FAADING MEMORY LEARNING IN THE COBWEB MODEL WITH RISK
AVERSE HETEROGENEOUS PRODUCERS

CARL CHIARELLA, XUE-ZHONG HE AND PEIYUAN ZHU

School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

August 1, 2003

ABSTRACT. This paper studies the dynamics of the traditional cobweb model with
risk averse heterogeneous producers who seek to learn the distribution of asset prices
using a geometric decay processes (GDP)—the expected mean and variance are es-

timated as a geometric weighted average of past observations—with either finite or
infinite fading memory. With constant absolute risk aversion, the dynamics of the
model can be characterized with respect to the length of memory window and the
memory decay rate of the learning GDP. The dynamics of such heterogeneous learn-

ing processes and capability of producers’ learning are discussed. It is found that the
learning memory decay rate of the GDP of heterogeneous producers plays a compli-
cated role on the pricing dynamics of the nonlinear cobweb model. In general, an
increase of the memory decay rate plays a stabilizing role on the local stability of
the steady state price when the memory is infinite, but this role becomes less clear
when the memory is finite. It shows a double edged effect of the heterogeneity on
the dynamics. It is shown that (quasi)periodic solutions and strange (or even chaotic)
attractors can be created through Neimark-Hopf bifurcation when the memory is in-
finit and through flip bifurcation as well when the memory is finite.

JEL classifications: D83; D84; E21; E32, C60

Keywords: Cobweb model, heterogeneity, bounded rationality, geometric decay learn-
ing dynamics, bifurcations

A short version of this paper was presented in 9th International Conference on Computing in Economics
and Finance, July 11-13, 2003, University of Washington, Seattle, USA.
1. INTRODUCTION

As an alternative paradigm of representative agent and rational expectation in finance and economics, research into the dynamics of heterogeneity and bounded rationality has flourished in recent years. In this paper, we investigate heterogeneous learning in the well-known cobweb model. In particular, we analyze the model under bounded rationality learning with heterogeneous fading memory and show that, on the one hand, heterogeneous learning can help agents to learn the rational equilibrium in some cases, on the other hand, such learning can lead to market instability and to periodic, or even chaotic, price fluctuations.

For the well-known cobweb model

\[
\begin{align*}
p_t &= \alpha q_t + \mu \\
p_t^e &= a q_t + b
\end{align*}
\]

(1.1)

here, \( q_t \) and \( p_t \) are quantities and prices, respectively, at period \( t \), \( p_t^e \) is the price expected at time \( t \) based on the information at \( t-1 \), and \( a, b, \mu ( > 0) \) and \( \alpha < 0 \) are constants, it is well known that, under the naive expectation scheme \( p_t^e = p_{t-1} \), the price either converges to the optimal market equilibrium (when \( |\alpha/\alpha| < 1 \) or explodes (when \( |\alpha/\alpha| > 1 \)). To obtain more realistic, oscillatory, price time paths, the literature has introduce non-linearities into the cobweb model and such nonlinearities can come from either nonlinear supply or demand curve, risk aversion (discussed as follows), or agents’ heterogeneity, bounded rationality and various learning processes.

When the producers are homogeneous, it has been shown that agents’ expectations and non-linearities in the supply or demand curves may lead the cobweb model to exhibit both stable periodic and chaotic behavior (i.e., Artsein (1983), Jensen and Urban (1984), Chiarella (1988), Holmes and Manning (1988), Hommes (1991, 1994, 1998), Puu (1991) and Day (1992)). These authors consider a variety of backward looking mechanisms for the formation of the expectations \( p_t^e \) ranging across the traditional naive expectation \( p_t^e = p_{t-1} \), learning expectations (e.g., learning by arithmetic mean \( p_t^e = (p_{t-1} + \cdots + p_{t-L})/L \) and adaptive learning expectation \( p_t^e = p_{t-1}^e + w(p_{t-1} - p_{t-1}^e) \) with \( 0 \leq w \leq 1 \). Given the bounded rationality of agents, Hommes (1998) even shows that such simple expectation schemes can be consistent with rational behaviour in the nonlinear cobweb model.

By assuming the bounded rationality, when producers are somewhat uncertain about the dynamics of the economic system in which they are to play out their roles, they need to engage in some learning scheme to update their beliefs. Among various learning schemes, the properties of recursive learning processes under homogeneous expectations have been studied extensively (e.g., Bray (1982, 1983), Evans and Ramey (1992), Balasko and Royer (1996), Evans and Honkapohja (1994, 1995, 1999), Barucci (2000, 2001)). In Bray (1982, 1983) and Evans and Honkapohja (1994, 1995), the agent’s expectation is computed as the arithmetic average of all the past observations with full memory (the same weight is employed for each observation). In Blasko and Royer (1996), agents’ expectations are updated by finite recursive least square (moving average of past \( h \) prices) processes and it is found that an equilibrium which is stable under learning with finite memory \( h \) is also stable for a finite memory \( h' \) with \( h' > h \). Their results are extended further in Chiarella and He (2003b) to a more general finite recursive \( \alpha_L \)-process with nonnegative weighting vector \( \alpha \) and it is found
that the stability of equilibrium depends on the weighting vector and complicated dynamics can be generated. In Barucci (2000, 2001), agent’s expectation is computed as a weighted average of all the past observation with no-full memory. The weights of the average are described by a geometric process with a ratio smaller than 1 and therefore, the weights for older observations are smaller than the weights for recent observations. As pointed by Barucci (p.234, 2001), these features of fading memory learning mechanism are “appealing because...the assumption of a constant weight for past observations are not fully plausible from a behavioral point of view. As a matter of fact, agents do not stop to learn as time goes on and they ‘forget’ remote observations”.

For a class of nonlinear deterministic forward-looking economic model under the fading memory learning, Barucci shows that the decay rate of the memory of the learning process plays a stabilizing role—an increase of the memory decay rate enlarges the local stability parameters region of the prefect foresight stationary equilibria.

In more recent work, Brock and Hommes (1997) studies heterogeneity in expectation formation by introducing the concept of adaptive rational equilibrium dynamics (ARED). They consider a cobweb model in which agents choose a predictor from a finite set of expectations functions of past information and update their beliefs over time according to a publically available ‘fitness’ measure. They show that a rational route to randomness. This framework has been extended further to heterogeneous cobweb model by allowing more types of agents (e.g. Branch (2002) and Onozaki et al (2000, 2003).) and various learning among heterogeneous agents (e.g. Chiarella and He (2003a)).

Nonlinearity can also come from risk and risk aversion (i.e., Boussard and Gerard (1991), Burton (1993) and Boussard (1996)). As pointed in Boussard (1996), with risk averse producers, the traditional linear cobweb model becomes nonlinear. By assuming that the actual price \( p_t \) is uncertain so that \( p_t^e \) has mean \( \tilde{p}_t \) and variance \( \tilde{v}_t \), Boussard (1996) shows that, under the simplest learning scheme \( \tilde{p}_t = \tilde{p} \) and \( \tilde{v}_t = (p_{t-1} - \tilde{p})^2 \) with constant \( \tilde{p} \), the nonlinear model may result in the market generating chaotic price series, and market failure, and therefore the source of risk is the risk itself (p.435, Boussard (1996)). Consequently, the study “casts a new light on expectations. Not only are expectations pertaining to mean values important for market outcomes. Those pertaining to variability can be just as crucial” (p.445, Boussard (1996)).

Apart from Boussard (1996), a great deal of attention in the expectations formation literature has been devoted to schemes for the mean, but very little to schemes for the variance. Chiarella and He (2000) extend Boussard’s framework in a way that takes account of the risk aversion of producers and allows them to estimate both the mean and variance via the arithmetic learning process (ALP) \( \tilde{p}_t = \frac{1}{L} \sum_{i=1}^{L} p_{t-i}, \quad \tilde{v}_t = \frac{1}{L} \sum_{i=1}^{L} [p_{t-i} - \tilde{p}_t]^2 \) with some integer \( L \geq 1 \). Chiarella and He (2000) show that the resulting cobweb dynamics form a complicated nonlinear expectations feedback structure whose dimensionality depends upon the length of the window of past prices (the lag length) used to estimate the moments of the price distributions. It is found that an increase of the window length \( L \) can enlarge the parameter region (in terms of \( |\alpha/\alpha| \) of the local stability of the steady state and, at the crossover from local stability to local instability, the dynamics exhibits resonance behavior which is indicative of quite complicated dynamical behavior, and even chaos (for the model with constant elasticity supply and demand functions).
Motivated largely by the above literature in heterogeneous expectations and learning, this paper aims to study the dynamics of the cobweb model with risk averse heterogeneous producers who following the fading memory learning processes. We first extend the homogeneous model in Chiarella and He (2000) to a heterogeneous model. By allowing the heterogeneous producers to follow geometric decay (learning) process (GDP, see Section 2 for definition), we then study the role of the memory decay rate on the price dynamics. It is found that, when the memory is infinite, an increase of the memory decay rate plays a stabilizing role on the local stability of the steady state price, which is also found in Barucci (2000, 2001) when agents are homogeneous. However, such role becomes less clear when the memories are finite. The heterogeneity has double edged effect on the price dynamics in the sense that heterogeneous learning can stabilize an otherwise unstable dynamics in some cases and destabilize an otherwise stable dynamics in other cases as well. It is shown that (quasi)periodic solutions and strange (or even chaotic) attractors can be created through Neimark-Hopf bifurcation when the memory is infinite and through flip bifurcation as well when the memories are finite. In addition, it is found that the source of risk is the risk itself, as pointed out in Boussard (1996), in the sense that the behaviour of producers in response to risk can generate market failure.

The paper is organised as follows. A general cobweb model with heterogeneous producers is established in Section 2. The heterogeneous geometric decay (learning) processes (GDP) is introduced, and the existence of steady-state (rational equilibrium) is also discussed in Section 2. As a special case of the GDP with finite memory, the dynamics of the heterogeneous model with standard arithmetic learning (ALP) is considered in Section 3. Then the dynamics of the model with heterogeneous GDP for both finite and infinite memories are analyzed in Sections 4 and 5, respectively. Section 6 concludes the paper.

2. COWEB MODEL WITH HETEROGENEOUS PRODUCERS

This section is intended to establish a cobweb model when producers are heterogeneous in their risk and expectation formulation on both the mean and variance. In the case of linear supply and demand functions, the model may be written as

\[
\begin{align*}
\text{Supply:} & \quad p_{i,t}^e = a_i q_{i,t} + b_i, \quad (i = 1, 2, \cdots, h); \\
\text{Demand:} & \quad p_t = \alpha q_t + \mu \quad (\alpha < 0),
\end{align*}
\]

where \(q_t\) is the aggregate supply, \(q_{i,t}\) and \(p_{i,t}^e\) are the quantity and price expected of producer \(i\) at time \(t\) based on the information set at \(t - 1\), and \(p_t\) is the price, and \(a_i, b_i, \mu (> 0)\) and \(\alpha < 0\) are constants.

Our approach to the formation of expectations will be somewhat different in that we assume that the actual price \(p_t\) is uncertain so that the heterogeneous producers treat \(p_{i,t}^e\) as a random variable drawn from a normal distribution whose mean and variance they are seeking to learn\(^1\).

\(\text{\(^1\)It would of course be preferable (and more in keeping with models of asset price dynamics in continuous time finance) to treat} p_{i,t}^e \text{as log-normally distributed. However this would then move us out of the mean-variance framework so we leave an analysis of this approach to future research.}\)
2.1. **Market Clearing Price and Heterogeneous Model.** Let $\tilde{p}_{i,t}$ and $\tilde{v}_{i,t}$ be, respectively, subjective mean and variance of price $p_{i,t}$ of producer $i$ formed at time $t$ based on the information set at $t-1$, and $q_t$ be quantity at time $t$. With constant absolute risk aversion $A_i$, the marginal revenue certainty equivalent of producer $i$ is given by

$$\tilde{p}_{i,t} = \tilde{p}_{i,t} - 2A_i \tilde{v}_{i,t}q_t.$$  \hspace{1cm} (2.2)

Suppose a linear marginal cost, as in (2.1), so that the supply equation, under marginal revenue certainty equivalent, becomes

$$\tilde{p}_{i,t} = a_i q_t + b_i$$  \hspace{1cm} (2.3)

It follows from (2.2) and (2.3) that

$$aq_{i,t} + b_i = \tilde{p}_{i,t} - 2A_i \tilde{v}_{i,t}q_t$$

and hence the supply for producer $i$ is given by

$$q_{i,t} = \frac{\tilde{p}_{i,t} - b_i}{a_i + 2A_i \tilde{v}_{i,t}}.$$  \hspace{1cm} (2.4)

Denote by $n_i$ the proportion of type $i$ producers, then the market clearing price is determined by

$$p_t = \mu + \alpha \sum_i n_i \frac{\tilde{p}_{i,t} - b_i}{a_i + 2A_i \tilde{v}_{i,t}}.$$  \hspace{1cm} (2.5)

In fact, it follows from (2.1) and (2.4) that the aggregated supply is given by

$$q_t = \sum n_i q_{i,t} = \frac{p_t - \mu}{\alpha}$$

and hence

$$\frac{p_t - \mu}{\alpha} = \sum n_i \frac{\tilde{p}_{i,t} - b_i}{a_i + 2A_i \tilde{v}_{i,t}},$$

from which (2.5) follows.

