}{\partial \varphi} < 0 \Rightarrow m \text{ decreases under } F_{\Delta}(Z) \end{cases}$$

$$\text{For } 0 < \rho < 1, \alpha > 1, \begin{cases} \frac{\partial G}{\partial Z} < 0 \Rightarrow \varphi \text{ decreases under } F_{\Delta}(Z) \\ \frac{\partial m}{\partial \varphi} > 0 \Rightarrow m \text{ decreases under } F_{\Delta}(Z) \end{cases}$$

$$\text{For } \rho > 1, 0 < \alpha < 1, \left| \begin{array}{l} \frac{\partial G}{\partial Z} > 0 \Rightarrow \varphi \text{ increases under } F_{\Delta}(Z) \\ \frac{\partial m}{\partial \varphi} > 0 \Rightarrow m \text{ increases under } F_{\Delta}(Z) \end{array} \right.$$

$$\text{For } \rho > 1, \alpha > 1, \left| \begin{array}{l} \frac{\partial G}{\partial Z} < 0 \Rightarrow \varphi \text{ decreases under } F_{\Delta}(Z) \\ \frac{\partial m}{\partial \varphi} < 0 \Rightarrow m \text{ increases under } F_{\Delta}(Z) \end{array} \right.$$

So regardless of the size of the relative risk aversion parameter, transformations of the returns distribution that are described by first order stochastic dominance result in a decrease in the optimal rate of drawdown whenever  $0 < \rho < 1$  and an increase in the optimal rate of drawdown when  $\rho > 1$ . Weil (1990) showed this result for log-normally distributed portfolio returns, but here we have generalised to the case of any well-behaved continuous returns distribution.

The former case  $0 < \rho < 1$  describes charities with high elasticities of intertemporal substitution, and the latter  $\rho > 1$ , agents with low elasticities of intertemporal substitution. For optimistic returns scenarios, and where  $0 < \rho < 1$ , the substitution effect dominates the income effect and the charity is willing to transfer a higher rate of consumption through time into the future and spending rates fall, whereas for  $\rho > 1$ , charities with low elasticities of intertemporal substitution find the prospect of good times now compelling, the income effect dominates the substitution effect, and they increase current spending rates. These effects are independent of tastes for risk.

#### 4.2. Empirical illustration

Figures 5 and 6 show the impact on optimal drawdown of a range of transformations of the distribution of  $\tilde{Z}_i$ , the portfolio return. The two panels in Figure 5 show graphs for optimal spending rate when  $\alpha = 0.5$  and  $2.6$ , and  $\rho$  ranges from  $0.4$  to  $1$ . A positive rescaling of the returns distribution of size  $0.02$  shifts  $2\%$  of the total probability mass from the left to the right tail of the distribution and matches an optimistic outlook for investment returns. In the same way, a negative rescaling of  $0.02$  shifts the same probability mass from the right to the left tail, when the investment outlook is bleak. Whenever  $\rho = 1$  the optimal spending rate is  $3\%$  p.a., but as  $\rho$  shrinks, EIS increases and spending rises with optimistic expectations and falls with pessimistic expectations.<sup>20</sup> For  $\rho = 0.8$  and  $\alpha = 0.5$ , the optimal spending rate based on historical returns is  $2.6\%$ . As optimism increases, and we shift probability mass towards the right tail spending declines so that when right tail returns are  $4$  percentage points more probable, spending is down to  $1.4\%$  and to  $0.05\%$  when the right tail probability is  $10$  percentage points higher. Similarly spending rises to  $3.8\%$  when the left tail returns are  $4$  percentage points more probable and reaches  $5.8\%$  at  $10$  percentage points. For  $\rho = 0.8$  and  $\alpha = 2.6$ , the pattern is very similar: beginning at  $3.4\%$ , as optimism increases the right tail by  $4$  percentage points, spending falls to  $1.9\%$  and to  $0.04\%$  for a  $10$  percentage point shift. When the left tail returns are  $4$  percentage points more probable, spending rises to  $4.3\%$  and reaches  $6.3\%$  at  $10$  percentage points.

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<sup>20</sup> The slightly jagged shape to the surface is caused by the bootstrap process: a different set of random draws is made at each combination of  $\rho$  and  $\Delta$ . Edges of the surface are not smooth because the feasibility and boundary conditions are not met for some extreme values of  $\rho$  and  $\alpha$ .

We see that a very flexible foundation facing better prospects does best by decreasing current spending rates in favour of future consumption, with substitution effects dominating income effects.

[INSERT FIGURE 5 HERE]

For charities with low elasticities of inter-temporal substitution, where  $\rho > 1$ , optimistic transformations of the portfolio returns distribution increase the optimal drawdown, as the charity enjoys higher income in the current period rather than favouring future consumption. Figure 6 graphs changing spending rates as optimism increases and EIS decreases. When  $\rho = 2$  (EIS = 0.5) and  $\alpha = 2.6$ , optimal spending at the historical average return is 2.4% p.a. Reducing the probability of left tail returns by 4 percentage points more than doubles optimal spending to 5.2% p.a. The same shift in the direction of pessimism reduces spending to 0.5% p.a. As inter-temporal substitution becomes even less attractive at say  $\rho = 4$ , (EIS = 0.25), a 4 percentage point positive shift raises spending from 2.1% to 6% and a 2 percentage point negative shift lowers spending to 0.4% p.a. When risk aversion is low, the same pattern of changes applies at higher overall consumption rates.

[INSERT FIGURE 6 HERE]

#### *4.3. Second order stochastic dominance*

Our first discussion considered changes in the mean of the returns distribution. We now consider changes in risk while allowing the mean to stay constant.

**Proposition 4.** If  $Z_i^\omega$  second order stochastic dominates (SSD)  $Z_i$  then  $\varphi$  is increased if  $0 < \alpha < 1$  and decreased if  $\alpha > 1$ .

**Proof.** Note that if  $Z_i^\omega$  SSD  $Z_i$  then  $\int_0^{\bar{Z}} F_\omega(s)ds \leq \int_0^{\bar{Z}} F(s)ds$  for all  $\bar{Z} \in [0, \infty]$  where  $F_\omega(Z)$  and  $F(Z)$  are the respective distribution functions. Denote expectations with respect to them by  $E_\omega(\cdot)$  and  $E(\cdot)$ . Second order stochastic dominance implies that  $E_\omega[G(Z)] \geq E[G(Z)]$  for  $G(\cdot)$ , any increasing, concave function.<sup>21</sup> If  $G(Z) = Z^{1-\alpha}$ ,  $0 < \alpha < 1$ , then  $G(Z)$  is increasing and concave in  $Z$ , hence under  $F_\omega(Z)$ ,  $\varphi$  is increased. If  $\alpha > 1$  we have  $G(\cdot)$  a decreasing and convex function in  $Z$  and under  $F_\omega(Z)$  the reverse happens,  $\varphi$  decreases.

