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The Stochastic Dynamics of Speculative Prices *

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Abstract

Within the framework of the heterogeneous agent paradigm, we establish a stochastic model of speculative price dynamics involving of two types of agents, fundamentalists and chartists, and the market price equilibria of which can be characterised by the invariant measures of a random dynamical system. By conducting a stochastic bifurcation analysis, we examine the market impact of speculative behaviour. We show that, when the chartists use lagged price trends to form their expectations, the market equilibrium price can be characterised by a unique and stable invariant measure when the activity of the speculators is below a certain critical value. If this threshold is surpassed, the market equilibrium can be characterised by more than two invariant measures, of which one is completely stable, another is completely unstable and the remaining ones may exhibit various types of stability. Also, the corresponding stationary measure displays a significant qualitative change near the threshold value. We show that the stochastic model displays behaviour consistent with that of the underlying deterministic model. However, when the time lag in the formation of the price trends used by the chartists approaches zero, such consistency breaks down. In addition, the change in the stationary distribution is consistent with a number of market anomalies and stylised facts observed in financial markets, including a bimodal logarithmic price distribution and fat tails.

Key Words: Heterogeneous agents, speculative behaviour, random dynamical systems, stochastic bifurcations, invariant measures, chartists.

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1 Introduction

Traditional economic and finance theory based on the paradigm of the representative agent with rational expectations has not only been questioned because of the strong (and some would argue - unrealistic) assumptions of agent homogeneity and rationality, but has also encountered great difficulties in explaining many market anomalies and stylised facts that show up in many empirical studies, including high trading volume, excess volatility, volatility clustering, long-range dependence, skewness, and excess kurtosis (see Pagan (1996) for a description of the various anomalies). Furthermore, survey data and empirical literature (see Frankel and Froot (1986), Shiller (1987), Allen and Taylor (1990) and Taylor and Allen (1992)) have provided evidence of heterogeneity and bounded rationality. As a result, there has been a rapid growth in the literature on heterogeneous agent models that is well summarised in the recent survey papers by Hommes (2006), LeBaron (2006) and Chiarella, Dieci and He (2008). These models characterise the dynamics of financial asset prices and returns resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices. For example, Brock and Hommes (1997, 1998) propose a simple *Adaptive Belief System* to model economic and financial markets. Agents' decisions are based upon their predictions of future values of endogenous variables, the actual values of which are determined by equilibrium equations. A key aspect of these models is that they exhibit feedback of expectations. Agents update their beliefs based on the market price which in turn is determined by agents' expectations. The resulting dynamical system is *nonlinear* and, as Brock and Hommes (1998) show, capable of generating complex behaviour from local stability to (a)periodic cycles and even chaos. It has been shown (see for instance Hommes (2002) and He and Li (2007)) that such simple nonlinear adaptive models are capable of capturing important empirically observed features of real financial time series, including fat tails, clustering in volatility and power-law behaviour (in returns). The analysis of the stylised simple evolutionary adaptive models and their numerical analysis provide insight into the connection between individual and market behaviour.

The current paper contributes to the development of this new paradigm of heterogeneous boundedly rational agents by modelling the stochastic price dynamics of speculative behaviour in financial markets in a continuous time framework. One of the most important issues for various heterogeneous asset pricing models is the interaction of the behaviour of the heterogeneous agents and the interplay of noise with the underlying nonlinear deterministic market dynamics. Indeed He and Li (2007) in their simulations found that these two effects interact in ways which are not yet understood at a theoretical level. The noise can be either fundamental noise (by which we mean noise in the underlying economic processes determining the fundamental market price) or market noise (this can best be thought of as noise impinging on the market-clearing mechanism, due perhaps to the arrival of news events causing stochastic shifts in demand and supply, and hence so-called noise traders could be incorporated under this rubric), or both. One simple approach, referred to as the *indirect approach* for convenience, is to first consider the corresponding deterministic "skeleton" of the stochastic models where the noise terms are set to zero and use is made of stability and bifurcation theory to investigate the dynamics of this nonlinear deterministic system; one then uses simulation methods to examine the interplay of various types of noise and the deterministic dynamics. In fact, this indirect approach has been used extensively in the economic dynamics and financial market modelling literature (we refer the reader to Hommes (2006) for related references). However it is well known that the dynamics of stochastic systems can be very different from the dynamics of the corresponding deterministic systems. Ideally we would like to deal directly with the dynamics of the stochastic systems, but this *direct approach* can be difficult. For example, in an agent-based financial market model with stochastic noise, we would like to know how the distributional properties of the model, which can be characterised by the stationary distribution of the market price process, change as agents' behaviour changes and how the market price distribution is influenced by the underlying deterministic dynamics. In particular, we can ask if there is a connection between different types of attractors and bifurcations of the underlying deterministic skeleton and different types of invariant measures of the stochastic system. By adding noise to the underlying deterministic system and using the simulation approach, many models are able to generate realistic time series (see, for example, Hommes (2002), Chiarella, He and Hommes (2006*a*), (2006*b*)), but few models have been developed that are able to give a qualitative characterisation of the stochastic nature of the dynamics.

In this paper, we extend the deterministic models of speculative price dynamics of Beja and Goldman (1980) and Chiarella (1992) to a stochastic model whose market price equilibria can be characterised by different types of invariant measures. We choose this very basic model of fundamentalist and speculative behaviour as it captures in a very simple way the essential aspects of the heterogeneous boundedly rational agent paradigm. Instead of the *indirect* approach, we use the *direct* approach to investigate the stochastic model. By using concepts and stochastic bifurcation techniques from the theory of random dynamical systems (see Arnold (1998)), we conduct a quantitative and qualitative analysis of the stochastic model and examine the existence and stability of invariant measures of the equilibrium market price. Because of the analytical limitations imposed by the state of the art in stochastic bifurcation theory for higher dimensional systems, we investigate the equilibrium distribution of the market price through a combination of theoretical analysis and numerical studies of the stochastic bifurcation. We show that the market price can display different forms of equilibrium distribution, depending on the speculative behaviour of the chartists. In particular, when the chartists use lagged price trends in their expectation, we show through a so-called dynamical (D)-bifurcation analysis that the market equilibrium price can be characterised by a unique and stable invariant measure when the activity of the speculators is below a certain critical value. If this threshold is surpassed, a new and stable invariant measure appears, while the original invariant measure becomes unstable. In addition, we show through a so-called phenomenological (P)-bifurcation analysis that the corresponding stationary measure displays a significant qualitative change near the threshold value from a single peak (unimodal) to crater-like (bimodal) joint distributions (and also marginal distributions) as the chartists become more active in the market. Using a stochastic approximation method, we confirm this qualitative change analytically and show that the stochastic approximation of the stochastic model displays a bifurcation of very similar nature to that of the underlying deterministic model. However, when the time lag of the price trends used by the chartists approaches zero, the stochastic model can display very different features from those of its underlying deterministic model and the resulting P-bifurcation may not be consistent with the D-bifurcation. Overall, the change of the stationary distributions is characterised by a number of market anomalies and stylised facts observed in financial markets, including a bimodal logarithm price distribution and fat tails.

A number of stochastic asset pricing models have been constructed in the heterogeneous agent literature. The earliest of which we are concern is that of Föllmer (1974) who allows agents' preferences to be random and governed by a law that depends on their interaction with the economic environment. Föllmer and Schweizer (1993) consider a model of fundamentalists and noise traders (akin to our chartists) in a stochastic environment and under certain assumptions show that the limiting distribution can display fat tail bahaviour. Brock, Hommes and Wagener (2005) study the evolution of a discrete financial market model with many types of agents by focusing on the limiting distribution over types of agents. They show that the evolution can be well described by the large type limit (LTL) and a simple version of LTL buffeted by noise is able to generate important stylised facts, such as volatility clustering and long memory, observed in real financial data. Föllmer, Horst and Kirman (2005) consider a discrete financial market model in which adaptive heterogeneous agents form their demands and switch among different expectations stochastically via a learning procedure. They show that, if the probability that an agent will switch to being a "chartist" is not too high, the limiting distribution of the price process exists, is unique and displays fat tails. Rheinlaender and Steinkamp (2004) study a one-dimensional continuously randomised version of Zeeman's (1974) model and show a stochastic stabilisation effect and possible sudden trend reversal. Other related works include Hens and Schenk-Hoppé (2005) who analyse portfolio selection rules in incomplete markets where the wealth shares of investors are described by a discrete random dynamical system, Lux and Schornstein (2005) who present an adaptive model of a two country foreign exchange market where agents learn by using genetic algorithms, Böhm and Chiarella (2005) who consider the dynamics of a general explicit random price process of many assets in an economy with overlapping generations of heterogeneous consumers forming optimal portfolios, and Böhm and Wenzelburger (2005) who provide a simulation analysis of the empirical performance of portfolios in a competitive financial market with heterogeneous investors and show that the empirical performance measure may be misleading. Most of the cited papers focus on the existence and uniqueness of limiting distributions of discrete time models, rather than the existence and stability of multiple limiting distributions of continuous time models, which are the focus of this paper. The only exception seems to be the one-dimensional continuously randomised version of Zeeman's (1974) stock market model studied by Rheinlaender and Steinkamp (2004) and the work of Föllmer and Schweizer (1993). To our knowledge, the current paper is the first to consider the existence and stability of multiple limiting distributions for a two-dimensional financial market model in a continuous time framework. We show that, as the chartists change their behaviour (through their speed of reaction coefficient), the market price can display different forms of equilibrium distribution. In particular, the market price can be driven away from the fundamental price when the chartists extrapolate the price trend strongly.

The paper unfolds as follows. Section 2 reviews the heterogeneous agent financial market models developed by Beja and Goldman (1980) and Chiarella (1992). Section 3 gives the deterministic dynamical behaviour of the model. Section 4 analyses the stochastic dynamics of the model, including the existence and/or stability of invariant measures and stationary distributions, and their bifurcations. In addition, through the method of stochastic approximation, the properties of stationary distributions are obtained analytically. In both Sections 3 and 4, we examine two different cases in which the time lag of the price trends used by the chartists is either positive or zero. The latter (which is a limiting case when the time lag goes to zero) corresponds to the case when the chartists place full weight on the most recent price change. Section 5 concludes with a discussion of the main results. All proofs are in the Appendix.

2 The Model

Consider a financial market consisting of two types of investors, *fundamentalists* and *chartists* and two types of assets, a *risky* asset (for instance a stock market index) and a *riskless* asset (typically a government bond). The fundamentalists base their investment decisions on an understanding of the fundamentals of the market, perhaps obtained through extensive statistical and economic analysis of market trends. The chartists base their investment decisions on recent price trends. In the market, the transactions and price adjustments occur simultaneously. The changes of the risky asset price $P(t)$ are brought about by aggregate excess demand $D(t)$ of investors at a finite speed of price adjustment. Accordingly, these assumptions may be expressed as

$$dp(t) = D(t)dt = [D_t^f + D_t^c]dt, \tag{2.1}$$

where $p_t = \ln P(t)$ is the logarithm of the risky asset price $P(t)$ at time t and $D(t)$ is the investors' excess demand for the risky asset at time t. The excess demand $D(t)$ is further decomposed as the excess demands of the fundamentalists (D_t^f) and of the chartists (D_t^c), defined below.

The excess demand of the fundamentalists is assumed to be given by

$$D_t^f(p(t)) = a[F(t) - p(t)], \tag{2.2}$$

where $F(t)$ denotes the logarithm of the fundamental price[1] that clears fundamental demand at time t so that $D_t^f(F(t)) = 0$ and $a > 0$ is a constant measuring the excess demand of the fundamentalists brought about by the market price deviation from the fundamental price.

Unlike the fundamentalists, the chartists consider the opportunities afforded by the existence of continuous trading *out of equilibrium*. Their excess demand is assumed to reflect the potential for direct speculation on price changes, including the adjustment of the price towards equilibrium. Let $\psi(t)$ denote the chartists' assessment of the current trend in $p(t)$. Then the chartists' excess demand is assumed to be given by

$$D_t^c(p(t)) = h(\psi(t)), \tag{2.3}$$

where h is a nonlinear continuous and differentiable function, satisfying

$$h(0) = 0; \quad h'(x) > 0 \text{ for } x \in \mathbb{R}, \lim_{x \to \pm\infty} h'(x) = 0; \tag{2.4a}$$

$$h''(x)x < 0 \text{ for } x \neq 0 \text{ and } h^{(3)}(0) < 0, \tag{2.4b}$$

where $h^{(n)}(\psi)$ denotes the nth order derivative of $h(\psi)$ with respect to ψ.

