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Optimal VWAP Trading Strategy and Relative Volume*

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Abstract

Volume Weighted Average Price (VWAP) for a stock is total traded value divided by total traded volume. It is a simple quality of execution measurement popular with institutional traders to measure the price impact of trading stock. This paper uses classic mean-variance optimization to develop VWAP strategies that attempt to trade at better than the market VWAP. These strategies exploit expected price drift by optimally 'front-loading' or 'back-loading' traded volume away from the minimum VWAP risk strategy.

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1 Introduction and Motivation

Volume Weighted Average Price (VWAP) trading is used by large (institutional) traders to trade large orders in financial markets. Implicit in the use of VWAP trading is the recognition that large orders traded in financial markets may trade at an inferior price compared to smaller orders. This is known as the liquidity impact cost or market impact cost of trading large orders.

VWAP orders attempt to address this cost by bench-marking the price of trading the large order against the volume weighted average price of all trades over a specific period of time (generally 1 trading day). This allows any liquidity impact costs associated with trading the large order to be quantified. VWAP trading also recognizes that the key to minimizing these costs is to breakup large orders up into a number of sub-orders executed over the VWAP period in such a way as to minimize instantaneous liquidity demand.

The VWAP price as a quality of execution measurement was first developed by Berkowitz, Logue and Noser [4]. They argue (page 99) that *'a market impact measurement system requires a benchmark price that is an unbiased estimate of prices that could be achieved in any relevant trading period by any randomly selected trader'* and then define VWAP as an appropriate benchmark that satisfies this criteria.

An important paper in modelling VWAP was written by Hizuru Konishi [15] who developed a solution to the minimum risk VWAP trading strategy for a price process modelled as Brownian motion without drift ($dP = \sigma_t dW_t$). In this paper the solution is generalized to a price process that is a continuous semimartingale, $P_t = A_t + M_t + P_0$, where A_t is price drift, M_t is a martingale and P_0 is the initial price. It is proved that price drift A_t does not contribute to VWAP risk. The relative volume process X_t is also introduced, defined as intra-day cumulative volume V_t divided by total final volume $X_t = V_t/V_T$. It is shown that VWAP is naturally defined using relative volume X_t rather than cumulative volume V_t.

The minimum VWAP risk trading problem is generalized into the optimal VWAP trading problem using a mean-variance framework. The optimal VWAP trading strategy $x_t^\star$ here becomes a function of a trader defined risk aversion coefficient λ. This is relevant because VWAP trades are often large institutional trades and the size of the VWAP trade itself may be price sensitive information that the VWAP trader can exploit for the benefit of his client. The optimal strategy is then obtained for VWAP trading which includes expected price drift $\mathbb{E}[A_t]$ over the VWAP trading period. This can be expressed in following mean-variance optimization (subject to constraints on strategy x_t) where $\mathcal{V}(x_t)$ is the difference between traded VWAP and market VWAP as a function of the trading strategy x_t:

$$x_t^\star \;=\; \max_{x_t} \left[\mathbb{E}\big[\mathcal{V}(x_t)\big] \;-\; \lambda \,\mathrm{Var}\big[\mathcal{V}(x_t)\big] \right]$$

It is shown that for all feasible VWAP trade strategies x_t there is always residual VWAP risk. This residual risk is shown to be proportional to the price variance σ^2 of the stock and variance the relative volume process $\mathrm{Var}[X_t]$. When the relative volume process variance is empirically examined in section 3 it is found to be proportional to the inverse of stock final trade count K raised to the power 0.44. This is of importance to VWAP traders because it formalizes the intuition that traded VWAP risk is lower for high turnover stocks.

$$\min_{x_t} \mathrm{Var}[\mathcal{V}(x_t)] \;\propto\; \sigma^2 \int_0^T \mathrm{Var}[X_t]\, dt \;\propto\; \frac{\sigma^2}{K^{0.44}}$$

Finally, a practical VWAP trading strategy using trading bins is examined. The additional bin-based VWAP risk from using discrete volume bins to trade VWAP is shown to be $\mathcal{O}(n^{-2})$ for a n bin approximation of the optimal continuous VWAP trading strategy $x_t^\star$.

2 Modelling VWAP

The stochastic VWAP model is based on the filtered probability space with the observed progressive filtration $\mathcal{F}_t$, $(\Omega, \mathcal{F}, \mathbb{F} = \mathcal{F}_{t\geq 0}, \mathbf{P})$. The model also defines a filtration $\mathcal{G}_t$ initially enlarged by knowledge of the final traded volume of the VWAP stock $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V_T)$. The resultant filtered probability space $(\Omega, \mathcal{F}, \mathbb{G} = \mathcal{G}_{t\geq 0}, \mathbf{P})$ is used to define VWAP using the relative volume process X_t.

2.1 A Stochastic Model of Price P_t

The price process P_t will be assumed to be a strictly positive, continuous (special) semimartingale with Doob-Meyer decomposition:

$$P_t = P_0 + A_t + M_t \qquad P_t > 0$$

Where A_t is price drift, M_t is a martingale and P_0 is the initial price.

2.2 A Stochastic Model of Relative Volume X_t

Cumulative volume arrives in the market as discrete trades, this suggests that the cumulative volume process V_t should be modelled as a marked point process. A very general model of point process is the Cox[1] point process (also called the doubly stochastic Poisson point process, a simple (no co-occurring points) point process with a general random intensity. The Cox process has been used to model trade by trade market behaviour by a number of financial market researchers including Engle and Russell [10], Engle and Lunde [10], Gouriéroux, Jasiak and Le Fol [11] and Rydberg and Shephard [18].

If trade count N_t is modelled as a Cox process, then intra-day trade count can be scaled to a relative trade count by the simple expedient of dividing the intra-day count ($N_t = a_t K$) by the final trade count ($N_T = K$). This defines the relative trade count process $R_{t,K} = N_t/N_T = a_t$. The resultant point process is no longer the Cox process as this has been transformed into a doubly stochastic binomial point process by knowledge of the final trade count enlarging the observed filtration $\mathcal{F}_t \vee \sigma(N_T)$ (McCulloch [16]).

But the object of interest when executing a VWAP trade is not relative trade count $R_{t,K}$ but the closely related relative volume X_t. This can be modelled by a marked point process where each occurrence or point is associated with a random value (the mark) representing trade volume. Thus each trade is specified by a pair of values on a product space, the time of occurrence and a mark (integer) value specifying the volume of the trade $\{t_i, v_i\} \in \mathbb{R}^+ \otimes \mathbb{Z}^+$.

[1]Named a Cox process in recognition of David Cox's 1955 [9] paper which he introduced the doubly stochastic Poisson point process.

$$V_t = \sum_{i=1}^{N_t} \Delta V_i$$

The relative volume X_t is then the ratio of a random sum specified by the doubly stochastic binomial point process as the 'ground process' over the non-random sum of all trade volumes.

$$X_t = \frac{V_t}{V_T} = \frac{\sum_{i=1}^{N_t} \Delta V_i}{\sum_{i=1}^{K} \Delta V_i}$$

The relative volume process X_t is the cumulative volume process transformed by knowledge of final volume (and thus final trade count) and is adapted to $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V_T)$. Note X_t is a semimartingale with respect to $\mathcal{G}_t$ because this filtration is enlarged by the sigma algebra generated by a random variable, final volume V_T, with a countable number of possible values (corollary 2, page 373 Protter [17]).

