A Benchmark Approach to Filtering in Finance

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Abstract. The paper proposes the use of the growth optimal portfolio for the construction of financial market models with unobserved factors that have to be filtered. This benchmark approach avoids any measure transformation for the pricing of derivatives. The suggested framework allows to measure the reduction of the variance of derivative prices for increasing degrees of available information.

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1 Introduction

In financial modelling it is sometimes the case that not all quantities, which determine the dynamics of security prices, can be fully observed. Some of the factors that characterize the evolution of the market are hidden. However, these unobserved factors may be essential to reflect in a market model the type of dynamics that one empirically observes. This leads naturally to filter methods. These methods determine the distribution, called filter distribution, of the unobserved factors, given the available information. This distribution allows then to compute the expectation of quantities that are dependent on unobserved factors, for instance, derivative prices.

There is a growing literature in the area of filtering in finance. To mention a few recent publications let us list Elliott & van der Hoek (1997), Fischer, Platen & Runggaldier (1999), Elliott, Fischer & Platen (1999), Fischer & Platen (1999), Landen (2000), Gombani & Runggaldier (2001), Frey & Runggaldier (2001), Elliott & Platen (2001), Bhar, Chiarella & Runggaldier (2001a, 2001b) and Chiarella, Pasquali & Runggaldier (2001). All these papers provide examples, where filter methods have been applied in the area of finance. Such applications involve optimal asset allocation, interest rate term structure modelling, estimation of risk premia, volatility estimation and hedging under partial observation.

A key problem that arises in most filtering applications in finance is the determination of a suitable risk neutral equivalent martingale measure for the pricing of derivatives. The resulting derivative prices and hedging strategies depend often significantly on the chosen measure. On the other hand it is obvious that in filtering one has to deal with the real world probability measure. It is therefore important to explore alternative methods that are based on the real world measure and allow consistent derivative pricing.

In this paper we suggest a *benchmark approach* to filtering, where the benchmark portfolio is chosen as the *growth optimal portfolio* (GOP), see Long (1990) and Platen (2002). The GOP has the important economic interpretation of being the portfolio that maximizes expected logarithmic utility. The dynamics of the growth optimal portfolio depends on the degree of available information. Given a certain information structure, one naturally obtains in this approach a *fair* price system, where benchmarked prices equal their expected future benchmarked prices. This avoids the involvement of a risk neutral equivalent martingale measure. All resulting prices, when expressed in units of the GOP, turn out to be local martingales under the given real world measure. In cases when benchmarked prices are strict local martingales the benchmark approach generalizes the standard risk neutral approach.

The paper is structured in the following way. It summarizes in Section 2 the general filtering methodology for multi-factor jump diffusion models with unobserved factors. Section 3 describes the proposed filtered benchmark model. The fair pricing and hedging of derivatives is then studied in Section 4. This section also demonstrates how to quantify the reduction of the variance for derivative prices using more information.

2 Filtered Multi-Factor Models

2.1 Factor Model

To build a financial market model with a sufficiently rich structure and high computational tractability we introduce a multi-factor model. This model provides the basis for the dynamics of financial quantities.

We consider a multi-factor model with $n \geq 2$ factors $z^1, z^2, \ldots, z^n$, forming the vector process

$$z = \left\{ z_t = \left(z_t^1, \ldots, z_t^k, z_t^{k+1}, \ldots, z_t^n \right)^\top, \, t \in [0,T] \right\}. \tag{2.1}$$

We shall assume that not all of the factors are observed. More precisely, only the first k factors are directly observed, while the remaining $n-k$ are not. Here k is an integer with $1 \leq k < n$ that we shall suppose to be fixed during most of this paper. However, in Section 4.4 we shall discuss the implications of a varying k. For fixed k we shall consider the following subvectors of z_t

$$y_t = (y_t^1, \ldots, y_t^k)^\top = (z_t^1, \ldots, z_t^k)^\top \quad \text{and} \quad x_t = (x_t^1, \ldots, x_t^{n-k})^\top = (z_t^{k+1}, \ldots, z_t^n)^\top \tag{2.2}$$

with y_t representing the *observed* and x_t the *unobserved factors*. For instance, y_t may represent the vector of logarithms of the continuous and jump parts of observed risky security prices.

Let there be given a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$, where $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0,T]}$ is a given filtration to which all the processes will be adapted. We assume that the observed and unobserved factors satisfy the system of *stochastic differential equations* (SDEs)

$$\begin{aligned} dx_t &= a_t(z_t)\, dt + b_t(z_t)\, dw_t + g_{t-}(z_{t-})\, dm_t \\ dy_t &= A_t(z_t)\, dt + B_t(y_t)\, dv_t + G_{t-}(y_{t-})\, dN_t \end{aligned} \tag{2.3}$$

for $t \in [0,T]$ with given initial value z_0. Here

$$w = \left\{ w_t = \left(w_t^1, \ldots, w_t^k, w_t^{k+1}, \ldots, w_t^n \right)^\top, \, t \in [0,T] \right\} \tag{2.4}$$

is an n-dimensional $(\underline{\mathcal{A}}, P)$-Wiener process and

$$v_t = \left(w_t^1, \ldots, w_t^k \right)^\top \tag{2.5}$$

is the subvector of its first k components. The process $m = \{m_t = (m_t^1, \ldots, m_t^k, m_t^{k+1}, \ldots, m_t^n)^\top,\ t \in [0,T]\}$ is an n-dimensional $(\underline{\mathcal{A}}, P)$-jump martingale defined as follows: Consider n counting processes $N^1, \ldots, N^n$ having no common jumps. These are at time $t \in [0,T]$ characterized by the corresponding vector of intensities $\lambda_t(z_t) = (\lambda_t^1(z_t), \ldots, \lambda_t^n(z_t))^\top$, where for $i \in \{1, 2, \ldots, k\}$

$$\lambda_t^i(z_t) = \tilde{\lambda}_t^i(y_t). \tag{2.6}$$

This means, we assume without loss of generality that the jump intensities of the first k counting processes are observed. Define the ith jump martingale by

$$dm_t^i = dN_t^i - \lambda_t^i(z_t)\, dt \tag{2.7}$$

for $t \in [0,T]$ and $i \in \{1, 2, \ldots, n\}$. Let

$$N_t = \left(N_t^1, \ldots, N_t^k\right)^\top \tag{2.8}$$

be the vector of the first k counting processes at time $t \in [0,T]$.

Concerning the coefficients in the SDE (2.3), we assume that the vectors $a_t(z_t)$, $A_t(z_t)$, $\lambda_t(z_t)$ and the matrices $b_t(z_t)$, $B_t(y_t)$, $g_t(z_t)$ and $G_t(y_t)$ are such that a unique strong solution of (2.3) exists that does not explode until time T, see Protter (1990). We shall also assume that the $k \times k$-matrix $B_t(y_t)$ is *invertible* for all $t \in [0,T]$. Finally, $g_t(z_t)$ may be any bounded function and the $k \times k$-matrix $G_t(y_t)$ is assumed to be a given function of y_t, *invertible* for each $t \in [0,T]$. This latter assumption implies that, since there are no common jumps among the components of N_t, by observing a jump of y_t we can establish which of the processes N^i, $i \in \{1, 2, \ldots, k\}$, has jumped.

In addition to the filtration $\underline{\mathcal{A}}$, which represents the *complete information*, we shall also consider the subfiltration

$$\tilde{\underline{\mathcal{A}}}^k = (\tilde{\mathcal{A}}_t^k)_{t \in [0,T]} \subseteq \underline{\mathcal{A}}, \tag{2.9}$$

where $\tilde{\mathcal{A}}_t^k = \sigma\{y_s = (z_s^1, \ldots, z_s^k)^\top,\ s \le t\}$ represents the *observed information* at time $t \in [0,T]$. Thus $\tilde{\underline{\mathcal{A}}}^k$ provides the structure of the actually available information in the market, which depends on the specification of the *degree of available information* k.