In the rest of this paper, the simplest heterogeneous model when there are two types of producers is considered. Then the population of heterogeneous producers can be measured by a single parameter. Let $n_1 = (1+w)/2, n_2 = (1-w)/2$. Then (2.5) can be rewritten in the following form

$$p_t = \mu + \frac{\alpha}{2} (1+w) \frac{\tilde{p}_{1,t} - b_1}{a_1 + 2A_1 \tilde{v}_{1,t}} + \frac{\alpha}{2} (1-w) \frac{\tilde{p}_{2,t} - b_2}{a_2 + 2A_2 \tilde{v}_{2,t}}.$$  \hspace{1cm} (2.6)

---

\[2\]With constant absolute risk aversion $A_i$, we assume the certainty equivalent of the receipt $r = pq$ is $R(q_t) = \tilde{p}_{i,t}q_t - A_i \tilde{v}_{i,t}q_t^2$. Then maximisation of this function with respect to $q_t$ leads to the marginal revenue certainty equivalent $\tilde{p}_t = \frac{\partial R}{\partial q_t} = \tilde{p}_{i,t} - 2A_i \tilde{v}_{i,t}q_t$. We recall that this objective function is consistent with producers having the utility of receipts function $U_i(r) = -e^{-A_i r}$.

\[3\]In general, the proportion $n_i$ is a function of time $t$, that is, $n_{i,t}$, which can be measured by certain fitness function and discrete choice probability, as in Brock and Hommes (1997). Because of the complexity of the dynamics, we consider only the case with fixed propitiation and leave the changing proportion problem to our future work.
2.2. **Heterogeneous Learning Processes.** The heterogeneous model (2.6) is incomplete unless producers’ expectations are specified. In this paper, geometric decay processes (GDP) with either finite and infinite memory are assumed. More precise, for type $i$ producers, the GDP with finite memory is defined by assuming that the conditional mean and variance of the price follows a geometric probability distribution with decay rate of $\delta_i$ over a window length of $L_i$, that is,

\[
\begin{aligned}
\bar{p}_{i,t} &\equiv m_{i,t-1} = B_i \sum_{j=1}^{L_i} \delta_i^{j-1} p_{t-j}, \\
\bar{v}_{i,t} &\equiv v_{i,t-1} = B_i \sum_{j=1}^{L_i} \delta_i^{j-1} [p_{t-j} - m_{i,t-1}]^2 ,
\end{aligned}
\]

(2.7)

where $B_i = 1/(1 + \delta_i + \delta_i^2 + \cdots + \delta_i^{L_i-1})$, $L_i \geq 1$ are integers, and $\delta_i \in [0, 1]$ are constants for $i = 1, 2$. Two special cases of the GDP are of particular interested. When $\delta_i = 0$, the expectation of the mean follows the naive expectation $\bar{p}_{i,t} = p_{t-1}$ and $\bar{v}_{i,t} = 0$. When $\delta_i = 1$, the GDP (2.7) is reduced to the standard arithmetic learning process (ALP),

\[
\begin{aligned}
\bar{p}_{i,t} = \frac{1}{L_i} \sum_{j=1}^{L_i} p_{t-j}, \\
\bar{v}_{i,t} = \frac{1}{L_i} \sum_{j=1}^{L_i} [\bar{p}_{i,t} - p_{t-j}]^2 .
\end{aligned}
\]

(2.8)

As memory becomes infinite, that is, as $L_i \to \infty$, it is shown (see Appendix A) that, as a limiting process of GDP with finite memory, the GDP with infinite memory satisfies

\[
\begin{aligned}
m_{i,t} &= \delta_i m_{i,t-1} + (1 - \delta_i) p_t, \\
v_{i,t} &= \delta_i v_{i,t-1} + \delta_i (1 - \delta_i) (p_t - m_{i,t-1})^2.
\end{aligned}
\]

(2.9)

2.3. **Existence of the Unique Steady State Price.** Denote by $p^*$ the state steady price of the GDP model with finite memory. Then it is found from (2.6) that $p^*$ satisfies

\[
p^* = \frac{\mu - \frac{a}{2}[(1+w)\frac{b_1}{a_1} + (1-w)\frac{b_2}{a_2}]}{1 - \frac{a}{2}[(1+w)\frac{1}{a_1} + (1-w)\frac{1}{a_2}]} .
\]

(2.10)

For the GDP model with infinite memory, the state steady is given by $(p_t, m_{i,t}, v_{i,t}) = (p^*, p^*, 0)$. Note that the steady state price $p^*$ is the same under GDP with both finite and infinite memory.

In the following sections, dynamics of the heterogeneous model (2.6) are studied when agents update their estimations on both mean and variance by using the ALP (2.8) first. The analysis is then generalised to the GDP (2.7) with finite memory and (2.9) with infinite memory.

3. **Dynamics of the Heterogeneous Cobweb Model with ALP**

As a special case of the heterogeneous model with finite GDP, this section focuses on the case where producers have finite full memory about the history prices, that is $\delta_1 = \delta_2 = 1$. Correspondingly, the GDP is reduced to ALP, which has been focused in the literature (e.g. Balasko and Royer (1996) and Chiarella and He (2003b)). Without loss of generality, we assume $L_1 \leq L_2$ and denote $L = \max\{L_1, L_2\} = L_2$. Because of the dependence of the subjective mean $\bar{p}_t$ and variance $\bar{v}_t$ on price lagged $L$ periods, equation (2.6) is a difference equation of order $L$ (see system (B.2) in Appendix B).

The local stability of the unique steady state $p_t = p^*$ is determined by the eigenvalues of the corresponding characteristic equation (equation (B.3) in Appendix B), which
is difficult to analyze in general. The following discussion first focuses on the case when \( L_1 = L_2 = L \) and then some special cases when \( L_1 \neq L_2 \) and \( L_1, L_2 = 1, 2, 3, 4 \).

Denote

\[
\beta_1 = -\frac{\alpha}{2a_1} (1 + w) \\
\beta_2 = -\frac{\alpha}{2a_2} (1 - w)
\]

(3.1)

and

\[
\gamma_1 = \frac{\beta_1}{L_1} > 0, \quad \gamma_2 = \frac{\beta_2}{L_2} > 0.
\]

As indicated from the following results, the local stability of the steady state depends on various parameters, including those from supply and demand functions \( a_1, a_2, \alpha \), the proportion difference of two types of producers \( w \), and the window lengths \( L_1 \) and \( L_2 \) used by the heterogeneous producers. The discussion here is focused on two different aspects. On the one hand, for a fixed window length combination of \( (L_1, L_2) \), we consider how the demand parameter \( \alpha \) and the proportion difference \( w \) of producers affect the local stability of the steady state and bifurcation. On the other hand, for a set of fixed parameters, we examine how these results on the local stability and bifurcation are affected by different combination of the window lengths. It is found from the following discussion that both the local stability region and bifurcation boundary are geometrically easy to construct by using parameters \( \beta_1 \) and \( \beta_2 \), instead of \( w \) and \( \alpha \). However, the one-one relation (3.1) between \( (w, \alpha) \) and \( (\beta_1, \beta_2) \) makes it possible to transform the results between different set of parameters, and in addition, to preserve the geometric relation of the local stability regions between the two sets of parameter.\(^4\)

In the following discussion, because of the geometric advantage, the results are formulated in terms of \( (\beta_1, \beta_2) \), although some of the stability regions are plotted using \( (w, \alpha) \) as well.

3.1. **Case 1**: \( L_1 = L_2 = L \). When both types of producer use the same window length, that is \( L_1 = L_2 = L \), using the Lemma in Chiarella and He (2003\(a \)), a relatively complete result on both the local stability region of the steady state and the types of bifurcation is obtained in Proposition 3.1 for general lag length \( L \) (see Appendix B for the proof).

**Proposition 3.1.** For the nonlinear system (2.6), assume producers follow ALP and \( L_1 = L_2 = L \). Then the steady state \( p^* \) is locally asymptotically stable (LAS) if

\[
0 \leq -\frac{\alpha}{2} \left( \frac{1 + w}{a_1} + \frac{1 - w}{a_2} \right) < L, \quad \text{i.e.,} \quad 0 \leq \beta_1 + \beta_2 < L.
\]

(3.2)

Furthermore, the boundary \( \beta_1 + \beta_2 = L \) defines a 1 : \( (L + 1) \) resonance bifurcation.\(^5\)

---

\(^4\)Note that the determinant of the Jacobian of the transformation (3.1) does not change the sign, implying the reservation of the transformation.

\(^5\)When \( \beta_1 + \beta_2 = L \), the eigenvalues are given by \( \lambda_k = e^{2k\pi i} \) with \( k = 1/(L + 1) \). Geometrically, the \( L \) eigenvalues correspond to the \( L + 1 \) unit roots distributed evenly on the unit circle, excluding \( \lambda = 1 \). When \( L = 1 \), a flip or period-doubling bifurcation occurs. When \( L = 2 \), according to Kaznetsov (1995), the bifurcation is a 1:3 strong resonance. For \( L \geq 2 \), according to Sonis (2000), the bifurcation is given by 1 : \( L + 1 \) periodic resonances. Theoretical analysis for such types of bifurcation of higher dimensional discrete system can be exceedingly complicated and not yet completely understood, (see Example 15.34 in Hale and Kocak (pp. 481-482, (1991))).
The local stability region and the resonance bifurcation boundary are plotted in Figure 3.1 in the parameter \((\beta_1, \beta_2)\) space for general lag length \(L\). In particular, there are two special cases are of interesting:

- when \(a_1 = a_2\), the local stability condition and the bifurcation boundary are independent of \(w\), as expected;
- when \(a_1 \neq a_2\), the local stability region for \(\alpha\) becomes (i) \(\alpha \in (-La_1, 0]\) for \(w = 1\); and (ii) \(\alpha \in (-La_2, 0]\) for \(w = -1\). In other word, the local stability depends more on the ratio \(a_1/a_2\) and less on the population distribution \(w\).

In order to see the bifurcation feature, numerical simulations are used to analyse the dynamics of the nonlinear system (2.6) for the cases \(L = 2, 3, 5\) and \(10\). When \(L = 2\) and \(3\), for fixed \(a_1 = 0.8 < a_2 = 1\), the stability regions and resonance bifurcation boundaries are plotted in terms of parameters \((\alpha, w)\) in Figure 3.2.

Assume \(a_1 \neq a_2\), for \(L = 2\), the bifurcation boundary \(\beta_1 + \beta_2 = 2\) becomes

\[
\frac{1 + w}{a_1} + \frac{1 - w}{a_2} = -\frac{4}{\alpha},
\]

which defines the 1:3 resonance bifurcation boundary \(w\) as a function of \(\alpha\)

\[
w = \frac{4a_1a_2}{\alpha(a_1 - a_2)} + \frac{a_1 + a_2}{a_1 - a_2} \equiv W(\alpha). \tag{3.3}
\]

Note that,

- for \(a_1/a_2 < 1\), \(W(\alpha)\) is an increasing function of \(\alpha\), and hence, as \(w\) increases, the local stability regions for \(\alpha\) become small (note that \(\alpha < 0\));
• for $a_1/a_2 > 1$, $W(\alpha)$ is a decreasing function of $\alpha$, and hence, as $w$ increases, the local stability regions for $\alpha$ become large.

In terms of the effect of lag length $L$ on the local stability region of the steady state, an analysis on the stability boundary $\beta_1 + \beta_2 = L$ leads to the following Corollary.

**Corollary 3.2.** For the nonlinear system (2.6), assume producers follow ALP and $L_1 = L_2 = L$. Then, in terms of the parameters $\alpha$ and $w$, increasing of $L$ can stabilise the otherwise unstable steady state.

![Bifurcation diagrams of the nonlinear system (2.6) for $\alpha$ and $w$ when the initial values are either close (upper panel), or not close (the lower panel) to the steady state with papersters $\beta = 11, a_1 = a_2 = 1, A = 0.005, w = 0, b_1 = b_2 = 0$ and $L_1 = L_2 = L = 2$.](image)

The above theoretical analysis on the local stability and bifurcation is verified by numerical analysis on the nonlinear system (2.6). Consider a special case when $a_1 = a_2 = 1$. In this case, the steady state is LAS for $\alpha \in (-2, 0]$ for any $w \in [-1, 1]$ and $\alpha = -2$ leads to a 1:3 resonance bifurcation. Bifurcation diagrams for the nonlinear system (2.6) are plotted in Figure 3.3 in terms of parameter $\alpha < 0$ with different initial values. It is found that,

• when the initial values are close to the steady state (within 1% interval of the steady state), the bifurcation value $\alpha_o$ is close to the theoretical bifurcation value $\alpha_o = -2$, as indicated in the upper panel in Figure 3.3;

• when the initial values are not close to the steady state (within 400% interval of the steady state), the bifurcation value $\alpha_o \approx -1.9$ moves away from the theoretical bifurcation value $\alpha_o = -2$, as indicated in the lower panel in Figure 3.3;

• in both cases, the nonlinear system (2.6) displays a simple type of bifurcation, which is a 3 cycles as indicated by the phase plot in Figure 3.4 and the time series plot in Figure 3.5, over a wide range of the parameter $\alpha$. 


In order to understand the nature of the resonance bifurcation, let $L = 2$, and then the instability of the steady state leads to a 1:3 resonance bifurcation. Consider the case as indicated in the lower panel in Figure 3.3(b) and let $\beta = 11, a_1 = a_2 = 1, A = 0.005, w = 0, b_1 = b_2 = 0$. For $\alpha = -1.9$ near the bifurcation value $\alpha_o$, a phase plot (in the space of $(x_{t-1}, x_t)$) for different initial values is plotted in Figure 3.4. In this case, a strong 1:3 periodic resonance bifurcation leads to two sets of period three cycles $P(p_1, p_2, p_3)$ and $S(s_1, s_2, s_3)$, having the following behaviour:

- when the initial values are close to the steady state, the solutions converge to the steady state $p^*$ and both $P$ and $S$ are unstable;
- when the initial values are not close to the steady state $p^*$, it becomes unstable and solutions converge to one of the two sets of the period three cycles, either $P$ or $S$, depending on the initial values.

The dynamics of the nonlinear system (2.6) is very similar to those found in Chiarella and He (2000).
For \( L_1 = L_2 = L = 3, 4 \), instability of the steady state leads to 1:4 and 1:5 periodic resonance bifurcations, respectively, and similar dynamics along the bifurcation boundary are also found. To illustrate the periodicity of different resonance bifurcation, time series for \( L = 2, 5 \) and 10 are plotted in Figure 3.5. Similar observation (not reported here) is also found when \( a_1 \neq a_2 \).

3.2. Case 2: \( L_1 \neq L_2 \). For \( L_1 < L_2 = L \), comparing with the case of \( L_1 = L_2 \), the local stability regions of the steady state and bifurcation boundaries for different combination of lag lengths have less clear feature and become very complicated and difficult to analyse in general. To be able to see how the window lengths of heterogeneous producers affect the stability of the steady state and bifurcation, a combination of analytical analysis and numerical simulation approach is used in the following discussion. Analytical results for \( L = \max\{L_1, L_2\} \leq 4 \) are summarized in Proposition 3.3, followed by a comparison on the local stability regions for various lags. Some numerical simulations on various types of bifurcation are employed to demonstrate the complicity of the heterogeneous ALP.

3.2.1. Local Stability and Bifurcation Analysis. For \( L_1 < L_2, L_1, L_2 = 1, 2, 3, 4 \), the local stability of the state steady and types of bifurcation\(^6\) are analysed in Appendix B and the results are summarised in the following Proposition 3.3.

**Proposition 3.3.** For the nonlinear system (2.6), assume the two types of the heterogeneous producers, with constant proportion difference \( w \), follow MAP with \( L_1 < L_2, L_1, L_2 = 1, 2, 3, 4 \).

(i) For \( (L_1, L_2) = (1, 2) \),
- the steady state \( x^* \) is LAS for
  \[ (\beta_1, \beta_2) \in D_{12} = \{(\beta_1, \beta_2); 0 \leq \beta_1 < 1, 0 \leq \beta_2 < 2\}; \]
- a flip bifurcation occurs along the boundary \( \beta_1 = 1 \); and
- a Neimark-Hopf bifurcation occurs along the boundary \( \beta_2 = 2 \) with two eigenvalues
  \[ \lambda_{1,2} = e^{\pm(2\pi\theta)i}, \quad \rho = 2 \cos(2\pi\theta) \in [-2, -1]. \]

(ii) For \( (L_1, L_2) = (1, 3) \),
- the steady state \( x^* \) is LAS for
  \[ (\beta_1, \beta_2) \in D_{13} = \{(\beta_1, \beta_2); 0 \leq \beta_1, \beta_2, \beta_1 + \beta_2/3 < 1\}; \]
- a flip bifurcation occurs along the boundary \( \beta_1 + \beta_2/3 = 1 \).

