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Valuing Guaranteed Minimum Death Benefit Options in Variable Annuities under a Benchmark Approach

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Abstract

Variable annuities (VAs) represent a marked change from earlier life products in the guarantees that they offer and it is no longer possible to manage the risks of these liabilities using traditional actuarial methods. Thinking about guarantees as options suggests applying risk neutral pricing in order to value the embedded guarantees, such as guaranteed minimum death benefits (GMDBs). However, due to the long maturities of contracts, stochastic volatility and many other reasons, VA markets are incomplete. In this paper we propose a methodology for pricing GMDBs under a benchmark approach which does not require the existence of a risk neutral probability measure. We assume that the insurance company invests in the growth optimal portfolio of its investment universe and apply real world pricing rather than risk neutral pricing. In particular, we consider the minimal market model and conclude that in this setup the fair price of a roll-up GMDB is lower than the price obtained by applying standard risk neutral pricing. Moreover, we take into account rational as well as irrational lapsation of the policyholder.

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1 Introduction

Variable annuities are insurance contracts which are designed to provide payments to the policyholder, usually after some specified deferment period. Unlike the traditional fixed annuities, variable annuities are fund-linked, i.e. their benefits are based on the performance of a portfolio of securities, which usually consists of equities and bonds.

Variable annuities are appealing to policyholders, not only because they are tax-deferred, but in particular because usually insurers include different types of guarantees and thus provide some protection against the downside movements in the market. Until the beginning of the 1990's these guarantees were simple guaranteed minimum death benefits (GMDB), meaning a return of the maximum of the account value and the invested premiums. However, due to the strong bullish market at that time, insurance companies started to design GMDBs with special features, such as roll-ups (original investment accrued at a pre-defined interest rate) or ratchets (death benefit based upon the highest anniversary - usual annual - account value). Therefore, no longer is it just the behaviour of the policyholder that the insurance company must worry about, but the behaviour of the implied options within VAs, and modern techniques have to be used to build advanced financial models of the embedded options. Particularly, insurers are faced with the challenging task of hedging against market downturn (and simultaneous occurance of death).

As the GMDB can be viewed as a put, floating put and/or look-back put option, respectively, risk managers apply option pricing theory to price and hedge the embedded guarantees in variable annuities ([3]). Provided that the number of policyholders is large enough, it is assumed that the market is complete under mortality risk and the option price is equal to the expected value of the payoff with respect to a risk neutral probability measure. However, as soon as one allows for e.g. stochastic volatility, the financial market becomes incomplete and a unique equivalent martingale measure for the equity price does not necessarily exist. Consequently, there is no obvious best choice when trying to find a price of the option.

In general, one problem that one often encounters when modeling life insurance contracts are the long maturities of these contracts. Furthermore, the

choice of the dynamics of the underlying makes substantial differences in prices of contracts. In this paper we assume that the market is complete under mortality risk (enough policies were sold), but incomplete under financial risk, e.g. due to discrete hedging (as a consequence of high transaction costs), stochastic volatility and/or the non-existence of a continuum of standard options as liquid hedge instruments.

Since there might not exist a unique equivalent risk neutral probability measure, in this paper we study the pricing and hedging of GMDB options under a benchmark approach as described in [9]. The pricing and hedging of derivatives under this approach does not require the existence of an equivalent risk neutral probability measure, but leads to a unique fair price, namely the minimal price, which is calculated by using real world expectations. We will show that for contingent claims with long term to maturity this may provide significantly lower prices than suggested under the classical approach. For insurance and finance a discrete time version of this benchmark approach has been considered in [1].

Applying the benchmark approach we work with local volatility models. This means that volatility is allowed to change as a function of the underlying security and time. One particular local volatility model we focus on is the minimal market model (MMM) described in [9]. Here the index dynamics follow a time-transformed squared Bessel process.

While it is possible to assume that the holder of an option in the financial markets will react rationally to its changing value, the same cannot always be said of the behaviour of policyholders towards the value of the options embedded in their variable annuity (VA) policies. The assumption of full and unconditional rationality in the behaviour of option investors cannot be transferred directly to the insurance market. Consequently, like with conventional insurance products one needs to make policyholder behaviour forecasts based on historical data. However, data of VA contracts on which to make forecasts with any certainty is very limited and moreover, lapsation rates will be sensitive to a wide range of market factors. We will show how in our financial model one can take into account rational but also irrational lapsation.

This paper focuses on the pricing and hedging of GMDBs with a roll-up feature. Guaranteed minimum living benefits, such as guaranteed minimum accumulation benefits (GMAB), guaranteed minimum income benefits (GMIB) and the rather complex guaranteed minimum withdrawal benefits (GMWB) will be considered in forthcoming work.

The article is organized as follows. In Section 2 we introduce the roll-up GMDB product we are going to price later. Then in Section 3 we introduce

the financial market under the benchmark approach. In particular, we define the growth optimal portfolio (GOP), local volatility models and as a special case discuss the MMM in further detail. Since for the MMM an equivalent risk neutral probability measure does not exist, we cannot apply risk neutral pricing. Therefore, in Section 4 we consider real world pricing under the benchmark approach and derive fair prices which are minimal prices, as well. Finally, in Section 5 we show how the roll-up GMDB of Section 2 can be priced in a fair way taking into consideration rational as well as irrational lapsation.

2 The Product

The VA product we consider in this paper contains a GMDB that guarantees a return of at least the original invested premium compounded at some annual growth rate g, i.e. the payout to the policyholder is

$$\max(e^{g\tau} AV_0, AV_\tau). \tag{2.1}$$

Here the time of death τ is a random variable, $g \geq 0$ is the guaranteed instantaneous growth rate, AV_0 is the initial account value, i.e. the invested principal and AV_τ is the value of the policyholder's account at the time of death τ.