2.3 A Stochastic Integral Model of VWAP

One the reasons for the popularity of VWAP as a measure of order execution quality is the simplicity of it's definition - the *total value* of all[2] trades divided by the *total volume* of all trades. If P_i and ΔV_i are the price and volume respectively of the N trades in the VWAP period, then VWAP is readily computed as:

$$\text{vwap} = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{\sum_{i=1}^{N} P_i \, \Delta V_i}{\sum_{i=1}^{N} \Delta V_i}$$

Alternatively the definition of VWAP can be written in continuous time notation. Let V_t be the cumulative volume traded at time t and P_t be the

[2] Not all trades are accepted as admissible in a VWAP calculation. Admissible trades are determined by market convention and are generally on-market trades. Off-market trades and crossings are generally excluded from the VWAP calculation because these trades are often priced away from the current market and represent volume in which a *'randomly selected trader'* [4] cannot participate.

time varying price on a market that trades on the time interval $t \in [0, T]$. Then VWAP is defined by the Riemann-Stieltjes integral.

$$\text{vwap} = \frac{\text{total traded value}}{\text{total traded volume}} = \frac{1}{V_T} \int_0^T P_t \, dV_t \tag{1}$$

Examining the integral above, it is intuitive that it relates to the relative volume process $X_t = V_t/V_T$. Using the theory of initial enlargement of filtration (see Jeulin [14], Jacod [12], Yor [19] and Amendinger [2]) VWAP can be expressed in terms of X_t:

$$\text{vwap} = \int_0^T P_t \, dX_t \tag{2}$$

Proof. The assertion that the vwap random variable is the same in equations 1 and 2 under filtrations $\mathcal{F}_t$ and $\mathcal{G}_t$ respectively is proved under the assumption that the price process P_t is independent of the final volume random variable, $\sigma(P_t) \cap \sigma(V_T) = \emptyset$, $\forall t \in [0, T]$. This implies that P_t is also a $\mathcal{G}_t$ semimartingale with the same Doob-Meyer decomposition as $\mathcal{F}_t$ (theorem 2, page 364, Protter [17]). Independence with V_T implies that the price process P_t is unchanged by the enlarged filtration $\mathcal{G}_t$.

Cumulative volume V_t arrives in the market as discrete trades and is modelled as a marked point process (see section 2.2 below). Noting that V_t as a pure jump process has finite variation under filtration F_t and the enlarged filtration G_t, it is readily shown that the Riemann-Stieltjes integrals of integrand Price P_t (unchanged by the enlarged filtration) and integrator volume V_t are equivalent with respect the filtration F_t and the enlarged filtration $\mathcal{G}_t$.

Let $\tau_i, i = 1, \ldots, N_t$ be the N_t jump times for the volume process V_t on the interval [0,t] and ΔV_i be the corresponding jump magnitudes. Then the Riemann-Stieltjes integrals with respect to the filtrations $\mathcal{F}_t$ and $\mathcal{G}_t$ are equivalent to the same Riemann-Stieltjes sum because the volume jump times and magnitudes ΔV_i are the same in both filtrations and the price process is the same in both filtrations (by assumption).

$$\int_0^t P_s \, dV_s \,|\, \mathcal{F}_t = \sum_{i=1}^{N_t} P_{\tau_i} \, \Delta V_i = \int_0^t P_s \, dV_s \,|\, \mathcal{G}_t$$

Noting that the term $(1/V_T)$ is adapted to $\mathcal{G}_0$.

$$\frac{1}{V_T} \int_0^t P_s \, dV_s \,|\, \mathcal{F}_t \;=\; \int_0^t P_s \, \frac{dV_s}{V_T} \,|\, \mathcal{G}_t \;=\; \int_0^t P_s \, dX_s \,|\, \mathcal{G}_t$$

□

This is a key insight, VWAP is naturally defined using relative volume X_t rather than actual volume V_t. One implication of using relative volume is that common *relative* intraday features in the daily trading of stocks with different absolute turnovers can be exploited for VWAP trading. Also, the difference between traded VWAP and market VWAP as a function of the trading strategy $\mathcal{V}(x_t)$ is conveniently defined using relative volume.

$$\mathcal{V}(x_t) \;=\; \int_0^T P_t \, dx_t \;-\; \int_0^T P_t \, dX_t \;=\; \int_0^T P_t \, d(x_t - X_t)$$

Using integration by parts[3], this integral can be transformed into a stochastic integral and quadratic covariation.

$$\mathcal{V}(x_t) \;=\; \int_0^T P_t \, d(x_t - X_t) \;=\; P_T \,(x_t - X_T) - \int_0^T (x_t - X_{t-}) \, dP_t - [x - X, P]_T$$

Where $[x - X, P]_t$ denotes the covariation process between $x_t - X_t$ and P_t. Since the price process P_t is continuous, the relative volume X_t is assumed to be a marked point (pure jump) process and x_t is deterministic, the quadratic covariation term is zero. Also noting that $P_T(x_T - X_T) = 0$ the integration by parts equation simplifies to:

$$\mathcal{V}(x_t) \;=\; \int_0^T (X_{t-} \;-\; x_t) \, dP_t \tag{3}$$

[3] The integrand of the stochastic integral X_{t-} is a left continuous (predictable) version of the relative volume process X_t where for $\forall t$ X_{t-} is defined as the left limit of X_t, $X_{t-} = \lim_{s \uparrow t} X_s$.

3 Empirical Properties of Relative Volume X_t

Relative volume as self-normalized trade counts was analyzed in McCulloch [16], where details of empirical data collection and analysis can be found. Briefly, New York Stock Exchange (NYSE) trade data from the TAQ database was used to collect relative trade volume data of all stocks that traded from 1 June 2001 to 31 August 2001 (a total of 62 trading days[4]) for a total of 203,158 relative trade volume sample paths for all stocks. The relative trade volume data was collected in a 391 × 253 2-D histogram with time in minutes (390 minutes + 1 end-point) in the x-axis and relative volume (a prime number 251 to avoid bin boundaries, plus two end-points) in the y-axis.

3.1 Expected Relative Volume $\mathbb{E}[X_t]$ is 'S' Shaped

All professional equity traders know that markets are, on average, busy on market open and market close and less busy during the middle of the trading day. This is the classic 'U' shape in trading intensity found in all major equity markets[5] and is, by definition, the derivative of the expectation of the relative volume $d\mathbb{E}[X_t]/dt$. Figure 1 plots the expected relative volume $\mathbb{E}[X_t]$ for four groups of stocks with different ranges of trade counts on the NYSE. The expectation of relative volume $\mathbb{E}[X_t]$ can be approximated with the the following polynomial.

$$\mathbb{E}\big[X_t\big] \;\approx\; \frac{5t}{3T} - \frac{2t^2}{T^2} + \frac{4t^3}{3T^3}, \qquad t \in [0, T]. \tag{4}$$

3.2 High Turnover Stocks have Lower Var$[X_t]$

The second feature of empirical data readily seen in Figure 2 is that the low turnover stock (SUS) appears to have a higher volatility around the mean relative volume (shown with red line) than the high turnover stock (TXN).

[4] 3 July 2001 (half day trading) and 8 June 2001 (NYSE computer malfunction delayed market opening) were excluded from the analysis.

[5] For further discussion and explanations of the causes of the 'U' shaped intraday market seasonality see Brock and Kleidon [5], Admati and Pfleiderer [1] and Coppejans, Domowitz and Madhavan [8].

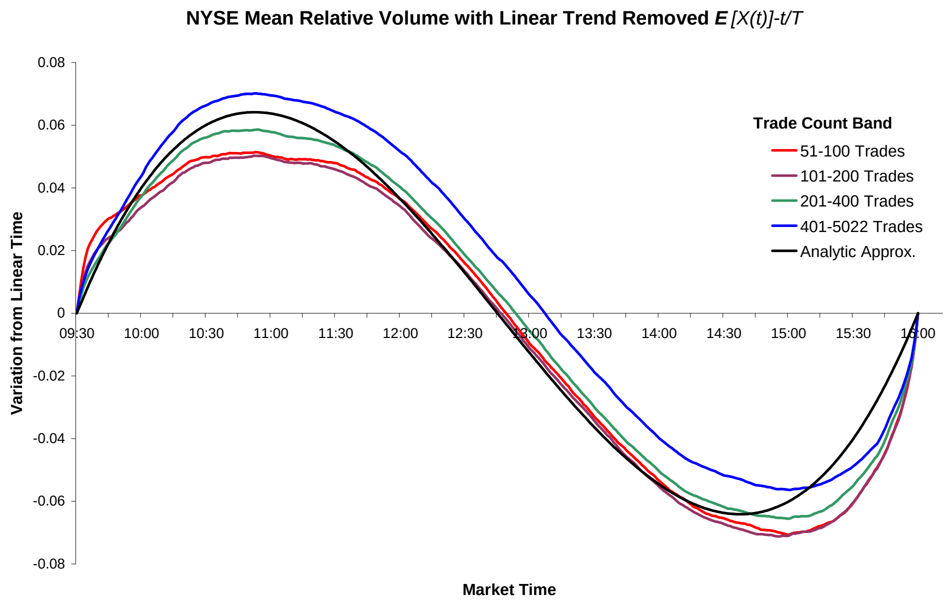

Figure 1: The mean of the relative volume $\mathbb{E}\big[X_t\big]$ for stocks within different average number of daily trades. Here the constant trade line has been subtracted, $\mathbb{E}\big[X_t\big] - t/T$ (so all means are monotonically increasing function of time). The polynomial approximation (eqn 4) is shown as the black line.