(iii) For \( (L_1, L_2) = (1, 4) \),
- the steady state \( x^* \) is LAS for
  \[ (\beta_1, \beta_2) \in D_{14} = \{(\beta_1, \beta_2); 0 \leq \beta_1, \beta_2, \beta_1 < 1, \]
  \[ \Delta \equiv [1 - \beta_2/4]^2 - \beta_1(\beta_2/4)[\beta_1 + \beta_2/4(\beta_1 + \beta_2/4 - 1)] > 0\}; \]
- \( \beta_1 = 1 \) is a flip boundary;
- \( \Delta = 0 \) is a Neimark-Hopf boundary.

\(^6\)A saddle-node bifurcation occurs when there is at least one of the eigenvalue \( \lambda_{i,a} = 1 \) among all the eigenvalues satisfying \( |\lambda_i| \leq 1 \); a flip bifurcation occurs when there is at least one of the eigenvalue \( \lambda_{i,a} = -1 \) among all the eigenvalues satisfying \( |\lambda_i| \leq 1 \); a Neimark-Hopf bifurcation occurs when there exists a pair of eigenvalues \( \lambda = e^{2\pi\theta i} \) among all the eigenvalues satisfying \( |\lambda_i| \leq 1 \).
(iv) For \((L_1, L_2) = (2, 3)\),
- the steady state \(x^*\) is LAS for
  \((\beta_1, \beta_2) \in D_{23} \equiv \{(\beta_1, \beta_2); 0 \leq \beta_1 < 2, 0 \leq \beta_2 < 3\};
- along \(\beta_1 = 2\), a Neimark-Hopf and flip bifurcation occurs with \(\lambda_1 = -1\) and \(\lambda_{2,3} = e^{\pm 2\pi/3}i\);
- along \(\beta_2 = 3\), a Neimark-Hopf bifurcation occurs with \(\lambda_1 \in [-1, 1]\) and \(\lambda_{2,3} = e^{\pm 2\pi \theta}\) with \(\rho \equiv 2\cos(2\pi \theta) \in [-1, 0]\).

(v) For \((L_1, L_2) = (2, 4)\),
- the steady state \(x^*\) is LAS for
  \((\beta_1, \beta_2) \in D_{14} \equiv \{(\beta_1, \beta_2); 2\gamma_1 > 2; \gamma_2 < 1, \gamma_2(\gamma_1 + \gamma_2 - 1)^2 < (1 - \gamma_2)(1 - \gamma_1 - \gamma_2^2)\};
- along \(\beta_2 = 4\), a flip bifurcation occurs;

(vi) For \((L_1, L_2) = (3, 4)\),
- the steady state \(x^*\) is LAS for
  \((\beta_1, \beta_2) \in D_{34} \equiv \{(\beta_1, \beta_2); 0 \leq \beta_1 < 3, 0 \leq \beta_2 < 4\};
- along \(\beta_2 = 4\), a Neimark-Hopf and flip bifurcation occurs;

The local stability regions and bifurcation boundaries implied by Proposition 3.3 are plotted in Figure 3.6. In all these cases, there is no saddle-node bifurcation, and the nature of the Neimark-Hopf bifurcation is characterized by the value of \(\theta\) and therefore of \(\rho\), indicated by the following discussion.

### 3.2.2. Comparison of the Local Stability Regions

To see the effect of the various learning process, the local stability regions for different \((L_1, L_2)\) are combined together in Figure 3.7.

Comparing the stability regions of the steady state for different combination of \((L_1, L_2)\) leads to the following observations.

(i) Let
\[
D_L = D_{LL} = \{(\beta_1, \beta_2); 0 < \beta_1, \beta_2, \beta_1 + \beta_2 < L\},
\]
then,
\[
D_L \subset D_{L'} \quad \text{for } L < L'.
\]
implies that an increase of the lag length enlarges the parameter region of the local stability of the steady state.

(ii) For \(L = 1, 2, 3\),
\[
D_{LL} \subset D_{L,L+1}.
\]
In addition, the local stability regions \(D_{L,L+1}\) is significantly enlarged compared with \(D_{LL}\).

(iii) In general,
\[
D_{11} \subset D_{12}, D_{13}, D_{22} \subset D_{23}
\]
\[
D_{13} \subset D_{14} \subset D_{24} \subset D_{34}, D_{44},
\]
however,
\[
D_{12} \notin D_{13}, \quad D_{23} \notin D_{24}.
\]
and

\[ D_{12} \not\subseteq D_{22}, \quad D_{23} \not\subseteq D_{33}, \quad D_{34} \not\subseteq D_{44}. \]

Numerical analysis on the local stability region of the steady state of the nonlinear system (2.6) for (i) fixed \( L_1 = 2 \) and \( L_2 = 1, 2, \ldots, 7 \); and (ii) fixed \( L_1 = 7 \), \( L_2 = 1, 2, \ldots, 10 \) in the parameter space \((\alpha, w)\) are given in Figure 3.8(a) and (b), respectively.

Based on these observations, regarding to the local stability region, one may draw the following conclusions.
Figure 3.7. The local stability of the steady state and bifurcation regions \((L_1, L_2)\) with \(L_1 \leq L_2 = 1, 2, 3\).

Figure 3.8. Local stability regions of the steady state of the nonlinear system (2.6) for (a) \(L_1 = 2, L_2 = 1, 2, \cdots, 7\); and (b) \(L_1 = 7, L_1 = 1, 2, 3, \cdots, 10\) in the parameter \((\alpha, w)\) plane (b) \(L = 5, \alpha = -4.5\); and (a) \(L = 10, \alpha = -9\) with parameters \(\beta = 11, a_1 = 0.8, a_2 = 1, A = 0.005, b_1 = b_2 = 0\), where the boundaries for different lag \(L_2\) move from right to left as \(L_2\) increases, indicated by \(\alpha = -L_2\) when \(w = -1\).

- When \(L_2 \neq L_1 + 1\), an increase of window length (either \(L_1\) or \(L_2\)) can enlarge the parameter region of the local stability of the steady state in general (e.g., \(D_{11} \subset D_{13} \subset D_{14} \subset D_{24}, D_{22} \subset D_{24}\)).
- When \(L_1 = L_2 - 1\), an increase of \(L_1\) to \(L_2\) does not necessarily stabilise an otherwise unstable steady state for certain region of the parameters (e.g., \(D_{12} \not\subset D_{22}, D_{23} \not\subset D_{33}, D_{34} \not\subset D_{44}\)). In other words, homogeneity of the lag length \((L_1 \text{ and } L_2)\) may not have stabilising effect.
- When the difference of the different lag lengths is small, in particular, when \(L_2 - L_1 = 1\) (e.g., \((L_1, L_2) = (1, 2), (2, 3) \text{ and } (3, 4)\)), the stability regions can be significantly enlarged, compared with the homogeneous case of \(L_1 = L_2\).
As indicated by Figure 3.8, an increase in both the lag length and the population proportion for type 2 producers enlarges the stability region of the parameter $\alpha$ in general. However, when $L_2 = L_1 \pm 1$ (e.g., $L_1 = 2, L_2 = 1, 3$ in Figure 3.8(a), and $L_1 = 7, L_2 = 6, 8$ in Figure 3.8(b)), an increase of the population proportion of the type 2 producers does not necessarily enlarge the stability region for $\alpha$. In those cases, there is an optimal value in $w$ leading to the largest stability region in $\alpha$.

### 3.2.3. Types of Bifurcation and Complexity of the Dynamics under Heterogeneous ALP

Proposition 3.3 indicates that heterogeneous ALP can lead various types of bifurcation, and the variety of types of bifurcation and complexity of the dynamics is demonstrated through the case $(L_1, L_2) = (1, 2)$ in the following discussion.

Let $(L_1, L_2) = (1, 2)$, with the ALP, the characteristic equation of the steady state is given by

$$
\Gamma(\lambda) \equiv \lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_2 = 0,
$$

where $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2/2$. Based on the analysis in Appendix B (i), along the boundary $\beta_1 = 1, \beta_2 \in [0, 2]$, one of the eigenvalue $\lambda = -1$, implying that a flip bifurcation occurs along this boundary.

Along the other boundary $\beta_2 = 2, \beta_1 \in [0, 1]$, the two eigenvalues $\lambda_{1,2} = e^{\pm 2\pi \theta_1}$, satisfying

$$
\rho \equiv \lambda_1 + \lambda_2 = 2 \cos(2\pi \theta) = -(\beta_1 + \beta_2/2), \quad \lambda_1 \lambda_2 = \beta_2/2 = 1,
$$

and hence, the Neimark-Hopf bifurcation boundary is defined by

$$
\beta_1 = -1 - \rho, \quad \beta_2 = 2.
$$

It follows from $\beta_1 \in [0, 1]$ that $\rho \in [-2, -1]$. The types of Neimark-Hopf bifurcation are determined by the value of $\theta$ and hence of $\rho$. If $\theta = p/q$ is a rational fraction, then so-called $p : q$-periodic resonance occur. If $\theta$ is an irrational number, then one obtains quasi-periodic orbits. Therefore, the types of Neimark-Hopf bifurcation along the boundary are determined by the values of $\rho \in [-2, -1]$. The corresponding values of $\rho$ to having $p : q$ resonances can be found from the table in Sonis (2000).

The local stability region $D_{12}$ is transformed from the parameter space $(\beta_1, \beta_2)$ in Figure 3.6 (a) to the parameter space $(\alpha, w)$ in Figure 3.9(a) with the corresponding flip and Neimark-Hopy boundaries indicated.

Along the Neimark-Hopf bifurcation boundary, types of periodic resonance (when $\theta = p/q$) and quasi-periodic resonance (when $\theta$ is irrational) are determined by $\rho = 2 \cos(2\pi \theta) \in [-1, -2]$. Note that, by solving (3.1), $(\alpha, w)$ is related to $(\beta_1, \beta_2)$ as follows:

$$
\begin{align*}
\alpha &= -[a_1\beta_1 + a_2\beta_2] \\
w &= \frac{a_1\beta_1 - a_2\beta_2}{a_1\beta_1 + a_2\beta_2},
\end{align*}
$$

(3.4)

Table 1 sets up the corresponding parameter values of $(w, \alpha)$ which give different types of resonances (with $(p, q) = (1, 2), (1, 3), (2, 5), (3, 5), (1, 5), (4, 5))$, and one quasi-periodic orbit (with $\theta = \sqrt{2}$).
Figure 3.9. The local stability regions of the steady state of the nonlinear system (2.6) for (a) \((L_1, L_2) = (1, 2)\) and (b) \((L_1, L_2) = (1, 3)\) in \((\alpha, \omega)\) plane with parameters \(\beta = 11, a_1 = 0.8, a_2 = 1, A = 0.005, b_1 = b_2 = 0\).

<table>
<thead>
<tr>
<th>((p, q))</th>
<th>(\rho)</th>
<th>((\beta_1, \beta_2))</th>
<th>((w, \alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 2))</td>
<td>-2</td>
<td>((1, 2))</td>
<td>((-0.43, -2.8))</td>
</tr>
<tr>
<td>((1, 3))</td>
<td>-1</td>
<td>((0, 2))</td>
<td>((-1, -2))</td>
</tr>
<tr>
<td>((2, 5), (3, 5))</td>
<td>-1.618</td>
<td>((0.618, 2))</td>
<td>((-0.60357, -2.49))</td>
</tr>
<tr>
<td>(\theta = \sqrt{2})</td>
<td>-1.7164</td>
<td>((0.7164, 2))</td>
<td>((-0.554517, -2.57))</td>
</tr>
</tbody>
</table>

Table 1. Parameter values for various resonance and quasi-periodic bifurcation for ALP with \((L_1, L_2) = (1, 2)\) and \(a_1 = 0.8, a_2 = 1\).

The above local bifurcation analysis and the variety types of bifurcation along the Neimark-Hopf boundary are confirmed by our numerical simulations on the nonlinear system (2.6) when the parameter values are selected as indicated by Table 1. Points \(D, B\) and \(C\) in Figure 3.9(a) correspond to a \(1 : 3\) and \(2 : 5\) resonances, and quasi-periodic closed orbit, respectively. For the initial values near the steady state, the corresponding time series for the parameter values indicated by \(D, B\) and \(C\) in Figure 3.9(a) converge to those three time series plotted in the left panel in Figure 3.10. Corresponding to point \(D\) and \(B\), \((p, q) = (1, 3)\) and \((2, 5)\) or \((3, 5)\), respectively, and the periodicity of the cycles of the time series are clearly identified by the time series (on the left panel) and phase plot (on the right panel) in Figure 3.10. In fact, corresponding to point \(B\), the phase plot indicates clearly a two sets of period 5 cycles. Corresponding to point \(C\), \(\theta = \sqrt{2}\), solutions with initial values near the steady state converge to the quasi-periodic time series, the bottom one on the left panel. The quasi-periodicity of the time series is identified by the closed orbit of the phase plot, the bottom one on the right panel in Figure 3.10.
As a further support our bifurcation analysis for other combinations of the lag lengths, the case \((L_1, L_2) = (2, 3)\) is analysed in Appendix B, and similar results on various types of bifurcation are also found.

For fixed \(\alpha = -2.494\) and \((L_1, L_2) = (1, 2)\) and \((1, 3)\), bifurcation diagrams in parameter \(w\) is given in Figure 3.11. For \((L_1, L_2) = (1, 3)\), the local stability region of the steady state of the nonlinear system (2.6) is given in Figure 3.9(b). One can see that, for \((L_1, L_2) = (1, 3)\) and the fixed \(\alpha\), as \(w\) increases, instability of the steady state leads to a flip type of bifurcation for a wider range of parameter of \(w\), indicated in the upper panel of Figure 3.11. However, for \((L_1, L_2) = (1, 2)\), one can see that, for the fixed \(\alpha\), as \(w\) decreases (from \(w = -0.5\)), instability of the steady state leads to more complicated and richer dynamics, indicated by the bifurcation diagram over the range of \(w \in (-1, -0.6)\) in the lower panel of Figure 3.11.

4. Dynamics of the Heterogeneous Model with Finite Memory GDP

This section focuses on the dynamics of (2.6) when producers follow the GDP with finite memory and different window lengths \(L_i\). In the following discussion, we consider the case \(L_1 = L_2 = L\) first and then the case \(L_1 \neq L_2\). Because of the geometric advantage, the results are formulated in terms of \((\beta_1, \beta_2)\).

4.1. Case 1: \(L_1 = L_2 = L\). Consider first the case when both types of producer use the same window length, that is \(L_1 = L_2 = L\), but different decay rates \((\delta_1, \delta_2)\).

4.1.1. Local Stability and Bifurcation Analysis. The simplest case of \(L = 1\) can be treated as special case of GDP when the decay rate \(\delta_i = 0\), that is, agents use the traditional naive expectation, taking the latest price as their expected price for the next period. In this case, the steady state becomes unstable through a flip bifurcation,
leading to a two-period cycle of two prices, one is above and one is below the steady state price. The proof of the following results can be found in Appendix C.1.