We shall be interested in the conditional distribution of x_t, given $\tilde{\mathcal{A}}_t^k$, that, according to standard terminology we call the *filter distribution* at time $t \in [0,T]$. There exist general filter equations for the dynamics described by the SDEs in (2.3). It turns out that these are SDEs for the conditional expectations of integrable functions of the unobserved factors x_t, given $\tilde{\mathcal{A}}_t^k$. Notice that, in particular, $\exp[\imath \nu\, x_t]$ is, for given $\nu \in \Re^k$ and with $\imath$ denoting the imaginary unit, a bounded and thus integrable function of x_t. Its conditional expectation leads therefore to the conditional characteristic function of the distribution of x_t, given $\tilde{\mathcal{A}}_t^k$. The latter characterizes completely the entire filter distribution. Considering conditional expectations of integrable functions of x_t is thus not a restriction for the identification of filter equations.

The general case of filter equations is beyond the scope of this paper. These are, for instance, considered in Liptser & Shiryaev (1977). We assume that the SDEs (2.3) are such that the corresponding filter distributions admit a representation of the form

$$P\left(z_t^{k+1} \leq z^{k+1}, \ldots, z_t^n \leq z^n \,\middle|\, \tilde{\mathcal{A}}_t^k\right) = F_{z_t^{k+1},\ldots,z_t^n}\left(z^{k+1}, \ldots, z^n \,\middle|\, \zeta_t^1, \ldots, \zeta_t^q\right) \tag{2.10}$$

for all $t \in [0, T]$. This means, that we have a *finite-dimensional filter*, characterized by the filter state process

$$\zeta = \left\{\zeta_t = \left(\zeta_t^1, \ldots, \zeta_t^q\right)^\top, \, t \in [0, T]\right\}, \tag{2.11}$$

which is an $\tilde{\mathcal{A}}_t^k$-adapted process with a certain dimension $q \geq 1$. We shall denote by $\tilde{z}_t^k$ the resulting $(k+q)$-vector of *observables*

$$\tilde{z}_t^k = \left(y_t^1, \ldots, y_t^k, \zeta_t^1, \ldots, \zeta_t^q\right)^\top, \tag{2.12}$$

which consists of the observed factors and the components of the filter state process. Furthermore, we assume that the filter state ζ_t satisfies an SDE of the form

$$d\zeta_t = C_t(\tilde{z}_t^k)\, dt + D_{t-}(\tilde{z}_{t-}^k)\, dy_t \tag{2.13}$$

with $C_t(\cdot)$ denoting a q-vector valued function and $D_t(\cdot)$ a $(q \times k)$-matrix valued function, $t \in [0, T]$.

There are various models of the type (2.3) that admit a finite-dimensional filter with ζ_t satisfying an equation of the form (2.13). In the following two subsections we recall two classical such models. These are the *conditionally Gaussian model*, which leads to a generalized Kalman-filter and the *finite-state jump model* for x, which is related to hidden Markov chain filters. Various combinations of these models have finite-dimensional filters and can be readily applied in finance, as demonstrated in the literature that we mentioned in the introduction.

Example 2.1 : Conditionally Gaussian Filter Model

Assume that in the system of SDEs (2.3) the functions $a_t(\cdot)$ and $A_t(\cdot)$ are linear in the factors and that $b_t(z_t) \equiv b_t$ is a deterministic function, while $g_t(z_t) \equiv G_t(y_t) \equiv 0$. This means the model (2.3) takes the form

$$\begin{aligned} dx_t &= \left[a_t^0 + a_t^1\, x_t + a_t^2\, y_t\right]\, dt + b_t\, dw_t \\ dy_t &= \left[A_t^0 + A_t^1\, x_t + A_t^2\, y_t\right]\, dt + B_t(y_t)\, dv_t, \end{aligned} \tag{2.14}$$

for $t \in [0,T]$ with given deterministic initial values x_0 and y_0. Here a_t^0 and A_t^0 are column vectors of dimensions $(n-k)$ and k respectively, and a_t^1, a_t^2, b_t, A_t^1, A_t^2, $B_t(y_t)$ are matrices of appropriate dimensions. Recall that w is an n-dimensional $(\underline{\mathcal{A}}, P)$-Wiener process and v the vector of its first k components.

In this case the filter distribution is a Gaussian distribution with vector mean $\mu_t = (\mu_t^1, \dots, \mu_t^{(n-k)})^\top$, where

$$\mu_t^i = E\left(x_t^i \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{2.15}$$

and covariance matrix $c_t = [c_t^{\ell,i}]_{\ell,i\in\{1,2,\dots,n-k\}}$, where

$$c_t^{\ell,i} = E\left(\left(x_t^\ell - \mu_t^\ell\right)\left(x_t^i - \mu_t^i\right) \,\middle|\, \tilde{\mathcal{A}}_t^k\right). \tag{2.16}$$

The dependence of μ_t and c_t on k is for simplicity suppressed in our notation. The above filter can be obtained from a generalization of the well-known Kalman filter, see Chapter 10 in Liptser & Shiryaev (1977), namely

$$\begin{aligned} d\mu_t &= \left[a_t^0 + a_t^1\,\mu_t + a_t^2\,y_t\right] dt + \left[\bar{b}_t\,B_t(y_t)^\top + c_t\,(A_t^1)^\top\right]\,\left(B_t(y_t)\,B_t(y_t)^\top\right)^{-1} \\ &\qquad\qquad\qquad\qquad \cdot\left[dy_t - \left(A_t^0 + A_t^1\,\mu_t + A_t^2\,y_t\right)dt\right] \\ dc_t &= \Big\{a_t^1\,c_t + c_t\,(a_t^1)^\top + (b_t\,b_t^\top) \\ &\quad - \left[\bar{b}_t\,B_t(y_t)^\top + c_t\,(A_t^1)^\top\right]\left(B_t(y_t)\,B_t(y_t)^\top\right)^{-1}\left[\bar{b}_t\,B_t(y_t)^\top + c_t\,(A_t^1)^\top\right]^\top\Big\}\,dt, \end{aligned} \tag{2.17}$$

where $\bar{b}_t$ is the k-dimensional vector obtained from the first k components of b_t, $t \in [0,T]$. We recall that $B_t(y_t)$ is assumed to be invertible.

Although for $t \in [0,T]$, c_t is defined as a conditional expectation, it follows from (2.17) that if $B_t(y_t)$ does not depend on the observable factors y_t, then c_t can be computed off-line. Notice that the computation of c_t is contingent upon the knowledge of the coefficients in the second equation of (2.17). These coefficients are given deterministic functions of time, except for $B_t(y_t)$ that depends also on observed factors. The value of $B_t(y_t)$ becomes known only at time t, however, this is sufficient to determine the solution of (2.17) at time t. Model (2.14) is in fact of the type of a *conditionally Gaussian filter model*, where the filter process ζ is given by the vector process $\mu = \{\mu_t,\, t \in [0,T]\}$ and the upper triangular array of the elements of the matrix process $c = \{c_t,\, t \in [0,T]\}$ with $q = (n-k)\,\frac{[3+(n-k)]}{2}$. Note by (2.16) that the matrix c_t is symmetric. Obviously, in the case when $B_t(y_t)$ does not depend on y_t for all $t \in [0,T]$, then we have a *Gaussian filter model*.