At time $t = 0$ the insurance company invests the premium AV_0 in a fund V_0 and we suppose that the insurer continuously deducts insurance charges $\xi \geq 0$ from the fund V, i.e. at time $t \geq 0$ the policyholder's account value satisfies $AV_t = e^{-\xi t} V_t$ and thus we have

$$\begin{aligned} \max(e^{g\tau} AV_0, AV_\tau) &= \max(e^{g\tau} V_0, e^{-\xi\tau} V_\tau) \\ &= e^{-\xi\tau} \max(e^{(g+\xi)\tau} V_0, V_\tau). \end{aligned}$$

In practice typical values for ξ are about 2%.

Note that rational investors will lapse when the embedded put-options are out of the money (OTM), but have to pay surrender charges.

Moreover, unlike standard financial options, GMDBs have stochastic maturity but are exercised involuntarily, i.e. they are triggered by involuntary death. This is the reason why in the literature they are refered to as Titanic options (see [8]).

Thus, the payoff H_t at time t of a variable annuity contract containing a roll-up GMDB option as above can be described as

$$H_t = \begin{cases} (1-\beta_t) AV_t, & \text{if lapsed at time } t, \\ \max(e^{g\tau} AV_0, AV_\tau), & \text{if death occurs at time } t = \tau. \end{cases} \tag{2.2}$$

Here β_t is the surrender charge at time t. Usually, the surrender charge is a piecewise step function, e.g.

$$\beta_t = \begin{cases} (8 - \lceil t \rceil)\%, & t \leq 7, \\ 0, & t > 7, \end{cases}$$

where $\lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, x \leq n\}$ is the ceiling function which returns the smallest integer not less than x.

We remark that in this paper we do not consider credit risk and assume that there is no accumulation phase.

3 The Financial Model

We work on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$, satisfying the usual conditions, where $\mathcal{A}_t$ denotes the market information that is available at time $t \in [0, T]$ and P denotes the real world probability measure. Here $T \in [0, \infty)$ is some final time point, e.g. the maximum term of the insurance contract.

For simplicity, we consider a market that contains only one underlying risky security with price process $S = \{S_t\}_{t\in[0,T]}$ which satisfies the stochastic differential equation (SDE)

$$dS_t = (\mu_t - \gamma)S_t dt + \sigma_t S_t \, dW_t, \qquad t \in [0, T], \tag{3.3}$$

with $S_0 > 0$. Here μ_t is the drift, σ_t is the volatility, $W = \{W_t\}_{t\in[0,T]}$ is a standard Brownian motion and $\gamma \geq 0$ models the management fee rates deducted from the underlying S_t on a continuous basis. Hence a payment $\gamma \, S_t$ flows to the insurer per unit of continuous time.

Furthermore, we model the riskless savings account process $B = \{B_t\}_{t\in[0,T]}$ by

$$dB_t = r_t B_t \, dt, \qquad t \in [0, T], \tag{3.4}$$

where $B_0 = 1$ and $r = \{r_t\}_{t\in[0,T]}$ is the (adapted) short rate process.

Now, introducing the so-called *market price of risk*

$$\theta_t = \frac{\mu_t - \gamma - r_t}{\sigma_t},$$

the SDE of the risky asset can be rewritten as

$$\frac{dS_t}{S_t} = r_t \, dt + \sigma_t \theta_t \, dt + \sigma_t \, dW_t.$$

We consider in our financial market only strategies that are self-financing.

Definition 1 *A financial strategy* $\delta = \{\delta_t = (\delta_t^0, \delta_t^1)^\top\}_{t\in[0,T]}$ *is called a self-financing strategy if* δ *is predictable,* $\int_0^t \delta_s^0\, dB_s < \infty$, $\int_0^t \delta_s^1\, dS_s < \infty$ *a.s. and for all* $t \in [0,T]$

$$dV_t = \delta_t^0\, dB_t + \delta_t^1 dS_t, \tag{3.5}$$

where

$$V_t = \delta_t^0 B_t + \delta_t^1 S_t, \quad t \in [0,T]$$

is the portfolio value.

Then, investing δ_t^0 units in the riskless savings account at time t and δ_t^1 units in the underlying security, we can define the corresponding fractions

$$\pi_t^0 = \delta_t^0 \frac{B_t}{V_t}, \qquad \pi_t^1 = \delta_t^1 \frac{S_t}{V_t}.$$

Obviously, these fractions add up to one, i.e.

$$\pi_t^0 + \pi_t^1 = 1$$

and the instantaneous portfolio return satisfies

$$\begin{aligned}
\frac{dV_t}{V_t} &= \frac{1}{V_t}\left(\delta_t^0 dB_t + \delta_t^1\, dS_t\right) \\
&= \pi_t^0 \frac{dB_t}{B_t} + \pi_t^1 \frac{dS_t}{S_t} \\
&= (1-\pi_t^1) r_t\, dt + \pi_t^1 r_t\, dt + \pi_t^1(\sigma_t \theta_t\, dt + \sigma_t\, dW_t) \\
&= r_t\, dt + \pi_t^1 \sigma_t(\theta_t\, dt + dW_t).
\end{aligned}$$

Now, denote by $\mathcal{V}$ the set of all nonnegative portfolios and by $\mathcal{V}^+$ the set of all strictly positive portfolios V. The growth optimal portfolio (GOP) V^* is said to be the portfolio which maximizes expected log-utility from terminal wealth, i.e. $V^* = \max_{V\in\mathcal{V}^+} E(\log V_T)$ for all strictly positive portfolios $V \in \mathcal{V}^+$. An application of Itô's formula gives

$$d\log V_t = G_t\, dt + \pi_t^1 \sigma_t\, dW_t$$

with growth rate

$$G_t = r_t + \pi_t^1(\mu_t - \gamma - r_t) - \frac{1}{2}(\pi_t^1)^2 \sigma_t^2, \quad t \in [0,T].$$

Then in the sense of Platen ([9]) we have the following definition.