**Proposition 4.1.** For \( L_1 = L_2 = 2 \), the local stability region \( D_{22} \) of the state steady is defined by \( D_{22} = \{(\beta_1, \beta_2) : \Delta_1 < 1, \Delta_2 < 1 \} \), where

\[
\Delta_1 = \frac{\delta_1}{1 + \delta_1} \beta_1 + \frac{\delta_2}{1 + \delta_2} \beta_2, \quad \Delta_2 = \frac{1 - \delta_1}{1 + \delta_1} \beta_1 + \frac{1 - \delta_2}{1 + \delta_2} \beta_2.
\]

Furthermore,

- a flip bifurcation occurs along the boundary \( \Delta_2 = 1 \) where two eigenvalues satisfy \( \lambda_1 = -1, \lambda_2 \in (-1, 1) \);
- a Neimark-Hopf bifurcation occurs along the boundary \( \Delta_1 = 1 \) where the two eigenvalues are given by \( \lambda_{1,2} = e^{\pm 2\pi \theta i} \), here \( \theta \) is determined by

\[
\rho \equiv 2 \cos(2\pi \theta) = - \left[ \frac{\beta_1}{1 + \delta_1} + \frac{\beta_2}{1 + \delta_2} \right].
\]  

One can see that the parameter (in terms of \((\beta_1, \beta_2)\)) region on the local stability of the steady state is enlarged as \( L \) increases from \( L = 1 \) to \( L = 2 \). This means that agents can learn the steady state price over a wide region of parameters as they follow the GDP with \( L = 2 \). However, as one can see from the following discussion, these learning process, in particular the decay rates \( \delta_i (i = 1, 2) \), can generate far more complicated dynamics when the steady state price becomes unstable. To understand
the effect of parameters $\beta_i$ and $\delta_i$ ($i = 1, 2$) on the stability of the state steady and types of bifurcation, we now undertake a more detailed analysis by considering various cases in terms of parameters $(\delta_1, \delta_2)$.

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4_1.png}
\caption{Stability region and bifurcation boundaries for $L_1 = L_2 = 2$, $\delta_1 = \delta_2 = \delta$ and $\beta = \beta_1 + \beta_2$.}
\end{figure}
\end{center}

The case $\delta_1 = \delta_2 = \delta$. In this case, it follows from Proposition 4.1 that the stability region of the state steady can be characterized by two parameters $\beta$ and $\delta$ with $D_{22} = \{(\beta_1, \beta_2) : 0 < \beta \equiv \beta_1 + \beta_2 < \bar{\beta}\}$ and $\bar{\beta} = \frac{1+\delta}{1-\delta}$ for $\delta \leq \frac{1}{2}$ and $\bar{\beta} = \frac{1+\delta}{1-\delta}$ for $\delta > \frac{1}{2}$. In this case, a flip bifurcation occurs along the boundary

$$\Gamma_1 : \beta = (1 + \delta)/(1 - \delta), \quad \delta \in [0, 1/2],$$

and a Neimark-Hopf bifurcation occurs along the boundary

$$\Gamma_2 : \beta = (1 + \delta)/\delta, \quad \delta \in (1/2, 1], \quad \rho = -1/\delta \in (-2, -1].$$

Note that functions $f(x) = \frac{1+x}{1-x}$, $g(x) = \frac{1+\delta x}{1-\delta x}$ satisfy $f'(x) > 0$, $f''(x) > 0$, $g'(x) < 0$, $g''(x) > 0$. The stability region $D_{22}$ is plotted in Figure 4.1 and it indicates that the different decay rate $\delta$ has different effects on the stability:

(i) for $\delta \in [0, \frac{1}{2}]$, the stability region $D_{22}$ in terms of the parameter $\beta$ is enlarged as $\delta$ increases, and the steady state price becomes unstable through flip bifurcation (implying a two-period cycle).

(ii) for $\delta \in [\frac{1}{2}, 1]$, the stability region $D_{22}$ in terms of the parameter $\beta$ is enlarged as $\delta$ decreases, and the steady state price becomes unstable through Neimark-Hopf bifurcation, which in turn generates either period cycle or aperiodic orbit.

(iii) for $\delta = 0$, we have the smallest parameter $\beta$ region for the local stability: $0 \leq \beta < 1$; while for $\delta = 1/2$, we have the largest parameter $\beta$ region for the local stability: $0 \leq \beta < 3$.

The case $0 \leq \delta_1, \delta_2 < 1/2$ and $\delta_1 \neq \delta_2$. In this case, it follows from Proposition 4.1 that the steady state becomes unstable through a flip bifurcation only, as indicated in Figure 4.2(a). Furthermore, as either $\delta_1$ or $\delta_2$ increases, the local stability region $D_{22}$ of the state steady with respect to parameters $(\beta_1, \beta_2)$ is enlarged, as indicated in Figure 4.3(b) where the stability region in $(\delta_1, \beta_1, \beta_2)$ is plotted for fixed $\delta_2 = 1/3$.

The case $\delta_1, \delta_2 > 1/2$ and $\delta_1 \neq \delta_2$. In this case, it follows from Proposition 4.1 that the steady state becomes unstable through a Neimark-Hopf bifurcation, as indicated in Figures 4.2(b) and 4.3(a) where the stability region in $(\delta_1, \beta_1, \beta_2)$ is plotted.
for fixed $\delta_2 = 2/3$. Along the bifurcation boundary, the nature of bifurcation is characterized by $\theta$ which satisfies (see Appendix C for the details) $\rho \equiv 2 \cos(2\pi \theta) \in (-1/\min(\delta_1, \delta_2), -1/\max(\delta_1, \delta_2))$. Say, for example, for fixed $\delta_2 = 2/3$, the region for the parameter $\rho$ varies for different $\delta_1$, as illustrated in Table 2. Different from the previous case, as either $\delta_1$ or $\delta_2$ increases, the local stability region of the parameters $(\beta_1, \beta_2)$ becomes smaller, as indicated in Figure 4.3(a).

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\rho$</th>
<th>$\delta_1$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>-2, -3/2</td>
<td>3/4</td>
<td>-3/2, -4/3</td>
</tr>
<tr>
<td>2/3</td>
<td>-3/2</td>
<td>1</td>
<td>-3/2, -1</td>
</tr>
</tbody>
</table>

**Table 2.** Parameter region for $\rho$ with fixed $\delta_2 = 2/3$ and different $\delta_1$.

The case either $0 < \delta_1 < 1/2, \delta_2 > 1/2$ or $0 < \delta_2 < 1/2, \delta_1 > 1/2$. In this case, the stability region is bounded by two bifurcation boundaries, as indicated in Figures 4.2(c), (d) and 4.3(a)-(b). The flip bifurcation boundary is defined by $\Delta_2 = 1$, while the Neimark-Hopf bifurcation boundary is defined by $\Delta_1 = 1$, along which the types of bifurcation are characterized by $\theta$ which satisfies $\rho \equiv 2 \cos(2\pi \theta) \in (-2, -1/\max(\delta_1, \delta_2))$. It is interesting to see that, unlike the previous case, the parameter $\rho$ is determined only by either $\delta_1$ (when $\delta_1 > 1/2$) or $\delta_2$ (when $\delta_2 > 1/2$). Also, the parameter region for $(\beta_1, \beta_2)$ on the local stability is enlarged as either $\delta_1$ increases and $\delta_2$ decreases or $\delta_2$ increases and $\delta_1$ decreases.

The previous Proposition 4.1 seems to indicate that as $L$ increases from 1 to 2, on the one hand, the stability region is enlarge and, on the other hand, instability leads to
a more complicated price dynamics through either flip or Hopf bifurcation. One may expect a similar dynamics would occur if we increase $L$ from 2 to 3. However, the following Proposition 4.2 indicates that this may not be the case.

**Proposition 4.2.** For $L_1 = L_2 = 3$, the local stability region $D_{33}(\beta_1, \beta_2)$ of the state steady is defined by $D_{33} = \{(\beta_1, \beta_2) : \Delta_3 < 1\}$, where

$$\Delta_3 = \frac{1 - \delta_1 + \delta_1^2}{1 + \delta_1 + \delta_1^2} \beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2} \beta_2.$$  

Furthermore, the steady state price becomes unstable through a flip bifurcation boundary defined by $\Delta_3 = 1$.

It is interesting to see that, similar to the case $L = 1$, but different from the case $L = 2$, the steady state becomes unstable only through flip bifurcation when $L = 3$. Moreover, the parameter region on the local stability is enlarged as the decay rates $\delta_i$
increase. The stability regions are plotted in Figure 4.4(a) for $\delta_1 = \delta_2 = \delta, \beta = \beta_1 + \beta_2$ and Figure 4.4(b) for $\delta_1 \neq \delta_2$ and fixed $\delta_2 = 1/2$.

A general comparison among $L = 1, 2$ and 3 may not be easy for various $\delta_1$ and $\delta_2$. However, such comparison when $\delta_1 = \delta_2 = \delta$ can lead to some insight regarding the role of the decay rate on the price dynamics. In such case, the stability condition for $L = 3$ is given by

$$\beta \equiv \beta_1 + \beta_2 < \frac{1 + \delta + \delta^2}{1 - \delta + \delta^2} \equiv H(\delta).$$

Note that $H(0) = 1, H(1) = 3, H' > 0, H'' > 0$. The stability regions for $L = 1, 2$ and 3 are plotted in Figure 4.4(c). One can see that: (i) for $\delta \in [0, 1/2]$, the parameter $\beta$ region on the local stability of the steady state is enlarged as $\delta$ increase, $L = 2$ leads to the largest stability region, and the steady state becomes unstable through a flip bifurcation; (ii) for $\delta \in (1/2, 1]$, $L = 2$ gives a larger stability region for $\delta \leq (\sqrt{5} - 1)/2$, while $L = 3$ gives a larger stability region for $\delta > (\sqrt{5} - 1)/2$. In addition, the steady state becomes unstable through a Neimark-Hopf bifurcation for $L = 2$, but a flip bifurcation for $L = 3$.

4.1.2. Dynamics of the Nonlinear System—Numerical Analysis. Guided by the above local analysis, numerical simulations are used to demonstrate the dynamics of the nonlinear system (2.6) and (2.7).

For $L = 1$, the GDP is reduced to the naive expectation and numerical simulations show the prices are either converge to the steady state price (when $\beta = \beta_1 + \beta_2 < 1$) or explode (when $\beta = \beta_1 + \beta_2 > 1$). The flip bifurcation does not lead to price oscillation and fluctuation.

For $L = 2$, the stability regions and bifurcation boundaries in terms of parameters $(\alpha, w)$ of the nonlinear system (2.6) are plotted in Figure 4.5.

- For $\delta_1 = \delta_2 = \delta$, the local stability region is bounded by a flip bifurcation boundary for $0 \leq \delta = 0.25, 0.5 \leq 1/2$ and a Neimark-Hopf bifurcation boundary for $\delta = 0.75, 1 > 1/2$ with $\rho \in [-2, -1]$, respectively, as indicated by Figure 4.5(a).
For $\delta_1 \neq \delta_2$ and a fixed $\delta_1 = 0.15$, the local stability region is bounded by a flip bifurcation boundary for $0 \leq \delta_2 = 0.15, 0.5 \leq 1/2$ and both flip and Neimark-Hopf bifurcation boundaries for $\delta_2 = 0.75 > 1/2$, as indicated in Figure 4.5(b).

![Figure 4.5](image-url)  

Figure 4.5. Local stability regions and bifurcation boundaries for $L = 2$ and (a) $\delta_1 = \delta_2 = 1/4, 1/2, 3/4, 1$; (b) $\delta_1 = 0.15, \delta_2 = 0.25, 0.5, 0.75$ with parameters $a_1 = 0.8 < a_2 = 1, A_1 = A_2 = 0.05, \beta = 11, b_1 = b_2 = 0$.

To illustrate the dynamics of the memory decay parameter, a bifurcation diagram for parameter $\delta_2$ is plotted in Figure 4.6 with parameters $\alpha = -2.5, w = -0.6, \delta_1 = 0.15, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.005, \mu = 11, b_1 = b_2 = 0$. In particular, for $\delta_2 = 0.2$ and $0.88$, the phase plots and the corresponding time series are illustrated in Figure 4.7. For $\delta_2 = 0.2$, the prices converge to a two-period cycle, characterized by the flip bifurcation, while for $\delta_2 = 0.88$, the prices converge to a closed orbit in the phase plot, which is characterized by the Neimark-Hopf bifurcation.

It is interesting to see that the local stability condition and bifurcation in Propositions 4.1-4.2 are independent of the risk aversion coefficients $A_i$ of the heterogeneous agents. This is because that they are associated with the variance, a higher order term of the linearised system of the nonlinear system at the steady state. In the above simulations in Figures 4.6 and 4.7, both the risk aversion coefficients are small, and hence the risk aversion and variance have no significant influence on the price dynamics induced from local stability analysis. When agents become more risk averse and willing to learn both mean and variance, the price dynamics are expected to be stabilized in the sense that irregular price patterns, such as quasi-periodic cycles, with higher variability may become regular, such as cycles, with lower variability. This can be verified (not reported here) for the case corresponding to the right panel in Figure 4.7, in which the steady state price becomes unstable through a Neimark-Hopf bifurcation and prices converge to aperiodic pattern characterized by the closed orbit on the phase plot for small risk aversion coefficients $A_1 = A_2 = 0.005$. As either $A_1$ or $A_2$ increases, the closed orbit becomes smaller (say for $A_1 = A_2 = A = 0.01$). However, as $A_i$
FIGURE 4.6. Bifurcation diagrams of the nonlinear system for δ₂ with parameters α = -2.5, w = -0.6, δ₁ = 0.15, a₁ = 0.8, a₂ = 1, A₁ = A₂ = 0.005, β = 11, b₁ = b₂ = 0..

FIGURE 4.7. Phase plot and time series of the nonlinear system for (a) δ₂ = 0.2 and (b) δ₂ = 0.88 with parameters α = -2.5, w = -0.6, δ₁ = 0.15, a₁ = 0.8, a₂ = 1, A₁ = A₂ = 0.005, μ = 11, b₁ = b₂ = 0..

increases further (say A = 0.05), prices converge to either aperiodic cycles (characterised by closed orbits for the phase plots) with lower variability for initial values near the steady state price or 3-period cycles with higher variability for initial values not near the steady state price. Similar price dynamics are also observed when δ₁, δ₂ > 1/2. This suggests that, when the steady state price becomes unstable through a Neimark-Hopf bifurcation, an increase in the risk aversion can stabilise otherwise unstable price patterns initially and leads to even simple price dynamics. However, this is not necessarily true when the steady state price becomes unstable through a flip bifurcation.

For a set of parameters: δ₁ = 0.15, δ₂ = 0.02, α = -2.5, β = 11, b₁ = b₂ = 0, w = -0.6, a₁ = 0.8, a₂ = 1, local stability analysis implies that the steady state price
FADING MEMORY LEARNING OF HETEROGENEOUS PRODUCERS

Figure 4.8. Bifurcation diagram in parameter $A = A_1 = A_2$ with parameters $\delta_1 = 0.15, \delta_2 = 0.02, a_1 = 0.8 < a_2 = 1, w = -0.6, \alpha = -2.5, \beta = 11, b_1 = b_2 = 0$.

becomes unstable through a flip bifurcation when $\delta_2$ is small. This can be verified for $A_1$ small (say $A_1 = 0.005$ or $0.05$), as indicated by the bifurcation diagram in parameter $A = A_1 = A_2$ in Figure 4.8. As $A$ increases, the prices converge to period-4 cycle for $A = 0.2$, period-8 cycle for $A = 0.35$, period-16 cycle for $A = 0.36$, and a strange attractor for $A_1 = 0.5$. This strange attractor and the corresponding chaotic time series generated through such flip bifurcation for $A_1 = 0.5$ are plotted in Figure 4.9.