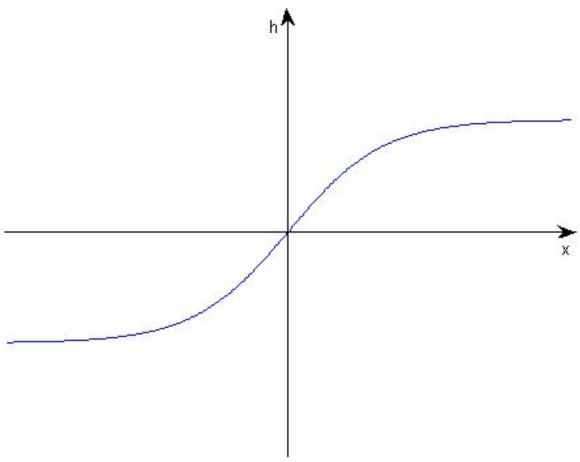


Figure 1: The chartists' demand function

These properties imply that h is an S-shaped function, as shown in Fig. 1, which indicates that when the expected trend in the price is above (below) zero, the chartists would like to hold a long (short) position in the risky asset. Since the chartists have

[1]In this paper, we consider the market price and the fundamental price to both be detrended by the risk-free rate or correspondingly, the risk-free rate is also assumed to be zero.

budget constraints, we impose upper and lower bounds on the chartists' demand function, that is

$$\sup_x |h(x)| < \infty. \tag{2.5}$$

In addition, $h'(\cdot)$ measures the intensity of the chartists' reaction to the long/short signal ψ. When $h'(\cdot)$ is small, the chartists react cautiously to the long (short) signals. Under the assumption of (2.4b), the intensity of the chartists' reaction is always bounded and especially, the chartists are most sensitive to the change at the equilibrium, that is $\max_x h'(x) = h'(0)$. We refer the reader to Chiarella (1992) and Chiarella, Dieci and Gardini (2002) for a more detailed economic justification of the nonlinear function $h(x)$ and the consistency of the demand function under the set up described above with the standard expected utility maximisation viewpoint.

Note that the chartists' speculation on the adjustment of the price toward equilibrium must primarily depend on an assessment of the state of the market as reflected in price trends. Typically, the assessment of the price trend will be based at least in part on recent price changes and will be an adaptive process of trend estimation. One of the simplest assumptions is that ψ is taken as an exponentially declining weighted average of past price changes[2], which can be expressed as the first order differential equation

$$d\psi(t) = c[dp(t) - \psi(t)dt], \tag{2.6}$$

where $c \in (0, \infty]$ is the decay rate, which can also be interpreted as the speed with which the chartists adjust their estimate of the trend to past price changes. Alternatively the quantity $\tau = 1/c$ may be viewed as the average time lag in the formation of expectations and it can be shown that[3] in a loose sense $\psi(t)dt \approx dp(t - \tau)$.

Summarising the above set up, we obtain the asset price dynamics

$$\begin{cases} dp(t) = a\Big[F - p(t)\Big]dt + h\big(\psi(t)\big)dt, \\ d\psi(t) = \dfrac{1}{\tau}\Big[-ap(t) - \psi(t) + h\big(\psi(t)\big) + aF\Big]dt. \end{cases} \tag{2.7}$$

In the earlier cited literature that focuses on the deterministic dynamics of the financial market model, it is usually assumed that F is constant. However, we would never expect that financial market behaviour can be characterised independently of randomness. There could exist at least two important random influences in our financial market model—a random fundamental price reflecting the many sources of uncertainty impinging on the underlying economy, and an unpredictable market noise that could for instance be due to noise traders or to the arrival of news events leading to unpredictable changes in demand/supply of the risky asset. For simplicity, this paper only considers the randomness from the fundamental price. When the fundamental price F follows a stochastic

[2]This means that $\psi(t) = c\int_{-\infty}^{t} e^{-c(t-s)}dp(s)$.

[3]Equation (2.6) can be written as $dp(t) = \tau d\psi(t) + \psi(t)dt \approx \psi(t+\tau)dt$. Hence $\psi(t)dt \approx dp(t-\tau)$. Alternatively one can calculate that the average time lag for the weighting function in footnote 2 is $\tau = 1/c$. So in an approximate sense ψ is based on the change in p evaluated at the average time lag τ.

process, for example a random walk, (2.7) becomes a stochastic dynamical system. We examine the dynamic behaviour of (2.7) when the average time lag τ of the chartists in the formation of expectations is either positive or zero, the latter can be treated as a limiting case of the former one as $\tau \to 0^+$.

3 The Deterministic Dynamical Behaviour

In this section, in order to later highlight the comparison between the deterministic and stochastic dynamics, we consider the situation in which the fundamental price is not perturbed by noise. Thus we assume that the fundamental price is constant with $F \equiv F^*$, in which case the system (2.7) has a unique steady-state $\bar{p} = F^*$, $\bar{\psi} = 0$. In the following discussion, we first consider the general case in which the average time lag τ of the chartists in the formation of expectations is positive. We also consider the limiting case $\tau \to 0^+$. This corresponds to the situation in which the chartists use the most recent price change to estimate the trend of the price. This limiting case is also of interest because it has a similar structure to the catastrophe theory model of Zeeman (1974), a structure that has been suggested by some empirical studies, such as Anderson (1989).

3.1 Dynamical Behaviour with Lagged Price Trend

Under the assumption that the chartists use a finite speed of adjustment in adapting to the price trend, that is $0 < c < \infty$, or equivalently that the chartists always estimate the price trend with a delay $\tau > 0$, Beja and Goldman (1980) perform a local linear analysis around the steady-state and show that the steady-state is locally stable if and only if

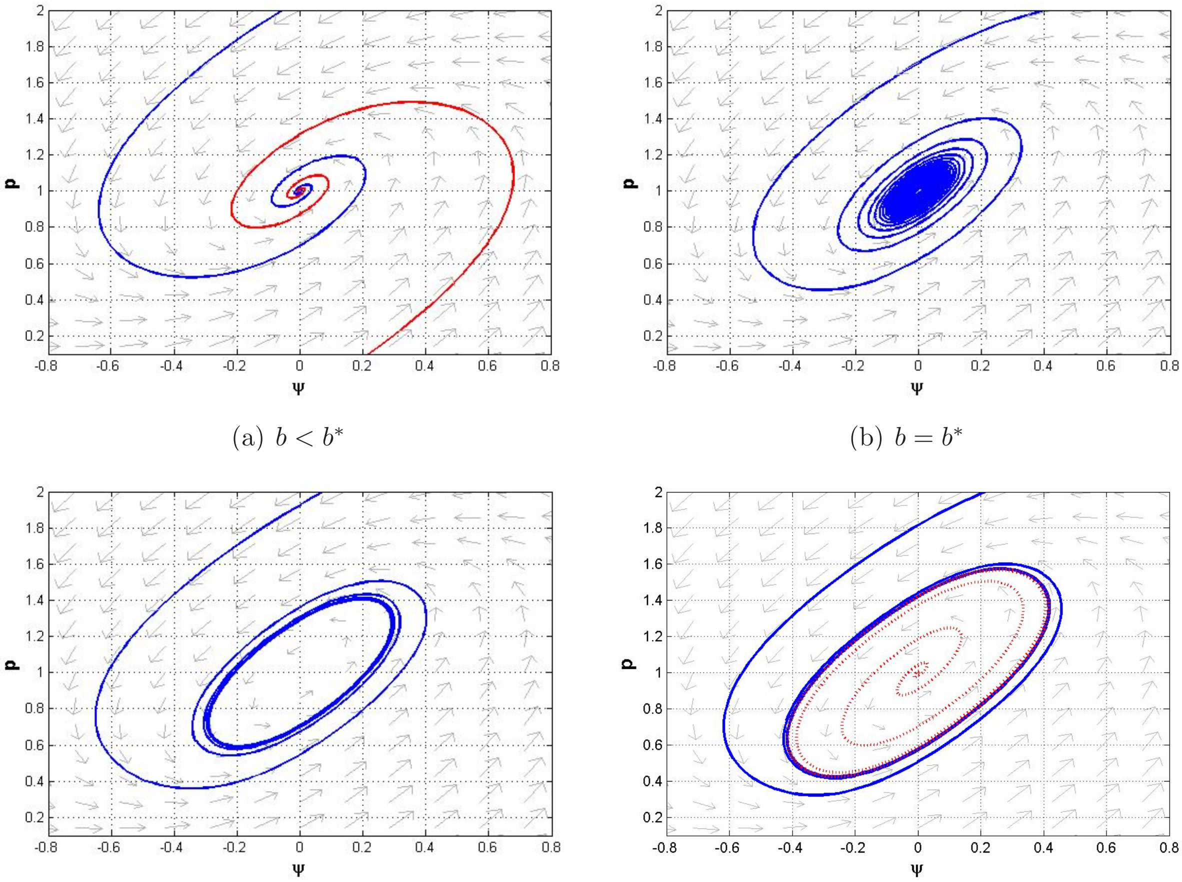
$$b < b^* = 1 + a\tau, \qquad b = h'(0), \tag{3.1}$$

where b represents the slope of the demand function at $\psi = 0$ for the chartists, which plays a very important role in determining the dynamics. In this paper, we take b as the key parameter through which to examine the role of the chartists in determining the market price. Beja and Goldman (1980) show that, at $b = b^*$, the system has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm\sqrt{-a/\tau}$. By considering the nonlinear nature of the function $h(x)$, Chiarella (1992) conducts a nonlinear analysis of the same model and demonstrates the occurrence of a Hopf bifurcation at $b = b^*$ if b is regarded as a bifurcation parameter. The basic result is summarised in the following theorem.

Theorem 3.1 (The Bifurcation of the Fundamental Equilibrium $(p^*, \psi^*) = (F^*, 0)$**)** *Let $b = h'(0)$ and $b^* = 1 + a\tau$.*

1. *When $b < b^*$, (p^*, ψ^*) is locally asymptotically stable.*

2. *At $b = b^*$, (p^*, ψ^*) undergoes a supercritical Hopf bifurcation.*



(a) $b < b^*$ (b) $b = b^*$

(c) $b \gtrapprox b^*$ (d) $b > b^*$

Figure 2: The phase plots from the stable fixed point (a) and (b) to the stable limit cycle (c) and (d) when the bifurcation parameter b increases near the bifurcation point $b^* = 1 + a\tau$. Here $h(x) = \alpha \tanh(\beta x)$, $a = 1$, $\alpha = 1$, $\tau = 1$, $F^* = 1$ (hence $b^* = 2$) and $b = \beta = 1.5$ (a), 2.0 (b), 2.2 (c), 2.5 (d), respectively.

3. *When $b > b^*$, (p^*, ψ^*) is unstable and a stable limit cycle exists.*

The behaviour of the dynamics in the phase plane of p and ψ for various values of b is illustrated in Fig. 2. In particular, by use of the Hopf bifurcation theorem (stated for example in Guckenheimer and Holmes (1983)), under the transformation $(p, \psi) = (F^* + u\tau - v\sqrt{\tau/a}, u)$, the stable limit cycle appearing for $b \to (1 + a\tau)^+$ near the steady state is well-approximated in the (u, v) coordinates by

$$u^2 + v^2 = -\frac{8}{h^{(3)}(0)}(b - b^*). \tag{3.2}$$

Therefore, when the chartists use lagged price trends to form their expectations, the strength of their activity, measured by $b = h'(0)$, can lead to different market behaviour. When they are less active (as measured by a low values of b), the market price converges to the fundamental price. However, when they are very active, the fundamental price becomes unstable and the market price oscillates periodically.

3.2 Dynamical Behaviour in the Limit $\tau \to 0^+$

We now consider the limiting case $\tau \to 0^+$. This corresponds to the situation in which the chartists react very quickly to the recent price changes, which might even be described as over-reaction. We show that, different from the previous case, the dynamics of the limiting case displays jump phenomena when the chartists are very active in the market.

The system (2.7) can be rewritten as

$$\Sigma : \begin{cases} \dot{p} &= f(p,\psi) \\ \tau\dot{\psi} &= g(p,\psi), \end{cases} \tag{3.3}$$

where

$$f(p,\psi) := a(F^* - p) + h(\psi), \quad g(p,\psi) := a(F^* - p) + h(\psi) - \psi.$$

When $\tau \to 0^+$, dynamical systems such as (3.3), where one of the time derivatives is multiplied by a small parameter that tends to zero, are known as singularly perturbed systems. As we show in the following they are characterised by the fact that the dynamics are fast in one direction (here the ψ-direction) and slow in the other direction (here the p-direction), see Yurkevich (2004) for a more detailed study of such systems.

In the limiting case $\tau = 0$, the system (3.3) becomes

$$\Sigma_s : \begin{cases} \dot{p} &= f(p,\psi), \\ 0 &= g(p,\psi), \end{cases} \tag{3.4}$$

which is a differential-algebraic equation (DAE)[4] with an algebraic constraint on the variables. The dynamical system Σ has apparently lost a dimension in the limit $\tau = 0$. However, the system Σ_s still has a two-dimensional phase plane.