Definition 2 *$V^* \in \mathcal{V}^+$ is a GOP if for all $t \in [0,T]$ and $V \in \mathcal{V}^+$*

$$G_t^* \geq G_t \qquad a.s.$$

The proof of the following proposition can be found in Platen ([9]), p. 373.

Proposition 3 *The GOP V^* is uniquely determined up to the initial value $V_0^* > 0$. The optimal fractions are*

$$\pi_t^{1*} = \frac{\mu_t - \gamma - r_t}{\sigma_t^2} = \frac{\theta_t}{\sigma_t}, \qquad \pi_t^{0*} = 1 - \pi_t^{1*}.$$

Therefore, the optimal growth rate is $G_t^* = r_t + \frac{1}{2}\theta_t^2 \geq G_t$ and the GOP satisfies the SDE

$$\frac{dV_t^*}{V_t^*} = r_t\,dt + \theta_t(\theta_t\,dt + dW_t), \quad t \in [0,T]$$

with $V_0^* > 0$.

Remark 4 (i) The GOP is defined in a pathwise sense and does not require any conditional expectation. In particular, the existence of an equivalent risk neutral probability measure, as needed under the risk neutral approach, is not required.

(ii) The growth rate characterizes the long term behaviour of a portfolio in the sense that

$$\lim_{T\to\infty} \frac{1}{T}\left(\log V_T - \int_0^T G_t\,dt\right) = 0 \quad a.s.$$

(iii) The GOP is the best performing portfolio in the sense that a.s.

$$\limsup_{T\to\infty} \frac{1}{T}\log\left(\frac{V_T^*}{V_0^*}\right) \geq \limsup_{T\to\infty} \frac{1}{T}\log\left(\frac{V_T}{V_0}\right), \quad \forall\, V \in \mathcal{V}^+,$$

where $\limsup_{T\to\infty} \frac{1}{T}\log\left(\frac{V_T}{V_0}\right)$ is the long term growth rate of V. This means that no strictly positive portfolio outperforms pathwise in the long run the growth optimal portfolio, that is, after sufficiently long time.

We refer to Platen ([9], Theorem 10.5.1) for a proof.

The GOP plays the central role in real world pricing under the benchmark approach. In the following section we will consider this pricing approach in further detail.

4 Real World Pricing

At time t, we call a price U_t that is expressed in units of the GOP, i.e. $\hat{U}_t = \frac{U_t}{V_t^*}$ a benchmarked price. It is important to note that under the benchmark approach all nonnegative benchmarked porfolios are $(\underline{\mathcal{A}}, P)$-supermartingales (see [9]), which means they have zero or negative trends.

This supermartingale property guarantees the absence of arbitrage in the following sense (see [9], Chapter 10).

Definition 5 (Arbitrage) *A nonnegative portfolio $V \in \mathcal{V}$ is an arbitrage if*

$$V_0 = 0 \quad a.s. \quad and \quad P(V_\tau > 0) > 0$$

at a later bounded stopping time $\tau > 0$.

In particular, Platen's no-arbitrage criterion is weaker than the usual NFLVR criterion (see [2]). Hence, some financial market models that exclude arbitrage in the sense of Platen may not admit the existence of an equivalent risk neutral probability measure. We will see that this holds true for the minimal market model.

Definition 6 *Benchmarked securities that form martingales are called fair.*

This means that we call a price process *fair*, when its current benchmarked value is the best forecast of its future benchmarked values.

Remark 7 (Fair prices are minimal prices) Consider a nonnegative fair portfolio $V \in \mathcal{V}$ and let $V' \in \mathcal{V}$ be a second nonnegative portfolio such that $V_\tau = V'_\tau$ a.s. at some bounded stopping time τ. Then it follows by the supermartingale property of benchmarked nonnegative portfolios that for all $t \in [0, T]$

$$V_{t\wedge\tau} \leq V'_{t\wedge\tau} \qquad a.s.$$

Hence, even if there are other portfolios that generate the same future payoff, they will have an initial value above that of the fair portfolio.

Since nonnegative portfolios when expressed in units of the GOP are supermartingales, the GOP is strongly related to the so-called numeraire portfolio which was originally introduced by Long ([7]).

Remark 8 (Relation between numeraire portfolio and GOP) The numeraire portfolio (see [7]) is defined to be the strictly positive portfolio

which generates a wealth process V^δ which makes the relative wealth processes $\frac{V_t}{V_t^\delta}$ of all other nonnegative portfolios denominated in units of it supermartingales under the real world probability measure P. In particular, if the numeraire portfolio exists, then it is growth-optimal in the sense that it maximises the growth rate of all non-negative portfolios, i.e. in this case the numeraire portfolio and the GOP are the same.

We will now consider the pricing of contingent claims under the real world probability measure P. In particular, we introduce the following real world pricing.