Example 2.2 : Finite-State Jump Model

Here we assume that the unobserved factors form a continuous time, $(n-k)$-dimensional jump process $x = \{x_t = (x_t^1, \ldots, x_t^{n-k})^\top,\ t \in [0,T]\}$, which can take a finite number M of values. More precisely, given an appropriate time t and z_t-dependent matrix $g_t(z_t)$, and an intensity vector $\lambda_t(z_t) = (\lambda_t^1(z_t), \ldots, \lambda_t^n(z_t))^\top$ at time $t \in [0,T]$ for the vector counting process $\bar{N} = \{\bar{N}_t = (N_t^1, \ldots, N_t^n)^\top,\ t \in [0,T]\}$, we consider the particular case of model (2.3), where in the x_t-dynamics we have $a_t(z_t) = g_t(z_t)\lambda_t(z_t)$ and $b_t(z_t) \equiv 0$. Thus, by (2.3) and (2.7) we have

$$dx_t = g_{t-}(z_{t-})\, d\bar{N}_t \tag{2.18}$$

for $t \in [0,T]$. Notice that the process x of unobserved factors has here only jumps and is therefore piecewise constant. On the other hand, for the vector y_t of observed factors we assume that it satisfies the second equation in (2.3) with $G_t(y_t) \equiv 0$. This means that the process of observed factors y is only perturbed by continuous noise and does not jump.

In this example, the filter distribution is completely characterized by the vector of conditional probabilities $p_t = (p_t^1, \ldots, p_t^M)^\top$, where M is the number of possible states $\eta^1, \ldots, \eta^M$ of the vector x_t and

$$p_t^j = P\left(x_t = \eta^j \,\middle|\, \tilde{\mathcal{A}}_t^k\right), \tag{2.19}$$

$j \in \{1, 2, \ldots, M\}$. Let $\bar{a}_t^{i,j}(y, \eta^h)$ denote the transition kernel for x at time t to jump from state i into state j given $y_t = y$ and $x_t = \eta^h$. The vector p_t satisfies the following dynamics

$$\begin{aligned} dp_t^j &= \left(\tilde{a}_t(y_t, p_t)^\top p_t\right)^j dt + p_t^j \left[A_t(y_t, \eta^j) - \tilde{A}_t(y_t, p_t)\right] \left(B_t(y_t)\, B_t(y_t)^\top\right)^{-1} \\ &\quad \cdot \left[dy_t - \tilde{A}_t(y_t, p_t)\, dt\right], \end{aligned} \tag{2.20}$$

see, Liptser & Shiryaev (1977), Chapter 9, where

$$\begin{aligned} \left(\tilde{a}_t(y_t, p_t)^\top p_t\right)^j &= \sum_{i=1}^M \left(\sum_{h=1}^M \bar{a}_t^{i,j}(y_t, \eta^h)\, p_t^h\right) p_t^i \\ A_t(y_t, \eta^j) &= A_t(y_t, x_t)\Big|_{x_t = \eta^j} \\ \tilde{A}_t(y_t, p_t) &= \sum_{j=1}^M A_t(y_t, \eta^j)\, p_t^j \end{aligned} \tag{2.21}$$

for $t \in [0,T]$, $j \in \{1,2,\dots,M\}$. The filter state process $\zeta = \{\zeta_t = (\zeta_t^1 \dots, \zeta_t^q)^\top$, $t \in [0,T]\}$ for the model of this example is thus given by the vector process $p = \{p_t = (p_t^1, \dots, p_t^q)^\top,\ t \in [0,T]\}$ with $q = M-1$. Since the probabilities add to one, we need only $M-1$ probabilities to compute.

2.2 Markovian Representation

As in the two previous examples we have, in general, in our filter setup to deal with the quantity $E(A_t(z_t) \,|\, \tilde{\mathcal{A}}_t^k)$ assuming that it exists. This is the conditional expectation of $A_t(z_t) = A_t(y_t^1, \dots, y_t^k, x_t^1, \dots, x_t^{n-k})$, given in (2.3), with respect to the filter distribution at time t for the unobserved factors x_t. Since the filter is characterized by the filter state process ζ, we obtain the representation

$$\tilde{A}_t(\tilde{z}_t^k) = E\left(A_t(z_t) \,\middle|\, \tilde{\mathcal{A}}_t^k\right), \tag{2.22}$$

where the vector $\tilde{z}_t^k$ is as defined in (2.12).

Notice that, in the case of Example 2.1, namely the conditionally Gaussian model, the expression $\tilde{A}_t(\tilde{z}_t^k)$ takes the particular form

$$\tilde{A}_t(\tilde{z}_t^k) = A_t^0 + A_t^1 \mu_t + A_t^2 y_t. \tag{2.23}$$

Furthermore, for Example 2.2, namely the finite-state jump model, $\tilde{\mathcal{A}}_t(\tilde{z}_t^k)$ can be represented as

$$\tilde{A}_t(\tilde{z}_t^k) = \tilde{A}_t(y_t, p_t) = \sum_{j=1}^{M} A_t(y_t, \eta^j)\, p_t^j \tag{2.24}$$

for $t \in [0,T]$, see (2.21).

We have now the following generalization of Theorem 7.12 in Liptser & Shiryaev (1977), which provides an important representation of the SDE for the observed factors.

Proposition 2.3 *Let $A_t(z_t)$ and the invertible matrix $B_t(y_t)$ in (2.3) be such that*

$$\int_0^T E\left(|A_t(z_t)|\right) dt < \infty \quad \textit{and} \quad \int_0^T B_t(y_t)\, B_t(y_t)^\top\, dt < \infty \tag{2.25}$$

P-a.s. Then there exists a k-dimensional $\tilde{\mathcal{A}}^k$-adapted Wiener process $\tilde{v} = \{\tilde{v}_t,\ t \in [0,T]\}$ such that the process $y = \{y_t,\ t \in [0,T]\}$ of observed factors in (2.3) satisfies the SDE

$$dy_t = \tilde{A}_t(\tilde{z}_t^k)\, dt + B_t(y_t)\, d\tilde{v}_t + G_{t-}(y_{t-})\, dN_t \tag{2.26}$$

with $\tilde{A}_t(\tilde{z}_t^k)$ as in (2.22).

The proof of Proposition 2.3 is given in Appendix A.

Instead of the original factors $z_t = (y_t^1, \dots, y_t^k, x_t^1, \dots, x_t^{n-k})^\top = (z_t^1, \dots, z_t^n)^\top$, where $x_t = (x_t^1, \dots, x_t^{n-k})^\top$ is unobserved, we may now base our analysis on the components of the vector $\tilde{z}_t^k = (y_t^1, \dots, y_t^k, \zeta_t^1, \dots, \zeta_t^q)^\top$, see (2.12), that are all observed. Just as was the case with $z = \{z_t,\, t \in [0,T]\}$, also the vector process $\tilde{z}^k = \{\tilde{z}_t^k,\, t \in [0,T]\}$ has a Markovian dynamics. In fact, replacing dy_t in (2.13) by its expression resulting from (2.26), we obtain

$$\begin{aligned} d\zeta_t &= \Big[C_t(\tilde{z}_t^k) + D_t(\tilde{z}_t^k)\, \tilde{A}_t(\tilde{z}_t^k) \Big]\, dt + D_t(\tilde{z}_t^k)\, B_t(y_t)\, d\tilde{v}_t + D_{t-}(\tilde{z}_{t-}^k)\, G_{t-}(y_{t-})\, dN_t \\ &= \tilde{C}_t(\tilde{z}_t^k)\, dt + \tilde{D}_t(\tilde{z}_t^k)\, d\tilde{v}_t + \tilde{G}_{t-}(\tilde{z}_{t-}^k)\, dN_t, \end{aligned} \tag{2.27}$$

whereby we implicitly define the vector $\tilde{C}_t(\tilde{z}_t^k)$ and the matrices $\tilde{D}_t(\tilde{z}_t^k)$ and $\tilde{G}_t(\tilde{z}_t^k)$ for compact notation.