Here we consider a mean-preserving spread of the distribution as a special case of SSD. For an arbitrary positive continuous density,  $pdf(x)$  where  $x_i = \mu_x + \varepsilon_i$ ,  $\varepsilon_i \sim iid(0, \sigma_\varepsilon^2)$  where  $\sigma_\varepsilon^2$ , we can construct a mean-preserving spread by the following transformation of  $x_i$ ,

$$x'_i = \mu_x + (1 + \omega)\varepsilon_i, \quad 0 < \omega < \infty \quad (25)$$

We can see that the mean of both distributions is

$$E(x_i) = E(x'_i) = \mu_x, \quad (26)$$

and that for  $0 < \omega < \infty$ , the variance of the transformed variable  $x'_i$  is greater than the variance of  $x_i$ ,

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<sup>21</sup> For proof see Huang and Litzenberger (1988).

$$\text{var}(x_i') = (1 + \omega)^2 \sigma_\varepsilon^2 > \text{var}(x_i) = \sigma_\varepsilon^2, \quad (27)$$

which are sufficient conditions for second order stochastic dominance of  $pdf(x)$  over  $pdf'(x)$ .

We can also shrink the variance of  $pdf(x)$  by choosing an optimistic transformation such that  $-1 < \omega < 0$ . A pessimistic transformation can be defined as an increase in risk when  $0 < \omega < \infty$ .

**Corollary.** If  $G(x)$  is a positive increasing, concave function then

$$\int_0^\infty G(x) pdf'(x) dx > (<) \int_0^\infty G(x) pdf(x) dx \quad (28)$$

for  $pdf'(x)$  the result of an optimistic (pessimistic) transformation. An opposite result applies to positive decreasing, convex functions. If we again apply the result for  $\varphi = E(\tilde{Z}_i^{1-\alpha})$ , we see that  $\tilde{Z}_i^{1-\alpha}$  is positive increasing and concave for  $0 < \alpha < 1$  hence  $\varphi_\omega > \varphi$ , and positive decreasing and convex for  $\alpha > 1$  so that  $\varphi_\omega < \varphi$ , where  $\omega$  is an optimistic transformation.

We consider now the change in  $m$  for each of four combinations of  $\alpha$  and  $\rho$ .

$$\text{For } 0 < \rho < 1, 0 < \alpha < 1, \begin{cases} \frac{\partial G}{\partial Z} > 0, \frac{\partial^2 G}{\partial Z^2} < 0 \Rightarrow \varphi \text{ increases under } F_\omega(Z) \\ \frac{\partial m}{\partial \varphi} < 0 \Rightarrow m \text{ decreases under } F_\omega(Z) \end{cases}$$

$$\text{For } 0 < \rho < 1, \alpha > 1, \left| \begin{array}{l} \frac{\partial G}{\partial Z} < 0, \frac{\partial^2 G}{\partial Z^2} > 0 \Rightarrow \varphi \text{ decreases under } F_\omega(Z) \\ \frac{\partial m}{\partial \varphi} > 0 \Rightarrow m \text{ decreases under } F_\omega(Z) \end{array} \right.$$

$$\text{For } \rho > 1, 0 < \alpha < 1, \left| \begin{array}{l} \frac{\partial G}{\partial Z} > 0, \frac{\partial^2 G}{\partial Z^2} < 0 \Rightarrow \varphi \text{ increases under } F_\omega(Z) \\ \frac{\partial m}{\partial \varphi} > 0 \Rightarrow m \text{ increases under } F_\omega(Z) \end{array} \right.$$

$$\text{For } \rho > 1, \alpha > 1, \left| \begin{array}{l} \frac{\partial G}{\partial Z} < 0, \frac{\partial^2 G}{\partial Z^2} > 0 \Rightarrow \varphi \text{ decreases under } F_\omega(Z) \\ \frac{\partial m}{\partial \varphi} < 0 \Rightarrow m \text{ increases under } F_\omega(Z) \end{array} \right.$$

If our transformation of  $\tilde{Z}_i$  shrinks the variance, then our drawdown ( $m$ ) decreases if  $0 < \rho < 1$ . If  $\rho > 1$ , SSD implies the opposite effect where  $m$  increases as risk shrinks and decreases as risk rises (for a constant expected return). This result confirms the reasoning in Weil (1990) that responses to mean-preserving spreads of the returns distribution depend only on the value of  $\rho$ .

#### 4.4. Empirical illustration

In Figures 7 and 8 we graph the optimal drawdown when the variance, but not the mean, of the distribution of  $\tilde{Z}_i$  is increased or decreased. In Figure 7 we optimistically shrink the standard deviation from its historical value to almost zero (rescaling to -1), or pessimistically raise it to twice the historical size (rescaling to 1), while setting  $\alpha = 0.5, 2.6$  and allowing  $\rho$  to range from 0.4 to 1. For  $\rho = 0.8$  and  $\alpha = 0.5$ , the optimal spending rate based on historical returns is around 2.7% and shrinking the volatility by 50% causes a small decline in drawdown towards 2.6%,



while increasing risk by 50% increases drawdown by about the same amount. The historical benchmark spending level for  $\rho = 0.4$  is around 1.2% p.a. and reducing volatility by 50% lowers this towards 0.8%, while a 50% increase moves spending towards 1.9%. When risk aversion is higher at 2.6 and  $\rho = 0.8$ , the benchmark spending rate is 3.1% p.a., and reducing(increasing) risk by 50% decreases (increases) drawdown to 2.7% (3.8%). These changes are small compared with the FSD scenarios, but support the analytical prediction that optimistic influences on the returns distribution decrease spending rates when the EIS is high.