For the system Σ_s in (3.4), if $h'(\psi) \neq 1$, we have

$$\dot{\psi} = -\frac{a\psi}{1 - h'(\psi)},$$

and

$$D_\psi g(p,\psi) = h'(\psi) - 1 \neq 0,$$

where $D_\psi g(p,\psi) = \partial g(p,\psi)/\partial_\psi$. Then the Implicit Function Theorem holds with respect to ψ in the function $g(p,\psi) = 0$. Therefore, we can denote ψ as a function of p, that is $\psi = \psi(p)$ satisfying $g(p,\psi(p)) = 0$. Furthermore the first equation of (3.4) can then be rewritten as

$$\dot{p} = f(p,\psi(p)) = a(F^* - p) + h(\psi(p)), \tag{3.5}$$

[4]For more information about DAEs, we refer the reader to the book of Brenan, Campbell and Petzold (1989).

and the Jacobian matrix of (3.5) is

$$\frac{df(p,\psi(p))}{dp} = D_p f(p,\psi)\Big|_{\psi=\psi(p)} - \Big[D_\psi f(p,\psi)\big(D_\psi g(p,\psi)\big)^{-1} D_p g(p,\psi)\Big]\Big|_{\psi=\psi(p)}$$

which is well defined provided $D_\psi g(p,\psi) = h'(\psi) - 1 \neq 0$. In this case, Σ_s reduces to two one-dimensional equations under the condition that $h'(\psi) \neq 1$.

However, if there is a ψ_* such that $h'(\psi_*) = 1$, then we cannot apply the Implicit Function Theorem. Those points where the Implicit Function Theorem cannot be applied are called *singular points*. Note that $h'(\cdot)$ attains its maximum value at $\psi = 0$, that is $\max_\psi h'(\psi) = h'(0) = b$. This implies that, depending on $b < 1$, $b = 1$ or $b > 1$, the number of singularity points is different. In fact, there is no singular point for $b < 1$. For $b = 1$, there is one singular point $\psi = 0$. While for $b > 1$, there are two singular points satisfying $D_\psi g(p,\psi) = h'(\psi) - 1 = 0$. In terms of the number of the singular points, $b = 1$ is the critical case. From the following discussion, we see that the different types of singularities affect the dynamics differently. Therefore, as in the previous case of $\tau > 0$, we continue to take b as the bifurcation parameter.

In order to analyse the effect of the singularity, we adopt the Singularity Induced Bifurcation Theorem, see Venkatasubramanian, Schättler and Zaborszky (1995), in the following Theorem 3.2. Rewriting $f(p,\psi)$ and $g(p,\psi)$ as a function of the parameter b, that is

$$f(p,\psi,b) := a(F^* - p) + h(\psi), \quad g(p,\psi,b) := a(F^* - p) + h(\psi) - \psi,$$

and letting EQ represent the set of all equilibria of Σ_s, that is

$$EQ := \{(p,\psi,b) \in \mathbb{R}^3 : f(p,\psi,b) = 0,\ g(p,\psi,b) = 0\},$$

and S the set containing all the singular points, that is

$$S := \{(p,\psi,b) \in \mathbb{R}^3 : g(p,\psi,b) = 0,\ D_\psi g(p,\psi,b) = 0\}.$$

Then the singular equilibrium points occur when $EQ \cap S \neq \varnothing$. In particular, $(p,\psi,b) = (F^*,0,1)$ is one such singular equilibrium point. The case of $b = 1$ corresponds to the appearance of the singularity of the fundamental equilibrium $(p,\psi) = (F^*,0)$ and the stability and type of singularity of the fundamental equilibrium $(p,\psi) = (F^*,0)$ depend on the parameter b. Therefore, the bifurcation of the fundamental equilibrium $(p,\psi) = (F^*,0)$ near the singular parameter $b = 1$ is called a singularity induced bifurcation and its dynamics are described by the following Theorem.

Theorem 3.2 *A singularity induced bifurcation (SIB) occurs at $b = 1$. That is to say, there exists a smooth curve of equilibria EQ in $\mathbb{R}^3$ which passes through $(F^*,0,1)$ and is transversal to the singular surface S at $(F^*,0,1)$. In addition, when b increases from $b < 1$ to $b > 1$, the eigenvalue of the matrix $D_p f - D_\psi f (D_\psi g)^{-1} D_p g$ at $(p^*,\psi^*) = (F^*,0)$ moves from $\mathbb{R}^-$ to $\mathbb{R}^+$.*

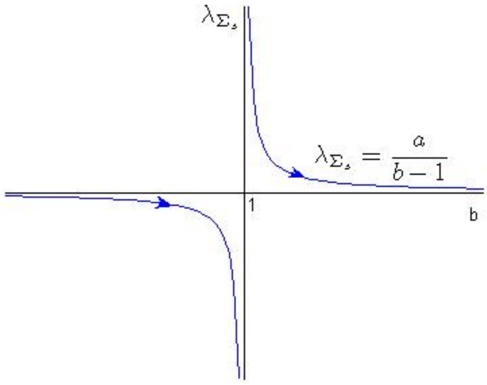


Figure 3: The eigenvalue of the system Σ_s at the fixed point as a function of the parameter b.

In fact, at $(p^*, \psi^*) = (F^*, 0)$, the eigenvalue $\lambda_{\Sigma_s} = D_p f - D_\psi f (D_\psi g)^{-1} D_p g = a/(b-1)$ when $b \neq 1$. The change of the eigenvalue at $(p^*, \psi^*) = (F^*, 0)$ in Theorem 3 is illustrated in Fig. 3.

Based on the above analysis and the fact that $(p^*, \psi^*) = (F^*, 0)$ is the unique fixed point of the system Σ_s, we obtain the following result. For $b < 1$, the eigenvalue at $(F^*, 0)$ is negative and hence $(F^*, 0)$ is stable. When b increases from 1^- to 1^+ through $b = 1$, λ_{Σ_s} moves from $\mathbb{R}^-$ to $\mathbb{R}^+$ due to the fact that the eigenvalue λ_{Σ_s} consists of two branches separated by $b = 1$, as shown in Fig. 3. This means that the steady state $(F^*, 0)$ loses its stability at $b = 1$ and becomes unstable as b increases from 1^- to 1^+.

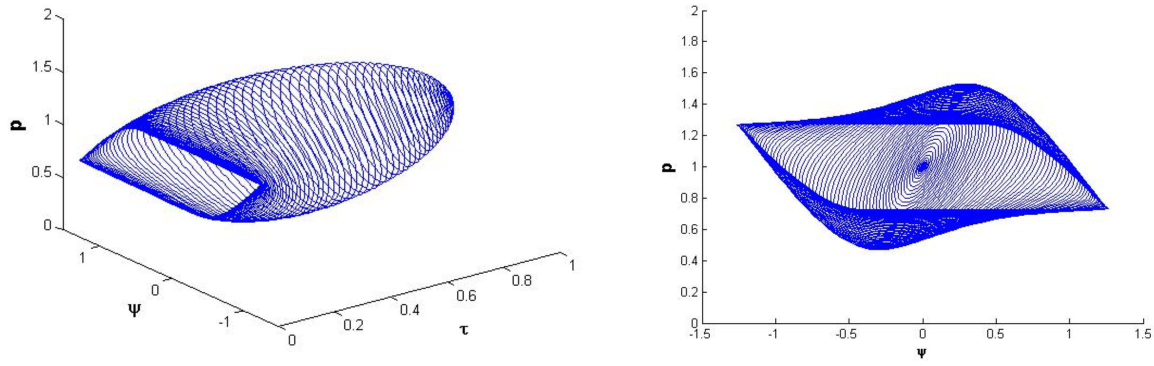
To understand the dynamics of the system Σ_s, we consider the limiting dynamics of the singularly perturbed system Σ as $\tau \to 0^+$. We show that the jump phenomena for the system Σ_s corresponds to the limiting case of the Hopf Bifurcation in the system Σ for $b^* = b^*(\tau)\,(\tau \neq 0)$ when $\tau \to 0^+$. Here $b^*(\tau)$ is a function in a small interval $[0, \tau_0)(\tau_0 > 0)$ with $b^*(0) = 1$.

Theorem 3.3 *Given any parameter interval (b_1, b_2) containing $b = 1$, there exists $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0)$, the Jacobian J of the system Σ,*

$$J = \begin{pmatrix} D_p f & D_\psi f \\ D_p g/\tau & D_\psi g/\tau \end{pmatrix}$$

evaluated along EQ has a pair of eigenvalues which cross the imaginary axis away from the origin within the parameter interval (b_1, b_2). In addition, there exists a unique smooth function $b^* = b^*(\tau)$ defined in the small interval $[0, \tau_0)$ with $b^*(0) = 1$ such that J at $(F^*, 0, b^*(\tau))$ for $\tau \neq 0$ has a pair of purely imaginary eigenvalues of the form $\lambda_{1,2}(\tau) = \pm\sqrt{\xi(\tau)/\tau}$ where $\xi(\tau)$ is smooth in $[0, \tau_0)$ and $\xi(0) = -a$.

Recall that, at $b^* = b^*(\tau) = 1 + \tau a$, the system Σ has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm\sqrt{-a/\tau}$, that is, $\xi(\tau) \equiv -a$. We have also shown in (3.2) that

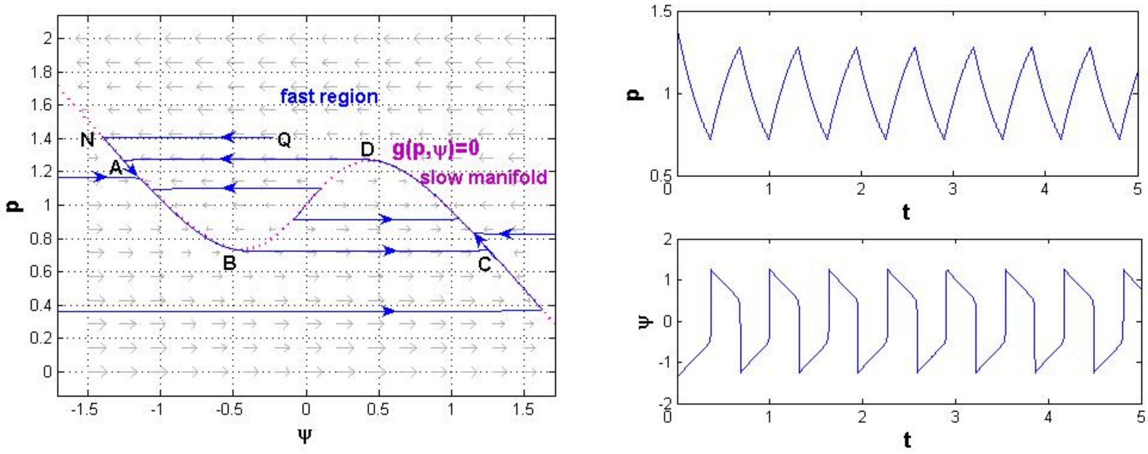


(a)

(b)

Figure 4: (a) The limit of the limit cycles as $\tau \to 0^+$; (b) The projection of the limiting cycles in the (ψ, p) plane.

with a fixed $b > b^*(\tau_0)\,(\tau_0 > 0)$, as $\tau \to 0^+$, the limit cycle persists. Figure 4(a) shows a sequence of limit cycles as $\tau \to 0^+$ and Fig. 4(b) gives a projection of Fig. 4(a) onto the (ψ, p) plane. In fact, in the (ψ, p) plane, the following observations are made about the system Σ. As illustrated in Fig. 5(a), for $g(p, \psi) \neq 0$, ψ moves infinitely rapidly toward the curve defined by $g(p, \psi) = 0$ since $\dot{\psi} \to \infty$ when $\tau \to 0^+$. We denote such a region as the fast region. For $g(p, \psi) = 0$, as $\tau \to 0^+$, the dynamics are governed by the differential equation for p, namely $\dot{p} = a(F^* - p) + h(\psi)$ along the curve $g(p, \psi) = 0$. We denote the regions of the phase plane where motion is governed by $\dot{p} = a(F^* - p) + h(\psi)$ as the slow manifold. Specifically, consider how the motion evolves from the initial point Q in the fast region in Fig. 5(a). The variable ψ moves instantaneously horizontally to the point N on the slow manifold $g(p, \psi) = 0$. Motion is then down the slow manifold under the influence of $\dot{p} = a(F^* - p) + h(\psi)$. When the singular point B is reached, ψ jumps instantaneously horizontally across to C on the opposite branch of the slow manifold. Motion is then up to another singular point D and the cycle then repeats itself. Therefore, the limit cycle with jump phenomena consisting of two slow movements along the manifold of Σ_s shown in Fig. 5(a), $A \to B$, and $C \to D$, and two jumps at the singular points B and D in Σ_s, namely $B \twoheadrightarrow C$ and $D \twoheadrightarrow A$. The corresponding time series in Fig. 5(b) clearly shows the periodic slow movement in price p and sudden jumps in ψ from time to time. In this way, the model is able to generate significant transitory and predictable fluctuations around the equilibrium. Using a different approach, this jump fluctuation phenomenon in the model of fundamentalists and chartists was studied by Chiarella (1992) who pointed out that it is merely the relaxation oscillation well-known in mechanics and expounded for example by Grasman (1987). Our analysis indicates that strong reaction to price changes by the chartists can make the fundamental price unstable, leading to predictable cycles for the market prices and jumps in their estimate of the price trends.



(a) (b)

Figure 5: (a)The jump fluctuation in the phase plane and (b) the corresponding time series of $p(t)$ and $\psi(t)$ at $\tau \to 0^+$.

4 The Stochastic Dynamical Behaviour

The model reviewed in the previous section is entirely deterministic. In this section, we examine the stochastic dynamics generated from the randomness of the fundamental price. As in the previous section, we consider the general case of $\tau > 0$ and its limiting case of $\tau \to 0^+$.