From equations (2.26) and (2.27) we immediately obtain the following result.

Corollary 2.4 *The dynamics of the vector $\tilde{z}_t^k = (y_t, \zeta_t)$ can be expressed by the system of SDEs*

$$\begin{aligned} dy_t &= \tilde{A}_t(\tilde{z}_t^k)\, dt + B_t(y_t)\, d\tilde{v}_t + G_{t-}(y_{t-})\, dN_t \\ d\zeta_t &= \tilde{C}_t(\tilde{z}_t^k)\, dt + \tilde{D}_t(\tilde{z}_t^k)\, d\tilde{v}_t + \tilde{G}_{t-}(\tilde{z}_{t-}^k)\, dN_t. \end{aligned} \tag{2.28}$$

From Corollary 2.4 it follows that the process $\tilde{z}^k = \{\tilde{z}_t^k,\, t \in [0,T]\}$ is Markov.

Due to the existence of a Markovian filter dynamics we have our original Markovian factor model, given by (2.3), projected into a Markovian model for the observed quantities. Here the driving observable noise $\tilde{v}$ is an $(\tilde{\mathcal{A}}^k, P)$-Wiener process and the observable counting process N is generated by the first k components $N^1, N^2, \dots, N^k$ of the n counting processes.

For efficient notation we write for the vector of observables $\tilde{z}_t^k = \bar{z}_t = (\bar{z}_t^1, \bar{z}_t^2, \dots, \bar{z}_t^{k+q})^\top$ the corresponding system of SDEs in the form

$$\begin{aligned} d\bar{z}_t^\ell &= \alpha^\ell(t, \bar{z}_t^1, \bar{z}_t^2, \dots, \bar{z}_t^{k+q})\, dt + \sum_{r=1}^{k} \beta^{\ell,r}(t, \bar{z}_t^1, \bar{z}_t^2, \dots, \bar{z}_t^{k+q})\, d\tilde{v}_t^r \\ &\quad + \sum_{r=1}^{k} \gamma^{\ell,r}\left(t-, \bar{z}_{t-}^1, \bar{z}_{t-}^2, \dots, \bar{z}_{t-}^{k+q}\right) dN_t^r \end{aligned} \tag{2.29}$$

for $t \in [0,T]$ and $\ell \in \{1, 2, \dots, k+q\}$. The functions, α^ℓ, $\beta^{\ell,r}$ and $\gamma^{\ell,r}$ follow directly from $\tilde{A}$, B, G, $\tilde{C}$, $\tilde{D}$ and $\tilde{G}$ appearing in (2.28).

We also have as an immediate consequence of the Markovianity of $\tilde{z}^k = \bar{z}$, as well as property (2.10), the following result.

Corollary 2.5 *Any expectation of the form $E(u(t,z_t)\,|\,\tilde{\mathcal{A}}^k_t) < \infty$ for a given function $u : [0,T]\times\Re^n \to \Re$ and given $k \in \{1,2,\ldots,n-1\}$ can be expressed as*

$$E\left(u(t,z_t)\,\middle|\,\tilde{\mathcal{A}}^k_t\right) = \tilde{u}^k(t,\tilde{z}^k_t) \tag{2.30}$$

with a suitable function $\tilde{u}^k : [0,T]\times\Re^{k+q} \to \Re$.

Relation (2.30) in Corollary 2.5 will be of importance for contingent claim pricing as we shall see later on.

3 Filtered Benchmark Model

On the basis of the Markovian dynamics for the prices, generated by the observed factors introduced above, we formulate a *filtered benchmark model*. As described in Platen (2002), we model the different denominations of the *growth optimal portfolio* (GOP), see Long (1990). We only use the observed factors to model the GOP. However, these factors evolve in conjunction with unobserved factors that influence the observed ones. The resulting filtered benchmark model has the key advantage that a consistent price system is automatically established without using any measure transformation.

3.1 Primary Security Accounts

We assume that there are $d+1$ primary assets in the market, where $d = 2\,k$. These are, for instance, currencies or shares. For the domestic currency as primary asset we express the time evolution of its value by the *savings account process* $B^0 = \{B^0(t),\, t \in [0,T]\}$. We call B^0 also the 0th *primary security account process*.

For the modelling of the time value of the jth primary asset, $j \in \{1,2,\ldots,d\}$, we introduce the jth *primary security account process* $S^j = \{S^j(t),\, t \in [0,T]\}$. For instance, in the case of currencies, $S^j(t)$ is the value of the savings account of the jth foreign currency, expressed in units of the domestic currency. If the jth asset is a share, then $S^j(t)$ is the cum-dividend share price, where all dividend payments are reinvested. We then denote by $\bar{S}^j(t)$ the *discounted* value at time t of the jth primary security account, that is

$$\bar{S}^j(t) = \frac{S^j(t)}{B^0(t)} \tag{3.1}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$. We assume that $\bar{S}^j$ is $\tilde{\mathcal{A}}^k$-adapted and the unique strong solution of the stochastic differential equation (SDE)

$$\begin{aligned} d\bar{S}^j(t) &= \bar{S}^j(t-)\sum_{r=1}^{k}\Big\{\left(\sigma^{0,r}(t)-\sigma^{j,r}(t)\right)\left(\sigma^{0,r}(t)\,dt+d\tilde{v}^r_t\right) \\ &\quad + \left(\frac{\varphi^{j,r}(t-)}{\varphi^{0,r}(t-)}-1\right)\left(-\varphi^{0,r}(t)\,\tilde{\lambda}^r_t(y_t)\,dt+dN^r_t\right)\Big\} \end{aligned} \tag{3.2}$$

for $t \in [0,T]$ with $\bar{S}^j(0) > 0$, $j \in \{1,2,\dots,d\}$.

We assume that $\sigma^{j,r}$ and $\varphi^{j,r}$ are $\tilde{\mathcal{A}}^k$-predictable with $\varphi^{j,r}(t) \geq 0$, $\varphi^{0,r}(t) > 0$ a.s. for $t \in [0,T]$ and

$$\int_0^T \left((\sigma^{j,r}(s))^2 + \tilde{\lambda}^r_s(y_s)\right) ds < \infty$$

for $j \in \{1,2,\dots,d\}$ and $r \in \{1,2,\dots,k\}$. The given parameterization of the above SDE (3.2) does not restrict its generality but is convenient for the benchmark approach.

3.2 Portfolios

Let us now form portfolios of primary security accounts. We say that an $\tilde{\mathcal{A}}^k$-predictable stochastic process $\delta = \{\delta(t) = (\delta^0(t),\dots,\delta^d(t))^\top,\ t \in [0,T]\}$ is a *self-financing strategy*, if δ is $\bar{S}$-integrable, see Protter (1990), the corresponding portfolio $V^0_\delta(t)$ has at time t the discounted value

$$\bar{V}^0_\delta(t) = \frac{V^0_\delta(t)}{B^0(t)} = \sum_{j=0}^{d} \delta^j(t)\,\bar{S}^j(t) \tag{3.3}$$

and it is

$$d\bar{V}^0_\delta(t) = \sum_{j=0}^{d} \delta^j(t-)\,d\bar{S}^j(t) \tag{3.4}$$

for all $t \in [0,T]$. The jth component $\delta^j(t)$, $j \in \{0,1,\dots,d\}$, of the self-financing strategy δ expresses the number of units of the jth primary security account held at time t in the corresponding portfolio. Under a self-financing strategy no outflow or inflow of funds occurs for the corresponding portfolio. All changes in the value of the portfolio are due to gains from trade in the primary security accounts.