[INSERT FIGURE 7 HERE]

For charities with low elasticities of inter-temporal substitution, when  $\rho > 1$ , increases in risk lower optimal spending rates with the effect becoming more dramatic as EIS shrinks. Figure 8 graphs changing spending rates as optimism over volatility increases and EIS decreases. When and EIS = 0.5 and  $\alpha = 2.6$ , optimal spending at the historical average return is 2.8% p.a. increasing to 3.5% as volatility is halved, and falling to 1.3% as volatility rises by 50%. When  $\rho = 4$ , (EIS =0.25), the same experiment sees spending rise from 2.7% to 3.8% for a halving of volatility and fall to 0.5% for a 50% increase in volatility. When the EIS is low but risk tolerance is high ( $\rho = 4$  and  $\alpha = 0.5$ ) benchmark spending is 3.9% p.a. If volatility is halved, spending rises to 4.1%, and falls to 3.5% for a 50% increase.

[INSERT FIGURE 7 HERE]

We conclude that lower current spending as a reaction to improved prospects is not necessarily irrational or irresponsible. On the contrary, such episodes could be evidence for high level of willingness to transfer disbursements into the future.

However if, as we expect, most charities favour smoother consumption, then unwillingness to shift consumption towards the future dominates and optimal spending rises and falls as the outlook brightens or blackens. Somewhat surprisingly, this is true whatever the charity's attitude to risk. Preferences for early or late resolution of uncertainty do not determine the direction of response. While the benchmark level of spending,  $m$ , will be sensitive to both risk aversion and the inter-temporal elasticity, whether spending decreases or increases from that level in response to scenario changes depends only on whether the elasticity of inter-temporal substitution is less than or greater than one.

## **5. Conclusion**

Charities whose trust deeds specify a very long (infinite) horizon and who generate independent and identically distributed returns from investment portfolios can operate optimally using simple, constant drawdown policies. The ideal rate of spending for a charitable trust is a function of preferences for safety and smoothness in expected consumption, tastes which can be parameterised in a Kreps-Porteus utility framework.

Our contribution is to investigate the responsiveness of these drawdown policies to changes in the shape of very general returns distributions and to tease out the empirical implications of such changes. We identify the effects of optimistic and pessimistic transformations of the returns distribution using the properties of stochastic dominance. Without assuming a specific functional form for the probability density, we derive the effects on optimal drawdown due to a transfer of probability mass from the lower to the upper tail (FSD), and vice versa, and the effects of mean-

preserving spread (SSD). This allows us to incorporate, into both analysis and estimation, the important idiosyncratic features of actual returns distributions.

While the optimal draw down rate depends on both tastes for risk and the charity's elasticity of inter-temporal substitution, scenario analysis shows that the whether optimal spending rates increase or decrease in response to first and second order dominance changes in returns depends entirely on the EIS. Whenever the EIS is less than one, income effects dominate substitution effects and optimistic changes to returns (FDS and/or SSD) raise current spending. The reverse holds when the EIS is greater than one, and when the EIS is unitary, spending rates are immune to revision and depend only on time preference.

While charitable trusts obviously make investment decisions, it is not clear that investment choices are always joint with choice over spending rates. By treating charities with different preferences as if they actually hold the same portfolio, we clarify the trade-off between income and substitution effects and demonstrate that an equivalent change in expectations can produce very different, but nevertheless equally optimal, reactions from trustees.

## Appendix A

The problem is to maximise utility defined by the aggregator function:

$$L_t = U[C_t, E_t L_{t+1}] \\ \equiv \frac{\{(1-\delta)C_t^{1-\rho} + \delta[1 + (1-\delta)(1-\alpha)E_t L_{t+1}]^{(1-\rho)/(1-\alpha)}\}^{(1-\alpha)/(1-\rho)} - 1}{(1-\delta)(1-\alpha)} \quad (29)$$

with respect to consumption  $C_t$  and subject to the wealth constraint

$$\tilde{W}_{t+1} = (1-m)W_t \tilde{Z}_t. \quad (30)$$

Following the well-known result for standard CRRA preferences, Weil proposes that

$$L(W) = \frac{(\psi W)^{1-\alpha} - 1}{(1-\delta)(1-\alpha)} \quad (31)$$

and that  $C_t = mW_t$  where  $\psi$  and  $m$  are to be determined

Substituting (31) into (29) and using the expressions for consumption and the wealth constraint gives

$$L_t = \frac{\left\{ (1-\delta)(mW_t)^{1-\rho} + \delta \left[ E_t \left( \psi \tilde{Z}_t (1-m)W_t \right)^{1-\alpha} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1-\alpha}{1-\rho}} - 1}{(1-\delta)(1-\alpha)} \quad (32)$$

Maximising (32) over  $m$  is the same as maximising over consumption, and gives the first order condition as a function of  $\psi$  :

$$\frac{\partial L_t}{\partial m} = (1-\rho)(1-\delta)m^{-\rho}W_t^{1-\rho} - (1-\rho)\delta\psi^{1-\rho}\left(E_t\tilde{Z}^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}(1-m)^{-\rho}W_t^{1-\rho} = 0 \quad (33)$$

Re-arranging (33) gives:

$$m = \left\{ 1 + \left[ \frac{\delta}{1-\delta} \psi^{1-\rho} \left( \left( E_t \tilde{Z}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right)^{1-\rho} \right]^{\frac{1}{\rho}} \right\}^{-1}. \quad (34)$$

If we follow Weil in setting

$$\psi = \left[ (1-\delta)m^{-\rho} \right]^{\frac{1}{1-\rho}}, \quad (35)$$

then (34) becomes

$$m = \left\{ 1 + \left[ \frac{\delta}{1-\delta} (1-\delta)m^{-\rho} \left( \left( E_t \tilde{Z}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right)^{1-\rho} \right]^{\frac{1}{\rho}} \right\}^{-1}, \quad (36)$$

and rearranging confirms that

$$m = 1 - \delta^{\frac{1}{\rho}} \left[ \left( E_t \tilde{Z}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right]^{\frac{1-\rho}{\rho}} \quad (37)$$

or,

$$m = 1 - \delta^{\frac{1}{\rho}} \left[ \varphi^{\frac{1}{1-\alpha}} \right]^{\frac{1-\rho}{\rho}}.$$

## Appendix B

Portfolio returns data are monthly from January 1990 to June 2006. A consistent series of returns to hedge funds are not available prior to January 1994, so from January 1990-December 1993, the allocation to each of U.K., Global, Emerging and Private Equity was increased by 0.9% and Hedge Funds set to zero. Total portfolio return is the weighted sum of log changes in each returns index and the cash rate (expressed

on a monthly basis), less the log change in the inflation rate:  $\dot{p} = 0.5 \ln\left(\frac{CPI_t}{CPI_{t-1}}\right) + 0.5 \ln\left(\frac{Earnings_t}{Earnings_{t-1}}\right)$ .