Figure 4.9. Phase plot and time series of the nonlinear system for $\alpha = -2.5, w = -0.6, \delta_1 = 0.15, \delta_2 = 0.02, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.5, \beta = 11, b_1 = b_2 = 0$.

Based on this analysis, one can see that, risk aversion has different effect on the price dynamics when the steady state price become unstable through different types of bifurcation. When the steady state price becomes unstable through a Hopf bifurcation, as agents become more risk averse, the price dynamics become less complicated and the variability of the prices is reduced. However, when the steady state price becomes unstable through a flip bifurcation, as agents become more risk averse, the price dynamics becomes more complicated, although the variability of prices is reduced. It is in this sense that, as claimed by Boussard (1996), the source of the risk is the risk.
itself. Market price fluctuation and market failure can be generated when agents become more risk averse. This result is unexpected and interesting, and it underlies the connection between price dynamics generated by agents’ risk and types of bifurcation.

For \( L = 3 \), numerical simulations (not reported here) show that parameter region on the stability of the steady state price is enlarged as \( \delta_i \) increases. The steady state price become unstable through a flip bifurcation only, as indicated by Proposition 4.2. Figure 4.10 illustrates the phase plot of price dynamics when the steady state price is unstable. For \( \alpha = -4 \), prices converge to a two-period cycle (as indicated by the flip bifurcation), as \( \alpha \) decreases further, the attractors become two coexisting closed orbits for \( \alpha = -5 \) and \(-6\). However, for \( \alpha = -7 \), prices converge to a 10-period cycle. Furthermore, there seems no chaotic attractor generated from the flip bifurcation, unlike the case of \( L = 2 \).

\[ \]

**Figure 4.10.** Phase plot of the nonlinear system for \( L_1 = L_2 = 3 \), \( \alpha = -4, -5, -6, -7 \) and \( w = -0.6, \delta_1 = 0.8, \delta_2 = 0.5, a_1 = 0.8, a_2 = 1, A_1 = A_2 = 0.005, \mu = 11, b_1 = b_2 = 0. \]

4.2. **Case 2:** \( L_1 \neq L_2 \). Consider now the case when both types of producer use the different window length \( L_1 \neq L_2 \) and decay rates \( (\delta_1, \delta_2) \).

4.2.1. **Local Stability and Bifurcation Analysis.** When \( \delta_1 = 0 \), the GDP with \((L_1, L_2) = (2, 2)\) and \((3, 3)\) are reduced to the GDP with \((L_1, L_2) = (1, 2)\) and \((1, 3)\), respectively. Therefore, one obtains the following Corollaries 4.3-4.4 from Propositions 4.1-4.2 by taking \( \delta_1 = 0 \).

**Corollary 4.3.** For \( L_1 = 1, L_2 = 2 \), the stability region \( D_{12}(\beta_1, \beta_2) \) of the state steady is defined by

\[
D_{12} = \{ (\beta_1, \beta_2) : \Delta_4 < 1 \}
\]

for \( \delta_2 \in [0, 1/2] \) and

\[
D_{12} = \{ (\beta_1, \beta_2) : \Delta_4 < 1, \Delta_5 < 1 \}
\]
for $\delta_2 \in (1/2, 1]$, where

$$
\Delta_4 = \beta_1 + \frac{1 - \delta_2}{1 + \delta_2} \beta_2; \quad \Delta_5 = \frac{\delta_2}{1 + \delta_2} \beta_2.
$$

In addition,

- a flip bifurcation occurs along the boundary $\Delta_4 = 1$ for $\delta_2 \in [0, 1/2]$;
- both flip and Neimark-Hopf bifurcations occur along the boundary $\Delta_4 = 1$ and $\Delta_5 = 1$, respectively, for $\delta_2 \in (1/2, 1]$. Furthermore, the nature of the Neimark-Hopf bifurcation is determined by

$$
\rho \equiv 2 \cos(2\pi \theta) = -[\beta_1 + 1/\delta_2].
$$

The stability region and the bifurcation boundaries in parameters $(\delta_2, \beta_1, \beta_2)$ space are plotted in Figure 4.11(a) with $(L_1, L_2) = (1, 2)$. One can see that the stability region is bounded by a flip bifurcation surface for $\delta_2 \leq 1/2$ and both flip and Neimark-Hopf bifurcation surfaces for $\delta_2 > 1/2$.

By applying Proposition 4.2, we obtain the following result for $(L_1, L_2) = (1, 3)$. The stability region and the flip bifurcation surface are plotted in Figure 4.11(b).

**Corollary 4.4.** For $L_1 = 1, L_2 = 3$, the stability region $D_{13}(\beta_1, \beta_2)$ of the state steady is defined by

$$
D_{13} = \{(\beta_1, \beta_2) : \Delta_6 < 1\},
$$

where

$$
\Delta_6 = \beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2} \beta_2.
$$

In addition, the stability region is bounded only by a flip bifurcation boundary defined by $\Delta_6 = 1$. 

**Figure 4.11.** Stability region and bifurcation boundaries for (a) $(L_1, L_2) = (1, 2)$, and (b) $(L_1, L_2) = (1, 3)$. 


For $L_1 = 1$, comparing the stability regions between $L_2 = 2$ and $L_2 = 3$, one can verify that $D_{13} \subset D_{12}$ for $\delta_2 \in [0, (\sqrt{5} - 1)/2]$. However, for $\delta_2 \in ((\sqrt{5} - 1)/2 - 1, 1]$, $D_{12} \subset D_{13}$ when $\beta_1 \leq \beta_1^*$ and $D_{13} \subset D_{12}$ when $\beta_1 \geq \beta_1^*$, where $\beta_1^* = 1 - (1 + \delta_2^*/(\delta_1(1 + \delta_2 + \delta_2^*)])$.

For $(L_1, L_2) = (2, 3)$, the following result can be obtained (see Appendix C.2 for the proof).

**Proposition 4.5.** For $L_1 = 2, L_2 = 3$, the stability region $D_{23}(\beta_1, \beta_2)$ of the state steady is defined by $D_{23} = \{(\beta_1, \beta_2) : \Delta_1 < 1\}$ for $\delta_1 \in [0, 1/2]$ and $D_{23} = \{(\beta_1, \beta_2) : \Delta_1 < 1, \Delta_8 < 1\}$ for $\delta_1 \in (1/2, 1]$, where

\[
\Delta_8 = \frac{\delta_1}{1 + \delta_1} \beta_1 + \frac{\delta_2}{1 + \delta_2 + \delta_2^*} \beta_2 - \frac{\delta_2 \beta_2}{1 + \delta_2 + \delta_2^*} \left( \frac{\beta_1}{1 + \delta_1} + \frac{(1 - \delta_2^*) \beta_2}{1 + \delta_2 + \delta_2^*} \right).
\]

Furthermore,

- a flip bifurcation occurs along the boundary $\Delta_7 = 1$ for $\delta_1 \in [0, 1/2]$;
- both flip and Neimark-Hopf bifurcations occur along the boundary $\Delta_7 = 1$ and $\Delta_8 = 1$, respectively, for $\delta_1 \in (1/2, 1]$.

Because of the nonlinearity of $\beta_1$ in $\Delta_8$, it is not easy to get a complete geometric relation for $L_1 = 2, L_2 = 3$ and related discussion is conduct by using numerical simulation in the following subsection.

4.2.2. Dynamics of the Nonlinear System—Numerical Analysis. For $(L_1, L_2) = (2, 3)$, we choose a set of parameters $\delta_1 = 0.15, \delta_2 = 0.3, \beta = 11, b_1 = b_2 = 0, \omega = -0.6, a_1 = 0.8, a_2 = 1$. Since $\delta_1 < 1/2$, the steady state become unstable through a flip bifurcation. It is found that the price dynamics generated through bifurcation parameter $\alpha$ is different from that through the risk aversion coefficients.

For fixed risk aversion coefficients $A_1 = A_2 = 0.005$, the price dynamics generated through the bifurcation parameter $\alpha$ is similar to the case of $(L_1, L_2) = (1, 3)$. That is, as $\alpha$ decreases, the steady state price becomes unstable and prices converge to 2-period cycle, and then to aperiodic cycles (characterised by two coexisting closed orbits), and then to simple periodic cycles again. In addition, the variability of the prices is also increasing as $\alpha$ decreases.

For fixed $\alpha = -4$, changing of the risk aversion coefficients can generate a very rich dynamics. For fixed $A_1 = 0.05$, the bifurcation diagram in parameter $A_2$ is plotted in Figure 4.12. One can see that various types of cycles and strange attractors can be generated as agents become more risk averse.

Instead of $\delta_1 = 0.15 < 1/2$, we can select $\delta_1 = 0.6 > 1/2$. In this case, the steady state price can become unstable through either a flip or Hopf bifurcation. A similar price pattern and bifurcation routine to complicated price dynamics can be observed for changing the risk aversion coefficients.

5. Dynamics of the Heterogeneous Model with Infinite Memory GDP

From the discussion in the previous section, we can see that the lags involved in the GDP can have different effect on the stability of the steady state price and price...
dynamics. In this section, we consider a limiting case when both lags tend to infinite.
Let \(\delta_i\) be the decay rate of agent \(i\)'s memory. Then it follows from (2.9) that the conditional mean \(m_{i,t}\) and variance \(v_{i,t}\) are given by

\[
\begin{align*}
    m_{1,t} &= \delta_1 m_{1,t-1} + (1 - \delta_1)p_{t-1} \\
    m_{2,t} &= \delta_2 m_{2,t-1} + (1 - \delta_2)p_{t-1} \\
    v_{1,t} &= \delta_1 v_{1,t-1} + \delta_1(1 - \delta_1)(p_t - m_{1,t-1})^2 \\
    v_{2,t} &= \delta_2 v_{2,t-1} + \delta_2(1 - \delta_2)(p_t - m_{1,t-1})^2.
\end{align*}
\]

(5.1)

Let \(x_t = m_{1,t}; y_t = m_{2,t}; z_t = v_{1,t}; u_t = v_{2,t}\). Then, under the GDP with infinite memory (5.1), the nonlinear system (2.6) is equivalent to the following 5-dimensional system

\[
\begin{align*}
    p_t &= f(p, x, y, z, u)_{t-1} \\
    x_t &= \delta_1 x_{t-1} + (1 - \delta_1)p_{t-1} \\
    y_t &= \delta_2 y_{t-1} + (1 - \delta_2)p_{t-1} \\
    z_t &= \delta_1 z_{t-1} + \delta_1(1 - \delta_1)(p_t - x_{t-1})^2 \\
    u_t &= \delta_2 u_{t-1} + \delta_2(1 - \delta_2)(p_t - y_{t-1})^2.
\end{align*}
\]

(5.2)

where

\[
f(p, x, y, z, u) = \beta + \frac{\alpha}{2} \left[ (1 + w) \frac{x - b_1}{a_1 + 2A_1z} + (1 - w) \frac{y - b_2}{a_2 + 2A_2z} \right].
\]

The proof of the following result on the local stability and bifurcation can be found in Appendix D.
Proposition 5.1. The steady state price $p^*$ is LAS if

$$\left[\delta_1\beta_2(1 - \delta_2) + \delta_2\beta_1(1 - \delta_1) - \frac{\delta_1 + \delta_2}{2}\right]^2 + \beta_2(1 - \delta_2) + \beta_1(1 - \delta_1) < 1 + \frac{(\delta_1 - \delta_2)^2}{4}. \quad (5.3)$$

Furthermore, the steady state becomes unstable through a Neimark-Hopf bifurcation and the nature of the Neimark-Hopf bifurcation is determined by

$$\rho = 2 \cos(2\pi\theta) = \delta_1[1 - \beta_2(1 - \delta_2)] + \delta_2[1 - \beta_1(1 - \delta_1)]. \quad (5.4)$$

In particular, when $\delta_1 = \delta_2 = \delta$, the steady state is stable if $\beta = \beta_1 + \beta_2 < 1/(1 - \delta)$ and the steady state becomes unstable through a Neimark-Hopf bifurcation with $\rho = \delta \in [0, 1)$.

It is interesting to see that, when the memory is infinite, the steady state become unstable through a Neimark-Hopf bifurcation only. It may not be easy to see the effect of the decay rates on the stability region from condition (5.3), but condition (5.4) when $\delta_1 = \delta_2 = \delta$ indicates that the parameter region for $\beta = \beta_1 + \beta_2$ on the local stability is enlarged as $\delta$ increases, as shown in Figure 5.1(a). In addition, the parameter region on the stability becomes unbounded as $\delta \to 1$. This general feature is also hold when $\delta_1 \neq \delta_2$ and this can be verified by numerical plot of the bifurcation surface (not reported here). Hence the stability region is enlarged as the decay rates increase.

For $\delta_1 = \delta_2 = \delta$, a comparison between $L_1 = L_2 = L = 1, 2, 3, \infty$ is plotted in Figure 5.1(b). One can see that, for small memory decay rate $\delta$, the stability region may not be enlarged as $L$ increased from finite to infinite. However, this is indeed the case as the memory decay rate $\delta$ is close to 1. Therefore, loosely speaking, high decay rate with long memory can improve the stability of the steady state price.

Numerical simulations can be used to show various price dynamics when the steady state price becomes unstable and it is found that the price dynamics is more dependent on the decay rates, rather than the risk aversion coefficients. For a set of parameters: $\beta = 11, w = 0, a_1 = 0.8, a_2 = 1, b_1 = b_2 = 0$, we have the following observations.
When both the decay rates are high, say $\delta_1 = 0.6, \delta_2 = 0.9$, the steady state price becomes unstable when $\alpha$ is small, say $\alpha = -8$. As $\alpha$ decreases further, prices oscillate quasi-periodically, characterised by closed orbits in the phase plot, with high variability, indicated by Figure 5.2(a). Also, for fixed $\alpha$, a sufficient high $\delta_1$ (close to 1) can lead an otherwise unstable price dynamics to converge to the steady state price, as indicated by the above local stability analysis.

For fixed $\alpha = -10, \delta_1 = 0.2, \delta_2 = 0.9$ and $A_1 = A_2 = 0.05, w = 0, a_1 = 0.8, a_2 = 1, b_1 = b_2 = 0$.

For the GDP with infinite memory and (a) $\alpha = -20, \delta_1 = 0.6$; (b) $\alpha = -10, \delta_1 = 0.2$ and $A_1 = A_2 = 0.05, w = 0, \delta_2 = 0.9, a_1 = 0.8, a_2 = 1, \beta = 11, b_1 = b_2 = 0$.