We assume that no primary security account is *redundant*. That means, no primary security account can be expressed as a self-financing portfolio of other primary security accounts. Let us set

$$b^{j,r}(t) = \begin{cases} \sigma^{0,r}(t) - \sigma^{j,r}(t) & \text{for} \quad r \in \{1,2,\ldots,k\} \\ \left(\frac{\varphi^{j,r-k}(t)}{\varphi^{0,r-k}(t)} - 1\right) \sqrt{\tilde{\lambda}_t^{r-k}(y_t)} & \text{for} \quad r \in \{k+1,\ldots,d\} \end{cases} \tag{3.5}$$

for $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$. We then define the matrix $b(t) = [b^{j,r}(t)]_{j,r=1}^{d}$ for $t \in [0,T]$ and assume that $b(t)$ is for Lebesgue-almost-every $t \in [0,T]$ *invertible*. Note that the observed market is *complete*, which means that the observed primary security accounts securitize the uncertainty generated by the Wiener processes $\tilde{v}^1,\ldots,\tilde{v}^k$ and the counting processes $N^1,\ldots,N^k$.

3.3 Growth Optimal Portfolio

The GOP is the self-financing portfolio that achieves maximum expected logarithmic utility. We denote by $\bar{V}^i_{\underline{\delta}}(t)$ the value of the GOP when it is expressed at time t in units of the ith primary security account. For the diffusion case without jumps the corresponding SDE is well known, see, for instance, Long (1990) or Karatzas & Shreve (1998). In the case with jumps the derivation of the SDE for the GOP is more involved and described in Platen (2002). It has the form

$$\begin{aligned} d\bar{V}^i_{\underline{\delta}}(t) &= \bar{V}^i_{\underline{\delta}}(t-) \sum_{r=1}^{k} \Bigg\{ \sigma^{i,r}(t)\,(\sigma^{i,r}(t) + d\tilde{v}^r(t)) + \left(\frac{1}{\varphi^{i,r}(t-)} - 1\right) \\ &\quad \cdot \left(-\varphi^{i,r}(t)\,\tilde{\lambda}^r_t(y_t)\,dt + dN^r_t\right)\Bigg\} \end{aligned} \tag{3.6}$$

for $t \in [0,T]$ and $i \in \{0,1,\ldots,d\}$.

To make the above framework computationally tractable, given our factor model, we specify $\bar{V}^i_{\underline{\delta}}(t)$ as a function of time t and the vector of observables $\tilde{z}^k_t$, that is

$$\bar{V}^i_{\underline{\delta}}(t) = \bar{V}^i_{\underline{\delta}}(t, \tilde{z}^k_t) = \bar{V}^i_{\underline{\delta}}(t, \bar{z}^1_t, \ldots, \bar{z}^{k+q}_t) \tag{3.7}$$

for $t \in [0,T]$ and $i \in \{0,1,\ldots,d\}$. Assuming sufficient smoothness of $\bar{V}^i_{\underline{\delta}}(\cdot,\cdot)$, by application of the Itô formula and using (2.29), we then obtain

$$\begin{aligned} \bar{V}^i_{\underline{\delta}}(t, \tilde{z}^k_t) &= \bar{V}^i_{\underline{\delta}}(0, \tilde{z}^k_0) + \int_0^t L^0\, \bar{V}^i_{\underline{\delta}}(s, \tilde{z}^k_s)\, ds + \sum_{r=1}^{k} \int_0^t L^r\, \bar{V}^i_{\underline{\delta}}(s, \tilde{z}^k_s)\, d\tilde{v}^r_s \\ &\quad + \sum_{r=1}^{k} \int_0^t \Delta^r_{\bar{V}^i_{\underline{\delta}}}(s-, \tilde{z}^k_{s-})\, dN^r_s \end{aligned} \tag{3.8}$$

for $t \in [0,T]$ and $i \in \{0,1,\ldots,d\}$. Here we use the operators

$$L^0 = \frac{\partial}{\partial t} + \sum_{\ell=1}^{k+q} \alpha^\ell\left(t, \bar{z}_t^1, \ldots, \bar{z}_t^{k+q}\right) \frac{\partial}{\partial \bar{z}^\ell} + \frac{1}{2} \sum_{\ell,p=1}^{k+q} \sum_{r=1}^{k} \beta^{\ell,r}\left(t, \bar{z}_t^1, \ldots, \bar{z}_t^{k+q}\right) \beta^{p,r}\left(t, \bar{z}_t^1, \ldots, \bar{z}_t^{k+q}\right) \frac{\partial^2}{\partial \bar{z}^\ell \, \partial \bar{z}^p}, \tag{3.9}$$

$$L^r = \sum_{\ell=1}^{k+q} \beta^{\ell,r}\left(t, \bar{z}_t^1, \ldots, \bar{z}_t^{k+q}\right) \frac{\partial}{\partial \bar{z}^\ell} \tag{3.10}$$

and

$$\Delta_F^r(t-, \tilde{z}_{t-}^k) = F\left(t, \bar{z}_{t-}^1 + \gamma^{1,r}\left(t-, \bar{z}_{t-}^1, \ldots, \bar{z}_{t-}^{k+q}\right), \ldots, \bar{z}_{t-}^{k+q} + \gamma^{k+q,r}\left(t-, \bar{z}_{t-}^1, \ldots, \bar{z}_{t-}^{k+q}\right)\right) - F\left(t-, \bar{z}_{t-}^1, \ldots, \bar{z}_{t-}^{k+q}\right) \tag{3.11}$$

for $t \in [0,T]$, $r \in \{1,2,\ldots,k\}$ and $F : [0,T] \times \Re^{k+q} \to \Re$ being any given function of time and vector of observables, where $\tilde{z}_t^k = \bar{z}_t = (\bar{z}_t^1, \ldots, \bar{z}_t^{k+q})^\top$ as introduced before (2.29).

By comparison of (3.6) and (3.8) it follows that the i,ℓth GOP-volatility $\sigma^{i,\ell}(t)$ has the form

$$\sigma^{i,\ell}(t) = \frac{L^\ell \, \bar{V}_{\underline{\delta}}^i(t, \tilde{z}_t^k)}{\bar{V}_{\underline{\delta}}^i(t, \tilde{z}_t^k)} \tag{3.12}$$

and the inverted i,ℓth GOP-jump ratio $\varphi^{i,\ell}(t)$, see (3.6) is given by the expression

$$\varphi^{i,\ell}(t-) = \frac{\bar{V}_{\underline{\delta}}^i(t-, \tilde{z}_{t-}^k)}{\Delta_{\bar{V}_{\underline{\delta}}^i}^\ell(t-, \tilde{z}_{t-}^k) + \bar{V}_{\underline{\delta}}^i(t-, \tilde{z}_{t-}^k)} \tag{3.13}$$

for $t \in [0,T]$, $i \in \{0,1,\ldots,d\}$, $\ell \in \{1,2,\ldots,k\}$.