Asset Class	Data	Source	Portfolio weight	Notes
UK Equity	FTSE All Share returns index	DataStream FTALLSH(RI)	32.2%	Capitalisation-weighted index, for London Stock Exchange representing at least 98% of eligible UK companies.
Global Equity	MSCI ex UK	DataStream MSWFUK\$(RI)~US\$, translated to BPN using BBGBPSP(ER)	32.0%	The MSCI World Index is a free float-adjusted market capitalization index consisting of the 22 developed market country indices where the UK market is excluded.
Overseas Equity	MSCI Emerging Markets	DataStream MSEMKF\$(RI) ~US\$, translated to BPN using BBGBPSP(ER)	5.0%	The MSCI Emerging Markets Index is a free float-adjusted market capitalization index consisting of 25 emerging market country indices.
UK Gilts	FTA	DataStream FTBGTTF(RI)~£	2.8%	FTA British Government fixed 10-15 years total returns index
Property	IPD: Total return index	DataStream UKIPDRI.F	7.5%	UK Investment Property Databank Index measures total returns to investment in commercial property investment.

Hedge Funds	CSFB/Tremont hedge fund index, net asset value	DataStream; CSFB/Tremont: <a href="http://www.hedgeindex.com">www.hedgeindex.com</a> CSTHEDG~£	3.6%	Credit Suisse/Tremont Hedge Fund Index is an asset-weighted hedge fund index and includes only funds, as opposed to separate accounts. The Index uses the Credit Suisse/Tremont database, which track over 4500 funds, and consists only of funds with a minimum of US\$50 million under management, a 12-month track record, and audited financial statements. It is calculated and rebalanced on a monthly basis, and shown net of all performance fees and expenses.
Private Equity	UK-DS Investment Trusts Private Equity total returns index.	DataStream ITVCAPT(RI)~£	11.5%	The index is constructed by DataStream to measure the performance of all UK listed investment trusts in the private equity sector.
Cash	3-month CD rate	Bank of England	5.4%	End month sterling certificate of deposit 3 month rate, mean of bid-offer.
Inflation	Average of CPI and Earnings	DataStream CPI: UKCPHARMF Wages: UKWAGES.E		Equally weighted log change in UK CPI- Harmonised European Union basis, 2005=100 and UK average earnings index, whole economy, seasonally adjusted.

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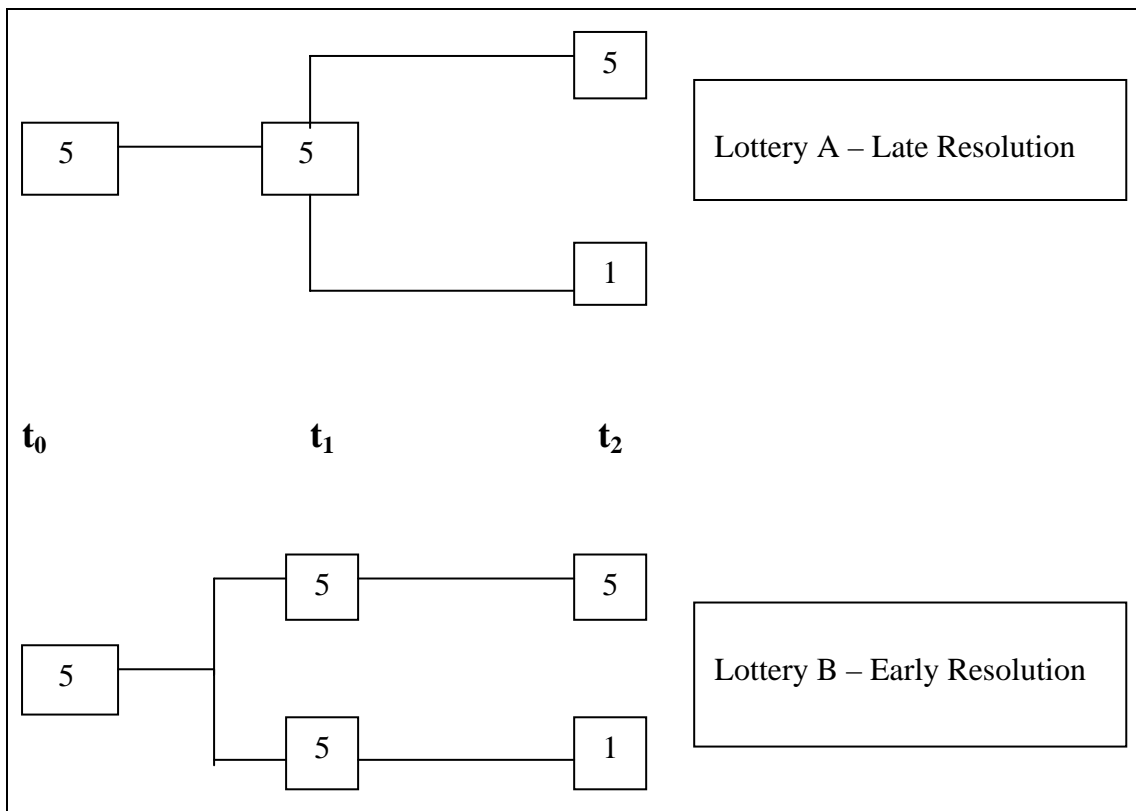
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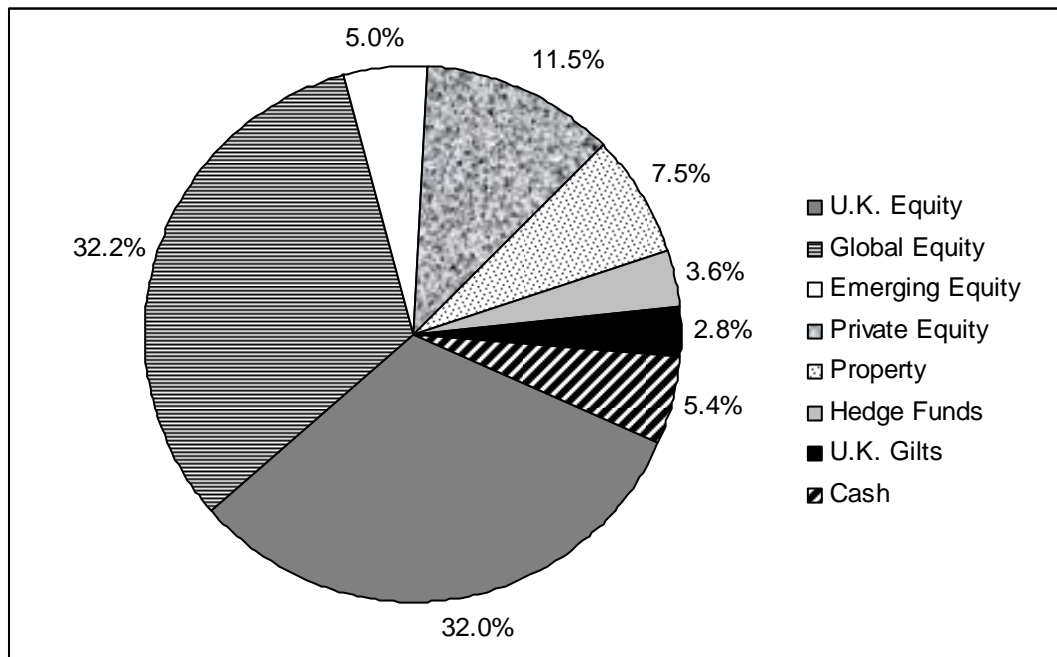
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**Figure 1: Timing of resolution of uncertainty.**