For the log fundamental price $F(t)$, in accordance with the theory of equilibrium prices in perfect markets, the successive changes in equilibrium values must be statistically independent. This proposition is usually formalised by the statement that $F(t)$ follows a random walk. That is, $F(t+h) - F(t)$ is normally distributed with mean 0 and variance $\sigma^2 h$, independently of values of $F(s)\,(s \leq t)$. Using the notation of stochastic differential equations, the log fundamental value $F(t)$ can be considered to follow the stochastic differential equation (SDE)

$$dF = \sigma \circ dW, \tag{4.1}$$

where W is a two-sided Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with zero drift and unit variance per unit time and $\sigma > 0$ is the standard deviation (volatility) of the fundamental returns. Here the circle $\circ$ indicates that the SDE (4.1) is to be interpreted in the Stratonovich sense, rather than the Itô sense[5].

By incorporating the random log fundamental price process $F(t)$ of (4.1) into the continuous model (2.7), we obtain the corresponding stochastic version of the financial market model. Letting $\phi dt = d\psi$, a nonlinear Stratonovich-SDE system in ψ and ϕ can

[5]The Stratonovich rather than the Itô SDE formalism is usually used in the theory of random dynamical systems, one reason being that it has the advantage of the validity of the chain rule without the addition of the second derivative terms.

be obtained, namely

$$\begin{cases} d\psi = \phi dt, \\ d\phi = \dfrac{1}{\tau}\Big[\big(\varsigma + h'(\psi) - b\big)\phi dt - a\psi dt + a\sigma \circ dW\Big], \end{cases} \tag{4.2}$$

where $\varsigma = b - b^*$ and $b^* = 1 + a\tau$. Once the dynamics of $\psi(t)$ have been obtained, the dynamics of the price $p(t)$ can be obtained by integrating the first equation in (2.7).

To clarify the dynamical behaviour of the model (4.2), it is necessary to deal with the random dynamical system (RDS) that it generates. We first establish some preliminary mathematical results in order to gain some insight into the (Hopf) bifurcation of the stochastic speculative behaviour generated by the system (4.2). We refer the reader to Arnold (1998) for a more detailed and systematic treatment of the theory of RDSs.

A random dynamical system is the stochastic analogue of a deterministic dynamical system. It consists of two ingredients: a model describing a dynamical system perturbed by noise and a model of the noise itself. Here the model of the noise is the standard two-sided Wiener process $\{W_t\}_{t\in\mathbb{R}}$, which consists of two independent Wiener processes, one with $t > 0$, the other with $t < 0$ and both pinned down at zero. Let Ω be the space of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ which satisfy $\omega(0) = 0$. Let $\mathcal{F}$ be the Borel σ-algebra on Ω, and let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F})$. Define the shift-mapping on Ω by $\vartheta_t\omega(s) := \omega(t + s) - \omega(t)$, reflecting the fact that the Wiener process has stationary increments rather than being stationary itself. Then ϑ is an ergodic metric dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ and $W_t(\omega) = \omega(t)$.

A dynamical system perturbed by noise, that is a random dynamical system, is characterised by the following *co-cycle property*. A local C^∞ random dynamical system on $\mathbb{R}^2$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is defined as a measurable mapping

$$\varphi : \mathcal{D} \to \mathbb{R}^2, \quad (t, \omega, \mathbf{x}) \mapsto \varphi(t, \omega, \mathbf{x}) \ (=: \varphi(t, \omega)\mathbf{x}),$$

where $\mathcal{D} \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$ and $\mathcal{B}(\mathcal{A})$ is the Borel σ-algebra generated by $\mathcal{A}$, such that (i) φ is a co-cycle, that is, $\varphi(0, \omega) = id$, the identity map, and $\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega)$; (ii) $(t, \mathbf{x}) \mapsto \varphi(t, \omega, \mathbf{x})$ is continuous and $\mathbf{x} \mapsto \varphi(t, \omega)\mathbf{x}$ is a C^∞ diffeomorphism.

4.1 Dynamical Behaviour with Lagged Price Trend

We use the method of Schenk-Hoppé (1996*a*) to show in Theorem 4.1 below that the solution $\varphi(t, \cdot)\mathbf{x}_0$ of (4.2) with the initial value $\mathbf{x}_0 = (\psi_0, \phi_0)^\top$ and the average time lag $\tau > 0$ defines a *global* random dynamical system, that is $\mathcal{D} = \mathbb{R} \times \Omega \times \mathbb{R}^2$.

Theorem 4.1 *(i) The (Stratonovich) SDE system* (4.2) *uniquely generates a local smooth RDS* φ *in* $\mathbb{R}^2$ *over the dynamical system* ϑ *modelling the Wiener process, that is there exists a local RDS* φ *such that* $(\psi_t, \phi_t) = \varphi(t, \cdot)\mathbf{x}_0$ *is the* $\mathbb{P}$*-a.s. unique maximal solution of* (4.2) *with any initial value* $\mathbf{x}_0 = (\psi_0, \phi_0)^\top \in \mathbb{R}^2$.

(ii)The SDEs (4.2) *are strictly (forward and backward) complete, that is the local diffeomorphism* $\varphi(t,\omega): \mathcal{D}_t(\omega) \to \mathcal{R}_t(\omega)(\subset \mathbb{R}^2)$ *has domain* $\mathcal{D}_t(\omega) = \mathbb{R}^2$ *for any* $t \in \mathbb{R}$. *Equivalently,*

$$\mathbb{P}(\omega : \iota^{\pm}(\omega, \mathbf{x}_0) = \pm\infty, \ \textit{for all } \mathbf{x}_0 \in \mathbb{R}^2) = 1,$$

where ι^+ *and* ι^- *are, respectively, the forward and backward explosion times*[6] *of the orbit* $\varphi(\cdot,\omega)\mathbf{x}_0$ *starting at time* $t = 0$ *in the initial position* $\mathbf{x}_0$.

In the following, the random dynamical behaviour of the solution of (4.2) is investigated through stochastic bifurcation methods by examining either the change of stability of invariant measures and the occurrence of new invariant measures, or the qualitative change of stationary measures. The first approach corresponds to the so-called *dynamical* (D)-bifurcation. For an n-dimensional SDE system, this approach examines the simultaneous behaviour of n (≥ 1) points forward and backward in time and characterises all of the stochastic dynamics of the SDEs. The second approach corresponds to the so-called *phenomenological* (P)-bifurcation. The stationary measure can be observed when studying the solution of the corresponding Fokker-Planck equation. In other words, the D-bifurcation examines the evolution of invariant measures from a dynamical sample paths point of view, while the P-bifurcation studies the stationary distribution from a distributional point of view. As indicated in Schenk-Hoppé (1996*b*) and the references cited therein, the difference between P-bifurcation and D-bifurcation lies in the fact that the P-bifurcation approach is, in general, not related to path-wise stability, whereas the D-bifurcation approach is based on invariant measures, the multiplicative ergodic theorem, Lyapunov exponents, and the occurrence of new invariant measures. The P-bifurcation has the advantage of allowing one to visualise the changes of the stationary density functions.

Stochastic bifurcation theory is a very powerful tool in helping us to understand the stochastic nature of random dynamical systems. In particular, the study of the limiting distribution of various stochastic models used in economics and finance is becoming more important. However stochastic bifurcation theory is still in its infancy and its application to heterogeneous agent models of financial markets presents many challenges. Here we confront this challenge by using a combination of numerical and approximate analytical

[6] The forward and backward explosion times $\iota^{\pm}(\omega, \mathbf{x}_0)$ are essentially the times taken for the orbit $\varphi(\cdot,\omega)\mathbf{x}_0$ to escape to ∞ based on the two sided Wiener process. Formally they are defined as below. Let $\iota_n^{\pm} := \iota_n^{\pm}(\omega, \mathbf{x}_0) = \pm(n \wedge \inf\{t : |\varphi(\pm t, \omega)\mathbf{x}_0| \geq n, \ t \geq 0\})$ where $a \wedge b$ stands for the smaller of a and b. Then $\iota^{\pm} := \iota^{\pm}(\omega, \mathbf{x}_0) = \lim_{n \uparrow +\infty} \iota_n^{\pm}(\omega, \mathbf{x}_0)$ is called the explosion time of the solution $\varphi(t,\omega)\mathbf{x}_0$ of (4.2) with the initial condition $\varphi(0,\omega)\mathbf{x}_0 = \mathbf{x}_0$. Explosion occurs on the set $ET := \{|\iota^{\pm}| < \infty\}$, because on this set, by the continuity of $\varphi(\pm t, \omega)\mathbf{x}_0$, $\varphi(\iota^{\pm}, \omega)\mathbf{x}_0 = \lim_{n \uparrow +\infty} \varphi(\iota_n^{\pm}, \omega)\mathbf{x}_0$. Thus,

$$\left|\varphi(\iota^{\pm}, \omega)\mathbf{x}_0\right| = \left|\lim_{n \uparrow +\infty} \varphi(\iota_n^{\pm}, \omega)\mathbf{x}_0\right| = \lim_{n \uparrow +\infty} n = +\infty,$$

and infinity is reached in finite time on ET.

tools to analyse the model. The numerical analysis of the stochastic bifurcation of our speculative market model is largely motivated by the work of Arnold, Sri Namachchivaya and Schenk-Hoppé (1996) and Schenk-Hoppé (1996*b*) on the noisy Duffing-van der Pol oscillator. Here we seek to use the combined analysis of D- and P-bifurcations to give a broader picture of the behaviour of the model, including existence and/or stability, of invariant measures and stationary distributions of the market prices as the chartist behaviour changes.

In the following discussion, we conduct first the D-bifurcation analysis of (4.2) in subsection 4.1.1. Based on the numerical results, we obtain the stochastic Hopf bifurcation discussed in Schenk-Hoppé (1996*b*). We then examine the P-bifurcation of (4.2) in subsection 4.1.2, followed by a summary of the overall picture of the stochastic dynamics that emerges from the analysis of the D- and P-bifurcations. In subsection 4.1.3, via a stochastic approximation method, we analytically confirm the picture obtained from the D- and P-bifurcation analysis.

4.1.1 D-bifurcation

The dynamical or D-bifurcation approach deals with invariant measures, the application of the multiplicative ergodic theorem, and random attractors. In stochastic models, random invariant measures are the corresponding concept for invariant sets in deterministic models. Invariant measures are of fundamental importance for an RDS as they encapsulate its long-run and ergodic behaviour. First, we present a definition of invariant measure.

Definition 4.1 *For a given RDS φ over ϑ, the measure $\mu \in \mathcal{P}r(\Omega \times \mathbb{R}^2)$, where $\mathcal{P}r(\Omega \times \mathbb{R}^2)$ is a set of all probability measures in $(\Omega \times \mathbb{R}^2, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2))$, is said to be an* invariant measure, *if*

(i) $\pi_\Omega \mu = \mathbb{P}$, *where $\pi_\Omega \mu$ means the marginal of μ on Ω, and*

(ii) $\Theta(t)\mu = \mu$ *for all $t \in \mathbb{R}$, where*

$$\Theta(t) : (\Omega, \mathbb{R}^2) \to (\Omega, \mathbb{R}^2), \quad (\omega, \mathbf{x}) \mapsto (\vartheta(t)\omega, \varphi(t,\omega)\mathbf{x}) =: \Theta(t)(\omega, \mathbf{x}).$$

In this definition, condition (i) indicates that the noise is an exogenous process, which means that the marginal of the invariant measure on the probability space Ω has to be the given measure $\mathbb{P}$. Condition (ii) implies that the stochastic process $\{\Theta(t)\}$ with the initial distribution μ has the same distribution μ at every time t, that is μ is invariant for $\{\Theta(t)\}$.

Every measure $\mu \in \mathcal{P}r(\Omega \times \mathbb{R}^2)$ with marginal $\mathbb{P}$ on Ω can factorise, that is, there exists a $\mathbb{P}$-a.s. unique measurable map $\mu : \omega \mapsto \mu_\omega$ (probability kernel) with $\omega \in \Omega$ and $\mu_\omega \in \mathcal{P}r(\mathbb{R}^2)$ such that $\mu(d\omega, d\mathbf{x}) = \mu_\omega(d\mathbf{x})\mathbb{P}(d\omega)$. In the following, we identify a measure μ with its factorisation $(\mu_\omega)_{\omega\in\Omega}$. Then, a measure $\mu \in \mathcal{P}r(\Omega \times \mathbb{R}^2)$ is invariant under φ if and only if for all $t \in \mathbb{R}$,

$$\varphi(t,\omega)\mu_\omega = \mu_{\vartheta_t\omega},\ \mathbb{P} - a.s. \qquad .$$

A D-bifurcation occurs if a reference invariant measure μ^γ depending on a parameter γ loses its stability at some point γ_D, and another invariant measure $\nu^\gamma \neq \mu^\gamma$ exists for some γ in each neighborhood of γ_D with ν^γ converging weakly to μ^{γ_D} as $\gamma \to \gamma_D$. Therefore, the D-bifurcation focuses on the loss of stability of invariant measures, which is determined by Lyapunov exponents, and on the occurrence of new invariant measures, which are characterised by random attractors, when a parameter varies. The following discussion examines these two aspects of the D-bifurcation. As in the deterministic case, we take b, the slope of the chartists' demand at $\psi = 0$, as the bifurcation parameter.