4 Fair Pricing and Hedging of Derivatives

4.1 Benchmarked Prices

In what follows we call prices that are expressed in units of the GOP, *benchmarked* prices. This means for $j \in \{0,1,\ldots,d\}$ that the jth benchmarked primary security account $\hat{S}^j = \{\hat{S}^j(t),\, t \in [0,T]\}$ with

$$\hat{S}^j(t) = \frac{S^j(t)}{V^0_{\underline{\delta}}(t)} = \frac{\bar{S}^j(t)}{\bar{V}^0_{\underline{\delta}}(t)} \tag{4.1}$$

satisfies by (3.2), (3.5), (3.6) and application of the Itô formula the SDE

$$d\hat{S}^j(t) = \hat{S}^j(t-) \sum_{r=1}^{k} \left\{ -\sigma^{j,r}(t)\, d\tilde{v}^r_t + (\varphi^{j,r}(t-) - 1)\, dm^r_t \right\} \tag{4.2}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$, see Platen (2002). Here m^r_t denotes the rth component of the jump martingale m defined in (2.7). Note that the jth benchmarked primary security account is an $(\tilde{\mathcal{A}}^k, P)$-local martingale. Moreover, as shown in Platen (2002), for any self-financing portfolio V^0_δ it follows by application of the Itô formula that its benchmarked value $\hat{V}_\delta(t) = \frac{V^0_\delta(t)}{V^0_{\underline{\delta}}(t)} = \frac{\bar{V}^0_\delta(t)}{\bar{V}^0_{\underline{\delta}}(t)}$ satisfies the SDE

$$\begin{aligned} d\hat{V}_\delta(t) \;&=\; \hat{V}_\delta(t-) \sum_{r=1}^{k} \left\{ - \sum_{j=0}^{d} \pi^j_\delta(t)\, \sigma^{j,r}(t)\, d\tilde{v}^r(t) \right. \\ &\quad \left. + \left(\sum_{j=0}^{d} \pi^j_\delta(t-)\, \varphi^{j,r}(t-) - 1 \right) dm^r_t \right\} \end{aligned} \tag{4.3}$$

for $t \in [0,T]$. This shows that $\hat{V}_\delta$ is an $(\tilde{\mathcal{A}}^k, P)$-local martingale too. Note that these processes are, in general, not $(\tilde{\mathcal{A}}^k, P)$-martingales. Since a nonnegative benchmarked portfolio process is here an $(\tilde{\mathcal{A}}^k, P)$-supermartingale, the resulting filtered benchmark model can be shown to exclude *standard arbitrage*. This means, it is impossible to generate, with strictly positive probability, strictly positive wealth from zero initial capital.

4.2 Derivative Prices

To provide an intuitive link between the benchmark framework and the standard risk neutral approach, let us discuss a situation where we assume for the moment that the following steps can be made and a standard equivalent risk neutral probability measure P^k exists. We underline that such assumptions will not be needed for our results. Then all prices, discounted by the domestic savings account B^0 would be $(\tilde{\mathcal{A}}^k, P^k)$-martingales. Denoting by E the expectation with respect to P and by E^k that with respect to P^k, we would have, taking into account (4.1), that

$$\begin{aligned} S^j(t) &= E^k\left(\frac{B^0(t)}{B^0(T)}\, S^j(T)\,\Big|\,\tilde{\mathcal{A}}^k_t\right) \\ &= E\left(\frac{\Lambda^k_T}{\Lambda^k_t}\,\frac{B^0(t)}{B^0(T)}\, S^j(T)\,\Big|\,\tilde{\mathcal{A}}^k_t\right) \\ &= V^0_{\underline{\delta}}(t)\, E\left(\frac{S^j(T)}{V^0_{\underline{\delta}}(T)}\,\Big|\,\tilde{\mathcal{A}}^k_t\right) \end{aligned} \tag{4.4}$$

for $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$. Here the Radon-Nikodym derivative $\Lambda^k_T = \frac{dP^k}{dP}$ would satisfy the expression

$$\Lambda^k_t = \frac{V^0_{\underline{\delta}}(0)}{V^0_{\underline{\delta}}(t)}\,\frac{B^0(t)}{B^0(0)} = \frac{\hat{S}^0(t)}{\hat{S}^0(0)} \tag{4.5}$$

for $t \in [0,T]$. Furthermore, by (4.5), the price of a self-financing portfolio V^0_δ would satisfy the relation

$$\begin{aligned} V^0_\delta(t) &= E^k\left(\frac{B^0(t)}{B^0(\tau)}\, V^0_\delta(\tau)\,\Big|\,\tilde{\mathcal{A}}^k_t\right) \\ &= E\left(\frac{\Lambda^k_\tau}{\Lambda^k_t}\,\frac{B^0(t)}{B^0(\tau)}\, V^0_\delta(\tau)\,\Big|\,\tilde{\mathcal{A}}^k_t\right) \\ &= V^0_{\underline{\delta}}(t)\, E\left(\frac{V^0_\delta(\tau)}{V^0_{\underline{\delta}}(\tau)}\,\Big|\,\tilde{\mathcal{A}}^k_t\right) \end{aligned} \tag{4.6}$$

for $t \in [0,\tau]$ and any $\tilde{\mathcal{A}}^k$-stopping time τ. Thus, under the above assumptions all benchmarked portfolio prices $\hat{V}_\delta(t) = \frac{V^0_\delta(t)}{V^0_{\underline{\delta}}(t)}$ would be $(\tilde{\mathcal{A}}^k, P)$-martingales, that is

$$\hat{V}_\delta(t) = E\left(\hat{V}_\delta(\tau)\,\big|\,\tilde{\mathcal{A}}^k_t\right) \tag{4.7}$$

for all $t \in [0,\tau]$.

In the benchmark framework we avoid the above steps and the assumption on the existence of an equivalent risk neutral measure by introducing the concept of a *fair price*. A price process is called *fair*, if its benchmarked values form an $(\tilde{\mathcal{A}}^k, P)$-martingale, see Platen (2002).

At a given maturity date τ, which is assumed to be an $\tilde{\mathcal{A}}^k$-stopping time, we consider a *benchmarked contingent claim* $U(\tau, y_\tau)$ as a function of τ and the corresponding values of observed factors y_τ, where we assume that

$$E(|U(\tau, y_\tau)|\,\big|\,\tilde{\mathcal{A}}^k_t) < \infty \tag{4.8}$$

a.s. for all $t \in [0, \tau]$. There is no point to let the payoff function depend on any other than observed factors, otherwise the payoff would not be verifiable at time τ. The *benchmarked fair price process* $\tilde{u}^k = \{\tilde{u}^k(t, \tilde{z}_t^k),\ t \in [0, \tau]\}$ for the benchmarked contingent claim $U(\tau, y_\tau)$ is then the $(\tilde{\mathcal{A}}^k, P)$-martingale, obtained by the conditional expectation

$$\tilde{u}^k(t, \tilde{z}_t^k) = E\left(U(\tau, y_\tau) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.9}$$

for $t \in [0, \tau]$. This means, we form directly the conditional expectation (4.7) without using any measure transformation. The corresponding fair price at time t for this contingent claim, when expressed in units of the domestic currency, is then

$$\tilde{u}^{0,k}(t, \tilde{z}_t^k) = V_{\underline{\delta}}^0(t)\, \tilde{u}^k(t, \tilde{z}_t^k) \tag{4.10}$$

for $t \in [0, \tau)$. The above concept of fair pricing generalizes the well-known concept of risk neutral pricing and avoids not only the assumption on the existence of an equivalent risk neutral measure but also some issues that arise from measure changes under different filtrations.

The vector of observables y_τ is a subvector, not only of $\tilde{z}_\tau^k$ but also of z_τ. This allows us to define the $(\underline{\mathcal{A}}, P)$-martingale $u = \{u(t, z_t),\ t \in [0, \tau]\}$ by the conditional expectation

$$u(t, z_t) = E\left(U(\tau, y_\tau) \,\middle|\, \mathcal{A}_t\right) \tag{4.11}$$

for $t \in [0, \tau]$, which at time t exploits the complete information characterized by the σ-algebra $\mathcal{A}_t$. The above derivation can be summarized in the following result.

Corollary 4.1 *The benchmarked fair price $\tilde{u}^k(t, \tilde{z}_t^k)$ for the benchmarked contingent claim $U(\tau, y_\tau)$ can be expressed as the conditional expectation*

$$\tilde{u}^k(t, \tilde{z}_t^k) = E\left(u(t, z_t) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.12}$$

for $t \in [0, \tau]$.