Definition 9 *Let H_τ be a nonnegative payoff of a contingent claim with maturity $\tau \in [0, T]$, where*

$$E\left[\frac{H_\tau}{V_\tau^*} \mid \mathcal{A}_t\right] < \infty$$

for all $t \in [0, \tau]$. Then the fair price $U_H(t)$ of H_τ at time $t \in [0, \tau]$ is

$$U_H(t) = V_t^* E\left[\frac{H_\tau}{V_\tau^*} \mid \mathcal{A}_t\right], \quad t \in [0, \tau]. \tag{4.6}$$

Note that here the conditional expectation is taken under the real world probability measure P and the GOP is the numeraire. No change of probability measure is necessary.

It is shown in [9] that for the fair benchmarked price $\hat{U}_H(t) = U_H(t)/V_t^*$ in a complete market a corresponding self-financing hedge portfolio can be constructed which perfectly replicates the contingent claim. By Remark 7 it turns out that the fair portfolio that matches the contingent claim at maturity is the minimal replicating portfolio.

A natural question that arises is how real world and risk neutral pricing are related. In order to answer this question we need the following propostion which can be found in e.g. [4]. We denote by a numeraire any strictly positive, non-dividend paying asset.

Proposition 10 *Assume V^* is a numeraire under the measure P such that the price of any benchmarked payoff H/V_τ^* is a martingale under P, i.e. is the fair price. Let D be an arbitrary numeraire such that when D is benchmarked it forms a martingale. Then there exists a probability measure Q^D such that any payoff normalized by D is a martingale under Q^D. The Radon-Nikodym derivative defining the measure Q^D is given by*

$$\left.\frac{dQ^D}{dP}\right|_{\mathcal{A}_\tau} = \frac{D_\tau V_0^*}{D_0 V_\tau^*}.$$

In standard risk neutral pricing the numeraire is the riskless savings account, i.e. $D_t = B_t$, the market is complete and the benchmarked savings account is assumed to be a martingale. Hence, the candidate Radon-Nikodym derivative process $\Lambda = \{\Lambda_t\}_{t\in[0,T]}$ for risk neutral pricing is given by

$$\Lambda_t = \left.\frac{dQ}{dP}\right|_{\mathcal{A}_t} = \frac{B_t V_0^*}{B_0 V_t^*}.$$

Since in general, the benchmarked savings account is only an $(\underline{\mathcal{A}}, P)$-supermartingale, we have

$$1 = \Lambda_0 \geq E[\Lambda_\tau \,|\, \mathcal{A}_0]. \tag{4.7}$$

Using this inequality we obtain from the real world pricing formula

$$\begin{aligned} U_H(0) &= E\left[\frac{V_0^*}{V_\tau^*} H_\tau \,|\, \mathcal{A}_0\right] \\ &= E\left[\Lambda_\tau \frac{B_0}{B_\tau} H_\tau \,|\, \mathcal{A}_0\right] \leq \frac{E\left[\Lambda_\tau \frac{B_0}{B_\tau} H_\tau \,|\, \mathcal{A}_0\right]}{E[\Lambda_\tau \,|\, \mathcal{A}_0]}. \end{aligned}$$

In the special case when Λ_τ is a martingale, then $E[\Lambda_\tau \,|\, \mathcal{A}_0] = 1$ and by Bayes' rule

$$U_H(0) = E_Q\left[\frac{B_0}{B_\tau} H \,|\, \mathcal{A}_0\right],$$

which is the risk neutral pricing formula with conditional expectation under the risk neutral probability measure Q. Consequently, if the Radon-Nikodym derivative $\Lambda_t = \left.\frac{dQ}{dP}\right|_{\mathcal{A}_t} = \frac{B_t V_0^*}{B_0 V_t^*}$ is an $(\underline{\mathcal{A}}, P)$-martingale, then the risk neutral price equals the fair price. Thus, the concept of fair pricing generalizes that of risk neutral pricing, however, only when the benchmarked savings account is a martingale under P.

Example 11 *Assuming a deterministic short rate $\{r_t\}_{t\in[0,T]}$, the fair price at time t of a zero coupon bond with maturity T is*

$$P(t,T) = V_t^* E\left[\frac{1}{V_T^*} \,|\, \mathcal{A}_t\right] = \exp\left\{-\int_t^T r_s \, ds\right\} E\left[\frac{\bar{V}_t^*}{\bar{V}_T^*} \,|\, \mathcal{A}_t\right], \tag{4.8}$$

where $\bar{V}^*$ denotes the discounted GOP. Obviously, the benchmarked zero coupon bond price

$$\hat{P}(t,T) = \frac{P(t,T)}{V_t^*} = E\left[\frac{1}{V_T^*} \,|\, \mathcal{A}_t\right]$$

is an $(\underline{\mathcal{A}}, P)$-martingale.

Recall from Proposition 3 that

$$V_t^* = V_0^* \exp\left\{ \int_0^t \left(r_s + \frac{\theta_s^2}{2} \right) ds + \int_0^t \theta_s \, dW_s \right\}.$$

Therefore,

$$P(t,T) = \exp\left\{ -\int_t^T r_s \, ds \right\} E\left[\exp\left\{ -\int_t^T \frac{\theta_s^2}{2} \, ds - \int_t^T \theta_s \, dW_s \right\} \mid \mathcal{A}_t \right].$$

In particualar, this shows that if the candidate Radon-Nikodym derivative is a strict supermartingale then

$$P(t,T) < \exp\left\{ -\int_t^T r_s \, ds \right\} = \frac{B_t}{B_T}.$$

The concept of real world pricing does not only generalize risk neutral pricing but also actuarial pricing, as the following remark shows.