(1) Lyapunov exponents

Similar to the stability analysis of a deterministic dynamical system, the stability of invariant measures of a random dynamical system is described by the Lyapunov exponents given by the multiplicative ergodic theorem (MET) (see Arnold (1998)).

Let μ be an invariant ergodic probability measure for the random dynamical system φ generated by (4.2). Consider the linearisation (variational equations) corresponding to (4.2), namely

$$\begin{cases} du = vdt, \\ dv = \dfrac{1}{\tau}\Big[\big(h''(\psi)\phi - a\big)u + \big(\varsigma + h'(\psi) - b\big)v\Big]dt, \end{cases} \tag{4.3}$$

where (ψ, ϕ) is the solution of (4.2) with initial value $\mathbf{x}$. By the MET, there exists an invariant set $\Gamma \subset \Omega \times \mathbb{R}^2$ with $\mu(\Gamma) = 1$ satisfying the conditions:

(i) there exist two Lyapunov exponents of the invariant measure μ, $\lambda_1 \geq \lambda_2$, which are a.e. constants in Γ, and

(ii) for any $(\omega, \mathbf{x}) \in \Gamma$, there exists an invariant splitting $E_1(\omega,\mathbf{x}) \oplus E_2(\omega,\mathbf{x}) = \mathbb{R}^2$, such that any solution $V_t(\omega,\mathbf{x})$ with initial value $V_0 \neq 0$ of the variational equation (4.3) has the exponential growth rates

$$\lambda_i \;=\; \lim_{t\to\infty} \frac{1}{t} log \parallel V_t(\omega,\mathbf{x})\| \qquad \text{if} \qquad V_0 \in E_i(\omega,\mathbf{x}), \qquad i = 1, 2.$$

Hence the Lyapunov exponent can be said to be the stochastic analogue of "the real part of an eigenvalue" of a deterministic system (at a fixed point) and E_1 and E_2 correspond to the eigenspaces. In addition, the stochastic analogue of "the imaginary part of the eigenvalue" is the rotation number $\kappa(\mu)$, which is defined as the average phase speed of $V_t(\omega,\mathbf{x})$, that is

$$\kappa(\mu) = \lim_{t\to\infty} \frac{1}{t} argV_t(\omega,\mathbf{x}).$$

When $\kappa(\mu) \neq 0$, the stochastic flow will converge to (diverge from) the attractor (repeller) in a spiralling fashion. Therefore the D-bifurcation approach is a natural generalisation of deterministic bifurcation theory, if one adopts the viewpoint that an invariant measure is the stochastic analogue of an invariant set, for instance a fixed point, and the MET is the stochastic equivalent of linear algebra[7].

To approximate the Lyapunov exponents and rotation number, we use stochastic numerical methods to solve the variational equations (4.3). In the following, we take

$$h(x) = \alpha \tanh(\beta x),$$

with $\alpha, \beta\,(>0)$, and note that $h(\cdot)$ satisfies the conditions (2.4)-(2.5). Then $b = h'(0) = \alpha\beta$. In the following discussion, we take $\alpha = 1$ and hence $b = \beta$. Therefore, we use β as the bifurcation parameter in our subsequent numerical analysis. The computational scheme proceeds as follows. We first solve the original SDEs (4.2) by the Euler-Maruyama scheme, substitute this solution into the variational equations (4.3), and solve this linear SDE system with the same numerical scheme as for (4.2). Then using the Gram-Schmidt Orthonormalisation method (see pages 74-80 in Parker and Chua (1989)), we can simultaneously estimate all Lyapunov exponents of a stable invariant measure. In addition, the rotation number can be calculated from the definition at the same time.

To detect the instability of an invariant measure, we calculate the Lyapunov exponents and rotation number for the time reversed SDEs. This means that we make a time transforation (time reversal) $t \to -t$. Letting $\widehat{\psi}(t) := \psi(-t)$, $\widehat{\phi}(t) := \phi(-t)$ and $\widehat{W}(t) := W(-t)$, then the SDE system (4.2) becomes

$$\begin{cases} d\widehat{\psi}(t) = d\psi(-t) = \phi(-t)d(-t) = -\widehat{\phi}(t)dt, \\ d\widehat{\phi}(t) = \dfrac{1}{\tau}\Big[\big(\varsigma + h'(\psi(-t)) - b\big)\phi(-t)d(-t) - a\psi(-t)d(-t) + a\sigma \circ dW(-t)\Big] \\ \qquad\;\; = \dfrac{1}{\tau}\Big[\big(-\varsigma - h'(\widehat{\psi}(t)) + b\big)\widehat{\phi}(t)dt + a\widehat{\psi}(t)dt + a\sigma \circ d\widehat{W}(t)\Big]. \end{cases} \tag{4.4}$$

We take $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$, the iteration time as 1000 time units with step size 0.001 for each time unit and vary β. Figure 6 shows the Lyapunov exponents of the invariant measures and their rotation numbers, as functions of the parameter β. Note that with $b = \beta$ increasing to $\beta_D \approx b^* = \beta^* = 1 + a\tau$, the Lyapunov exponents change from negative to positive, which means that the stability of the reference measure μ transfers from stable to unstable. Near β_D, because of $\kappa(\mu) < 0$, the convergence to (for $\beta < \beta_D$) and divergence from (for $\beta > \beta_D$) the measure μ occurs in a spiralling fashion. When $\lambda_{1,2}(\mu)$ becomes positive, we obtain two other Lyapunov exponents, denoted by $\lambda_{1,2}(\nu)$, satisfying $\lambda_2(\nu) < \lambda_1(\nu) \leq 0$, which indicates the appearance of a new stable invariant measure ν. This means that there always exists an invariant

[7] For deterministic systems we use linear algebra to determine the stable and unstable eigenspaces. For stochastic systems we use the MET to determine the so-called Oseledets spaces as described below (4.3).

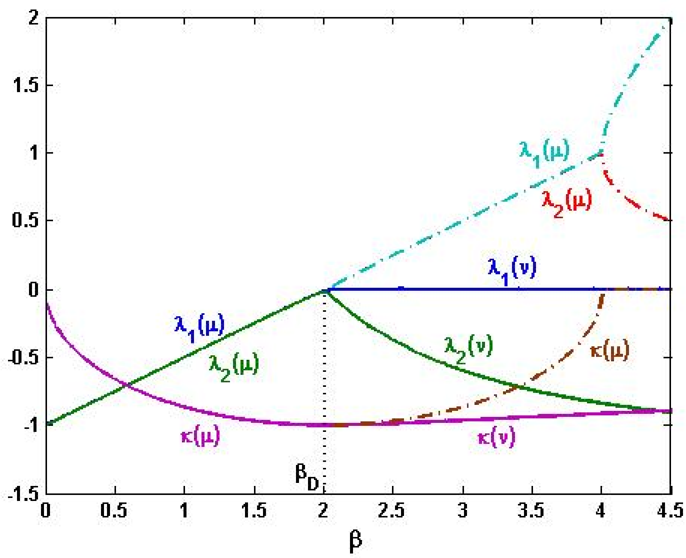


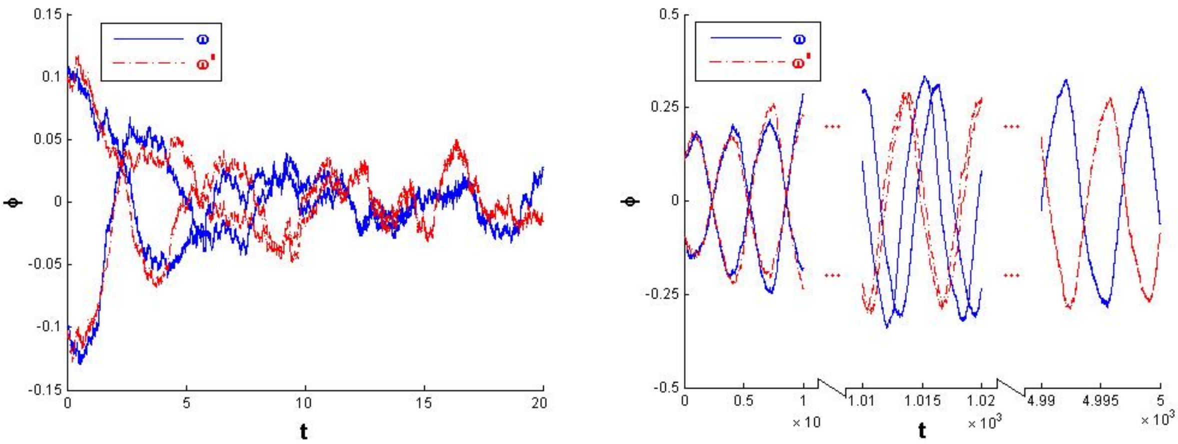
Figure 6: Lyapunov exponents and rotation number as a function of β for $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$ and hence $b = \beta$. The negative values $\lambda_{1,2}(\mu)$ indicate that the invariant measure μ is stable for $\beta < \beta_D$. For $\beta > \beta_D$ the invariant measure becomes unstable since $\lambda_{1,2}(\mu)$ are now positive. The appearance from $\beta = \beta_D$ of $\lambda_2(\nu) < \lambda_1(\nu) \leq 0$ indicates the emergence of new stable measure ν.

measure μ in the market. However, when the chartists react to their demand signal weakly (so that $\beta < \beta_D$), this invariant measure is unique and stable; when the chartists react to their demand signal strongly (so that $\beta > \beta_D$), there exists a new invariant measure ν such that the original invariant measure μ becomes unstable and the new invariant measure ν is stable and converges in a spiralling fashion (since $\kappa(\nu) \neq 0$). We note that the bifurcation value β_D is very close to the Hopf bifurcation value $b^* = \beta^*$ for the deterministic case discussed in the previous section. In the deterministic case, the Hopf bifurcation occurs when $b = \beta = \beta^*$, leading to the appearance of the limit cycle. In the following section, we characterise the stochastic Hopf bifurcation of the invariant measure by using the concept of random fixed points and random attractors.

(2) Random fixed points and random attractors

Changes in the Lyapunov exponents indicate the changes of invariant measures. In order to detect the changes of invariant measures, we first consider a random fixed point (that is a random variable $\mathbb{x}(\omega)$ satisfying $\varphi(t, \omega)\mathbb{x}(\omega) = \mathbb{x}(\vartheta_t\omega)$ for almost all $\omega \in \Omega$ and $t \in T$), because a random fixed point corresponds to a random invariant Dirac measure $\delta_{\mathbb{x}(\omega)}$.

Figure 7 displays sample paths, with two different parameter values of $\beta = \beta_1 = 1.5$ and $\beta = \beta_2 = 2.2$ satisfying $\beta_1 < \beta_D < \beta_2$, for two different initial values and two orbits of the Wiener process, ω and ω'. As $t \to \infty$, Fig. 7 shows that, for a fixed Wiener



(a) $\beta = 1.5$

(b) $\beta = 2.2$

Figure 7: Convergence of the process $\varphi(t, \vartheta_{-t}\omega)\mathbf{x}_0$ when $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$ from two different initial values and different orbits of the Wiener process ω and ω' for $\beta = 1.5$ (a) and $\beta = 2.2$ (b).

orbit, the sample paths converge to each other for two different initial values, however with a different Wiener orbit, the limiting sample paths are different. This implies the existence of a random fixed point which depends on the orbits of the Wiener process. In the case of $\beta = 1.5$, there is no particular structure to the random fixed point but in the case of $\beta = 2.2$, the sample paths converge with some fluctuating pattern. The change in the structure of the dynamical behaviour of the random fixed points indicates that the random invariant measures undergo some change as the parameter β varies.

In order to obtain more information about invariant measures, we need to examine global random attractors. A global random attractor is a measurable map $\omega \to A(\omega)$ such that $A(\omega)$ is compact in $\mathbb{R}^2$, invariant (that is $\varphi(t, \omega)A(\omega) = A(\vartheta_t\omega)$ for any t, ω), and it attracts any bounded deterministic set $D \subset \mathbb{R}^2$, that is $d(\varphi(t, \vartheta_{-t}\omega)D, A(\omega)) \to 0$ when $t \to \infty$, where $d(S_1, S_2) := \sup_{s_1 \in S_1} \inf\{d(s_1, s_2), s_2 \in S_2\}$. Random attractors are of particular importance since on them the long-term behaviour of the system takes place. In addition, global random attractors are connected and they support all invariant measures (see Crauel and Flandoli (1994) and Crauel (1999)).

Note that in the definition of a global random attractor, a pullback process $\varphi(t, \vartheta_{-t}\omega)$ is used, which is based on the concept of moving points from time $-t$ to time 0 (and not from time 0 to time t). This enables us to study the asymptotic behaviour as $t \to \infty$ in the fixed fibre at time 0. By increasing t the mapping is made to start at an ever earlier time, corresponding to a *pullback* in time, which is illustrated in Fig. 8.