For fixed $\alpha = -10, \delta_1 = 0.2, \delta_2 = 0.9$ and $A_1 = 0.05$, prices converge to some strange attractors for a wide range of $A_2$ (say $A_2 \in (0.05, 2)$), as shown in Figure 5.2(b) for $A_1 = 0.05$. However, for fixed $A_2$, say $A_2 = 0.05$, as $A_1$ increases from 0.05 up to 2, prices in the phase plane converge to strange attractors for $A_1$ small (say, $(A_1 = 0.05, 0.8)$), and then to a 5-period cycle for $A_1 = 1.2$, and then to a strange attractor for $A_1 = 1.5$. The bifurcation diagram for the parameter $A_1$ is plotted in Figure 5.3. This indicates that when agents have infinite memory, the risk aversion coefficient has no significant influence on the price dynamics when agents have high decay rate (and in particular, when agents have almost full memory over the whole history of price). However such influence can be significant when agents have a low decay rate.

For the GDP with finite memory discussed in the previous section, some of the regular or strange attractors are generated through bifurcation with certain period cycles. However, for the GDP with infinite memory, such attractor may have no connection with such periodic-cycle-induced bifurcation, as shown in Figure 5.2(a).

6. CONCLUSIONS

In this paper we have introduced a heterogeneous GDP learning mechanism into the traditional cobweb model with risk averse heterogeneous agents by allowing producers to learn both mean and variance with different geometric decay rate and different
Figure 5.3. Bifurcation diagram of the nonlinear system for GDP with infinite memory for parameter $A_1$, here $\alpha = -10, \delta_1 = 0.2, \delta_2 = 0.9, A_2 = 0.05, w = 0, a_1 = 0.8, a_2 = 1, \beta = 11, b_1 = b_2 = 0$.

memory. For a class of nonlinear forward-looking models with homogeneous agents, Barucci (2000, 2001) shows that, when the memory is infinite, the memory decay rate plays a stabilizing role in the sense that increasing the decay rate of the learning process the parameters stability region of a stationary rational expectation equilibrium becomes larger and eliminate cycles and chaotic attractors created through flip bifurcation, but not Hopf bifurcation. We have shown in this paper that the memory decay rate plays a similar stabilizing role and complicated price dynamics can be created through Neimark-Hopf bifurcation, not flip bifurcation, when memory is infinite and agents are heterogeneous. However, when memory is finite, we show that the decay rate of the GDP of heterogeneous producers plays a complicated role on the pricing dynamics. When both the lag lengths are odd, increasing of the decay rate enlarges the parameters region of the stability of the steady state and complicated price dynamics can only be created through flip bifurcation. However when both the lag lengths are not odd, there exists a critical value (between 0 and 1) such that, when the decay rate is below the critical value, the decay rate plays the stabilizing role and, for the decay rate is above the critical value, the decay rate plays a destabilizing role in the sense that the parameters region of the local stability of the steady state becomes smaller as the decay rate increases. In addition, (quasi)periodic cycles and strange attractors can be created through flip bifurcations when the decay rate is below the critical value and through Neimark-Hopf bifurcations when the decay rate is above the critical value. It is also found that the source of risk is the risk itself in the sense that the behaviour of producers in response to risk can generate complicated price dynamics and market failure.

The heterogeneous GDP considered in this paper are some of the simplest learning processes and the analysis has shown how they yield very rich dynamics in terms of the stability, bifurcation and routes to complicated dynamics. It is found that the market fractions of heterogeneous agents plays an important role. It would be very interesting
to see how the price dynamics are changed when different types of learning schemes
(such as naive expectation, ALP and GDP) are competing each other and agents update
their beliefs based on certain fitness measures, as in Brock and Hommes (1997). In
practice, agents revise their expectations by adapting the decay rate in accordance to
observations. How the GDP learning affects the dynamics in general is a question left
for future work.

Appendix A. MEAN AND VARIANCE OF GDP WITH INFINITE MEMORY

Let \( m_t \) and \( v_t \) be the mean and variance of the GDP with infinite length \( L \), that is

\[
\begin{align*}
\{ m_{t-1} &= B[p_{t-1} + \delta p_{t-2} + \cdots + \delta^{L-1} p_{t-L}], \\
\quad v_{t-1} &= B((p_{t-1} - m_{t-1})^2 + \delta(p_{t-2} - m_{t-1})^2 + \cdots + \delta^{L-1}(p_{t-L} - m_{t-1})^2),
\end{align*}
\]

(A.1)

where \( B = (1 - \delta)/(1 - \delta^L) \) for \( \delta \in [0, 1) \) and \( B = 1/L \) for \( \delta = 1 \). The mean process
\( m_t \) can be rearranged as follows:

\[
m_t = B[p_t - \delta^{L} p_{t-L}] + \delta m_{t-1}.
\]

Then for \( \delta \in [0, 1) \), as \( L \to \infty \), the limiting mean process is given by

\[
m_t = (1 - \delta)p_t + \delta m_{t-1},
\]

which can be written as follows

\[
m_t - m_{t-1} = (1 - \delta)(p_t - m_{t-1}) \tag{A.2}
\]

or

\[
m_t - p_t = \delta(m_{t-1} - p_t). \tag{A.3}
\]

For the variance process, from

\[
v_t = B[(p_t - m_t)^2 + \delta(p_{t-1} - m_{t})^2 + \cdots + \delta^{L-1}(p_{t-(L-1)} - m_{t})^2],
\]

we have

\[
v_t - \delta v_{t-1} = B[(p_t - m_t)^2 + \delta[(p_{t-1} - m_t)^2 - (p_{t-1} - m_{t-1})^2] + \cdots + \delta^{L-1}[(p_{t-(L-1)} - m_t)^2 - (p_{t-(L-1)} - m_{t-1})^2] - \delta^{L}(p_{t-L} - m_{t-1})^2],
\]

which can be rewritten as follows:

\[
v_t - \delta v_{t-1} = B(p_t - m_t)^2 - B\delta^{L}(p_{t-L} - m_{t-1})^2 + B\delta[(p_{t-1} - m_t) + (p_{t-2} - m_{t-1})]m_{t-1} - m_t + \cdots + \delta^{L-1}[(p_{t-(L-1)} - m_t) + (p_{t-(L-1)} - m_{t-1})]m_{t-1} - m_t
\]

\[
= B(p_t - m_t)^2 - B\delta^{L}(p_{t-L} - m_{t-1})^2 + (m_{t-1} - m_t)[B\delta(p_{t-1} - m_t) + \delta^2(p_{t-2} - m_t) + \cdots + \delta^{L-1}(p_{t-(L-1)} - m_t)]
\]

\[
= B(p_t - m_t)^2 - B\delta^{L}(p_{t-L} - m_t)^2 + (m_{t-1} - m_t)[-B(p_t - m_t) - B\delta^{L}(p_{t-L} - m_t)].
\]
Note that, for $\delta \in [0, 1)$, as $L \to \infty$, $B = (1 - \delta)/(1 - \delta^L) \to 1 - \delta$ and, using (A.3),

$$p_{t-L} - m_t = \delta(p_{t-L} - m_{t-1}) = \delta^2(p_{t-L} - m_{t-2}) = \cdots = \delta^L(p_{t-L} - m_{t-L}) \to 0.$$  

Therefore the limiting variance process is given by

\begin{align*}
v_t - \delta v_{t-1} &= (1 - \delta)(p_t - m_t)^2 + (m_{t-1} - m_t)[-(1 - \delta)(p_t - m_t)] \\
&= (1 - \delta)(p_t - m_t)[(p_t - m_t) + (m_t - m_{t-1})] \\
&= (1 - \delta)(p_t - m_t)(p_t - m_{t-1}),
\end{align*}

that is,

$$v_t = \delta v_{t-1} + (1 - \delta)(p_t - m_t)(p_t - m_{t-1}). \tag{A.4}$$

Based on the above argument, for $\delta \in [0, 1)$, the limiting process (as $L \to \infty$) of the mean and variance are given by

\begin{align*}
m_t &= \delta m_{t-1} + (1 - \delta)p_t \\
v_t &= \delta v_{t-1} + (1 - \delta)(p_t - m_t)(p_t - m_{t-1}) \\
&= \delta v_{t-1} + \delta(1 - \delta)(p_t - m_{t-1})^2. \tag{A.5}
\end{align*}

Appendix B. Characteristic Equation and Stability Analysis of the Heterogeneous Model with ALP

B.1. Characteristic Equation of the Heterogeneous GDP Model with Finite Memory. When the memory is finite, the heterogeneous GDP can be written as follows:

\begin{align*}
\bar{p}_{i,t} &= \sum_{j=1}^{L_i} w_{ij} p_{t-j}, \\
\bar{v}_{i,t} &= \sum_{j=1}^{L_i} w_{ij} [\bar{p}_{i,t} - p_{t-j}]^2,
\end{align*} \tag{B.1}

in which, $w_{ij} = B_i \delta^{j-1} (i = 1, 2$ and $j = 1, \cdots, L_i)$. Let

\begin{align*}
x_{1,t} &= p_t \\
x_{2,t} &= p_{t-1} \\
x_{3,t} &= p_{t-2} \\
&\vdots \\
x_{L,t} &= p_{t-(L-1)},
\end{align*}

where $L = \max\{L_1, L_2\}$. Then, (2.6) with finite memory GDP is equivalent to the following $L$-dimensional difference system

\begin{align*}
x_{1,t+1} &= f(x_t) \\
x_{2,t+1} &= x_{1,t} \\
&\vdots \\
x_{L,t+1} &= x_{L-1,t}, \tag{B.2}
\end{align*}

where

\begin{align*}
f(x_t) &= \beta + \frac{a}{2} (1 + w) \frac{x_{1,t} - b_1}{a_1 + 2A_1 v_{1,t}} + \frac{a}{2} (1 - w) \frac{x_{2,t} - b_2}{a_2 + 2A_2 v_{2,t}} \\
x_t &= (x_{1,t}, x_{2,t}, \cdots, x_{L,t}) \\
\bar{x}_{i,t} &= \sum_{j=1}^{L_i} w_{ij} x_{j,t} \\
\bar{v}_{i,t} &= \sum_{j=1}^{L_i} w_{ij} [\bar{x}_{i,t} - x_{j,t}]^2.
\end{align*}
At the steady state $p^*, \bar{x}_1 = \bar{x}_2 = p^*$ and $\bar{v}_1 = \bar{v}_2 = 0$. Without loss generality, it is assumed that $L_1 \leq L_2$ and then $L = L_2$. Evaluating function $f(x_t)$ at the steady state, one obtain that
\[
\frac{\partial f}{\partial x_j} = \frac{\alpha}{2} \left[ (1 + w) \frac{1}{a_1} w_{1j} + (1 - w) \frac{1}{a_2} w_{2j} \right] - [w_{1j} \beta_1 + w_{2j} \beta_2]
\]
for $j = 1, \ldots, L_1$ and $\frac{\partial f}{\partial x_j} = -w_{2j} \beta_2$ for $j = L_1 + 1, \ldots, L$. Therefore the corresponding characteristic equation is given by
\[
\Gamma(\lambda) \equiv \lambda^L + \sum_{j=1}^{L_1} [w_{1j} \beta_1 + w_{2j} \beta_2] \lambda^{L-j} + \sum_{j=L_1+1}^{L} w_{2j} \beta_2 \lambda^{L-j}. \tag{B.3}
\]
In particular, for the GDP, it follows from $w_{ij} = B_t \delta^{i-1}$ with $B_t = (1 - \delta_t)/(1 - \delta_t L_t)$, $L_1 \leq L_2$ and (B.3) that
\[
\Gamma(\lambda) \equiv \lambda^L + \sum_{j=1}^{L_1} \left[ \beta_1 B_t \delta^{i-1} + \beta_2 B_2 \delta^{i-1} \right] \lambda^{L-j} + \sum_{j=L_1+1}^{L} \beta_2 B_2 \delta^{i-1} \lambda^{L-j} = 0. \tag{B.4}
\]
As a special case, the corresponding characteristic equation with ALP is given by
\[
\Gamma(\lambda) \equiv \lambda^L + \sum_{j=1}^{L_1} \left[ \beta_1 \frac{1}{L_1} + \beta_2 \frac{1}{L_2} \right] \lambda^{L-j} + \sum_{j=L_1+1}^{L} \lambda^{L-j} = 0. \tag{B.5}
\]

B.2. Proof of Proposition 3.1. For $L_1 = L_2 = L$, let
\[
\gamma = \gamma_1 + \gamma_2 = -\frac{\alpha}{2} \left( \frac{1+w}{a_1} + \frac{1-w}{a_2} \right) \frac{1}{L},
\]
Then
\[
\Gamma(\lambda) = \lambda^L + \gamma [\lambda^{L-1} + \cdots + \lambda + 1] = 0.
\]
The result is then follows from Lemma in Chiarella and He (2003a).

B.3. Proof of Corollary 3.2. With ALP, for $L_1 = L_2 = L$, the steady state is stable for $0 \leq \beta_1 + \beta_2 < L$ and bifurcation boundary is given by
\[
\beta_1 + \beta_2 = L,
\]
which can be written, in terms of $\alpha$ and $w$ (and $a_1, a_2$ as well), as follows:
\[
\alpha = F(w) \equiv -\frac{2L}{(1+w)/a_1 + (1-w)/a_2}.
\]
The relation $\alpha = F(w)$ defines a nonlinear function of $w$. Note that
\[
F'(w) = \frac{2L}{[(1+w)/a_1+(1-w)/a_2]^2} \frac{1}{a_1 - a_2},
\]
\[
F''(w) = -\frac{4L}{[(1+w)/a_1+(1-w)/a_2]^3} \frac{1}{a_1 - a_2}^2 < 0.
\]
Hence the bifurcation boundary has the following shapes on $(w, \alpha)$ plane:
- the boundary is defined by $\alpha = -L$ for $w \in [-1, 1]$ if $a_1 = a_2$;
- the boundary is an increasing concave function of $w$ for $a_1 < a_1$;
- the boundary is a decreasing concave function of $w$ for $a_1 > a_1$.
In addition, $\alpha = -a_2 L$ for $w = -1$ and $\alpha = -a_1 L$ for $w = 1$. Hence, for fixed $a_1, a_2$, the parameter $\alpha$ region for the local stability of the state steady is enlarged as the lag length $L$ increases. In other word, increase of window lag can stabilise an otherwise unstable steady state.

**B.4. Proof of Proposition 3.3.** The characteristic equation for the ALP is given by equation (B.5) for general lag lengths $L_1$ and $L_2$.

(i) For $(L_1, L_2) = (1, 2)$,

$$
\gamma_1 = -\frac{\alpha}{2a_1} (1 + w) = \beta_1, \quad \gamma_2 = -\frac{\alpha}{4a_2} (1 - w) = \frac{\beta_2}{2},
$$

and the characteristic equation is given by

$$
\Gamma(\lambda) = \lambda^2 + (\gamma_1 + \gamma_2) \lambda + \gamma_2 = 0.
$$

- A saddle-node bifurcation would occur if there is at least one of the eigenvalue $\lambda_i = 1$ among all the eigenvalues satisfying $|\lambda_i| \leq 1$. For $\lambda = 1$, $\Gamma(1) = 1 + (\gamma_1 + \gamma_2) + \gamma_2 = 1 + \gamma_1 + 2\gamma_2 > 0$ and hence one can conclude that there is no saddle-node bifurcation.