This intuition is correct and is the second important insight into VWAP trading - the volatility of the relative volume process X_t of low turnover stocks is higher than high turnover stocks.

Figure 3 shows the empirical time indexed variance of the relative volume process $\mathrm{Var}[X_t]$ for different ranges of number of daily trades. It has an inverted 'U' shape where the variance is zero at $t = 0$ and $t = T$, similar to the time indexed variance of a Brownian bridge. Stocks with a lower number of daily trades have higher variance. The variances of the relative volume process for stocks with a different final trade count K can be empirically scaled to fit a single curve by multiplying them by final trade count raised to the power 0.44 ($K^{0.44}$). Figure 4 plots the scaled empirical variances.

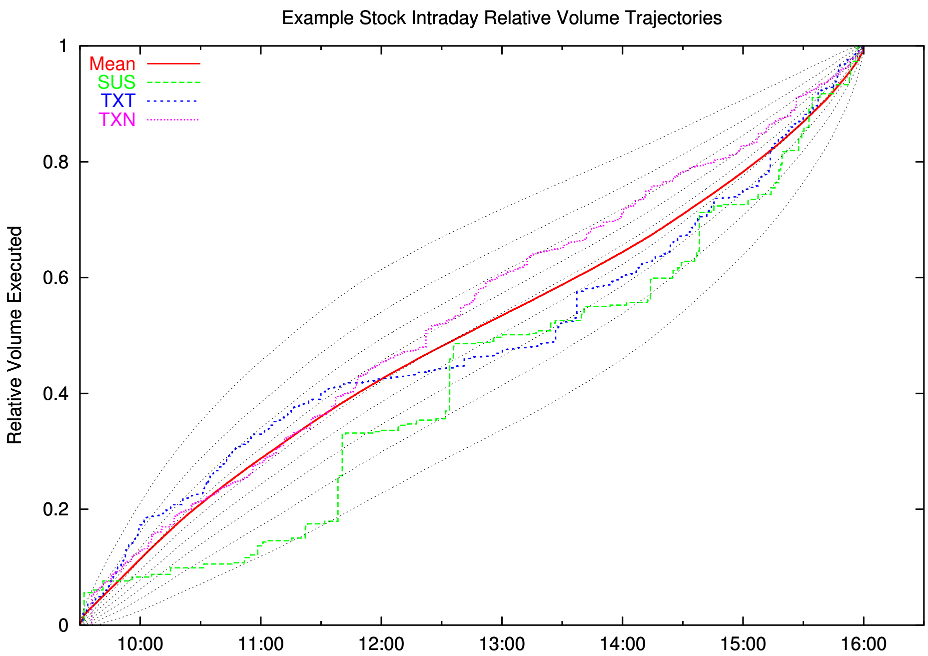

Figure 2: This graph shows typical relative volume trajectories for 3 stocks representing low, medium and high turnover stocks. The red line is the expected relative volume $\mathbb{E}[X_t]$ for all stocks trading more than 50 trades a day on the NYSE over the data period. *SUS* is *Storage USA*, *TXT* is *Textron Incorporated* and *TXN* is *Texas Instruments*. On 2 Jul 2001 these stocks recorded 101, 946 and 2183 trades correspondingly.

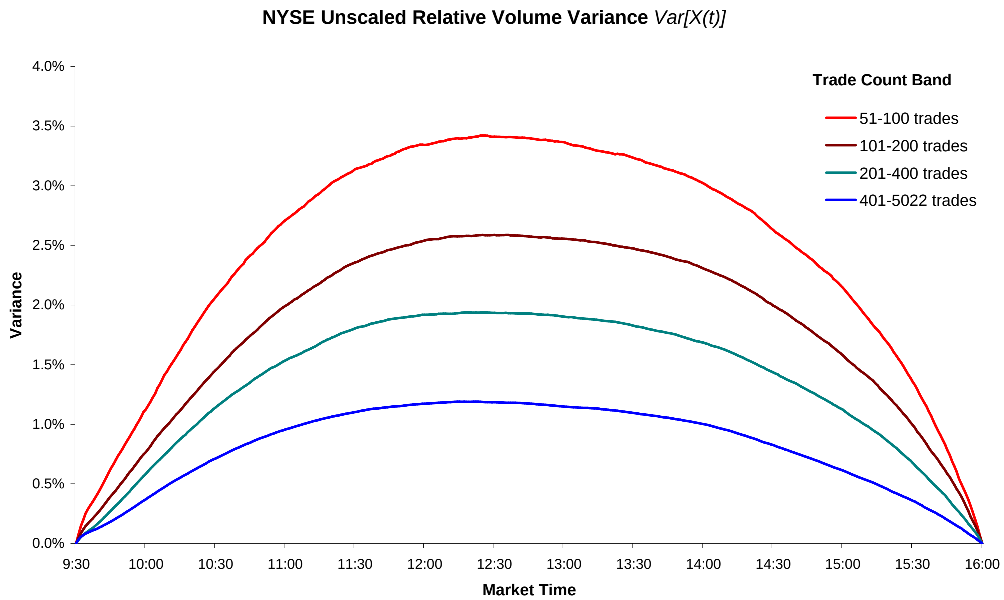

Figure 3: The inverse 'U' shaped time-indexed variance for relative volume $\mathrm{Var}\left[X_t\right]$. Lower trade count stocks have a higher variance for $\mathrm{Var}\left[X_t\right]$.

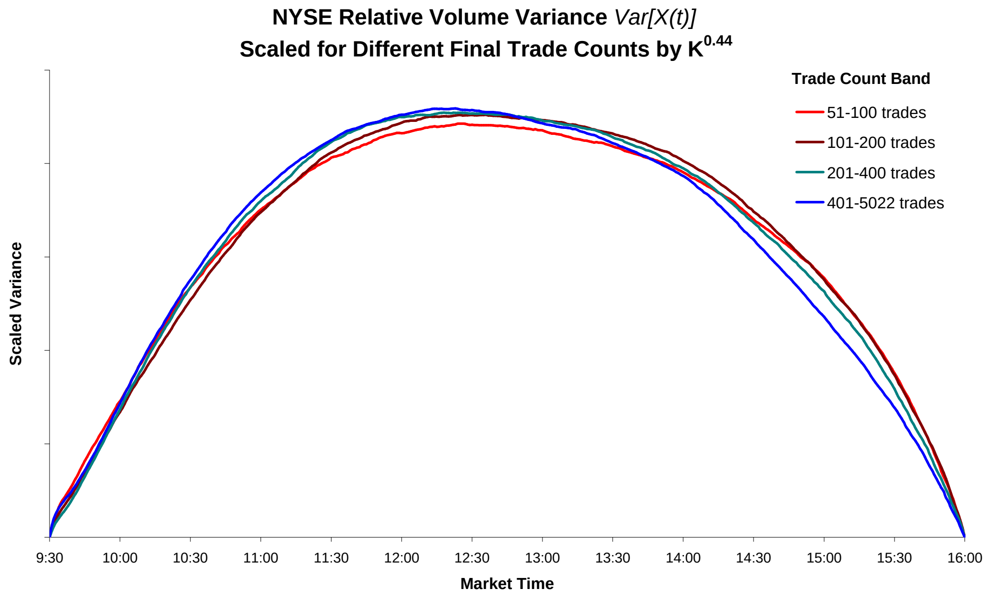

Figure 4: The scaled relative volume variances $\mathrm{Var}\left[X_t\right]K^{0.44}$ for stocks with different ranges of final trade counts K.