**Figure 2. Asset allocation of simulated charity**



This figure shows the proportions of total funds invested in each asset class for simulated portfolio returns. Weights are fixed for the whole sample period. A consistent series of returns to hedge funds are not available prior to January 1994, so from January 1990-December 1993, the allocation to each of U.K., Global, Emerging and Private Equity was increased by 0.9% and Hedge Funds set to zero.

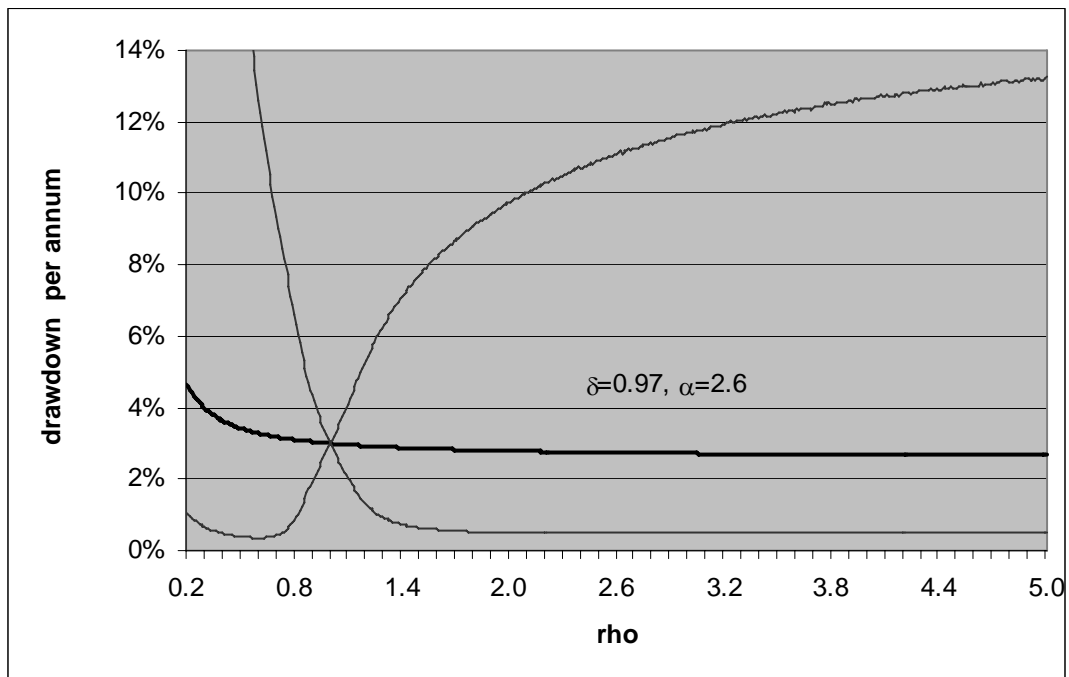
Appendix B lists data sources for each returns series.

**Table 1: Summary statistics, real annualised portfolio returns**

Mean	4.75%
Standard Deviation	13.02%
Skewness	-0.69
Kurtosis	3.88
Jarque-Bera	21.87
(p-value)	(0.000)

This table shows summary statistics for monthly log returns to the portfolio constructed using weights and asset classes from Figure 1. Portfolio returns are the weighted sum of the log changes in indexes for each asset class, less the log change in inflation. See Appendix B for a complete description of data sources. Data are monthly from January 1990 to June 2006: 198 observations.

**Figure 3: Optimal drawdown rate and error range**

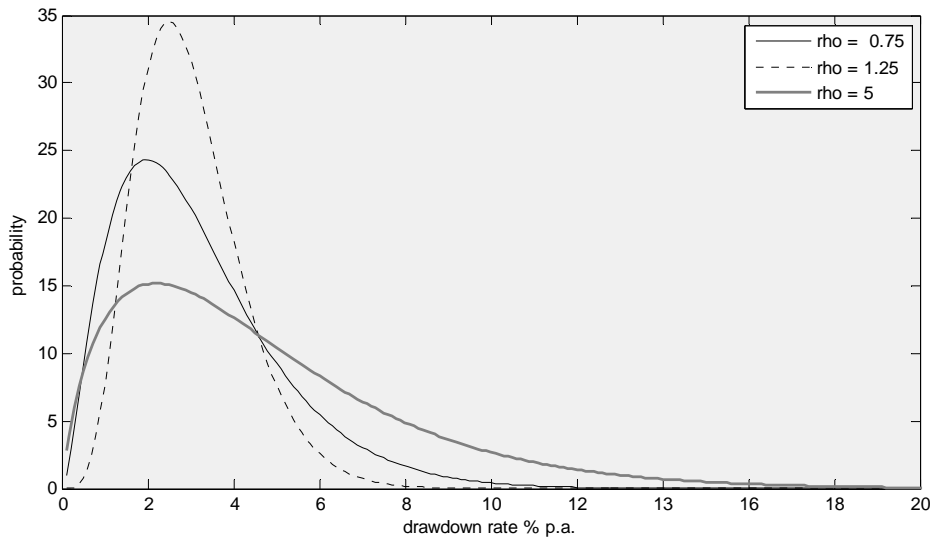


This Figure graphs the estimated optimal rate of drawdown for a charity with assets invested as for Figure 1 and assuming that relative risk aversion,  $\alpha = 2.6$ , where the optimal drawdown is given by

$$\hat{m} = 1 - \left( \delta \hat{\varphi}^{\frac{(1-\rho)}{(1-\alpha)}} \right)^{1/\rho}, \quad \hat{\varphi} = \frac{1}{10,000} \sum_{i=1}^{10,000} \tilde{Z}_i^{1-\alpha}$$

where the gross portfolio returns  $\tilde{Z}_i$  are computed using 10,000 random draws from the historical portfolio returns series, the rate of time preference is  $\delta = 0.97$  and  $0.2 < \rho < 5$ . The fine grey lines represent beta distribution approximations to a 95% error range for  $\hat{m}$ .