To view the global random attractor of (4.2), we need to calculate $\varphi(t, \vartheta_{-t}\omega)D$ for a given parameter β and given Wiener process ω. Here again, we use the Euler-Maruyama Scheme to simulate φ with a step size $\Delta t = 0.001$. As in the calculation of the Lyapunov exponents, we fix the parameters $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$ and choose two different

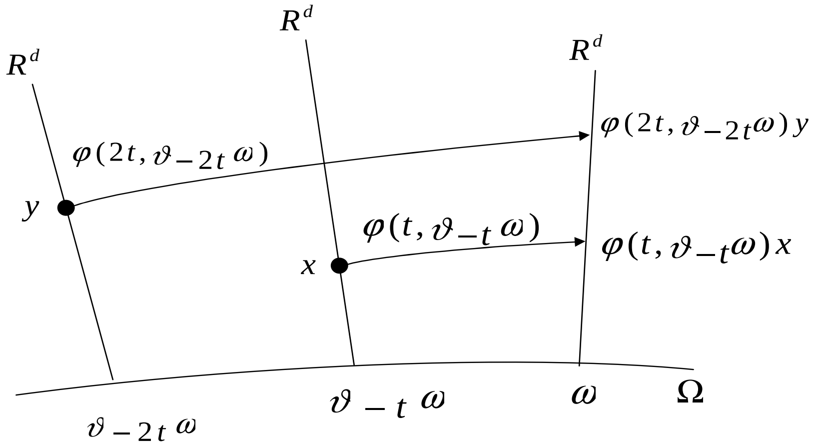
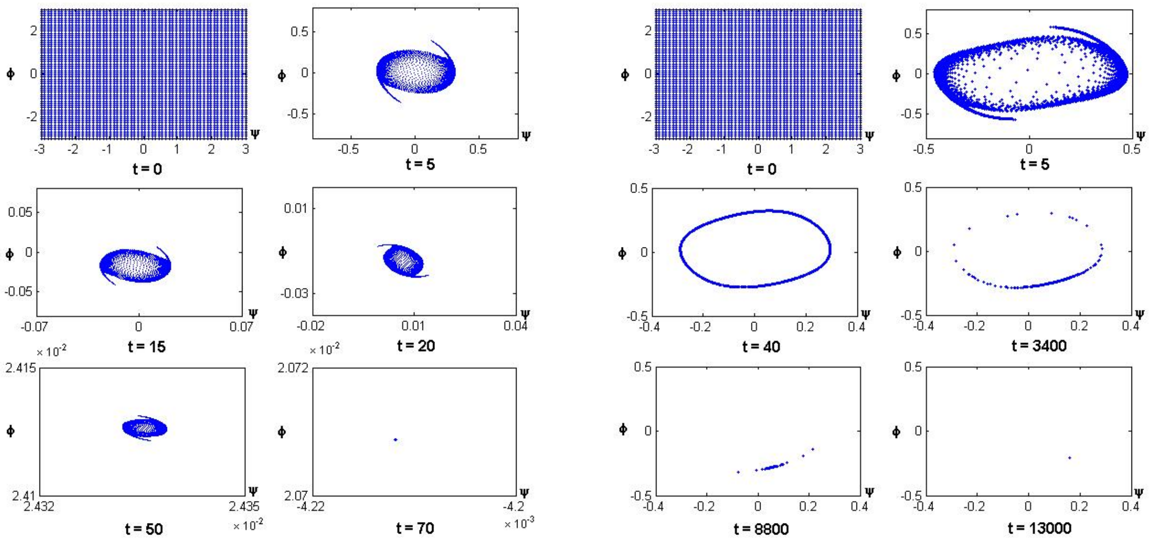


Figure 8: The pullback process $\varphi(t, \vartheta_{-t}\omega)$. By starting the process at earlier times allows us to study the asymptotic behaviour in the fixed fibre at time 0.

values of β, namely $\beta = 1.5$ and 2.2. At the same time, we choose a uniform distribution as the initial value set D. The outcome of the pullback operation is shown in Fig. 9 with different initial values and different parameters β at different times t.

For $\beta = 1.5$, through the pullback process, we can see from Fig. 9(a) that $\varphi(t, \vartheta_{-t}\omega)D$ shrinks to a random point $\mathbf{x}^*(\omega)$ which is distinct from zero. This is because the small additive noise perturbs the invariance of the zero fixed point of the deterministic system. Moreover, numerically it can be observed that, under time reversal, the solution of the system satisfies $\varphi(-t, \vartheta_t\omega)\mathbf{x}_0 \to \infty$ $(t \to \infty)$ for any $\mathbf{x}_0 \neq \mathbf{x}^*(\omega)$, which implies that there is no other invariant measure. Linking the pullback calculation with the calculation of the Lyapunov exponents in Fig. 6, we know that the system is stable at this value of β because of the negative largest Lyapunov exponents, and the system has a unique and stable invariant measure which is a random Dirac measure $\mu_\omega = \delta_{\mathbf{x}^*(\omega)}$ whilst the global random attractor is $A(\omega) = \{\mathbf{x}^*(\omega)\}$. This is exactly the stochastic analogue for the corresponding deterministic case discussed in the previous section. Of course if ω' rather than ω is used, then the Dirac measure and the global random attractor would be different.

However, when $\beta = 2.2$, we observe from Fig. 6 the occurrence of positive Lyapunov exponents. Applying the pullback operation again, we see from Fig. 9(b) that a different behaviour emerges, compared to the case of $\beta = 1.5$, during the convergence of $\varphi(t, \vartheta_{-t}\omega)D$. A random circle becomes visible (at $t = 40$ in Fig. 9(b)) and further convergence takes place on this circle. At last, $\varphi(t, \vartheta_{-t}\omega)D$ converges to a random point $\mathbf{x}^\sharp(\omega)$. We find that, again, the invariant measure is a random Dirac measure $\nu_\omega = \delta_{\mathbf{x}^\sharp(\omega)}$ which is stable with the nonpositive largest Lyapunov exponent. However, through the time reversed solution $\varphi(-t, \vartheta_t\omega)\mathbf{x}_0(t \to \infty)$, we show that the invariant measure $\mu_\omega = \delta_{\mathbf{x}^*(\omega)}$ exists in the interior of the circle, which is illustrated in Fig. 10. Also, the invariant measure $\mu_\omega = \delta_{\mathbf{x}^*(\omega)}$ is unstable and has two Lyapunov exponents that are positive. In addition, under time reversal, $\mathbf{x}^\sharp(\omega)$ is not attracting. As suggested in Schenk-Hoppé



(a) $\beta = 1.5$ (b) $\beta = 2.2$

Figure 9: Random attractors of $\varphi(t, \vartheta_{-t}\omega)D$ when $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$, where the initial value set D comes from a uniform distribution.

(1996b), another invariant measure, say ν'_ω, on the random circle exists as shown in Fig. 10. This analysis implies that, for $\beta = 2.2$, there exist more than two invariant measures, one is completely stable and one is completely unstable (which is stable under time reversal for the SDE system (4.2)), and the global random attractor $A(\omega)$ which supports all invariant measures is a random disc whose boundary is a random circle shown in Fig. 9(b) ($t = 40$). Again if ω' rather than ω is used, then the Dirac measures and the global random attractor would be different.

In summary, our analysis on the D-bifurcation gives us insights into the significant impact of the chartists on the market equilibria. These equilibria can be characterised by the invariant measures of the SDE system. We show that there exists a unique stable invariant measure in the market. However, the stable invariant measure when the chartists react to their demand weakly is quantitatively different from the one when the chartists react to their demand strongly. The change in the stable invariant measure can be described by the stochastic Hopf bifurcation. We have observed that the Hopf bifurcation manifests itself on the level of the invariant measures as the loss of stability of a measure and occurrence of a new stable measure, and on the level of the global attractor as the change from a random point to a random disc.

4.1.2 P-bifurcation

The analysis of the D-bifurcation gives us a perspective from the dynamical systems viewpoint by focusing on the evolution of the random dynamical system. However with SDE systems, there is also a distributional viewpoint. To illustrate the distributional

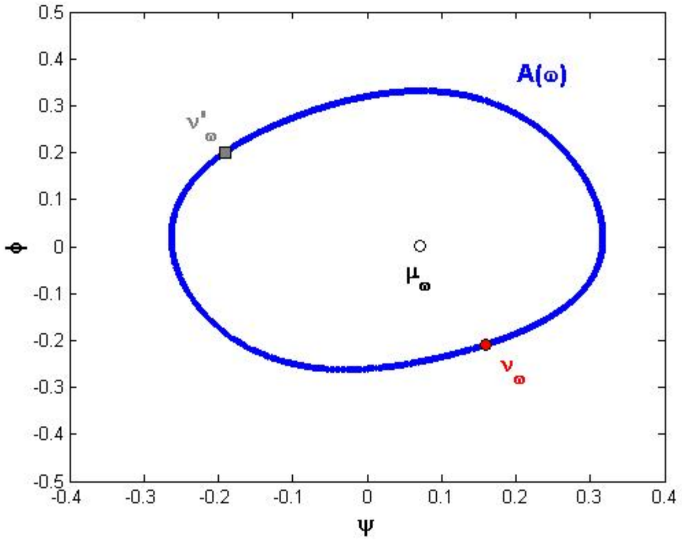


Figure 10: Global random attractor for $\beta = 2.2$, $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$. The blank round point is the unstable invariant measure μ_ω, the full round point represents the stable invariant measure ν_ω and the full square point is the additional invariant measure ν'_ω.

characteristics of a random dynamical system, a stationary measure is an appropriate choice to describe the long term distributional behaviour of solutions of differential equations with random perturbations. The *P-bifurcation* approach to stochastic bifurcation theory examines the *qualitative* changes of the stationary measure when a parameter, in this case β, varies. In the following, we first review the concept of stationary measures and connect stationary distributions to invariant measures. We then show, using a numerical approach, the existence of a stationary measure for our stochastic model. By calculating the joint and marginal stationary densities, we show the qualitative changes of the densities as the chartist behaviour changes. The combination of D- and P-bifurcations gives us insights into the market equilibrium measures from both quantitative and qualitative perspectives.

Definition 4.2 *A probability measure ρ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ is called* stationary *if*

$$\int_{\mathbb{R}^2} P(t, \mathbf{x}, B)\rho(d\mathbf{x}) = \rho(B) \; for\ all\ t \geq 0,\ B \in \mathcal{B}(\mathbb{R}^2),$$

where $P(t, \mathbf{x}, B) = \mathbb{P}(\varphi(t, \omega)\mathbf{x} \in B)$ is generated by φ for time $\mathbb{R}_+$.

There is a one-to-one correspondence between the stationary measure ρ and the invariant measure μ_ω which is measurable with respect to the past $\mathcal{F}_{-\infty}^0 := \sigma(W_s,\ s \leq 0)$ (the σ-algebra generated by $\{W_s\}_{s\leq 0}$). This correspondence is given by (see Arnold (1998))

$$\mu_\omega \to \mathbb{E}\mu_\omega = \rho \qquad \text{and} \qquad \rho \to \lim_{t\to\infty} \varphi(t, \vartheta_{-t}\omega)\rho := \mu_\omega. \tag{4.5}$$

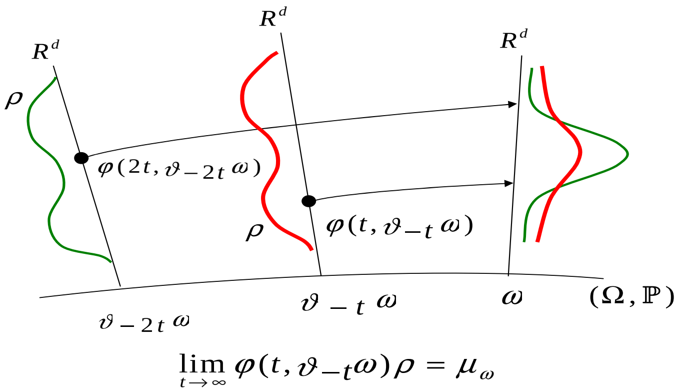


Figure 11: The relationship between the invariant measure μ_ω and the stationary measure ρ.

This relationship is illustrated in Fig. 11.

If ρ has a density $\mathbb{p}$, the stationarity of ρ is equivalent to the statement that $\mathbb{p}$ is a stationary (i.e. time independent) solution of the Fokker-Planck equation, that is

$$L^*\mathbb{p} = 0,$$

where L^* is the formal adjoint of the generator L of $P(t, \mathbf{x}, B)$ given by

$$L = \sum_{i=1}^{2} \mathbb{f}_i(\mathbf{x})\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{k,l=1}^{2} \left(\mathbb{g}(\mathbf{x})\mathbb{g}^\top(\mathbf{x})\right)_{k,l}\frac{\partial^2}{\partial x_k \partial x_l},$$

in which $\top$ represents a transpose operator and $\mathbf{x} = (\psi, \phi)^\top$,

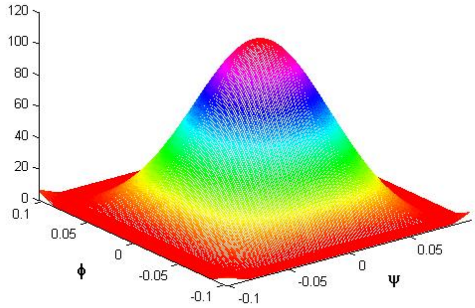
$$\mathbb{f}(\mathbf{x}) = \begin{pmatrix} \phi \\ \frac{1}{\tau}\Big[\big(\varsigma + h'(\psi) - b\big)\phi - a\psi\Big] \end{pmatrix}, \quad \mathbb{g}(\mathbf{x}) = \begin{pmatrix} 0 \\ \frac{a\sigma}{\tau} \end{pmatrix}.$$

The hypoellipticity[8] of L^* and L is given in Theorem 4.2.