- A flip bifurcation would occur if there is at least one of the eigenvalue $\lambda_i = -1$ among all the eigenvalues satisfying $|\lambda_i| \leq 1$. When $\lambda = -1$, $\Gamma(-1) = 1 - (\gamma_1 + \gamma_2) + \gamma_2 = 1 - \gamma_1 = 0$ is equivalent to $\beta_1 = 1$. Hence, along the boundary $\beta_1 = 1$, flip bifurcations occur.

- A Neimark-Hopf bifurcation would occur if there exists a pair of eigenvalues $\lambda = e^{2\pi i \theta}$ among all the eigenvalues satisfying $|\lambda_i| \leq 1$. Let $\lambda_{1,2} = \cos(2\pi \theta) \pm i \sin(2\pi \theta)$, and hence $\gamma_2 = 1$, which is equivalent to $\beta_2 = 2$. Let $\rho = 2 \cos(2\pi \theta)$.

Then,

$$
\rho = -(\gamma_1 + \gamma_2) = -(\gamma_1 + 1) = 2 \cos(2\pi \theta)
$$

$$
\gamma_1 = -1 - 2 \cos(2\pi \theta) = -1 - \rho.
$$

Since $\gamma_1 > 0$ and $\gamma_1 = 1$ corresponds to a flip bifurcation boundary, $\gamma_1 \in [0, 1]$, and hence it follows from $\rho = -(1 + \gamma_1)$ that $\rho \in [-2, -1]$. Therefore, along the Neimark-Hopf boundary,

$$
\lambda_{1,2} = e^{\pm (2\pi i \theta)}, \quad \rho = 2 \cos(2\pi \theta) \in [-2, -1].
$$

(ii) For $(L_1, L_2) = (1, 3)$,

$$
\gamma_1 = -\frac{\alpha}{2a_1} (1 + w) = \beta_1, \quad \gamma_2 = -\frac{\alpha}{6a_2} (1 - w) = \frac{\beta_2}{3}.
$$

The characteristic equation has the following form:

$$
\Gamma(\lambda) = \lambda^3 + (\gamma_1 + \gamma_2) \lambda^2 + \gamma_2 \lambda + \gamma_2
$$

- Since $\Gamma(1) = 1 + (\gamma_1 + \gamma_2) + \gamma_2 + 2\gamma_2 > 0$, there is no saddle-node bifurcation.
– For $\lambda = -1$, $(-1)^3 \Gamma(-1) = 1 - (\gamma_1 + \gamma_2) + \gamma_2 - \gamma_2 = 1 - (\gamma_1 + \gamma_2) = 0$ corresponds to $\gamma_1 + \gamma_2 = 1$, or equivalently, $\beta_1 + \frac{\beta_2}{3} = 1$, which leads to a flip bifurcation boundary.
– The Neimark-Hopf boundary would occur if $\lambda_{1,2} = e^{\pm 2\pi i}$ and $\lambda_3 = r_o \in (-1, 1)$. This implies that
\[
\begin{align*}
\gamma_2 &= -(\lambda_1 + \lambda_2 + \lambda_3) = -2 \cos(2\pi \theta) - r_0 = -\rho - r_0 \\
\gamma_1 + \gamma_2 &= 1 + r_0 \rho \\
\gamma_2 &= -r_0
\end{align*}
\]
leading to $\rho = 0$. Hence there is no Neimark-Hopf bifurcation.

(iii) For $L_1, L_2) = (1, 4)$,
\[
\gamma_1 = \beta_1, \quad \gamma_2 = \frac{\beta_2}{4}
\]
and
\[
\Gamma(\lambda) = \lambda^4 + (\gamma_1 + \gamma_2)\lambda^3 + \gamma_2(\lambda^2 + \lambda + 1).
\]
Following Jury’s test, $|\lambda_i| < 1$ if
(i) $\Gamma(1) = 1 + \beta_1 + \beta_2 > 0$;
(ii) $(-1)^3 \Gamma(-1) = 1 - \gamma_1 > 0$;
(iii) $|B_3^+| > 0, |B_3^-| > 0$ and $1 + \gamma_2 = 1 \pm \beta_2/4 > 0$, where
\[
B_3^\pm = \begin{bmatrix}
1 & 0 & 0 \\
\gamma_1 + \gamma_2 & 1 & 0 \\
\gamma_2 & \gamma_1 + \gamma_2 & 1
\end{bmatrix} \pm \begin{bmatrix}
0 & 0 & \gamma_2 \\
0 & \gamma_2 & \gamma_2 \\
\gamma_2 & \gamma_2 & \gamma_2
\end{bmatrix}.
\]
Condition (i) implies that there is no saddle-node bifurcation; condition (ii) implies that the flip bifurcation boundary is given by $\gamma_1 = \beta_1 = 1$. From condition (iii), $1 + \gamma_2 = 1 \pm \beta_2/4 > 0$ is equivalent to $\beta_2 < 4$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, it can be shown that $|B_3^+| > 0$ implies $|B_3^-| > 0$. Note that $|B_3^-| > 0$ if
\[
\Delta \equiv (1 - \gamma_2)^2 - \gamma_1 \gamma_2 (\gamma_1 + \gamma_2 - 1) > 0.
\]
Therefore, the steady state is LAS if
\[
\gamma_1 < 1, \quad \gamma_2 < 1, \quad \Delta > 0.
\]
Furthermore, $\beta_1 = 1$ defines a flip bifurcation boundary and $\Delta = 0$ defines a Neimark-Hopf bifurcation boundary.

(iv) For $(L_1, L_2) = (2, 3)$,
\[
\gamma_1 = -\frac{\alpha}{4a_1} (1 + w) = \frac{\beta_1}{2}, \quad \gamma_2 = -\frac{\alpha}{6a_2} (1 - w) = \frac{\beta_2}{3}
\]
and
\[
\Gamma(\lambda) = \lambda^3 + (\gamma_1 + \gamma_2)\lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_2 = 0
\]
– Since $\Gamma(1) = 1 + 2(\gamma_1 + \gamma_2) + \gamma_2 > 0$, there is no saddle-node bifurcation.
– It follows from $(-1)^3 \Gamma(-1) \equiv 1 - (\gamma_1 + \gamma_2) + (\gamma_1 + \gamma_2) - \gamma_2 = 0$ that $\gamma_2 = 1$. Hence $\gamma_2 = 1$ defines a flip bifurcation boundary.
Along the Neimark-Hopf boundary,
\[
\begin{align*}
\gamma_1 + \gamma_2 &= -(\lambda_1 + \lambda_2 + \lambda_3) = -2 \cos(2\pi\theta) - r_0 = -\rho - r_0 \\
\gamma_1 + \gamma_2 &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 1 + r_0 \rho \\
\gamma_2 &= -\lambda_1 \lambda_2 \lambda_3 = -r_0
\end{align*}
\]

implying
\[
\gamma_2 = -r_0, \quad \gamma_1 = 1 + r_0 (1 + \rho) = 1 - \gamma_2 (1 + \rho).
\]

Hence
\[
\begin{align*}
\gamma_1 + \gamma_2 &= -\rho - r_0 = -\rho + \gamma_2 \\
\gamma_1 &= 1 - \gamma_2 (1 + \rho),
\end{align*}
\]
leading to
\[
\begin{align*}
\gamma_1 &= -\rho \\
(1 + \rho)(\gamma_2 - 1) &= 0.
\end{align*}
\]

Hence, for \(\rho = -1\), \(\gamma_1 = 1\); for \(\rho \neq -1\), \(\gamma_2 = 1\). Therefore, there are two Neimark-Hopf boundaries:

* Along \((F_1)\): \(\gamma_1 = 1, \rho = -1\), there exist 1:3 resonance bifurcation.
* Along \((F_2)\): \(\gamma_2 = 1, \gamma_1 = -\rho, \rho \in [-1, 0]\), implying that
\[
\lambda_{1,2} = e^{\pm 2\pi i \theta} \quad \text{with} \quad \rho = 2 \cos(2\pi \theta) \in [-1, 0].
\]

(v) For \((L_1, L_2) = (2, 4)\),
\[
\begin{align*}
\gamma_1 &= \frac{\beta_1}{2}, \\
\gamma_2 &= \frac{\beta_2}{4}
\end{align*}
\]
and
\[
\Gamma(\lambda) = \lambda^4 + (\gamma_1 + \gamma_2)(\lambda^3 + \lambda^2) + \gamma_2(\lambda + 1).
\]

Using Jury’s test, \(|\lambda_1| < 1\) if
- \(\Gamma(1) = 1 + 2\gamma_1 + 4\gamma_2 > 0\);
- \((-1)^4 \Gamma(-1) = 1 > 0\);
- \(1 + \gamma_2 > 0 \iff \gamma_2 < 1\);
- \(|B_{3}^\pm| > 0\), where
\[
B_{3}^\pm = \begin{bmatrix}
1 & 0 & 0 \\
\gamma_1 + \gamma_2 & 1 & 0 \\
\gamma_1 + \gamma_2 & \gamma_1 + \gamma_2 & 1
\end{bmatrix} \pm \begin{bmatrix}
0 & 0 & \gamma_2 \\
0 & \gamma_2 & \gamma_2 \\
\gamma_2 & \gamma_2 & \gamma_1 + \gamma_2
\end{bmatrix}
\]

Note that \(|B_{3}^-| > 0\) implies \(|B_{3}^+| > 0\) and \(|B_{3}^-| > 0\) if
\[
\gamma_2(\gamma_1 + \gamma_2 - 1)^2 < (1 - \gamma_2)(1 - \gamma_1 - \gamma_2^2).
\]

The above analysis also indicates that there is no saddle-node and flip bifurcation and Neimark-Hopf bifurcation is the only type of bifurcation in this case.

(vi) For \((L_1, L_2) = (3, 4)\),
\[
\begin{align*}
\gamma_1 &= \frac{\beta_1}{3}, \\
\gamma_2 &= \frac{\beta_2}{4}
\end{align*}
\]
and
\[
\Gamma(\lambda) = \lambda^4 + (\gamma_1 + \gamma_2)(\lambda^3 + \lambda^2 + \lambda) + \gamma_2.
\]

Using Jury’s test, \(|\lambda_1| < 1\) if
- \(\Gamma(1) = 1 + 3\gamma_1 + 4\gamma_2 > 0\);
- \((-1)^4 \Gamma(-1) = 1 - \gamma_2 > 0 \iff \gamma_2 < 1\);
- \(|B_{3}^\pm| > 0\), where
\[
B_{3}^\pm = \begin{bmatrix}
1 & 0 & 0 \\
\gamma_1 + \gamma_2 & 1 & 0 \\
\gamma_1 + \gamma_2 & \gamma_1 + \gamma_2 & 1
\end{bmatrix} \pm \begin{bmatrix}
0 & 0 & \gamma_2 \\
0 & \gamma_2 & \gamma_2 \\
\gamma_2 & \gamma_2 & \gamma_1 + \gamma_2
\end{bmatrix}
\]

Note that \(|B_{3}^-| > 0\) implies \(|B_{3}^+| > 0\) and \(|B_{3}^-| > 0\) if
\[
\gamma_2(\gamma_1 + \gamma_2 - 1)^2 < (1 - \gamma_2)(1 - \gamma_1 - \gamma_2^2).
\]
\[ B_3^\pm = \begin{bmatrix} 1 & 0 & 0 \\ \gamma_1 + \gamma_2 & 1 & 0 \\ \gamma_1 + \gamma_2 & \gamma_1 + \gamma_2 & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & 0 & \gamma_2 \\ 0 & \gamma_2 & \gamma_1 + \gamma_2 \\ \gamma_2 & \gamma_1 + \gamma_2 & \gamma_1 + \gamma_2 \end{bmatrix} \]

Note that
\[ |B_3^+| = (1 - \gamma_1)(1 - \gamma_2)|2(\gamma_1 + \gamma_2) + (1 + \gamma_2)| \]
and
\[ |B_3^-| = (1 - \gamma_1)(1 - \gamma_2)^2. \]

Hence \(|\lambda_i| < 1\) if \(\gamma_1 < 1, \gamma_2 < 1\).

The above analysis also indicates that there is no saddle-node bifurcation and \(\gamma_2 = 1\) defines both flip and Neimark-Hopf bifurcations.

**B.5. Bifurcation Analysis for the ALP with \((L_1, L_2) = (2, 3)\).**

With the MAP, for \((L_1, L_2) = (2, 3)\), based on the previous analysis, along the boundary \(\beta_2 = 3, \beta_1 \in [0, 2], \lambda_1 = -1, \) and \(\lambda_{2,3} = e^{2\pi \rho i}\) with \(\rho = 2\cos(2\pi \theta) \in [-1, 0]\), implying that both flip and Neimark-Hopf bifurcations occur along this boundary. Along the boundary \(\beta_1 = 2, \beta_2 \in [0, 3], (p, q) = (1, 3)\) resonance bifurcation occurs.

The stability region \(D_{23}\) is transformed from the parameter space \((\beta_1, \beta_2)\) in Figure 3.6 (c), to the parameter space \((\alpha, w)\) in Figure B.1 with the corresponding flip and Neimark-Hopf boundaries indicated.

**Figure B.1.** Local stability regions of the steady state of the nonlinear system (2.6) for \(L_1 = 2, L_2 = 3\) on \((\alpha, w)\) plane with parameters \(\beta = 11, a_1 = 0.8, a_2 = 1, A = 0.005, b_1 = b_2 = 0\).

Table 3 sets up the corresponding parameter values for different types of resonances and quasi-periodic bifurcation along the boundary \(\beta_2 = 3\) where \(\beta_1 = -2\rho\) and hence \(\rho \in [-1, 0]\).

Time series are plotted for \((p, q) = (1, 3), (2, 7)\) and \(\rho = 1 - \sqrt{3}\) in Figure B.2(a). For \((p, q) = (1, 3)\) and \((2, 7)\), the periodicity of the cycles are clearly identified. For
\[ \theta = \sqrt{11} \]

<table>
<thead>
<tr>
<th>((p, q))</th>
<th>(\rho)</th>
<th>((\beta_1, \beta_2))</th>
<th>((w, \alpha))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 3))</td>
<td>(-1)</td>
<td>((2, 3))</td>
<td>((-0.3043, -4.6))</td>
</tr>
<tr>
<td>((1, 4))</td>
<td>(0)</td>
<td>((0, 3))</td>
<td>((-1, -3))</td>
</tr>
<tr>
<td>((2, 7), (5, 7))</td>
<td>(-0.4450)</td>
<td>((0.89, 3))</td>
<td>((-0.616351, -3.7120))</td>
</tr>
<tr>
<td>(\theta = \sqrt{11})</td>
<td>(-0.81299)</td>
<td>((1.62598, 3))</td>
<td>((-0.395093, -4.300788))</td>
</tr>
</tbody>
</table>

**TABLE 3.** Parameter values for various resonance bifurcation for MAP with \((L_1, L_2) = (2, 3)\) and \(\alpha_1 = 0.8, \alpha_2 = 1\).

\(\rho = 1 - \sqrt{3}\), aperiodic time series is obtained and a closed orbit is obtained from the phase plot of the time series in Figure B.2(b).