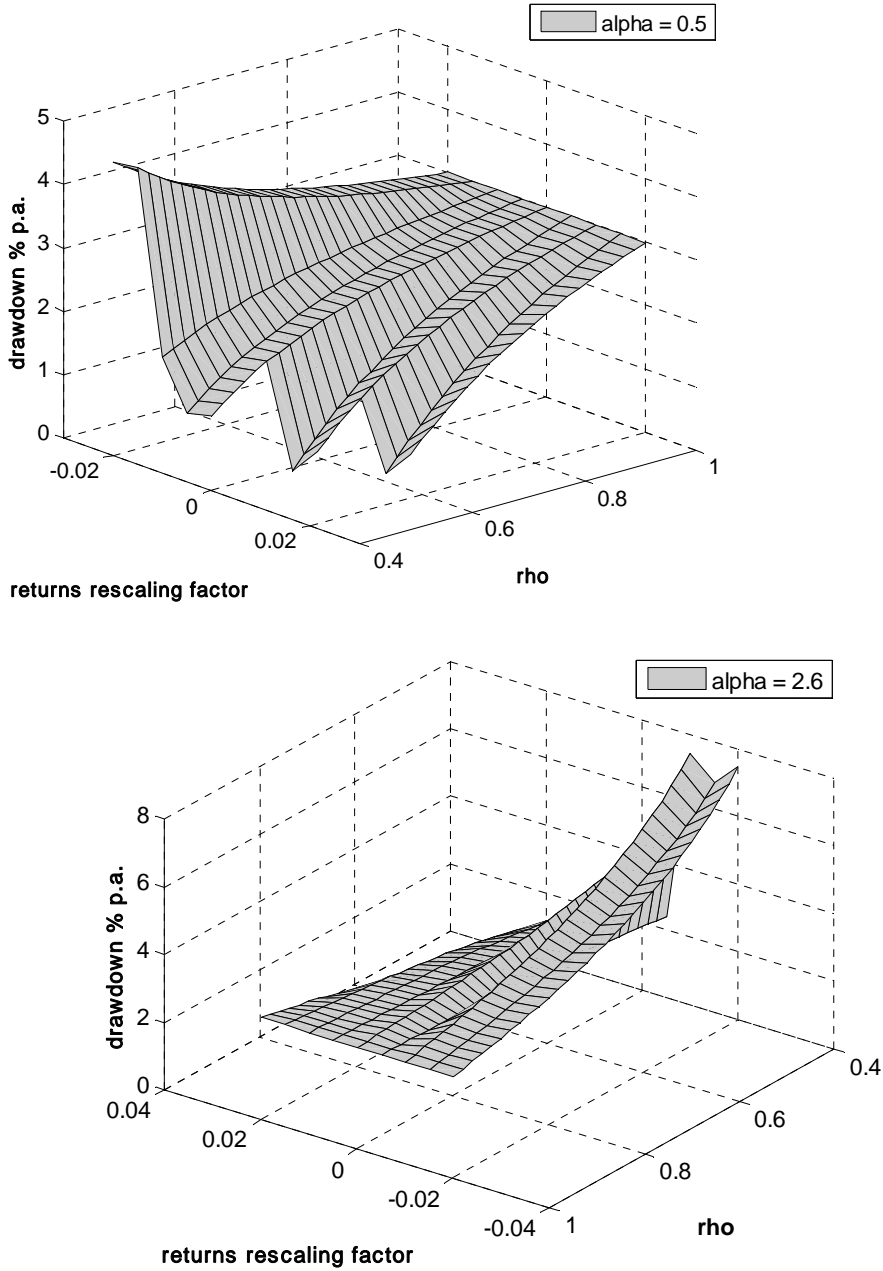
**Figure 4: Drawdown error distribution**



This figure graphs the estimated beta distributions of  $\hat{m} = 1 - \left( \delta \hat{\varphi}^{\frac{(1-\rho)}{(1-\alpha)}} \right)^{1/\rho}$  when relative risk aversion,  $\alpha = 2.6$  and  $\rho = 0.75, 1.25, 5.0$ . The simulated values of the optimal drawdown are computed as for Figure 3.