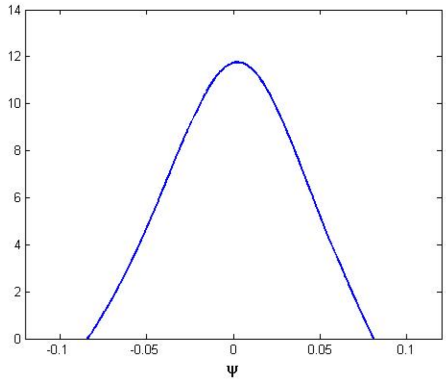
Theorem 4.2 *Consider the SDE system (4.2) in the form $d\mathbf{x} = \mathbb{f}(\mathbf{x})dt + \mathbb{g}(\mathbf{x}) \circ dW$. Then L and L^* are hypoelliptic and hence any solution of $L^*\mathbb{p} = 0$ is smooth.*

The P-bifurcation approach studies qualitative changes of densities of stationary measures ρ when a parameter varies. Hence, for the P-bifurcation, we are only interested in the changes of the shape of the stationary density. We continue to take β as the bifurcation parameter and use the Euler-Maruyama scheme to calculate one sample path up to time $500,000$ with step size 0.001. Since the average amount of time this solution path spends in each sample set is approximately equal to the measure of this set, we obtain a histogram as an estimate of the density of the stationary measure. To enable a better visualisation, the density functions are then smoothed by using a standard procedure.

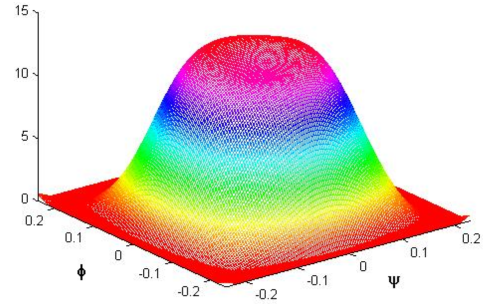
[8] L is hypoelleptic if solutions υ of $L\upsilon = q$ are smooth whenever q is smooth.



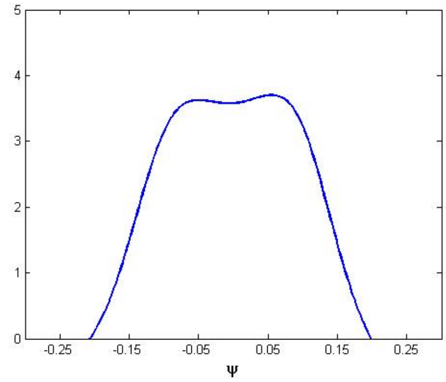
(a) $\beta = 1.5$



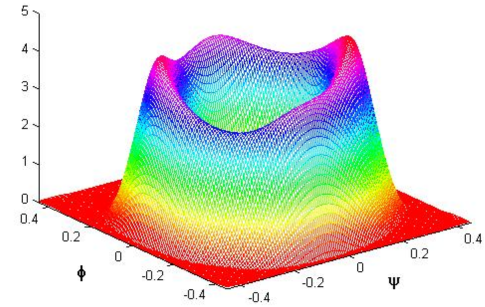
(b) $\beta = 1.5$



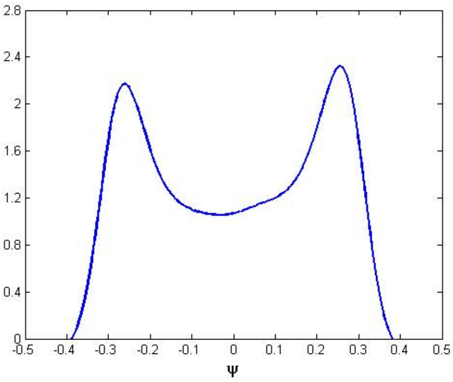
(c) $\beta = 2.0$



(d) $\beta = 2.0$

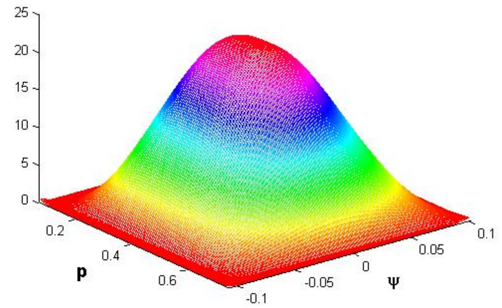


(e) $\beta = 2.2$

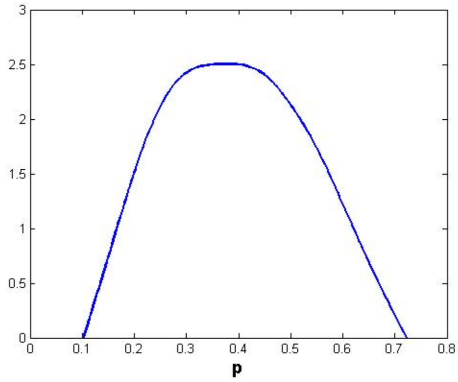


(f) $\beta = 2.2$

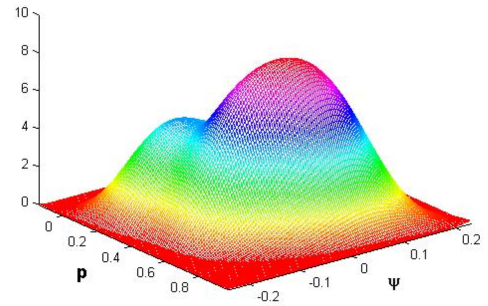
Figure 12: Joint stationary densities of ψ and ϕ and the corresponding marginal distributions for ψ for $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$.



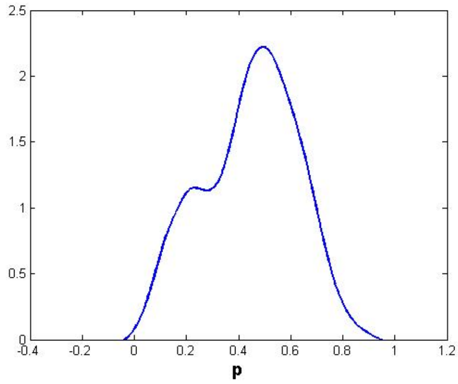
(a) $\beta = 1.5$



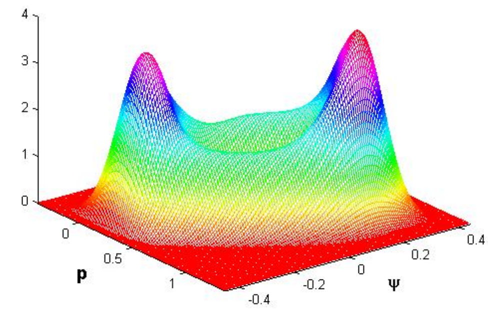
(b) $\beta = 1.5$



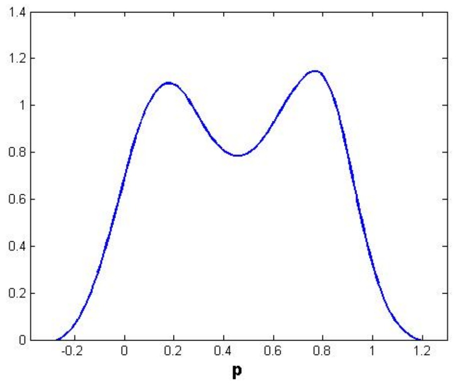
(c) $\beta = 2.0$



(d) $\beta = 2.0$



(e) $\beta = 2.2$



(f) $\beta = 2.2$

Figure 13: Joint stationary densities of the log price p and the assessment of the price trend ψ and the corresponding marginal distributions for p for $a = 1$, $\alpha = 1$, $\tau = 1$, $\sigma = 0.02$.

For $a = 1$, $\alpha = 1$, $\tau = 1$, and $\sigma = 0.02$, Figs 12 and 13 show qualitatively different joint and marginal stationary densities for the different β values of $1.5, 2$ and 2.2. Figure 12 shows the joint stationary densities for variables ψ and ϕ (the left panel) and the marginal densities for the variable ψ (the right panel); while Fig. 13 shows the joint stationary densities for the variables p and ψ (the left panel) and the marginal densities for the logarithm price p (the right panel). We can see from Figs 12 and 13 that, for $\beta = 1.5$, the joint densities in either the (ψ, ϕ) or (ψ, p) planes have one peak and the marginal densities for either ψ or p are unimodal, which correspond to the stable Dirac invariant measure $\mu_\omega = \delta_{\mathbf{x}^*}(\omega)$ with the global random attractor $A(\omega) = \{\mathbf{x}^*(\omega)\}$ under the D-bifurcation analysis. However, for $\beta = 2.2$, the joint density in either the (ψ, ϕ) or (ψ, p) planes has a crater-like shape and the marginal densities for either ψ or p are bimodal. This change is underlined by the stable Dirac invariant measure $\nu_\omega = \delta_{\mathbf{x}^\sharp}(\omega)$ with the global random attractor of a random disc under the D-bifurcation analysis. For $\beta = 2$, the joint (marginal) densities can be regarded as the transition from single peak to crater-like (from unimodal to bimodal) densities. Therefore, as the strength of reaction to the demand signal of the chartists increases, the qualitative change of the stationary densities indicates the occurrence of a P-bifurcation.

In summary, our analysis shows that D- and P-bifurcations characterise the stochastic behaviour in different ways. The relation (4.5) between the invariant measure, used for D-bifurcation analysis, and stationary measure, used for P-bifurcation analysis, highlights the connection between D- and P-bifurcations. Based on our analysis, we know that when $\beta < \beta_D$, the system only has one invariant measure $\delta_{\mathbf{x}^*(\omega)}$ which is stable. In this case, $\mathbf{x}^*(\omega)$ has a stationary measure which has one peak as shown in Figs 12(a) and 13(a). However, when $\beta > \beta_D$, a new stable random Dirac measure $\delta_{\mathbf{x}^\sharp(\omega)}$ appears and the corresponding stationary measure has a crater-like density as shown in Figs 12(e) and 13(e). Quantitative changes under the D-bifurcation can help us to obtain a better view of the qualitative changes under the P-bifurcation, but the combined analysis of both D- and P-bifurcations certainly gives us a relatively complete picture of the stochastic behaviour of the model. Economically, our analysis indicates the significant role the chartists play on the distributional outcomes of the financial market model. When chartists change from less active (so that $\beta < \beta_D$) to more active (so that $\beta > \beta_D$), the joint (marginal) density function of the market equilibrium changes from single peak to crater-like (from unimodal to bimodal) density function.

4.1.3 Stochastic approximation

To complete our understanding of the stochastic behaviour of the SDE system (4.2), ideally we would like to obtain its solution. However, this SDE system cannot be solved explicitly. In subsections 4.1.1 and 4.1.2, we have used numerical methods to detect the D- and P-bifurcations of (4.2). In this subsection, we shall use an approximate method to obtain some analytical properties of our stochastic model. Under certain assumptions concerning the nonlinearity and the noise terms, we obtain an approximate solution near the steady state of the corresponding deterministic system. We show that the approximate solution of our stochastic model shares the characteristics of the corresponding dynamics of the deterministic model. Our analysis also verifies the quantitative and qualitative changes of the invariant measures and stationary measures detected by our D- and P-bifurcation analysis in subsections 4.1.1 and 4.1.2.

In the following, we first need to rescale the parameters and transform the Kolmogorov backward equation of the SDEs (4.2) to a standard form to which the stochastic method of averaging (see Khas'minskii (1963)) can be applied. We then calculate the corresponding stationary probability density in polar coordinates. As a result we are able to show how the density function behaves as the parameter b varies.