Remark 12 In the important case when the contingent claim H_τ is independent of V_τ^* and $\tau = T$ is fixed, i.e. $H_\tau = H_T = H$, one obtains from the real world pricing formula (4.6) the *actuarial pricing formula*

$$U_H(t) = P(t,T) E[H \mid \mathcal{A}_t], \quad t \in [0,T]$$

with the zero coupon bond $P(t,T) = V_t^* E\left[\frac{1}{V_T^*} \mid \mathcal{A}_t \right]$. This provides a bridge between actuarial and real word pricing.

From now on, in this paper it is assumed that the underlying risky asset S is a diversified accumulation index that approximates the GOP V^*. This assumption is supported by a result in [9], where it is shown that the GOP is approximated by any well-diversified accumulation index. For example in the case of the world stock market one can use the MSCI accumulation world stock index as proxy for the GOP. Therefore, we interpret the GOP V^* as the accumulation index of the market.

Moreover, for the remainder of this paper we assume that the time t value of the GOP V_t^* follows a *local volatility (LV) model*, i.e. it satisfies an SDE of the form

$$\frac{dV_t^*}{V_t^*} = (r_t + \sigma^2(t, V_t^*)) \, dt + \sigma(t, V_t^*) \, dW_t, \qquad t \in [0,T] \tag{4.9}$$

for fixed initial value $V_0^* > 0$. This is equivalent to assuming

$$\theta_t = \sigma(t, V_t^*) > 0$$

for the market price of risk.

Note that we suppose that the local volatility function $\sigma : [0,T] \times (0,\infty) \to (0,\infty)$ is such that a unique strong solution of the SDE (4.9) exists.

Example 13 *If*

$$\sigma(t, V_t^*) = (V_t^*)^{\alpha-1}\varphi$$

for some constant $\alpha < 1$ and scaling parameter $\varphi > 0$, then the resulting LV model is the modified CEV model considered in [6]. It is well-known that the modified CEV model does not allow for an equivalent risk neutral probability measure and thus risk neutral pricing is not applicable.

The LV model we are going to work with within this paper is the so-called *minimal market model (MMM)*, described in [9]. The MMM interpretes the GOP as the accumulation index of the market and hence supposes that the drift of the GOP provides a link to the long term growth of the macro economy.

Now, introducing the discounted GOP

$$\bar{V}_t^* = \frac{V_t^*}{B_t}, \quad t \in [0,T]$$

and applying Itô's rule one obtains that its value satisfies

$$\frac{d\bar{V}_t^*}{\bar{V}_t^*} = \theta_t \left(\theta_t \, dt + dW_t\right), \quad t \in [0,T].$$

The drift of the discounted GOP equals, therefore, $\alpha_t^* := \bar{V}_t^* \theta_t^2$ and its volatility is

$$\theta_t = \sqrt{\frac{\alpha_t^*}{\bar{V}_t^*}}, \quad t \in [0,T].$$

Thus, if the index increases, then the volatility decreases. This provides a natural explanation for the leverage effect (negative correlation between index value and volatility), which is often encountered in practice.

Since historical records suggest that the world economy (in a long-term sense) has been growing exponentially, we assume that the discounted GOP drift is an exponentially growing function of time and model α_t^* as

$$\alpha_t^* = \alpha_0 \exp\{\eta t\}, \qquad t \in [0,T], \tag{4.10}$$

with scaling parameter $\alpha_0 > 0$ and net growth rate $\eta > 0$.

Now, the next step is to introduce the normalised GOP

$$Y_t = \frac{\bar{V}_t^*}{\alpha_t^*}, \quad t \in [0, T].$$

Then applying the Itô formula again, it follows

$$dY_t = (1 - \eta Y_t)\, dt + \sqrt{Y_t}\, dW_t, \qquad Y_0 = \frac{\bar{V}_0^*}{\alpha_0}. \tag{4.11}$$

Hence, for working with the MMM, one only needs to specify initial values $\bar{V}_0^*$, α_0 and $\eta > 0$. Furthermore, in the MMM the discounted GOP is the product of a square root process and an exponential function.

Remark 14 For the transformed time (see [9] for details), given by

$$\varphi(t) = \frac{1}{4} \int_0^t \alpha_s^*\, ds = \frac{\alpha_0}{4\eta} \left(e^{\eta t} - 1 \right), \quad t \in [0, T],$$

it can be shown that

$$\varphi(t) - \varphi(0) = \left[\sqrt{\bar{V}^*} \right]_t, \qquad t \in [0, T].$$

Since the quadratic variation of the square root of the GOP is an observable quantity, the transformed time is observable. This result can be used for calibration of α_0 and η if we take a world stock index as a proxy for the GOP.

Remark 15 Note that

$$\sigma(t, V_t^*) = \frac{1}{\sqrt{Y_t}}, \qquad t \in [0, T]. \tag{4.12}$$

Consequently, the volatility of the GOP is propotional to the inverse square root of a square root process. Therefore, the MMM is characterized by a volatility that has a stationary density. This is a particular feature of the MMM, which most other local volatility models do not share. For instance, for the CEV model the volatility is a fixed function of the index and thus - similar to the index - changes its average value over long periods of time drastically.

Figure 1 displays a trajectory of a simulated GOP under the MMM for $\eta = 0.05$, $\alpha_0 = 0.05$, constant $r = 0.05$ and $Y_0 = 20$. To visualize the negative correlation of the GOP with its volatility, i.e. the leverage effect, we also plot the corresponding local volatility $\sigma(t, V_t^*)$.