**Figure 5: Optimal drawdown under transformations of the portfolio returns distribution,  $0 < \rho \leq 1$ .**

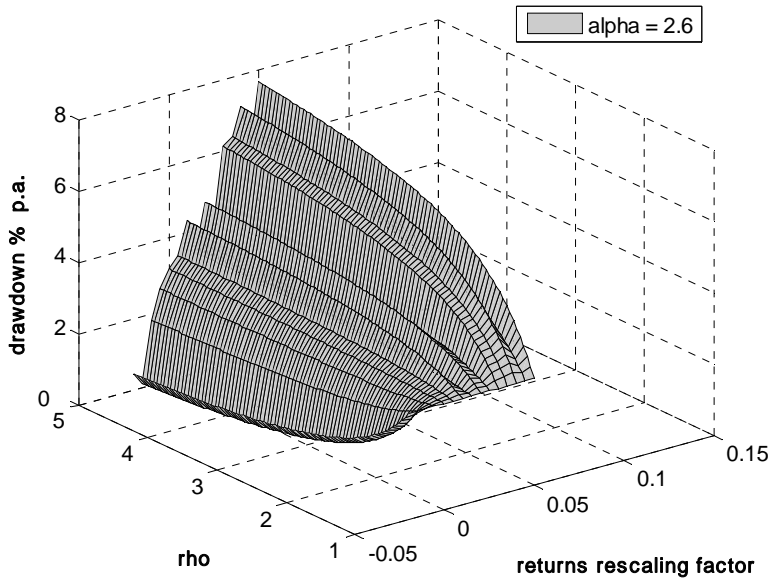
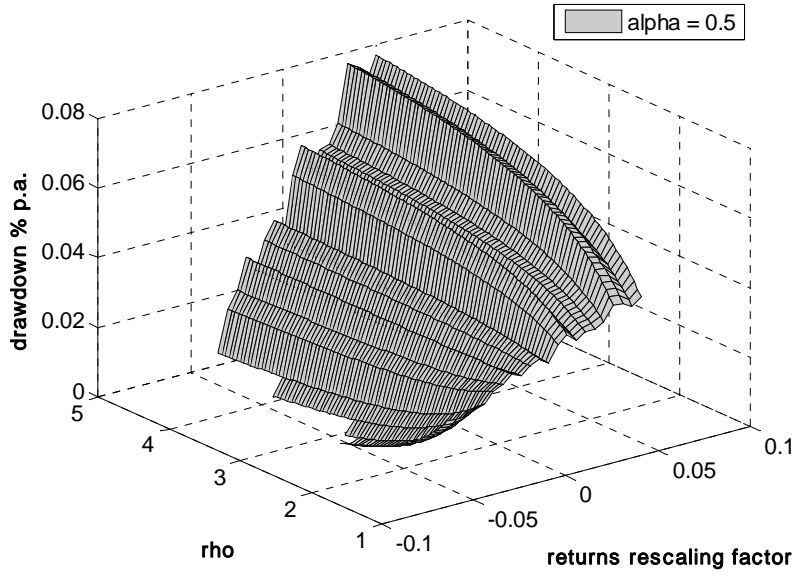


This Figure shows optimal rates of drawdown,  $m$ , when the distribution of portfolio returns is re-weighted by the factor  $\Delta$ , relative risk aversion  $\alpha = 2.6, 0.5$ , and  $0 < \rho \leq 1$ . We sort the original 198 real returns into percentiles and divide the dataset into three sections:  $d_1 = 40$  lowest observations representing the 1-20th percentiles,  $d_2 = 118$  observations representing the middle 20-80<sup>th</sup> percentiles, and  $d_3 = 40$  observations representing the 80-100<sup>th</sup> percentiles. We rescale the probability of low (high) returns by increasing the probability

of an extreme event so that  $P_i = \frac{d_i}{\sum_j d_j}$ ,  $P_u' = P_u + \Delta$ ,  $P_l' = P_l - \Delta$  and the probability of draws from the

middle stays constant. We draw returns from the low, middle and high range randomly with replacement and in proportion to the assigned probability, and compute  $\tilde{Z}_i^{1-\alpha}$ ,  $\varphi_\Delta$ , and  $m$ .

**Figure 6: Optimal drawdown under transformations of the portfolio returns distribution,  $1 \leq \rho \leq 5$ .**

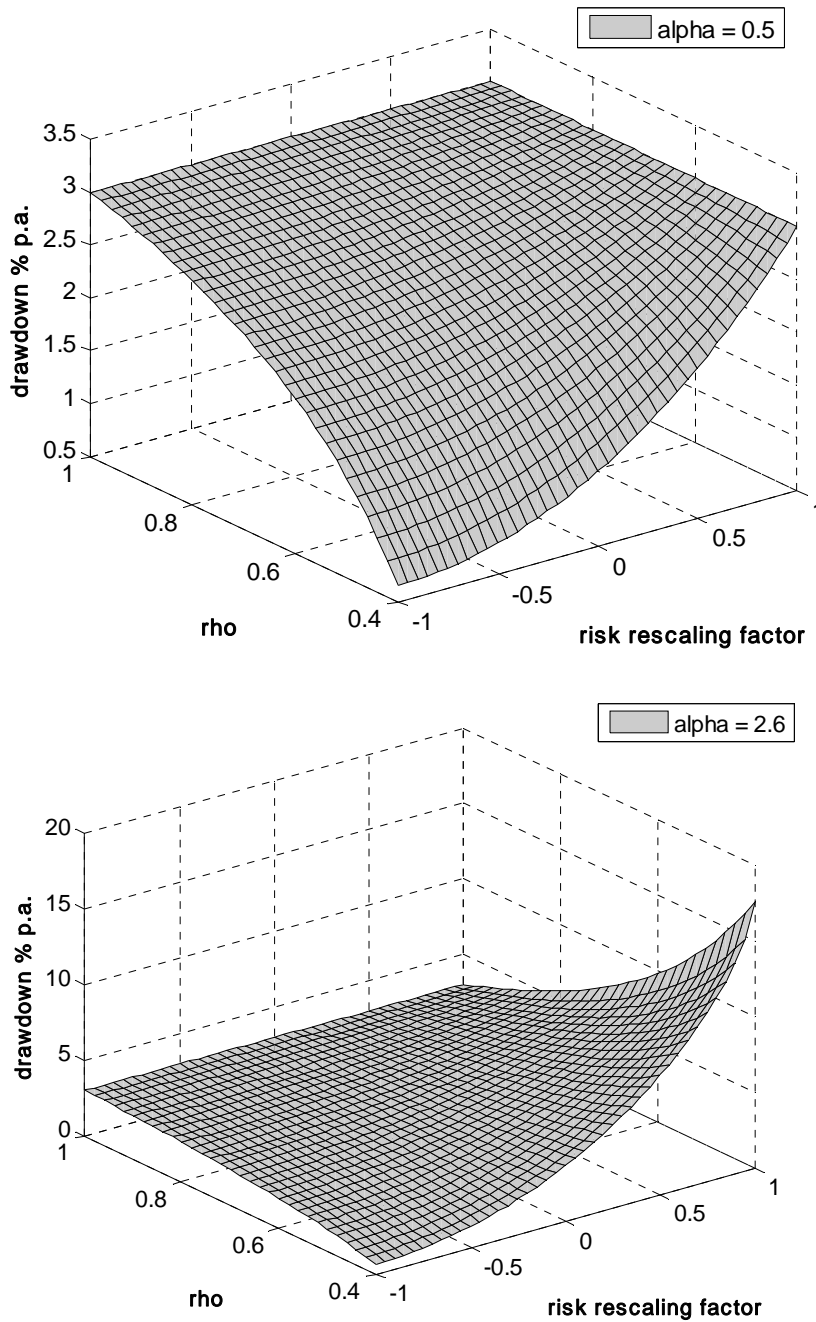


This Figure shows optimal rates of drawdown,  $m$ , when the distribution of portfolio returns is re-weighted by the factor  $\Delta$ , relative risk aversion is equal to 0.5 or 2.6, and  $1 \leq \rho \leq 5$ . We sort the original 198 real returns into percentiles and divide the dataset into three sections:  $d_1 = 40$  lowest observations representing the 1-20th percentiles,  $d_2 = 118$  observations representing the middle 20-80<sup>th</sup> percentiles, and  $d_3 = 40$  observations representing the 80-100<sup>th</sup> percentiles. We rescale the probability of low (high) returns by increasing the probability

of an extreme event so that  $p_i = \frac{d_i}{\sum_j d_j}$ ,  $p_u' = p_u + \Delta$ ,  $p_l' = p_l - \Delta$  and the probability of draws from the