4 VWAP Trading Strategies

4.1 Feasible Trading Strategies

Any deterministic trading strategy x_t is feasible only if it conforms to the first constraint below. The second and third constraints are not strictly necessary but enforce a uni-directional strategy where buy VWAP traders only buy stocks and sellers only sell stocks.

1. Trader starts trading the VWAP strategy at $t = 0$ when $x_0 = 0$ and has traded the whole strategy at $t = T$ when $x_T = 1$.
2. The relative volume for the strategy must always be between zero (nothing has been traded) and one, all order's volume was traded, $0 \leq x_t \leq 1, \quad \forall t \in [0, T]$.
3. The strategy must be monotonically non-decreasing, $0 \leq x_t \leq x_{t+\delta} \leq 1$.

4.2 VWAP Trade Size

It is intuitive and true that the greater percentage of trading that the VWAP trader controls, the easier it is to trade at the market VWAP price. In the limit, the trader controls 100% of traded volume and exactly determines the market VWAP irrespective of trading strategy. It seems clear that VWAP risk is proportional to the traded volume that the VWAP trader does not control and this intuition is quantified below. The relative volume process of other market traders $\bar{X}_t$ will be assumed to be independent of the trading strategy x_t adopted by the VWAP trader. Market relative volume process X_t can be written as a weighted sum of the relative volume of other market participants $\bar{X}_t$ and the VWAP trader x_t. If $\bar{V}_t$ is the cumulative volume process of that does not include VWAP trader volume, then the relative volume of other market participants $\bar{X}_t$ is defined:

$$\bar{X}_t = \frac{\bar{V}_t}{\bar{V}_T}$$

Similarly the relative volume strategy of the VWAP trader is simply the trader final cumulative volume v_T divided by cumulative volume at time t, v_t.

$$x_t = \frac{v_t}{v_T}$$

The proportion[6] β of the total market traded by the VWAP trader can be calculated.

$$\beta = \frac{v_t}{\bar{V}_T + v_T}$$

The expected total relative volume (known in $\mathcal{G}_t$) can be decomposed into the relative volume process of other market participants $\bar{X}_t$ and the deterministic trading strategy of the VWAP trader.

$$X_t = (1-\beta)\bar{X}_t + \beta x_t$$

Using the definitions above, $\mathcal{V}(x_t)$ can be rewritten as:

$$\mathcal{V}(x_t) = \int_0^T (\bar{X}_t - x_t)\, dP_t = (1-\beta) \int_0^T (X_t - x_t)\, dP_t$$

In the following exposition it is assumed that $\beta << 1$ and all $O(\beta)$ terms are ignored.

4.3 The Risk of VWAP Strategies

The risk of traded VWAP with trading strategy x_t is readily expressed using equation 3.

$$\mathrm{Var}\big[\mathcal{V}(x_t)\big] = \mathrm{Var}\bigg[\int_0^T (X_{t-} - x_t)\, dP_t \bigg]$$

[6]Note that β is known under the enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(V_T)$ and a random variable under $\mathcal{F}_t$.

Using the semimartingale generalization of Ito's isometry this variance can be written as:

$$\mathrm{Var}\left[\int_0^T (X_{t-} - x_t)\, dP_t\right] = \mathbb{E}\left[\int_0^T (X_t - x_t)^2\, d[P,P]_t\right]$$

Since the price semimartingale P_t is assumed continuous, the drift term A_t is continuous and it is proved below that the drift term does not contribute to VWAP risk and that the VWAP risk can be written just using the martingale component of the Doob-Meyer decomposition.

$$P_t = M_t + A_t + P_0$$

$$\mathrm{Var}\left[\int_0^T (X_{t-} - x_t)\, dP_t\right] = \mathbb{E}\left[\int_0^T (X_t - x_t)^2\, d[M,M]_t\right] \tag{5}$$

Proof. The integrands of eqn 5 are identical, so by the properties of the Riemann-Stieltjes integral, the equality of eqn 5 is established if the two integrating processes, the quadratic variations, are equal (a.e) $[M,M]_t = [P,P]_t$. Using the polarization identity for quadratic covariation.

$$[A,M]_t = \frac{1}{2}\big([A+M, A+M]_t - [M,M]_t - [A,A]_t\big)$$

The drift process A_t is continuous by assumption and therefore the quadratic covariation term is zero (Jacod and Shiryaev [13], page 52) $[A,M]_t = 0$. Also the drift process A_t is predictable, continuous and of bounded variation so the drift quadratic variation term is zero (Protter [17], theorem 22, page 66) $[A,A]_t = 0$ and the polarization identity simplifies to:

$$[P,P]_t = [A+M, A+M]_t = [M,M]_t$$

$\square$

Since the martingale term of the price process is continuous the martingale representation theorem (Protter [17], theorem 43, page 188) can written as follows for a continuous predictable process σ_t.

$$M_t \;=\; \int_0^t \sigma_s \, dW_s$$

Using this representation, the VWAP variance of equation 5 can be further simplified:

$$\mathrm{Var}\big[\mathcal{V}(x_t)\big] \;=\; \mathbb{E}\bigg[\int_0^T (X_t - x_t)^2 \, d[M,M]_t\bigg] \;=\; \mathbb{E}\bigg[\int_0^T (X_t - x_t)^2 \, \sigma_t^2 \, dt\bigg] \quad (6)$$

4.4 Minimum Risk VWAP Strategy

It seems reasonable that an optimal trading strategy $x_t^\star$ is a strategy that is close to X_t without any knowledge of the actual outcome of X_t. Thus the optimal trading strategy should be, by intuition, close to the expectation of relative volume $x_t^\star = \mathbb{E}[X_t]$. This is shown below. Following Konishi [15] the equation can be decomposed as:

$$\begin{aligned}
x_t^\star &= \min_{0 \le x_t \le 1} \mathrm{Var}\big[\mathcal{V}(x_t)\big] \\
&= \min_{0 \le x_t \le 1} \bigg[\int_0^T \mathbb{E}\Big[\big(X_t^2 - 2x_t X_t + x_t^2\big)\,\sigma_t^2\Big]\,dt\bigg] \\
&= \min_{0 \le x_t \le 1} \bigg[\int_0^T x_t^2 \, \mathbb{E}\big[\,\sigma_t^2\,\big] \;-\; 2\,x_t\,\mathbb{E}\big[X_t\,\sigma_t^2\,\big]\,dt\bigg] \\
&= \min_{0 \le x_t \le 1} \bigg[\int_0^T \mathbb{E}[\,\sigma_t^2\,]\bigg(x_t^2 \;-\; 2\,x_t \frac{\mathbb{E}[X_t\,\sigma_t^2\,]}{\mathbb{E}[\,\sigma_t^2\,]} + \frac{\mathbb{E}[X_t\,\sigma_t^2\,]^2}{\mathbb{E}[\,\sigma_t^2\,]^2}\bigg) \;-\; \frac{\mathbb{E}[X_t\,\sigma_t^2\,]^2}{\mathbb{E}[\,\sigma_t^2\,]}\,dt\bigg] \\
&= \min_{0 \le x_t \le 1} \bigg[\int_0^T \bigg(x_t \;-\; \frac{\mathbb{E}[X_t\,\sigma_t^2\,]}{\mathbb{E}[\,\sigma_t^2\,]}\bigg)^2\,dt\bigg]
\end{aligned}$$

This is minimized when:

$$x_t \;=\; \frac{\mathbb{E}[X_t\,\sigma_t^2\,]}{\mathbb{E}[\,\sigma_t^2\,]} \;=\; \mathbb{E}\big[X_t\big] \;+\; \frac{\mathrm{Cov}\big[X_t,\sigma_t^2\big]}{\mathbb{E}[\,\sigma_t^2\,]}$$