**Figure B.2.** (a) Time series of periodic resonances of the nonlinear system (2.6) with \((p, q) = (1, 3)\) and \((2, 7)\), and quasi-periodic resonance with \(\rho = 1 - \sqrt{3}\); (b) Phase plot of the quasi-periodic resonance for \(\theta = \sqrt{2}\) for \(L_1 = 2, L_2 = 3\) and \(\beta = 11, \alpha_1 = 0.8, \alpha_2 = 1, A = 0.005, b_1 = b_2 = 0\).

**Appendix C. Local Stability and Bifurcation Analysis of GDP with Finite Memory**

C.1. **The case \(L_1 = L_2 = L\).** When \(L_1 = L_2 = L\), one can see from (B.4) that the corresponding characteristic equation is given by

\[
\Gamma_L(\lambda) \equiv \lambda^L + \sum_{j=1}^L \left[ \beta_1 B_1 \delta_1^{j-1} + \beta_2 B_2 \delta_2^{j-1} \right] \lambda^{L-j} = 0. \tag{C.1}
\]

For \(L = 1\), \(\Gamma_1(\lambda) \equiv \lambda + [\beta_1 + \beta_2] = 0\). Hence, \(|\lambda| < 1\) holds if and only if \(\beta \equiv \beta_1 + \beta_2 < 1\). Furthermore, \(\lambda = -1\) when \(\beta = 1\), which leads to a flip bifurcation.
For $L = 2$, the characteristic equation has the form
\[ \Gamma_2(\lambda) \equiv \lambda^2 + [\beta_1 B_1 + \beta_2 B_2] \lambda + [\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2] = 0, \]
where $B_i = 1/(1 + \delta_i)$ $(i = 1, 2)$. It follows from Jury’s test that $|\lambda_i| < 1$ if and only if
(i). $\Gamma_2(1) = 1 + \beta_1 + \beta_2 > 0$;
(ii). $\Gamma_2(-1) = 1 - [\beta_1 B_1 + \beta_2 B_2] + [\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2] > 0$, which can be rewritten as
\[ \Delta_2 \equiv \frac{1 - \delta_1}{1 + \delta_1} \beta_1 + \frac{1 - \delta_2}{1 + \delta_2} \beta_2 < 1. \]
(C.2)
(iii). $\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2 < 1$, which can be rewritten as
\[ \Delta_1 \equiv \beta_1 + \frac{\delta_1}{1 + \delta_1} \beta_1 + \frac{\delta_2}{1 + \delta_2} \beta_2 < 1. \]
(C.3)

Therefore, $|\lambda_i| < 1$ if and only if (C.2) and (C.3) hold. Note that $\Gamma_2(-1) = 0$ implies that a flip bifurcation occurs when $\Delta_2 = 1$. Also, when $\lambda_1, \lambda_2 = e^{\pm 2\pi i}$, we have $\lambda_1 \lambda_2 = \beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2 = \Delta_1 = 1$ and $\lambda_1 + \lambda_2 = -[\beta_1 B_1 + \beta_2 B_2] = 2 \cos(2\pi \theta) \equiv \rho$, which implies that $\Delta_1 = 1$ leads to a Neimark-Hopf bifurcation. In addition, the nature of the bifurcation is characterised by the parameter $\theta$, which is determined by (4.1).

When the local stability region is bounded by Neimark-Hopf bifurcation, the nature of the bifurcation is characterised by values of $\rho$ which have different region for different combination of $(\delta_1, \delta_2)$.

For $1/2 \leq \delta_1, \delta_2 \leq 1$, the stability region is bounded only by the Neimark-Hopf bifurcation boundary $\Delta_1 = 1$. Then, $\rho = -1/\delta_2$ for $(\beta_1, \beta_2) = (0, [1 + \delta_2]/\delta_2)$ and $\rho = -1/\delta_1$ for $(\beta_1, \beta_2) = ([1 + \delta_1]/\delta_1, 0)$. Hence
\[ \rho \equiv 2 \cos(2\pi \theta) \in \left(-2, \frac{1}{\min(\delta_1, \delta_2)}\right). \]

For $0 \leq \delta_1 \leq 1/2, 1/2 \leq \delta_2 \leq 1$, the stability region is bounded by both flip and Neimark-Hopf bifurcation boundaries. The Neimark-Hopf bifurcation boundary corresponds to the line segment between $A : (\beta_1, \beta_2) = (0, [1 + \delta_2]/\delta_2)$ and $B$ which is the interaction point between $\Delta_1 = 1$ and $\Delta_2 = 1$, leading to $\rho = -2$. Therefore,
\[ \rho \equiv 2 \cos(2\pi \theta) \in \left(-2, \frac{1}{\max(\delta_1, \delta_2)}\right). \]

For $L = 3$, the characteristic equation has the form $\Gamma_3(\lambda) \equiv \lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0$, where
\[ c_1 = [\beta_1 B_1 + \beta_2 B_2], \quad c_2 = [\beta_1 B_1 \delta_1 + \beta_2 B_2 \delta_2], \]
\[ c_3 = [\beta_1 B_1 \delta_1^2 + \beta_2 B_2 \delta_2^2], \quad B_i = 1/[1 + \delta_i + \delta_i^2] \quad (i = 1, 2). \]

It follows from Jury’s test that $|\lambda_i| < 1$ if and only if
(i). $\Gamma_3(1) = 1 + \beta_1 + \beta_2 > 0$;
(ii). $(-1)^3 \Gamma_3(-1) > 0$, which is equivalent to
\[ \Delta_3 \equiv \frac{1 - \delta_1 + \delta_1^2}{1 + \delta_1 + \delta_1^2} \beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2} \beta_2 < 1. \]
(C.4)
(iii). \( c_2 + c_3(c_3 - c_1) < 1 \), which is equivalent to
\[
\delta_1 \gamma_1 + \delta_2 \gamma_2 + (\delta_1^2 \gamma_1 + \delta_2^2 \gamma_2)[(\delta_1^2 - 1) \gamma_1 + (\delta_2^2 - 1) \gamma_2] < 1,
\]
where \( \gamma_i = \frac{\beta_i}{1 + \delta_i + \delta_i^2} \).

(iv). \( c_2 \equiv \delta_1 \gamma_1 + \delta_2 \gamma_2 < 3 \).

It follows from \( \beta_i > 0 \), \( \delta_i \in [0, 1] \) and \( \delta_i < 1 - \delta_i + \delta_i^2 \) that condition (i) is satisfied and condition (ii) implies conditions (iii) and (iv). Hence the only condition for \( |\lambda| < 1 \) is \( \Delta_3 < 1 \). In addition, \( \lambda = -1 \) when \( \Delta_3 = 1 \), implying that the stability region is bounded by the flip bifurcation boundary defined by \( \Delta_3 = 1 \).

C.2. The Case \( (L_1, L_2) = (2, 3) \). For \( L_1 = 2, L_2 = 3 \), the characteristic equation is given by \( \Gamma_{2,3}(\lambda) \equiv \lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0 \), where
\[
c_1 = [\gamma_1 + \gamma_2], \quad c_2 = \gamma_1 \delta_1 + \gamma_2 \delta_2, \quad c_3 = \gamma_2 \delta_2^2, \quad \gamma_1 = \beta_1/[1 + \delta_1], \quad \gamma_2 = \beta_2/[1 + \delta_2 + \delta_2^2].
\]

It follows from Jury’s test that \( |\lambda| < 1 \) if and only if
\[(i). \quad \Gamma_{2,3}(1) = 1 + \beta_1 + \beta_2 > 0; \]
\[(ii). \quad (-1)^3 \Gamma_{2,3}(-1) > 0, \]
which is equivalent to
\[
\Delta_7 \equiv \frac{1 - \delta_1}{1 + \delta_1} \beta_1 + \frac{1 - \delta_2 + \delta_2^2}{1 + \delta_2 + \delta_2^2} \beta_2 < 1. \tag{C.6}
\]

(iii). \( c_2 + c_3(c_3 - c_1) < 1 \), which is equivalent to
\[
\Delta_8 \equiv \frac{\delta_1}{1 + \delta_1} \beta_1 + \frac{\delta_2}{1 + \delta_2 + \delta_2^2} \beta_2 - \delta_2 \beta_2 \left( \frac{\beta_1}{1 + \delta_1} + \frac{(1 - \delta_2^2) \beta_2}{1 + \delta_2 + \delta_2^2} \right) < 1. \tag{C.7}
\]

(iv). \( c_2 \equiv \delta_1 \gamma_1 + \delta_2 \gamma_2 < 3 \).

Note that \( \beta_i > 0, \delta_i \in [0, 1] \) and \( \delta_2 < 1 - \delta_2 + \delta_2^2 \), one can see that \( \Delta_7 < 1 \) implies condition (iv). In addition \( \lambda = -1 \) when \( \Delta_7 = 1 \) is satisfied and \( \Delta_7 < 1 \) implies \( \Delta_8 < 1 \) for \( \delta_1 \leq 1/2 \).

Appendix D. Proof of Proposition 5.1

For the system
\[
\begin{align*}
    p_t &= f_1(p, x, y, z, u)_{t-1} \\
    x_t &= \delta_1 x_{t-1} + (1 - \delta_1) p_{t-1} = f_2 \\
    y_t &= \delta_2 y_{t-1} + (1 - \delta_2) p_{t-1} = f_3 \\
    z_t &= \delta_1 z_{t-1} + \delta_1 (1 - \delta_1) (p_t - x_{t-1})^2 = f_4 \\
    u_t &= \delta_2 u_{t-1} + \delta_2 (1 - \delta_2) (p_t - y_{t-1})^2 = f_5
\end{align*}
\]
with
\[
f_1 = \beta + \frac{\alpha}{2} \left[ (1 + w) \frac{x - b_1}{a_1 + 2 A_1 z} + (1 - w) \frac{y - b_2}{a_2 + 2 A_2 z} \right],
\]
evaluating at the unique fixed point \((p_t, x_t, y_t, z_t, u_t) = (p^*, p^*, p^*, 0, 0)\):

\[
\begin{align*}
\frac{\partial f_1}{\partial p} &= 0, \\
\frac{\partial f_1}{\partial x} &= 2 \left(1 + \frac{w}{a_1}\right) - \beta_1, \\
\frac{\partial f_1}{\partial y} &= 2 \left(1 - \frac{w}{a_2}\right) - \beta_2, \\
\frac{\partial f_1}{\partial z} &= \frac{1}{2} \left(1 + w\right) - \frac{2A_1(p^* - b_1)}{a_1^2} \equiv \Delta_1^*, \\
\frac{\partial f_1}{\partial u} &= \frac{1}{2} \left(1 + w\right) - \frac{2A_2(p^* - b_2)}{a_2^2} \equiv \Delta_2^*,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial f_2}{\partial p} &= 1 - \delta_1, \quad \frac{\partial f_2}{\partial x} = \delta_1, \quad \frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial x} = \frac{\partial f_2}{\partial y} = 0, \\
\frac{\partial f_3}{\partial p} &= 1 - \delta_2, \quad \frac{\partial f_3}{\partial x} = \delta_2, \quad \frac{\partial f_3}{\partial y} = \delta_1, \quad \frac{\partial f_3}{\partial u} = 0, \\
\frac{\partial f_4}{\partial p} &= 0 = \frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial y} = \frac{\partial f_4}{\partial z} = \frac{\partial f_4}{\partial u} = 0.
\end{align*}
\]

Hence the Jacobian matrix \(J\) is given by

\[
J = \begin{bmatrix}
0 & -\beta_1 & -\beta_2 & \Delta_1^* & \Delta_2^* \\
1 - \delta_1 & \delta_1 & 0 & 0 & 0 \\
1 - \delta_2 & 0 & \delta_2 & 0 & 0 \\
0 & 0 & 0 & \delta_1 & 0 \\
0 & 0 & 0 & 0 & \delta_2
\end{bmatrix}.
\]

Thus the characteristic equation is given by

\[
\Gamma(\lambda) \equiv |\lambda I - J| = (\lambda - \delta_1)(\lambda - \delta_2)h(\lambda),
\]

where \(h(\lambda) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3\) and

\[
\begin{align*}
c_1 &= - (\delta_1 + \delta_2), \\
c_2 &= \delta_1\delta_2 + \beta_2(1 - \delta_2) + \beta_1(1 - \delta_1), \\
c_3 &= -\delta_1\beta_2(1 - \delta_2) - \delta_2\beta_1(1 - \delta_1).
\end{align*}
\]

For \(\delta_1, \delta_2 \in (0, 1)\), applying Jury’s test to \(h(\lambda) = 0\), one can see that \(|\lambda_i| < 1\) if and only if \(\pi_i > 0\), where

\[
\begin{align*}
\pi_1 &= 1 + c_1 + c_2 + c_3, \\
\pi_2 &= 1 - c_1 + c_2 - c_3, \\
\pi_3 &= 1 - c_2 + c_3(c_1 - c_3), \\
c_2 &< 3.
\end{align*}
\]

Note that

\[
\begin{align*}
\pi_1 > 0 &\iff (1 - \delta_1)(1 - \delta_2)[1 + \beta_1 + \beta_2] > 0, \\
\pi_2 > 0 &\iff \left[\frac{1 - \delta_1 \beta_1}{1 + \delta} + \frac{1 - \delta_2 \beta_2}{1 + \delta}\right] < 1, \\
\pi_3 > 0 &\iff \left[\delta_1\beta_2(1 - \delta_2) + \delta_2\beta_1(1 - \delta_1) + \frac{\delta_1 + \delta_2}{2}\right]^2 \\
&\quad + \beta_2(1 - \delta_2) + \beta_1(1 - \delta_1) < 1 + \frac{(\delta_1 - \delta_2)^2}{4}
\end{align*}
\]

and \(c_2 < 3\) is implied by \(\pi_3 > 0\). Therefore, the only condition we need for the local stability is \(\pi_3 > 0\). Furthermore, from \(h(1) = \pi_1, (-1)^3h(-1) = \pi_2\), there is no saddle-node and flip bifurcation and the only boundary of the stability region is given
by Neimark-Hopf bifurcation boundary, defined by $\pi_3 = 0$. Along the bifurcation boundary, let $\lambda_{1,2} = e^{\pm 2\pi \theta i}$, $\lambda_3 = r \in (-1, 1)$. Then it follows from

$$\begin{align*}
[\lambda_1 + \lambda_2 + \lambda_2] &= -[r + r] = -[\delta_1 + \delta_2], \\
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 &= 1 + r \rho \\
&= \delta_1 \delta_2 + \beta_1 (1 - \delta_1) + \beta_2 (1 - \delta_2), \\
\lambda_1 \lambda_2 \lambda_3 &= -r = -[\delta_1 \beta_2 (1 - \delta_2 + \delta_2 \beta_1 (1 - \delta_1)]
\end{align*}$$

that $\rho = \delta_1 [1 - \beta_2 (1 - \delta_2)] + \delta_2 [1 - \beta_1 (1 - \delta_1)]$. In particular, for $\delta_1 = \delta_2 = \delta$, the stability condition becomes $[1 - \beta (1 - \delta)] [\delta^2 \beta (1 - \delta) + (1 - \delta^2)] > 0$, which is equivalent to $\beta < 1/(1 - \delta)$, where $\beta = \beta_1 + \beta_2$. Along the bifurcation boundary, $\beta (1 - \delta) = 1$, and hence $\rho = \delta$.

REFERENCES