We recall that $\tilde{\mathcal{A}}_t^k$ denotes in Corollary 4.1 the information, which is available at time t, whereas $\mathcal{A}_t$ is the complete information at time t that determines the original model dynamics including also the unobserved factors.

Note that the benchmarked fair price, given in Corollary 4.1, fits perfectly the expression of our result for the filtered factor model given in (2.30). The advantage of the representation (4.12) is that it allows us to express the benchmarked fair price $\tilde{u}^k(t, \tilde{z}_t^k)$ as conditional expectation with respect to $\tilde{\mathcal{A}}_t^k$. The actual computation of the conditional expectation in (4.12) is equivalent to the solution of the filtering problem for the unobserved factors.

4.3 Hedging Strategy

Assume that the above benchmarked pricing function $\tilde{u}^k(\cdot,\cdot)$ in (4.9) and (4.12) is differentiable with respect to time and twice differentiable with respect to the observables. Then we obtain by the Itô formula the representation

$$\begin{aligned} U(\tau, y_\tau) &= \tilde{u}^k(\tau, \tilde{z}^k_\tau) \\ &= \tilde{u}^k(t, \tilde{z}^k_t) + \sum_{\ell=1}^{k} \int_t^\tau \tilde{u}^k(s, \tilde{z}^k_s) \frac{L^\ell\, \tilde{u}^k(s, \tilde{z}^k_s)}{\tilde{u}^k(s, \tilde{z}^k_s)}\, d\tilde{v}^\ell_s \\ &\quad + \sum_{\ell=1}^{k} \int_t^\tau \tilde{u}^k(s-, \tilde{z}^k_{s-}) \frac{\Delta^\ell_{\tilde{u}^k}(s-, \tilde{z}^k_{s-})}{\tilde{u}^k(s-, \tilde{z}^k_{s-})}\, dm^\ell_s \end{aligned} \tag{4.13}$$

for $t \in [0,\tau]$. Let us search for a fair benchmarked price process $\hat{V}_{\delta_U}$, with self-financing *hedging strategy* δ_U, that possibly matches $\tilde{u}^k$. This means, we consider $\hat{V}_{\delta_U}(t)$ with

$$d\hat{V}_{\delta_U}(t) = \sum_{j=0}^{d} \delta^j_U(t-)\, d\hat{S}^j(t) \tag{4.14}$$

for $t \in [0,\tau]$. By (4.3) we then have

$$\begin{aligned} \hat{V}_{\delta_U}(\tau) = \hat{V}_{\delta_U}(t) &- \sum_{\ell=1}^{k} \int_t^\tau \hat{V}_{\delta_U}(s) \sum_{j=0}^{d} \pi^j_{\delta_U}(s)\, \sigma^{j,\ell}(s)\, d\tilde{v}^\ell_s \\ &+ \sum_{\ell=1}^{k} \int_t^\tau \hat{V}_{\delta_U}(s-) \left(\sum_{j=0}^{d} \pi^j_{\delta_U}(s-) \varphi^{j,r}(s-) - 1 \right) dm^\ell_s. \end{aligned} \tag{4.15}$$

Note that the volatilities and jump ratios in (4.15) are those identified in (3.12) and (3.13). Above we used the jth *proportion*

$$\pi^j_{\delta_U}(t) = \frac{\delta^j_U(t)\, \hat{S}^j(t)}{\hat{V}_{\delta_U}(t)} \tag{4.16}$$

of the value of the corresponding hedging portfolio that has to be invested into the jth primary security account at time $t \in [0,\tau]$. To replicate the benchmarked contingent claim $U(\tau, y_\tau)$ we can start at a given time $t \in [0,\tau]$ by forming a portfolio with fair benchmarked price

$$\hat{V}_{\delta_U}(t) = \tilde{u}^k(t, \tilde{z}^k_t) = E\left(U(\tau, y_\tau) \,\middle|\, \tilde{\mathcal{A}}^k_t \right). \tag{4.17}$$

By comparison of (4.13) and (4.15) the proportions must satisfy the system of linear equations

$$-\frac{L^\ell\, \tilde{u}^k(t, \tilde{z}^k_t)}{\tilde{u}^k(t, \tilde{z}^k_t)} = \sum_{j=0}^{d} \pi^j_{\delta_U}(t)\, \sigma^{j,\ell}(t) \tag{4.18}$$

and

$$\frac{\Delta^{\ell}_{\tilde{u}^k}(t-,\tilde{z}^k_{t-})}{\tilde{u}^k(t-,\tilde{z}^k_{t-})} + 1 = \sum_{j=0}^{d} \pi^j_{\delta}(t-)\,\varphi^{j,\ell}(t-) \tag{4.19}$$

for $\ell \in \{1,2,\ldots,k\}$ and $t \in [0,T]$. Let us use the d-dimensional vector $c(t-) = (c^1(t-), c^2(t-), \ldots, c^d(t-))^\top$ with components

$$c^r_{\tilde{u}^k}(t-) = \begin{cases} \frac{L^r\,\tilde{u}^k(t-,\tilde{z}^k_{t-})}{\tilde{u}^k(t-,\tilde{z}^k_{t-})} + \sigma^{0,r}(t) & \text{for} \quad r \in \{1,2,\ldots,k\} \\ \left(\frac{1}{\varphi^{0,\ell}(t-)}\left(\frac{\Delta^{r-k}_{\tilde{u}^k}(t-,\tilde{z}^k_{t-})}{\tilde{u}^k(t-,\tilde{z}^k_{t-})} + 1\right) - 1\right)\sqrt{\tilde{\lambda}^{r-k}_t(y_{t-})} & \text{for} \quad r \in \{k+1,\ldots,d\}, \end{cases} \tag{4.20}$$

$t \in [0,\tau)$. By involving the matrix $b(t)$ given in (3.5), we can then rewrite the system of equations (4.18) - (4.19) in the form

$$c_{\tilde{u}^k}(t-)^\top = \pi_{\delta_U}(t-)^\top\, b(t-) \tag{4.21}$$

for $t \in [0,T]$. Now, we obtain the following result.

Proposition 4.2 *For a given benchmarked contingent claim $U(\tau, y_\tau)$ with corresponding vector $c_{\tilde{u}^k}(t)$ given in (4.20) the proportions of the corresponding hedging portfolio are of the form*

$$\pi_{\delta_U}(t-) = \left(c_{\tilde{u}^k}(t-)^\top\, b^{-1}(t-)\right)^\top \tag{4.22}$$

for $t \in [0,\tau)$.

Note that the invertibility of the matrix $b(t)$ is not linked to a specific contingent claim. Thus, one can form a perfectly replicating hedging portfolio for all benchmarked contingent claims $U(\tau, y_\tau)$. The introduced filtered benchmark model forms a complete market despite the fact that the original model involves unobserved factors. The benchmarked pricing functions can always be obtained from the conditional expectation (4.12) on the basis of the filter distribution.

We did not consider the case $d < 2k$. In such a case the market is *incomplete*. Incomplete markets of this type can be handled by a generalization of the above described filtered benchmark approach.

4.4 Variance of Benchmarked Prices

Let us now investigate the impact of varying degrees of information concerning the factors $z_t = (z^1_t, \ldots, z^n_t)^\top$ that underly our model dynamics, see (2.2) - (2.3). As already mentioned in Section 2.1, the degree of available information is indexed by the parameter k. A larger value of k means that more factors are observed, providing thus more information in $\tilde{\mathcal{A}}^k$. Again we use the notation $\tilde{z}_t^k$ for the vector of observables defined in (2.12), where we stress its dependence on k and recall that, by (2.28), the process $\tilde{z}^k$ is Markovian.