Thus the constrained solution is:

$$x_t^\star \;=\; \begin{cases} \text{if} \quad \mathbb{E}\big[X_t\big] \;+\; \dfrac{\mathrm{Cov}\big[X_t,\sigma_t^2\big]}{\mathbb{E}[\,\sigma_t^2\,]} \geq 1, & 1 \\[2ex] \text{if} \quad \mathbb{E}\big[X_t\big] \;+\; \dfrac{\mathrm{Cov}\big[X_t,\sigma_t^2\big]}{\mathbb{E}[\,\sigma_t^2\,]} \leq 0, & 0 \\[2ex] \mathbb{E}\big[X_t\big] \;+\; \dfrac{\mathrm{Cov}\big[X_t,\sigma_t^2\big]}{\mathbb{E}[\,\sigma_t^2\,]}, & \text{otherwise.} \end{cases} \tag{7}$$

Where $\mathrm{Cov}\big[X_t,\sigma_t^2\big]$ is the covariance between relative volume X_t and stock price variance σ_t^2. In financial markets literature the positive relationship between trading volume and volatility is a 'stylized fact', see Cont [7], Clark [6] and Ané and Geman [3]. Therefore, since the expectation of relative volume $\mathbb{E}[X_t]$ is monotonically increasing and the covariance between relative volume and variance is non-negative $\mathrm{Cov}\big[X_t,\sigma_t^2\big] \geq 0$, the minimum risk solution (eqn 7) is feasible. Note that under the assumption that the relative volume and stock price variance are independent or stock price variance is a deterministic function then the covariance term is zero and the minimum risk strategy reduces to the expectation of the relative volume $x_t^\star = \mathbb{E}\big[X_t\big]$.

4.5 Non-removable residual risk of VWAP trading

Residual risk is the lower bound of VWAP risk that cannot be eliminated by choosing a trading strategy x_t. Substitution of eqn 7 into eqn 6 gives the following bound on the residual VWAP variance:

$$\min_{x_t}\ \mathrm{Var}[\mathcal{V}(x_t)] \;=\; \int_0^T \mathbb{E}[\,X_t^2\sigma_t^2\,] - \frac{\mathbb{E}[X_t\,\sigma_t^2\,]^2}{\mathbb{E}[\,\sigma_t^2\,]}\,dt$$

If price volatility is assumed constant $\hat{\sigma}^2 = \sigma_t^2$, then the expression above simplifies to the following:

$$\min_{x_t} \mathrm{Var}[\mathcal{V}(x_t)] \;=\; \hat{\sigma}^2 \int_0^T \mathrm{Var}[X_t] \, dt$$

Using the scaling property of $\mathrm{Var}[X_t]$ found above in the NYSE data (see section 3) then residual VWAP risk is proportional to the estimated stock variance divided by the final trade count K to the power 0.44.

$$\min_{x_t} \mathrm{Var}[\mathcal{V}(x_t)] = \text{ Const } \frac{\hat{\sigma}^2}{K^{0.44}}$$

So a stock with 100 times the trade count of another stock with similar price variance has approximately one-tenth the residual VWAP risk.

4.6 Optimal VWAP Strategy with Expected Drift

In practise a trader may wish to 'beat' VWAP. This is reasonable because the VWAP trader may have price sensitive information about a stock. A broker can exploit this private information for the benefit of his client by adopting a VWAP trading strategy x_t that is riskier than minimum variance strategy. This drift optimal strategy $x_t^\star$ can be found using mean-variance approach. *For definiteness the VWAP order is assumed to be a buy order in this paper.* Thus 'beating' market is defined as a positive expectation $\mathbb{E}[\mathcal{V}(x_t)] \geq 0$. Expanding the expectation and noting that the martingale transform has zero expectation:

$$\begin{aligned}
\mathbb{E}[\mathcal{V}(x_t)] \;&=\; \mathbb{E}\Big[\int_0^T \big(X_t - x_t\big) \; dA_t \Big] \;+ \mathbb{E}\Big[\int_0^T \big(X_{t-} - x_t\big) \, dM_t \Big] \\
&=\; \mathbb{E}\Big[\int_0^T \big(X_t - x_t\big) \; dA_t \Big]
\end{aligned}$$

The quadratic covariation between the continuous price drift A_t and the relative volume process is zero $[X, A]_t = 0$ therefore without loss of generality the covariance between price drift and relative volume can be assumed to be zero, $\mathrm{Cov}[A_t, X_t] = 0$. Denoting $\mu_t \equiv \mathbb{E}[A_t]$, the expectation of the VWAP return can be simplified to the following:

$$\mathbb{E}[\mathcal{V}(x_t)] \;=\; \int_0^T \big(\mathbb{E}[X_t] - x_t\big)\, \mu_t \, dt \tag{8}$$

In general, the optimal VWAP strategy is not the minimum VWAP risk strategy of section 4.4 because this strategy does not include the expected return of the VWAP trade. A strategy that includes expected return can be specified as a classic mean-variance optimization using a trader specified risk aversion constant λ.

$$x_t^\star \;=\; \max_{0 \le x_t \le 1} \Big[\mathbb{E}\big[\mathcal{V}(x_t)\big] \;-\; \lambda \,\mathrm{Var}\big[\mathcal{V}(x_t)\big] \Big]$$

Solving for this optimization problem:

$$\begin{aligned} x_t^\star \;&=\; \max_{0 \le x_t \le 1} \; \mathbb{E}\Big[\int_0^T (X_{t-} - x_t)\, dP_t \Big] \;-\; \lambda \,\mathrm{Var}\Big[\int_0^T (X_{t-} - x_t)\, dP_t \Big] \\ &=\; \min_{0 \le x_t \le 1} \Big[\lambda \int_0^T \mathbb{E}\big[(X_t - x_t)^2\, \sigma_t^2 \;-\; (X_t - x_t)\, \frac{\mu_t}{\lambda} \big]\, dt \Big] \\ &=\; \min_{0 \le x_t \le 1} \Big[\int_0^T \Big(x_t \;-\; \Big\{ \frac{\mathbb{E}[X_t\, \sigma_t^2\,]}{\mathbb{E}[\,\sigma_t^2\,]} - \frac{\mu_t}{2\lambda \mathbb{E}[\,\sigma_t^2\,]} \Big\} \Big)^2 dt \Big] \end{aligned}$$

The above is minimized when:

$$x_t \;=\; \frac{\mathbb{E}[X_t\, \sigma_t^2\,]}{\mathbb{E}[\,\sigma_t^2\,]} - \frac{\mu_t}{2\lambda \mathbb{E}[\,\sigma_t^2\,]} \;=\; \mathbb{E}\big[X_t\big] \;+\; \frac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}[\sigma_t^2]} \;-\; \frac{\mu_t}{2\lambda \mathbb{E}[\,\sigma_t^2\,]} \tag{9}$$

The constrained solution to optimal VWAP strategy with drift:

$$x_t^\star = \begin{cases} \text{if} \quad \mathbb{E}\big[X_t\big] \; + \; \dfrac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}\big[\,\sigma_t^2\,\big]} \; - \; \dfrac{\mu_t}{2\lambda\mathbb{E}\big[\,\sigma_t^2\,\big]} \; \geq \; 1, & 1 \\ \text{if} \quad \mathbb{E}\big[X_t\big] \; + \; \dfrac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}\big[\,\sigma_t^2\,\big]} \; - \; \dfrac{\mu_t}{2\lambda\mathbb{E}\big[\,\sigma_t^2\,\big]} \; \leq \; 0, & 0 \\ \mathbb{E}\big[X_t\big] \; + \; \dfrac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}\big[\,\sigma_t^2\,\big]} \; - \; \dfrac{\mu_t}{2\lambda\mathbb{E}\big[\,\sigma_t^2\,\big]}, & \text{otherwise.} \end{cases} \tag{10}$$