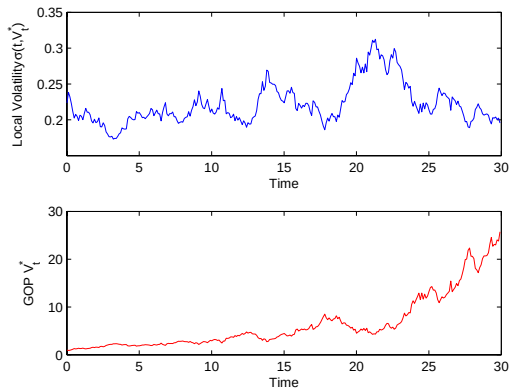

Figure 1: Simulated trajectory of volatility and the corresponding GOP under the MMM for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$ and $Y_0 = 20$.

Example 16 (Zero coupon bond under the MMM) *For simplicity, we assume in this example that the short rate r_t is again deterministic and the net growth rate η is constant. Since it can be shown, see [9], that the first negative moment of a squared Bessel process of dimension $\delta = 4$ has the form*

$$E\left[(\bar{V}_T^*)^{-1} \,|\, \mathcal{A}_t\right] = (\bar{V}_t^*)^{-1}\left(1 - \exp\left\{-\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))}\right\}\right),$$

we obtain under the MMM by (4.8) the price of the fair zero coupon bond as

$$P(t,T) = \exp\left\{-\int_t^T r_s\, ds\right\}\left(1 - \exp\left\{-\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))}\right\}\right) \tag{4.13}$$

for $t \in [0,T)$. Hence, $P(t,T) < \frac{B_t}{B_T}$ which means that for the MMM the fair zero coupon bond has a lower price than the savings bond $\frac{B_t}{B_T}$. This demonstrates that the MMM does not have an equivalent risk neutral probability measure and the candidate Radon-Nikodym derivative Λ is here a strict $(\underline{\mathcal{A}}, P)$-supermartingale.

Since we have just seen that an equivalent risk neutral probability measure does not exist for the MMM, we shall apply the real world pricing formula to obtain derivative prices and do not rely on risk neutral pricing.

Using the results above, the fair price $p(t, V_t^*, \tau, K, r)$ of a European put option with strike K and maturity $\tau \in [0,T]$ at time t, assuming a constant short rate r is given by

$$p(t, V_t^*, \tau, K, r) = V_t^* E\left[\frac{(K - V_\tau^*)^+}{V_\tau^*} \,|\, \mathcal{A}_t\right].$$

It is furthermore shown in [9], p. 502, that this put price can be calculated explicitly via the formula

$$p(t, V_t^*, \tau, K, r) = -V_t^* \chi^2(d_1; 4, l_2) + K e^{-r(T-t)} \left(\chi^2(d_1; 0, l_2) - \exp\{-l_2/2\} \right), \tag{4.14}$$

with

$$d_1 = \frac{4\eta K \exp\{-r(T-t)\}}{B_t \alpha_t^* (\exp\{\eta(T-t)\} - 1)}$$

and

$$l_2 = \frac{4\eta V_t^*}{B_t \alpha_t^* (\exp\{\eta(T-t)\} - 1)},$$

where $\chi^2(x; n, l)$ is the non-central chi-square distribution function with $n \geq 0$ degrees of freedom and non-centrality paramter $l > 0$, i.e.

$$\chi^2(x; n, l) = \sum_{k=0}^{\infty} \frac{\exp\left\{-\frac{l}{2}\right\} \left(\frac{l}{2}\right)^k}{k!} \left(1 - \frac{\Gamma\left(\frac{x}{2}; \frac{n+2k}{2}\right)}{\Gamma\left(\frac{n+2k}{2}\right)} \right).$$

In comparison the put price according to formal application of the risk neutral pricing rule would be of the form

$$p(t, V_t^*, \tau, K, r) + K e^{-r(T-t)} \exp\left\{ -\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))} \right\}, \tag{4.15}$$

since the discounted GOP is under the hypothetical risk neutral probability measure Q a squared Bessel process of dimension zero. This process is absorbed with strictly positive risk neutral probability until time T. Since the discounted GOP follows under the real world probability measure P a squared Bessel process of dimension four the real world probability to become absorbed until time T is zero. This demonstrates that events that have probability zero are different under P and Q and these probability measures are not equivalent. This means there does not exist an equivalent risk neutral probability measure. Note that the exponential in formula (4.15) represents the risk neutral probability to become absorbed at zero until time T.

Remark 17 It can be seen from (4.14) that when the GOP becomes very small, the put value also becomes small. In particular, a put price derived under the standard risk neutral pricing rule (4.15) would be larger than the fair put price and would under the MMM not become very small when the GOP becomes small.

5 Pricing a Roll-up GMDB

We shall now apply the theoretical results of the previous sections in order to price a roll-up GMDB. For simplicity, we first assume a deterministic life time $\tau = T$ of the policyholder. Recall from Section 2 that the payoff of a roll-up GMDB with annual growth rate g is given by

$$\begin{aligned} GMDB_T &= \max(e^{gT} AV_0, AV_T) \\ &= (e^{gT} AV_0 - AV_T)^+ + AV_T, . \end{aligned}$$

where AV_t denotes the policyholder's account value at time t. Here we note that $(e^{gT} AV_0 - AV_T)^+$ is nothing but a weighted European put option with strike $K = e^{gT} AV_0$ and underlying asset AV_T, i.e. the effective exercise price of the put option increases at a rate g for increasing T.

Now, recall that $AV_t = e^{-\xi t} V_t$, where ξ models the insurance charges which are deducted from the fund value V_t at time t.