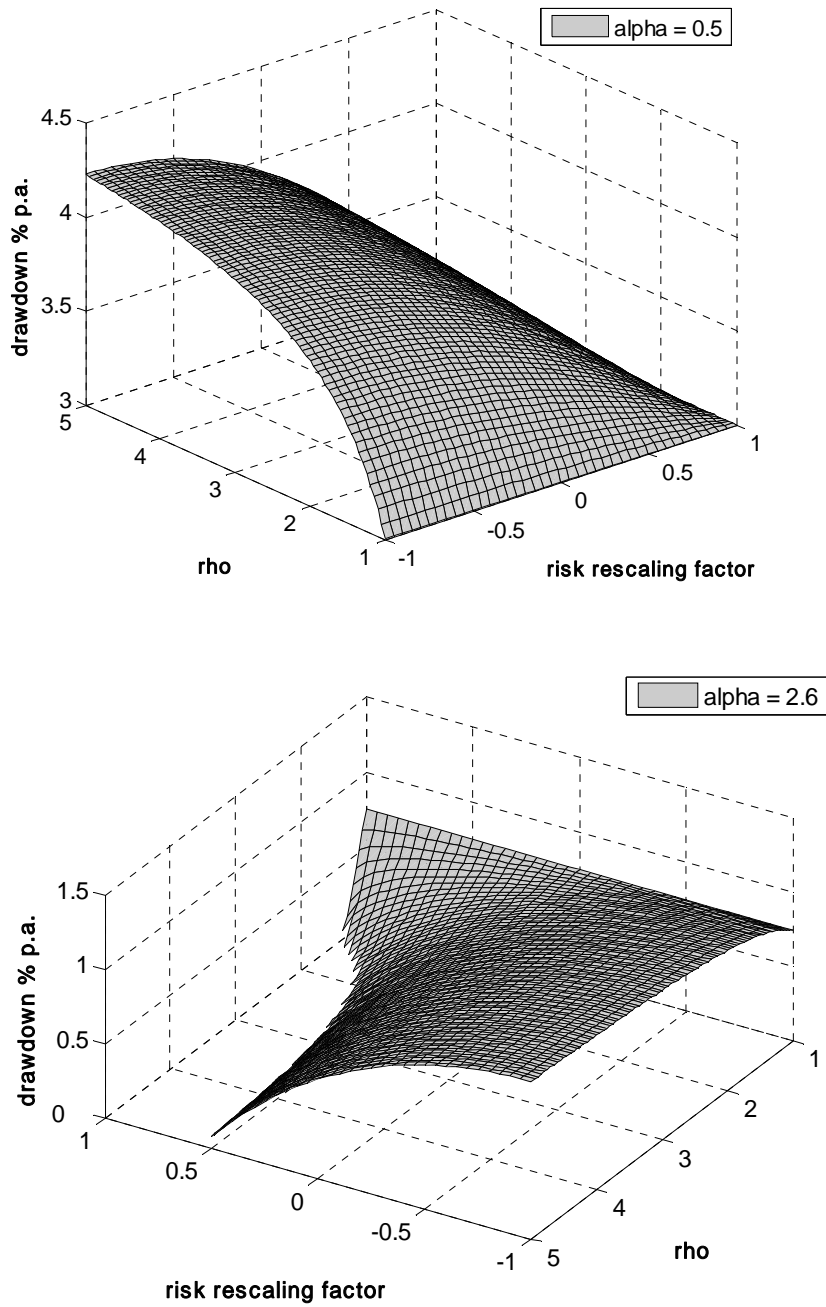
middle stays constant. We draw returns from the low, middle and high range randomly with replacement and in proportion to the assigned probability, and compute  $\tilde{Z}_i^{1-\alpha}$ ,  $\varphi_\Delta$ , and  $m$ .

**Figure 7: Optimal drawdown under mean-preserving spread transformations of the portfolio returns distribution,  $0 \leq \rho \leq 1$ .**



This Figure shows optimal rates of drawdown,  $m$ , when the variance of the distribution of portfolio returns is reweighted by the factor  $\omega$ , relative risk aversion is 2.6 or 0.5,  $0 \leq \rho \leq 1$ . We draw 120,000 of the original 198 real returns with replacement  $\tilde{Z}_i$  to compute 10,000 annual gross portfolio returns  $\tilde{Z}_i$ . We then compute the mean-zero errors  $\varepsilon_i = (Z_i - \bar{Z})$ , resample these without replacement and compute  $\tilde{Z}_i^{1-\alpha} = (\bar{Z} + (1 + \omega)\varepsilon_i)^{1-\alpha}$  for  $-1 \leq \omega \leq 1$ ,  $\varphi_\Delta$ , and  $m$ , for  $0 \leq \rho \leq 1$ .

**Figure 8: Optimal drawdown under mean-preserving spread transformations of the portfolio returns distribution,  $1 \leq \rho \leq 5$**



This Figure shows optimal rates of drawdown,  $m$ , when the variance of the distribution of portfolio returns is re-weighted by the factor  $\omega$ , and relative risk aversion is 0.5 or 2.6,  $1 \leq \rho \leq 5$ . We draw 120,000 of the original 198 real returns with replacement to compute 10,000 annual gross portfolio returns  $\tilde{Z}_i$ . We then compute the mean-zero errors  $\varepsilon_i = (Z_i - \bar{Z})$ , resample these without replacement and compute  $\tilde{Z}_i^{1-\alpha} = (\bar{Z} + (1 + \omega)\varepsilon_i)^{1-\alpha}$  for  $-1 \leq \omega \leq 1$ ,  $\varphi_\Delta$ , and  $m$ , for  $1 \leq \rho \leq 5$ .