4.6.1 An Example of Drift Optimal VWAP Trading

A simple example of optimally 'front-loading' and 'back-loading' the VWAP trading strategy to exploit expected price drift is illustrated by example optimizing strategies with both positive and negative expected price drift. In these examples the VWAP period is one day $T = 1$. The expected drift $\mathbb{E}[A_t]$ is assumed to be a simple linear function of time such that the stock has either lost 2% or gained 2% by the end of the trading day $\mu_t = \pm\, t\, 0.02$. The stock volatility (std dev.) is a constant 2% ($\sigma_t^2 = \hat{\sigma}^2 = 0.02^2$). Risk-aversion coefficient $\lambda = 17.5$. With these assumptions the optimal drift trading policies of eqn 10 are:

$$x_t^\star = \begin{cases} \text{if} \quad \mathbb{E}\big[X_t\big] \; \pm \; \dfrac{t}{0.7} \; \geq \; 1, & 1 \\ \text{if} \quad \mathbb{E}\big[X_t\big] \; \pm \; \dfrac{t}{0.7} \; \leq \; 0, & 0 \\ \mathbb{E}\big[X_t\big] \; \pm \; \dfrac{t}{0.7}, & \text{otherwise.} \end{cases}$$

It is clear from the example above that the optimal strategies for drift shift the optimal strategy upwards ('front-loading') for a positive expected drift $\mathbb{E}[X_t] > 0$ and downwards ('back-loading') for a negative expected drift $\mathbb{E}[X_t] < 0$.

These optimal strategies have discontinuities at $t = 0$ and $t = 1$ where volume is instantly acquired. This is unrealistic because it assumes that the market can supply instant liquidity and eliminates the central virtue of VWAP trading, distributing liquidity demand over the VWAP period in such a way so as to minimize instantaneous liquidity demand.

4.6.2 Optimal VWAP Trading with Constrained Trading Rate

The solution is add an additional constraint to the optimization problem by setting an upper bound to the instantaneous liquidity demand $\nu_t^{\max}$. This liquidity constraint can be specified as follows:

$$\frac{dx_t}{dt} \leq v_t^{\max}$$

The optimal strategy here is constructed using the set D of feasible strategies x_t as a rectangular in (x, t) space with upper left point at $(1, 0)$ and upper right-point at $(1, T)$, see figure 5. The left x_t^L and right x_t^R boundaries for region D are defined as integrals of the maximum trading rate $v_t^{\max}$.

$$x_t^L = \int_0^t v_s^{\max} ds$$

$$x_t^R = 1 - \int_t^T v_s^{\max} ds$$

All points to the right of x_t^R and to the left of x_t^L are outside the feasible region D. The optimal strategy is to trade following unconstrained strategy (9) inside D until one of the boundaries of D is encountered and then trade at the maximum allowable rate.

$$x_t^\star = \begin{cases} \text{if} \quad \mathbb{E}\big[X_t\big] + \dfrac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}\big[\,\sigma_t^2\,\big]} - \dfrac{\mu_t}{2\lambda\mathbb{E}\big[\,\sigma_t^2\,\big]} \geq x_t^L, & x_t^L \\ \text{if} \quad \mathbb{E}\big[X_t\big] + \dfrac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}\big[\,\sigma_t^2\,\big]} - \dfrac{\mu_t}{2\lambda\mathbb{E}\big[\,\sigma_t^2\,\big]} \leq x_t^R, & x_t^R \\ \mathbb{E}\big[X_t\big] + \dfrac{\mathrm{Cov}\big[X_t, \sigma_t^2\big]}{\mathbb{E}\big[\,\sigma_t^2\,\big]} - \dfrac{\mu_t}{2\lambda\mathbb{E}\big[\,\sigma_t^2\,\big]}, & \text{otherwise.} \end{cases} \tag{11}$$

Proof that (11) is the optimal strategy for VWAP trading problem with constrained liquidity is given in appendix.

The example above is re-considered now for time-dependent constrained liquidity, where the maximal rate of trading is assumed to be proportional to the expectation of the trading rate of the market (time-derivative of $\mathbb{E}\big[X_t\big]$)

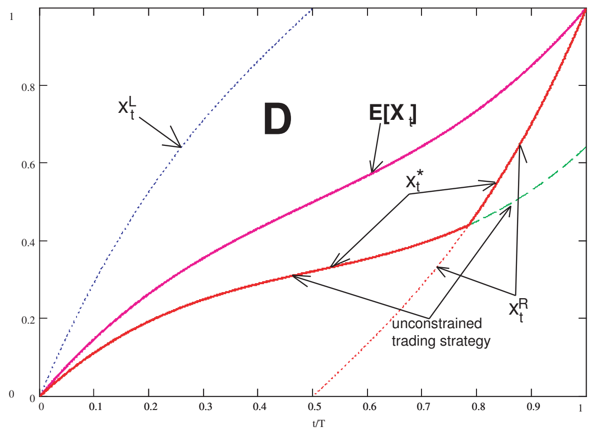

Figure 5: The optimal back-loading VWAP strategy for liquidity constrained trading in example.

$$v_t^{\max} \;=\; 2\frac{d}{dt}\,\mathbb{E}\big[X_t\big]$$

The resultant optimal VWAP trading strategy 'back-loads' volume along $x_t^\star$, shown in Figure 5.

4.7 'Bins' - VWAP Strategy Implementation

The optimal strategies $x_t^\star$ discussed previously are continuous. That is, it is assumed that the VWAP trader has complete control over trading trajectory at any moment of time during trading. This is unrealistic, traders need time to implement strategy and find trading counter-parties to provide liquidity. In order to model VWAP with uncertain liquidity a weaker assumption is adopted that trading can be divided into number of periods where trader has control over the average trading rate during each period. That is, the trader has sufficient control over trading to guarantee that the traded volume at beginning and the end of every period is equal to $x_t^\star$. These periods are called time 'bins'. The actual trajectory $x_t^\diamond$ is generated by a random liquidity process and could deviate from x_t^* inside the bin but will always coincide at its boundaries.

4.7.1 The Cost of a Suboptimal VWAP Trading Strategy

The VWAP bin trajectory $x_t^\diamond$ is suboptimal and the mean-variance 'cost' of suboptimal VWAP trading strategies $C(x_t^\diamond)$ is formulated below.

$$
\begin{aligned}
C(x_t^\diamond) &= \Big(\mathbb{E}[\mathcal{V}(x_t^\diamond)] + \lambda \mathrm{Var}[\mathcal{V}(x_t^\diamond)]\Big) - \Big(\mathbb{E}[\mathcal{V}(x_t^\star)] + \lambda \mathrm{Var}[\mathcal{V}(x_t^\star)]\Big) \\
&= \mathbb{E}\Big[\int_0^T (x_t^\diamond - x_t^\star)\,\mu_t + \lambda\left[(X_t - x_t^\diamond)^2 - (X_t - x_t^\star)^2\right]\sigma_t^2\,dt\Big] \\
&= \int_0^T (x_t^\diamond - x_t^\star)(\mu_t - 2\lambda\mathbb{E}[\sigma_t^2 X_t] + 2\lambda\,\mathbb{E}[\sigma_t^2]\,x_t^\star) + \lambda\,(x_t^\diamond - x_t^\star)^2\,\mathbb{E}[\sigma_t^2]\,dt
\end{aligned}
$$

Noting that the when the actual trading trajectory coincides with unconstrained optimal solution with drift (eqn 9) then the first term in the integral is eliminated and the cost of a suboptimal strategy is simplified.

$$
C(x_t^\diamond) = \lambda \int_0^T (x_t^\diamond - x_t^\star)^2\,\mathbb{E}[\sigma_t^2]\,dt \qquad (12)
$$

4.7.2 The Bounded Cost of a Bin Trading Strategy

Bins are designed by dividing the VWAP trading period $[0, T]$ into b time periods with the bin boundary times for bin i denoted as τ_{i-1} and τ_i.

$$0 \;=\; \tau_0 < \tau_1 < \cdots < \tau_i < \tau_{i+1} < \cdots < \tau_b \;=\; T$$