We suppose that the insurance company invests the entire fund value V in the GOP of its investment universe, i.e. $V_t = e^{\xi t} AV_t = V_t^*$ and thus

$$GMDB_T = e^{-\xi T} \left[(e^{(g+\xi)T} V_0^* - V_T^*)^+ + V_T^* \right]. \tag{5.16}$$

Therefore, the fair present value $GMDB_0$ of the total claim at time zero that the policyholder has on the insurance company equals, according to the real world pricing formula (4.6),

$$GMDB_0 = V_0^* E \left[\frac{GMDB_T}{V_T^*} \mid \mathcal{A}_0 \right]. \tag{5.17}$$

We emphasize again that expectation is taken under the real world probability measure P. Consequently, under the MMM,

$$\begin{aligned} GMDB_0 &= e^{-\xi T} V_0^* \left(E \left[\frac{(e^{(g+\xi)T} V_0^* - V_T^*)^+}{V_T^*} \right] + 1 \right) \\ &= e^{-\xi T} \left[p(0, V_0^*, T, e^{(g+\xi)T} V_0^*, r) + V_0^* \right], \end{aligned} \tag{5.18}$$

where $p(0, V_0^*, T, e^{(g+\xi)T} V_0^*, r)$ is given in (4.14) with $K = e^{(g+\xi)T} V_0^*$.

Figure 2 shows a comparison of the present value of a roll-up GMDB under real world and risk neutral pricing as a function of the time to maturity and for the growth rates $g = 0$ and $g = 0.025$, where $\xi = 0.01$. Obviously, the guaranteed growth rate g increases the value of the GMDB option. Moreover, we would like to stress that under the MMM the fair price (4.14) is always lower than the price obtained by using the standard risk neutral pricing formula (4.15)

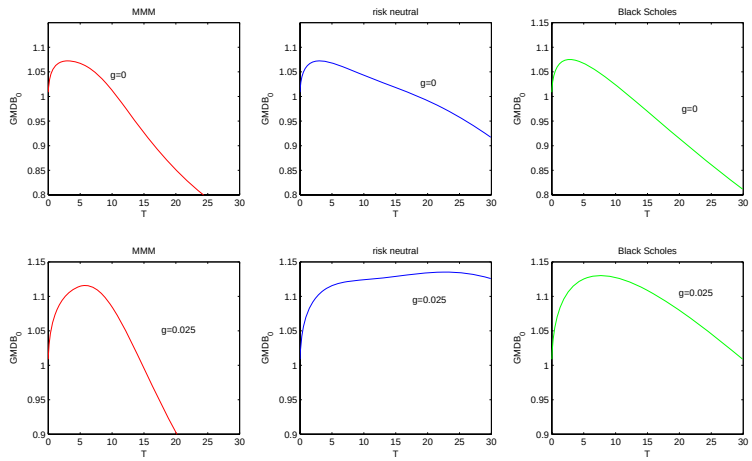

Figure 2: Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$, $\xi = 0.01$ and $Y_0 = 20$.

for a European put. Additionally, we show in Figure 2 also those prices obtained under the Black and Scholes model, when the volatility is simply set to $\frac{1}{\sqrt{Y_0}}$. We note that in the long term the MMM prices are also here again lower than comparable prices under the Black Scholes model.

Now, we assume that the lifetime τ is stochastic and independent of V_t^*. Denote by $\mathcal{F}_\tau = \mathcal{A}_0 \cup \mathcal{G}_\tau$ the joint σ-algebra generated by $\mathcal{A}_0$ and the information $\mathcal{G}_\tau$ about the outcome for the stopping time τ. Then the fair value of the GMDB at time $t = 0$ is given by

$$GMDB_0 = V_0^* E_\tau \left[E \left[\frac{GMDB_\tau}{V_\tau^*} \middle| \mathcal{F}_\tau \right] \middle| \mathcal{A}_0 \right]. \tag{5.19}$$

Here the second conditional expectation is taken with respect to $\mathcal{A}_0 \cup \mathcal{G}_\tau$, which practically means that τ is known under this σ-algebra. Consequently, the present value of the stochastic maturity GMDB is given by

$$GMDB_0 = \int_0^T (p(0, V_0^*, t, e^{(g+\xi)t} V_0^*, r) + V_0^*) e^{-\xi t} f_\tau(t)\, dt, \tag{5.20}$$

where T denotes the maximum term of the contract and $f_\tau(\cdot)$ is the probability density function of the future lifetime random variable τ. It is obvious that for a fixed issue age, a higher value of T increases the probability that the policyholder will die and use the embedded option.

Example 18 *Let τ denote the future lifetime random variable having distribution F_τ and density f_τ.*

(i) If τ is exponentially distributed, i.e. $f_\tau(t) = \lambda e^{-\lambda t}$, then $E[\tau] = \frac{1}{\lambda}$ and the force of mortality at time t of a policyholder aged x defined by $\lambda(x+t) = \frac{f_\tau(t)}{1-F_\tau(t)}$, is given by $\lambda(x) = \lambda$ for all x, i.e. in this case the probability of death is constant throughout life.

(ii) If we assume a Gompertz mortality, see [5], then we have

$$\lambda(x) = \frac{1}{b} \exp\left(\frac{x-m}{b}\right)$$

with modal value m and dispersion parameter b. The Gompertz specification, when calibrated to mortality tables, is very accurate at higher ages. Note that higher ages arise usually from the demographic in markets for variable annuities. Younger investors are less likely to die when the GMDBs are most valuable. Moreover, GMDBs are much more valuable for middle aged to senior investors compared to younger investors because of the management fees charged.

Alternatively, one can use mortality tables to discretize the integral in (5.20). This gives for a policyholder aged x at the inception of the contract

$$GMDB_0 = \sum_{j=1}^{\max_{age}-x} q(x;j) GMDB_0^j,$$

where $GMDB_0^j$ is the fair present value of a GMDB with known death date j and can be computed by (5.18). Moreover, $q(x;j)$ is the probability that the death benefit option is exercised in the j-th year, i.e.

$$q(x;j) = \prod_{i=0}^{j-2} (1 - q_{x+i}) q_{x+j-1}, \quad q(x;1) = q_x.$$

Here we assume that policyholders do not live beyond $\max_{age}$ years.