Consider from now on a benchmarked contingent claim

$$U(\tau, y_\tau) = U(\tau, y_\tau^1, y_\tau^2, \ldots, y_\tau^r) \tag{4.23}$$

for some fixed $r \in \{1, 2, \ldots, n-1\}$, where we assume that the number of observed factors that influence the claim equals r. For $k \in \{r, r+1, \ldots, n-1\}$ let $\tilde{u}^k(t, \tilde{z}_t^k)$ be the corresponding benchmarked fair price under the information $\tilde{\mathcal{A}}_t^k$, as given by (4.12). Recall by (2.30) that $\tilde{u}^k(t, \tilde{z}_t^k)$ can be computed as conditional expectation via the filter distribution. Then

$$\mathrm{Var}_t^k(u) = E\left(\left(u(t, z_t) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.24}$$

is the corresponding *conditional variance* at time $t \in [0, \tau)$. Note that for larger k we have more information available, which naturally should reduce the conditional variance.

For each degree of available information one obtains, in general, different equivalent risk neutral probability measures. The complexity of working with different pricing measures can be significant. This is avoided by using the suggested filtered benchmark model. All conditional expectations can be taken under the real world probability measure P. Furthermore, it is clear that filtering itself is always performed under the real world measure.

We can prove the following proposition, which expresses the reduction in conditional variance and can also be seen as a generalization of the celebrated Rao-Blackwell theorem towards filtering.

Proposition 4.3 *For $m \in \{0, 1, \ldots, n-k\}$ and $k \in \{r, r+1, \ldots, n-1\}$ we have*

$$E\left(Var_t^{k+m}(u) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) = Var_t^k(u) - R_t^{k+m}, \tag{4.25}$$

where

$$R_t^{k+m} = E\left(\left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.26}$$

for $t \in [0, \tau)$.

Proof: For $t \in [0, \tau)$ and $k \in \{r, r+1, \ldots, n-1\}$ we have

$$\begin{aligned}\left(u(t, z_t) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 = {} & \left(u(t, z_t) - \tilde{u}^{k+m}(t, \tilde{z}_t^{k+m})\right)^2 + \left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 \\ & + 2\left(u(t, z_t) - \tilde{u}^{k+m}(t, \tilde{z}_t^{k+m})\right)\left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right).\end{aligned} \tag{4.27}$$

By taking conditional expectations with respect to $\tilde{\mathcal{A}}_t^k$ on both sides of the above equation it follows that

$$\begin{aligned} \mathrm{Var}_t^k(u) &= E\left(\mathrm{Var}_t^{k+m}(u)\,\middle|\,\tilde{\mathcal{A}}_t^k\right) + R_t^{k+m} + 2\,E\Big(\left(\tilde{u}^{k+m}(t,\tilde{z}_t^{k+m}) - \tilde{u}^k(t,\tilde{z}_t^k)\right) \\ &\qquad \cdot E\left(\left(u(t,z_t) - \tilde{u}^{k+m}(t,\tilde{z}_t^{k+m})\right)\,\middle|\,\tilde{\mathcal{A}}_t^{k+m}\right)\,\Big|\,\tilde{\mathcal{A}}_t^k\Big). \end{aligned} \tag{4.28}$$

Since the last term on the right hand side is equal to zero by definition, we obtain (4.25). □

5 Conclusions

We constructed a filtered benchmark model by specifying the growth optimal portfolio for a given degree of available information. A consistent price system has been established. Benchmarked fair derivative prices are obtained as martingales under the real world probability measure. In general, benchmarked security prices are not forced to be martingales. They may be just local martingales. The reduction of the conditional variance of fair derivative prices under increased information is quantified via a generalization of the Rao-Blackwell theorem.

A Appendix

Proof of Proposition 2.3

Denote by y^c the continuous part of the observation process y, that is

$$y_t^c = y_t - \sum_{\tau_j \le t} G_{\tau_j -}(y_{\tau_j -})\,\Delta\, N_{\tau_j}, \tag{A.1}$$

where the τ_j denote the jump times of $N = \{N_t,\, t \in [0,T]\}$ and $\Delta\, N_{\tau_j} = N_{\tau_j} - N_{\tau_j -}$ is the vector $(\Delta\, N^1_{\tau_j -}, \ldots, \Delta\, N^k_{\tau_j -})^\top$. Let us now define the k-dimensional $\tilde{\mathcal{A}}^k$-adapted process $\tilde{v} = \{\tilde{v}_t,\, t \in [0,T]\}$ by

$$B_t(y_t)\, d\tilde{v}_t = dy_t^c - \tilde{A}_t(\tilde{z}_t^k)\, dt. \tag{A.2}$$

From (2.3), (A.1) and (A.2) it follows that

$$d\tilde{v}_t = dv_t + B_t(y_t)^{-1}\left[A_t(z_t) - \tilde{A}_t(\tilde{z}_t^k)\right] dt. \tag{A.3}$$

From this we find, by the multi-variate Itô formula with $\nu \in \Re^k$ a row vector and $\imath$ the imaginary unit, that

$$\begin{aligned}\exp\left[\imath\nu\left(\tilde{v}_t - \tilde{v}_s\right)\right] &= 1 + \imath\nu \int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] dv_u \\ &\quad + \imath\nu \int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]\, B_u^{-1}(y_u)\left(A_u(z_u) - \tilde{A}_u(\tilde{z}_u^k)\right) du \\ &\quad - \frac{\nu\,\nu^\top}{2} \int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] du. \end{aligned} \tag{A.4}$$

Recalling that v is an $\tilde{\mathcal{A}}^k$-measurable Wiener process, notice that

$$E\left(\int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]\, dv_u \,\middle|\, \tilde{\mathcal{A}}_s^k\right) = 0 \tag{A.5}$$

and that, by our assumptions and by the boundedness of $\exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]$,

$$\begin{aligned} E\left(\int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]\, B_u^{-1}(y_u)\left(A_u(z_u) - \tilde{A}_u(\tilde{z}_u^k)\right)\, du \,\middle|\, \tilde{\mathcal{A}}_s^k\right) &= \\ E\left(\int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] B_u^{-1}(y_u) E\left(\left(A_u(z_u) - \tilde{A}_u(\tilde{z}_u^k)\right) \,\middle|\, \tilde{\mathcal{A}}_u\right)\, du \,\middle|\, \tilde{\mathcal{A}}_s^k\right) &= 0. \end{aligned} \tag{A.6}$$

Taking conditional expectations on the left and the right hand sides of (A.4) we end up with the equation

$$E\left(\exp\left(\imath\nu\left[\left(\tilde{v}_t - \tilde{v}_s\right)\right]\right) \,\middle|\, \tilde{\mathcal{A}}_s^k\right) = 1 - \frac{\nu\,\nu'}{2} \int_s^t E\left(\exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] \,\middle|\, \tilde{\mathcal{A}}_s^k\right) du, \tag{A.7}$$

which has the solution

$$E\left(\exp\left[\imath\nu\left(\tilde{v}_t - \tilde{v}_s\right)\right] \,\middle|\, \tilde{\mathcal{A}}_s^k\right) = \exp\left[-\frac{\nu\,\nu^\top}{2}(t-s)\right] \tag{A.8}$$

for $0 \le s \le t \le T$. We can conclude that $(\tilde{v}_t - \tilde{v}_s)$ is a k-dimensional vector of independent $\tilde{\mathcal{A}}_t^k$-measurable Gaussian random variables, each with variance $(t-s)$ and independent of $\tilde{\mathcal{A}}_s^k$. By Levy's theorem, $\tilde{v}$ is thus a k-dimensional $\tilde{\mathcal{A}}^k$-adapted standard Wiener process. □

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