By construction $x^{\diamond}_{\tau_{i-1}} = x^{*}_{\tau_i}$ and $x^{\diamond}_{\tau_i} = x^{*}_{\tau_i}$. Since $x^{\diamond}_t$ and x^{*}_t are non-decreasing functions that are less than or equal to 1 the deviation between them is bounded.

$$|x^{\diamond}_t - x^{*}_t| \;\leq\; x^{\star}_{\tau_i} - x^{\star}_{\tau_{i-1}} \qquad \forall t \in [\tau_i, \tau_{i-1}] \tag{13}$$

Using (13) we get from (12) the following bound of additional cost from bins

$$C(\tau_1, \ldots, \tau_b) \;\leq\; \sum_{i=1}^{b} (x^{\star}_{\tau_i} - x^{\star}_{\tau_{i-1}}) \int_{\tau_{i-1}}^{\tau_i} (\mu_t - 2\lambda(\mathbb{E}[\sigma_t^2 X_t] \;-\; \mathbb{E}[\sigma_t^2]\, x^{\star}_t))dt$$

$$+ \;\sum_{i=1}^{b} (x^{\star}_{\tau_i} - x^{\star}_{\tau_{i-1}})^2 \int_{\tau_{i-1}}^{\tau_i} \lambda \mathbb{E}[\sigma_t^2]\, dt \tag{14}$$

4.7.3 Equal Volume Bins

Equal volume bins are often used by practitioners. They are defined as

$$x^{\star}(\tau_i) - x^{\star}(\tau_{i-1}) = \frac{1}{b} \qquad \forall i \in \{1, \ldots, b\}$$

The bin cost bound (14) for trading with unconstrained rate then takes the form:

$$C(\tau_1, \ldots, \tau_b) \;\leq\; \frac{1}{b^2}\, \lambda \int_0^T \mathbb{E}[\sigma_t^2]\, dt \tag{15}$$

Thus the additional VWAP risk from using discrete volume bins to trade VWAP depends on the number of bins b as $\mathcal{O}(b^{-2})$.

4.7.4 Optimal VWAP Bin Strategy

The optimal bins are obtained by minimizing the bound (14) on vector in bin boundary times τ. The first order condition of optimality is.

$$\frac{\partial C(\tau_1, \ldots, \tau_b)}{\partial \tau_k} = 0$$

Differentiating equation 14 with respect to the vector in bin boundary times τ gives:

$$\begin{aligned}
&(2x^\star_{\tau_i} - x^\star_{\tau_{i-1}} - x^\star_{\tau_{i+1}})\,(\mu_{\tau_i} - 2\lambda(\mathbb{E}[\sigma^2_{\tau_i} X_{\tau_i}] \ - \mathbb{E}[\sigma^2_{\tau_i}]x^\star_{\tau_i}))\\
&\quad + \frac{d}{d\tau} x^\star_{\tau_i} \Bigg[\int_{\tau_{i-1}}^{\tau_i} (\mu_t - 2\lambda(\mathbb{E}[\sigma^2_t X_t] \ - \mathbb{E}[\sigma^2_t]\, x^\star_t))\, dt\\
&\quad - \int_{\tau_i}^{\tau_{i+1}} (\mu_t - 2\lambda(\mathbb{E}[\sigma^2_t X_t] \ - \mathbb{E}[\sigma^2_t]\, x^\star_t)) dt \Bigg]\\
&\quad + \lambda \sigma^2_{\tau_i} \left[x^\star_{\tau_{i-1}} (x^\star_{\tau_{i-1}} - 2x^\star_{\tau_i}) - x^\star_{\tau_{i+1}} (x^\star_{\tau_{i+1}} - 2x^\star_{\tau_i}) \right]\\
&\quad + 2\lambda \frac{d}{d\tau} x^\star_{\tau_i} \left[(x^\star_{\tau_i} - x^\star_{\tau_{i-1}}) \int_{\tau_{i-1}}^{\tau_i} \mathbb{E}[\sigma^2_t]\, dt - (x^\star_{\tau_{i+1}} - x^\star_{\tau_i}) \int_{\tau_i}^{\tau_{i+1}} \mathbb{E}[\sigma^2_t]\, dt \right] \ = \ 0
\end{aligned} \tag{16}$$

Solving this equation for τ_i can be viewed as a computational operation which reduces bin-based additional cost by varying τ_i conditional on (as a function of fixed) τ_{i-1} and τ_{i+1}. It is applied recursively to the initial set of bins' times (eg equal-volume bins) until convergence to the optimal bins.

The example in figure 5 plots the bin boundaries of 10 equal volume bins for the liquidity-constrained VWAP strategy and 10 optimal bin boundaries obtained by applying recursively improving operation are shown in Figure 6. The reduction in the additional bin-based risk from the use of optimal instead of equal-volume bins is 4.65%.

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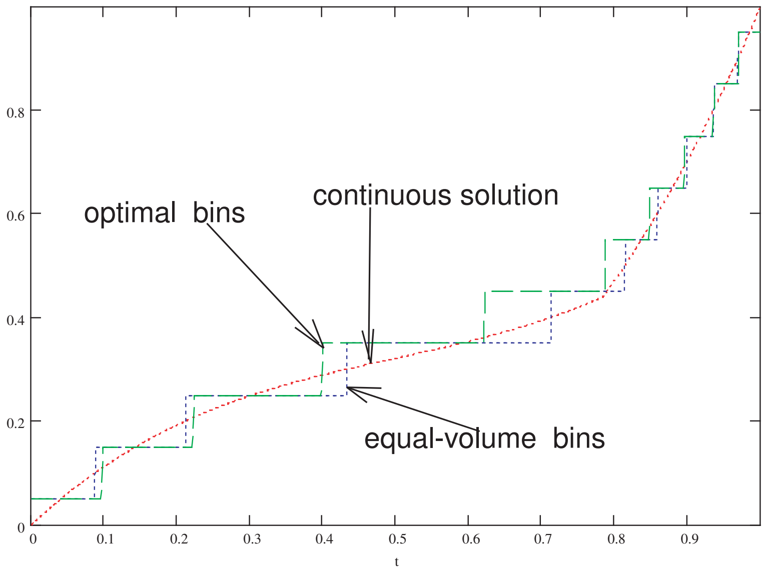

Figure 6: The optimal strategy the example with constrained liquidity and its corresponding 10 equal-volume bins and 10 optimal bins.

5 Conclusion and Summary

This paper builds on the paper by Hizuru Konishi [15] by developing a solution to an optimal minimum risk VWAP trading problem. The volume process is assumed to be marked point process and the price process to be a continuous semimartingale. It is shown that VWAP is naturally defined using the relative volume process X_t which is intra-day cumulative volume V_t divided by total final volume $X_t = V_t/V_T$. The novel expression for the risk of VWAP trading is derived. It is proven that this risk does not depend on the price drift.

The minimum risk strategy of VWAP trading is generalized into a mean-variance optimal strategy. This is useful when VWAP traders have price sensitive information that can be exploited by a VWAP strategy. The cost of exploiting price sensitive information is deviation from the minimum risk VWAP trading strategy by 'front-loading' or 'back-loading' traded volume to exploit the expected price movement.

It is shown that even with a minimum risk VWAP trading strategy is implemented there is always a residual risk. This residual risk is shown to be proportional to the price variance $\hat{\sigma}^2$ of the stock and the inverse of final trade count K raised to the power 0.44. Higher trade count stocks have lower residual VWAP risk because the variance of the relative volume process is lower for these stocks.

A practical VWAP trading strategy using trading bins is constructed. The additional VWAP risk from using discrete volume bins to trade VWAP is estimated. It is shown that it depends on the number of bins b as $\mathcal{O}(b^{-2})$.

References

[1] Anat Admati and Paul Pfleiderer, *A Theory of Intraday Patterns: Volume and Price Variability*, Review of Financial Studies **1** (1988), 3–40.

[2] Jürgen Amendinger, *Initial enlargement of filtrations and additional information in financial markets*, Ph.D. thesis, Berlin Technical University, Berlin, Germany, 1999.