Since lapsation statistics show that at least 1% of the policyholders irrationally (that is non-optimally) exercise the GMDB option every year, in a last step we will now assume an irrational lapsation rate $l = 1\%$. Then the probability that in the years $1, \ldots, j-1$, the policyholder will neither irrationally surrender nor die is given by

$$p_l(x;j) = \prod_{i=0}^{j-2} (1 - q_{x+i} - l), \quad p_l(x;1) = 1.$$

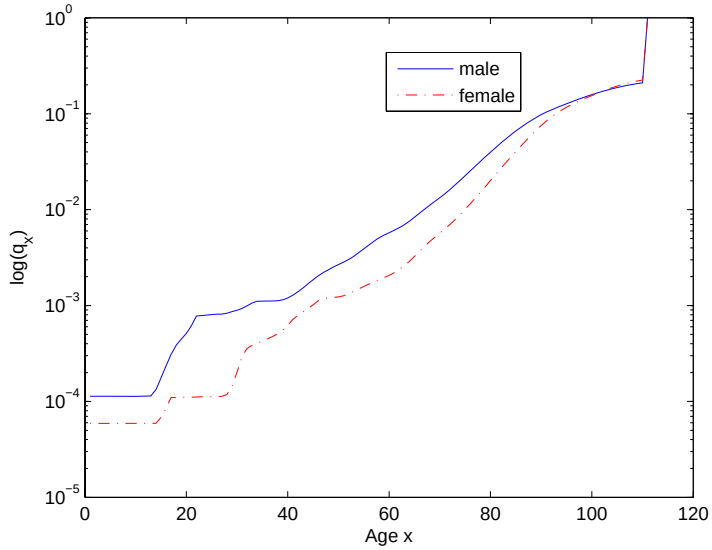

Figure 3: German mortality data

Hence, the GMDB present value of a person aged x assuming an annual lapsation rate l is

$$GMDB_0 = \sum_{j=1}^{\max_{age}-x} p_l(x;j)\left[q_{x+j}GMDB_0^j + l(1-\beta_j)V_0^*\right], \tag{5.21}$$

where β_j is the surrender charge in year j (see (2.2)) and $GMDB_0^j$ is again the present value of a GMDB with maturity j.

Example 19 *In this example we shall use standard mortality data shown in Figure 3. The data shows the death probability q_x (on a* log*-scale) for a male (female) aged x to die within the next year, i.e. prior to age $x+1$ for $x = 1,\ldots,111$. Moreover, we assume $\max_{age} = 111$, $\eta = 0.05$, $\alpha_0 = 0.05$, $Y_0 = 20$, $r = 0.05$ and $g = 0.025$. Using formula (5.21) we calculate the present value of the roll-up death benefit option for various ages x, assuming an irrational lapsation rate $l = 1\%$. Figure 4 shows the results. Our results indicate that the death benefit options are much more valuable to senior investors compared to middle aged and young investors. This is because older investors are more likely to die (and exercise the embedded option) when the GMDBs are most valuable.*

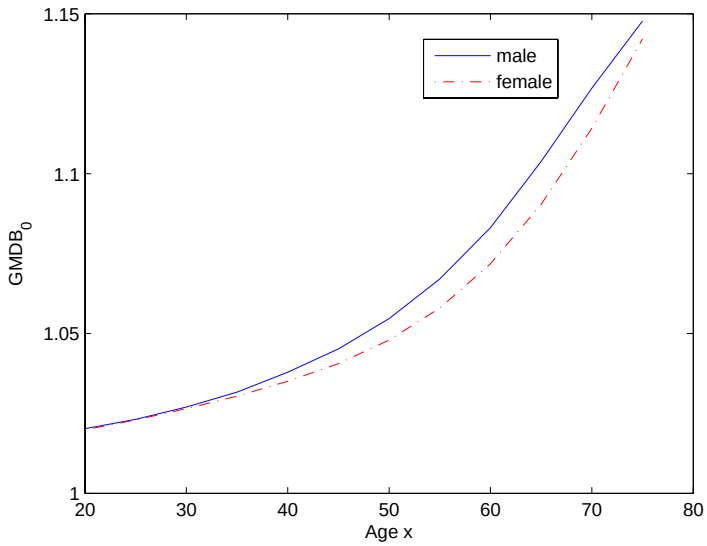

Figure 4: Present value of the GMDB under the MMM for male and female policyholders aged x, assuming an irrational lapsation of $l = 1\%$.

Furthermore, the GMDBs are less valuable to female policyholders due to their lower death probability.

6 Conclusion

In this paper we proposed a methodology for pricing guaranteed minimum death benefit (GMDB) contracts under a benchmark approach, where we assumed that the insurance company invested in the growth optimal portfolio, which turned out to be the portfolio that maximizes expected logarithmic utility. In particular, we replaced the classical risk neutral pricing by real world pricing, which for contingent claims with long term to maturity (such as the guarantees embedded in variable annuities) may provide significantly lower prices.

We showed that under the minimal market model the fair price of a roll-up GMDB was lower than the price obtained by applying standard risk neutral pricing. Moreover, we took into account rational as well as irrational lapsation of the policyholder.

Further research will expand on the methodology outlined in this paper in order to deal with more complex products (e.g. guaranteed minimum living

benefits) and more realistic market parameters such as stochastic interest rates.

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