[3] Thierry Ané and Helyette Geman, *Order flow, transaction clock, and normality of asset returns.*, The Journal of Finance. **55** (2000), no. 5, 2259–2284.

[4] Stephen Berkowitz, Dennis Logue, and Eugene Noser, *The Total Cost of Transactions on the NYSE*, Journal of Finance **43** (1988), 97–112.

[5] William Brock and Allan Kleidon, *Periodic Market Closure and Trading Volume: A Model of Intraday Bids and Asks*, Journal of Economic Dynamics and Control **16** (1992), 451–490.

[6] Peter Clark, *Subordinated stochastic process model with finite variance for speculative prices*, Econometrica **41** (1973).

[7] Rama Cont, *Empirical properties of asset returns: stylized facts and statistical issues*, Quantitative Finance **1** (2001), 223–236.

[8] Mark Coppejans, Ian Domowitz, and Ananth Madhavan, *Liquidity in an Automated Auction*, Working Paper. March 2001 version.

[9] David Cox, *Some Statistical Methods Connected with Series of Events (With Discussion)*, Journal of the Royal Statistical Society, B **17** (1955), 129–164.

[10] Robert Engle and Jeff Russell, *The Autoregressive Conditional Duration Model*, Econometrica **66** (1998), 1127–1163.

[11] Christian Gouriéroux, Joanna Jasiak, and Gaëlle Le Fol, *Intra-Day Market Activity*, Journal of Financial Markets **2** (1999), 193–216.

[12] Jean Jacod, *Grossissement Initial, Hypothèse et Théorème de Girsanov, Séminaire de Calcul Stochastique 1982/83*, Lecture Notes in Mathematics 1118, Springer (1985), 15–35.

[13] Jean Jacod and Albert Shiryaev, *Limit Theorems of Stochastic Processes*, Springer, Berlin, 2003.

[14] Thierry Jeulin, *Semi-martingales et grossissement d'une filtration*, Lecture Notes in Mathematics 920, Springer (1980).

[15] Hizuru Konishi, *Optimal slice of a VWAP trade*, Journal of Financial Markets **5** (2002), 197–221.

[16] James McCulloch, *Relative Volume as a Doubly Stochastic Binomial Point Process*, Quantitative Finance **7** (2007), 55–62.

[17] Phillip Protter, *Stochastic Integration and Differential Equations*, Springer, 2005.

[18] Tina Rydberg and Neil Shephard, *BIN Models for Trade-by-Trade Data. Modelling the Number of Trades in a Fixed Interval of Time*, Unpublished Paper. Available from the Nuffield College, Oxford Website; http://www.nuff.ox.ac.uk.

[19] Marc Yor, *Grossissement de filtrations et absolue continuité de noyaux*, Lecture Notes in Mathematics 1118, Springer (1985), 6–14.

A Optimal VWAP Trading Strategy with Constrained Trading Rate

Proof. That eqn 11 is the solution the the optimal VWAP trading problem with liquidity constrained trading rate $v_t \leq v_t^{max}$.

$$\min_{x_t, v_t} \left[\int_0^T (\mu_t x_t + \lambda \sigma_t^2 (x_t^2 - 2x_t \mathbb{E}[X_t]))\, dt \right] \tag{17}$$

Subject to

$$\frac{dx_t}{dt} = v_t, \quad v_t \leq v_t^{max}, \quad \forall t \in [0, T], \quad x_0 = 0, \quad x_T = 1.$$

The case in Figure 7 is considered where the unconstrained trading strategy of eqn 9 passes through the origin and intersects with the maximal trading line x_t^R at $t_R < T$. The proof for other cases when the unconstrained strategy ξ_t intersects with other the boundaries of D is identical.

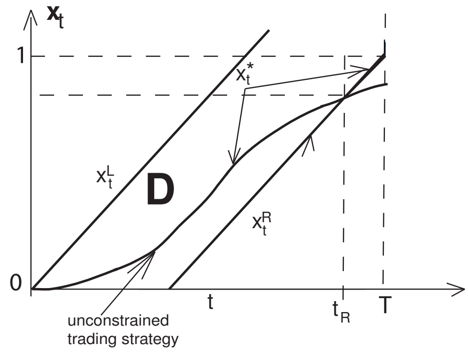

Figure 7: The feasible set D defined by constraints on the rate of trading and boundary conditions.

The adjoint variable Ψ_t, $\forall t \in [0, T]$ is calculated by solving following the equation:

$$\frac{d\Psi_t}{dt} = -\mu_t - 2\lambda\sigma^2(x_t^\star - 2\lambda\sigma_t^2 \mathbb{E}[X_t]), \qquad \Psi_{t_R} = 0. \tag{18}$$

Using integration by parts:

$$-\Psi_T x_T + \Psi_0 x_0 + \int_0^T \left[\Psi_t v_t^\star + \frac{d\Psi_t}{dt} x^\star \right] dt = 0.$$

After adding this identity's left side to VWAP mean-variance cost and dropping terms that depend on fixed x_0 and x_T the problem of eqn 17 is transformed to the following:

$$\min_{x_t, v_t} \left[\int_0^T (\mu_t x_t + \lambda \sigma_t^2 (x_t^2 - 2x_t \mathbb{E}[X_t]))\, dt \right] = \min_{x_t, v_t} \left[\int_0^T R(\Psi_t, x_t, v_t)\, dt \right] \quad (19)$$

Where:

$$R(\Psi_t, x_t, v_t) = \mu_t x_t + \lambda \sigma_t^2 (x_t^2 - 2x_t \mathbb{E}[X_t]) + \Psi_t v_t + \frac{d\Psi_t}{dt} x_t$$

Consider the left arc in $x_t^\star$, when $v_t^\star = dx_t^\star/dt < v_t^{max}$, and $t \in (0, t_R)$. Here the rhs of equation in eqn 18 is zero and therefore $\Psi_t = 0$. It is easy to check that:

$$\frac{\partial R}{\partial x_t}(\Psi_t, x_t = x_t^\star, v_t = v_t^*) = 0, \qquad \frac{\partial R}{\partial v_t}(\Psi_t, x_t = x_t^\star, v_t = v_t^*) = 0, \quad \forall t \in (0, t_R).$$

Thus R has a minimum on $x_t \in D_t$ at $x_t = x_t^*$ and on $v_t \in [0, v_t^{max}]$ at $v_t = v_t^\star < v_t^{max}$ everywhere along left arc of $x_t^\star$.

Consider the right arc of $x_t^\star$, when $v_t^\star = v_t^{max}$ and $t \in (t_R, T)$. Here x_t^* is higher than the unconstrained trading strategy ξ_t defined by eqn 9. After decomposing $x_t^\star = \xi_t + (x_t^\star - \xi_t)$ eqn 18 becomes:

$$\frac{d\Psi_t}{dt} = \big[-\mu_t - 2\lambda \sigma_t^2 (\xi_t - \mathbb{E}[X_t]) \big] - 2\lambda \sigma_t^2 (x_t^\star - \xi_t) = -2\lambda \sigma_t^2 (x_t^\star - \xi_t) < 0$$

Since $\Psi_{t_R} = 0$, $\Psi_t < 0$, $\forall t \in (t_R, T)$. It is easy to check that:

$$\frac{\partial R}{\partial x_t}(\Psi_t, x = x_t^\star, v = v_t^*) = 0, \qquad \frac{\partial R}{\partial v_t}(\Psi_t, x = x_t^\star, v = v_t^*) = \Psi_t < 0, \quad \forall t \in (t_R, T)$$

Thus R has minimum on $x_t \in D_t$ at $x_t = x_t^\star$. By inspection the function R is a linear function of v_t, so on $v_t \in [0, v_t^{max}]$ it has minimum on v_t at $v_t = v_t^\star = v_t^{max}$ everywhere along right arc of $x_t^\star$. Therefore $x_t^\star$ defined by eqn 11 and $v_t^\star = dx_t^\star/dt$ obey constraints in eqn 17 and minimize the integral of the equivalent mean-variance cost criterion R on x_t and v_t at every moment of time $t \in [0, T]$ and so is the optimal solution of eqn 17.

□