Assume that $h'(\psi) - b = h'(\psi) - h'(0)$ is small, that is ψ is near 0. We change parameters by rescaling and introducing the (small) parameter ε so that

$$h'(\psi) - b \to \varepsilon^2(h'(\psi) - b), \quad \varsigma \to \varepsilon^2\varsigma, \quad \sigma \to \varepsilon\sigma.$$

Then (4.2) can be rewritten as

$$\begin{cases} d\psi = \phi dt, \\ d\phi = \varepsilon^2 \dfrac{1}{\tau}\big[\varsigma + h'(\psi) - b\big]\phi dt - \dfrac{a\psi}{\tau}dt + \dfrac{\varepsilon a\sigma}{\tau} \circ dW, \end{cases} \tag{4.6}$$

the Kolmogorov backward equation of which is

$$\frac{\partial p_\varepsilon}{\partial t} = \phi\frac{\partial p_\varepsilon}{\partial \psi} - \eta^2\psi\frac{\partial p_\varepsilon}{\partial \phi} + \varepsilon^2\left[\frac{a^2\sigma^2}{\tau^2}\frac{\partial^2 p_\varepsilon}{\partial \phi^2} + \mathcal{K}(\psi,\phi)\frac{\partial p_\varepsilon}{\partial \phi}\right],$$

where

$$\mathcal{K}(\psi,\phi) = \frac{1}{\tau}\big[\varsigma + h'(\psi) - b\big]\phi,$$

$\eta^2 = a/\tau$ and $p_\varepsilon = p_\varepsilon(t;\psi,\phi;\psi_1,\phi_1)$ denotes the probability density of a transition from the point (ψ,ϕ) to the point (ψ_1,ϕ_1) in time t for a trajectory of (4.6). We apply the polar coordinate transformation

$$\psi = \frac{r}{\eta}\sin(\theta - \eta t), \quad \phi = r\cos(\theta - \eta t),$$

and set

$$\mathbb{u}_\varepsilon(t;r,\theta;r_1,\theta_1) = p_\varepsilon(t;\frac{r}{\eta}\sin(\theta-\eta t), r\cos(\theta-\eta t); \frac{r_1}{\eta}\sin(\theta_1-\eta t), r_1\cos(\theta_1-\eta t)).$$

Then it can be verified that $\mathbb{u}_\varepsilon$ satisfies

$$\frac{\partial \mathbb{u}_\varepsilon}{\partial t} = \varepsilon^2 L_\varepsilon(r,\theta,t)\mathbb{u}_\varepsilon, \tag{4.7}$$

where

$$
\begin{aligned}
L_\varepsilon(r,\theta,t) \;&=\; \frac{a^2\sigma^2}{\tau^2}\Big[\cos^2(\theta-\eta t)\frac{\partial^2}{\partial r^2} - \frac{\sin 2(\theta-\eta t)}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2(\theta-\eta t)}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&\quad + \frac{\sin^2(\theta-\eta t)}{r}\frac{\partial}{\partial r} + \frac{\sin 2(\theta-\eta t)}{r^2}\frac{\partial}{\partial\theta}\Big] \\
&\quad + \mathcal{K}\Big(\frac{r}{\eta}\sin(\theta-\eta t), r\cos(\theta-\eta t)\Big)\Big[\cos(\theta-\eta t)\frac{\partial}{\partial r} - \frac{\sin(\theta-\eta t)}{r}\frac{\partial}{\partial\theta}\Big],
\end{aligned}
$$

which is of the standard form to which the stochastic method of averaging may be applied. In fact, for partial differential equations of the form $\frac{\partial \mathbb{u}}{\partial t} = \varepsilon^2\mathcal{L}(\mathbf{x},t)\mathbb{u}$, the averaging principle has been studied in Khas'minskii (1963), where $\mathcal{L}$ is an elliptical or parabolic second-order differential operator. According to this principle, the solution of the Cauchy problem for this equation as $\varepsilon \to 0$ may be uniformly approximated over an interval of time which is $O(1/\varepsilon^2)$ by the solution of the equation

$$\frac{\partial \mathbb{v}}{\partial t} = \varepsilon^2 L^0 \mathbb{v},$$

where L^0 is an operator whose coefficients are obtained from those of $\mathcal{L}$ by averaging with respect to time, that is

$$L^0(\mathbf{x}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{L}(\mathbf{x},t)dt.$$

We now apply the averaging principle to (4.7). First let $p_0(t;r,\theta;r_1,\theta_1)$ be the probability density of a transition of the random process by the method of averaging, which is described in polar coordinates corresponding to (ψ,ϕ) by the partial differential equation

$$\frac{\partial p_0}{\partial t} = \frac{a^2\sigma^2}{2\tau^2}\left[\frac{\partial^2 p_0}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 p_0}{\partial\theta^2} + \frac{1}{r}\frac{\partial p_0}{\partial r}\right] + \mathscr{U}(r)\frac{\partial p_0}{\partial r} - \frac{\mathscr{V}(r)}{r}\frac{\partial p_0}{\partial\theta}, \tag{4.8}$$

where

$$\mathscr{U}(r) = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{K}\Big(\frac{r}{\eta}\sin t, r\cos t\Big)\cos t dt, \quad \mathscr{V}(r) = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{K}\Big(\frac{r}{\eta}\sin t, r\cos t\Big)\sin t dt.$$

Then for any $R>0$ and $T>0$

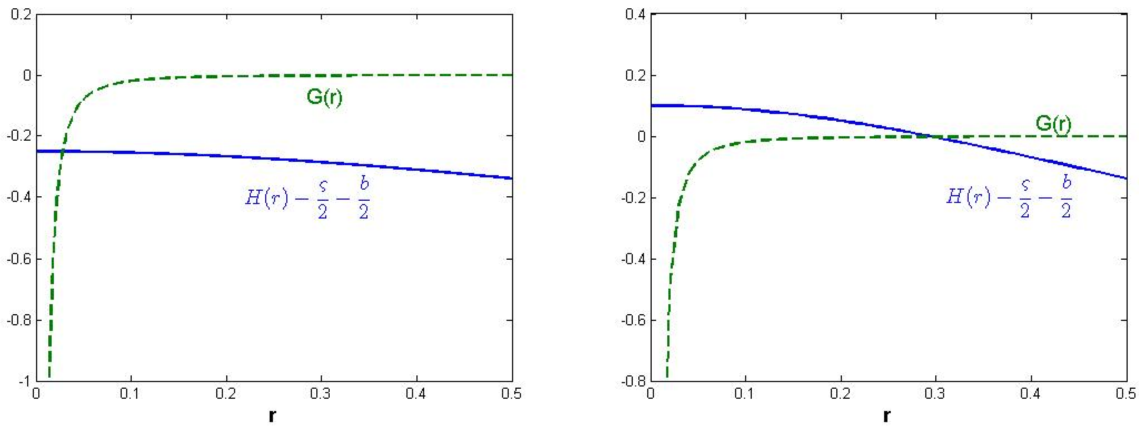
$$\mathbb{u}_\varepsilon(t;r,\theta,r_1,\theta_1) - p_0(t\varepsilon^2;r,\theta,r_1,\theta_1) \to 0 \quad \text{as} \quad \varepsilon\to 0,$$

uniformly with respect to $r,\ \theta,\ r_1,\ \theta_1$ in the region $r<R,\ r_1<R$ and with respect to t in the region $0\le t\le T/\varepsilon^2$.

Making use of the fact that the stationary density of the two-dimensional process is 2π periodic in θ, we can assert that the stationary density $\mathbb{p}(r,\theta)$ corresponding to the solution of (4.8) is independent of θ and has the form $\mathbb{p}(r,\theta) = \frac{\mathbb{p}(r)}{2\pi}$, where $\mathbb{p}(r)$ is the solution of the partial differential equation

$$\frac{a^2\sigma^2}{2\tau^2}\left[\frac{\partial^2\mathbb{p}(r)}{\partial r^2} - \frac{\partial}{\partial r}\frac{\mathbb{p}(r)}{r}\right] - \frac{\partial}{\partial r}(\mathscr{U}(r)\mathbb{p}(r)) = 0, \quad r\in[0,+\infty). \tag{4.9}$$

Furthermore, we can obtain the following result.



(a) b small (b) b large

Figure 14: The functions G and H.

Theorem 4.3 *There exists a unique stationary probability density of* (4.9)*, which has the form*

$$\mathbb{p}(r) = Cr\exp\{\frac{2\tau^2}{a^2\sigma^2}\int_0^r \mathscr{U}(s)ds\},$$

where C is the normalisation constant.

Clearly, $\mathbb{p}(r)$ attains its extremum at the point $r = r_e$ satisfying

$$\mathscr{U}(r_e) = -\frac{a^2\sigma^2}{2\tau^2 r_e}. \tag{4.10}$$

Let $H(r) = \frac{1}{2\pi}\int_0^{2\pi} h'(\frac{r}{\eta}\sin t)\cos^2 tdt$ and $G(r) = -\frac{a^2\sigma^2}{2\tau r^2}$, so that $\mathscr{U}(r) = \frac{r}{\tau}(H(r)+\frac{\varsigma}{2}-\frac{b}{2})$. Then (4.10) may be written as

$$H(r) + \frac{\varsigma}{2} - \frac{b}{2} = G(r). \tag{4.11}$$

The function $H(r)$ has previously been studied in Chiarella (1992). Note that $H(0) = \frac{b}{2}$ and $\lim_{r\to+\infty} H(r) = 0$. In addition, $H'(r) < 0$ for $r > 0$. However, it is obvious that $G(r)$ is monotonically increasing with $r \in (0,\infty)$. Hence (4.11) only has one solution $r = r_e$ and in particular, $\mathbb{p}(\cdot)$ attains its maximum value at $r = r_e$.

Note that

$$\mathbb{H}_{max} \equiv \max_r\{H(r) + \frac{\varsigma}{2} - \frac{b}{2}\} = \frac{\varsigma}{2} = \frac{b-b^*}{2}.$$

When b is very small, in particular $b < b^* = 1 + a\tau$, then $\varsigma < 0$, $\mathbb{H}_{max} < 0$ and (4.11) is approximated by

$$-\frac{a^2\sigma^2}{2\tau r^2} = \frac{\varsigma}{2},$$

Then $\mathbb{p}(r)$ attains its maximum value near

$$r_e \approx \frac{a\sigma}{\sqrt{-\varsigma\tau}},$$

which is close to zero, see Fig. 14(a). In particular, when $\sigma \to 0$, we have $r_e \to 0$. This analysis recovers the corresponding results for the deterministic case studied in Chiarella (1992).

When b is large, in particular $b > b^* = 1 + a\tau$, we have $\mathbb{H}_{max} > 0$ and $\mathbb{H}_{min} \equiv \min_r\{H(r) + \frac{\varsigma}{2} - \frac{b}{2}\} = -\frac{a\tau}{2} - \frac{1}{2} < 0$. In this case, the solution of (4.11) is far away from zero, as shown in Fig. 14(b). This indicates a crater-like density whose maximum is located on a circle (around the steady state of the deterministic system) with a large radius. If we treat this stationary density (when b is large) as a bifurcation from the case when b is small, the maximum radius of the density function corresponds to a Hopf bifurcation. It is in this sense that we argue that the stochastic model shares the corresponding dynamics to those of the deterministic model.

4.2 Dynamical Behaviour in the Limit of $\tau \to 0^+$

Corresponding to the analysis of the deterministic case in subsection 3.2, in this section, we analyse the behaviour of the stochastic dynamics as the chartists put greater and greater weight on the most recent price changes in forming their estimate of the trend, and in the limit $\tau \to 0^+$ relying on just the most recent price change.

As $\tau \to 0^+$, we see from (2.6) that $dp \to \psi dt$ is governed by

$$dp = \big[a(F-p) + h(\psi)\big]dt,$$

whilst from (2.7) and (4.1) the dynamics of ψ are driven by

$$d\psi = \frac{-a\psi}{1-h'(\psi)}dt + \frac{a\sigma}{1-h'(\psi)} \circ dW. \tag{4.12}$$

Let $m(\psi) = \frac{-a\psi}{1-h'(\psi)}$ and $\mathfrak{s}(\psi) = \frac{a\sigma}{1-h'(\psi)}$, then the Itô stochastic differential equation corresponding to (4.12) is

$$\begin{aligned} d\psi &= \mathbb{M}(\psi)dt + \mathfrak{s}(\psi)dW \\ &:= (m(\psi) + \frac{1}{2}\mathfrak{s}'(\psi)\mathfrak{s}(\psi))dt + \mathfrak{s}(\psi)dW \\ &= (\frac{-a\psi}{1-h'(\psi)} + \frac{a^2\sigma^2 h''(\psi)}{2(1-h'(\psi))^3})dt + \frac{a\sigma}{1-h'(\psi)}dW. \end{aligned} \tag{4.13}$$

Hence, if there exists ψ_* such that $h'(\psi_*) = 1$, then (4.13) (or (4.12)) is singular at $\psi = \psi_*$. Similarly to the deterministic case, for the different cases, $b < 1$, $b = 1$ and $b > 1$, the stochastic differential equation (4.13) (or (4.12)) will have a different number of singularity points and therefore exhibit different behaviour. We will discuss each case in turn in the following analysis.

To simplify the analysis, in this subsection, we assume[9] that there exists $x_1 < 0,\ x_2 > 0$ such that

$$\text{for } \psi \in (x_1, x_2),\ h^{(3)}(\psi) < 0; \text{ otherwise, } h^{(3)}(\psi) > 0 \tag{4.14a}$$

[9]The conditions (4.14) are general conditions, which are applicable to many functions. It is not difficult to show that the hyperbolic tangent function satisfies these conditions.

$$\psi h^{(4)}(\psi) > 0, \text{ for } \psi \in (x_1, x_2). \tag{4.14b}$$

Figure 15 illustrates the graphs of h'', $h^{(3)}$ and $h^{(4)}$ satisfying the above conditions.

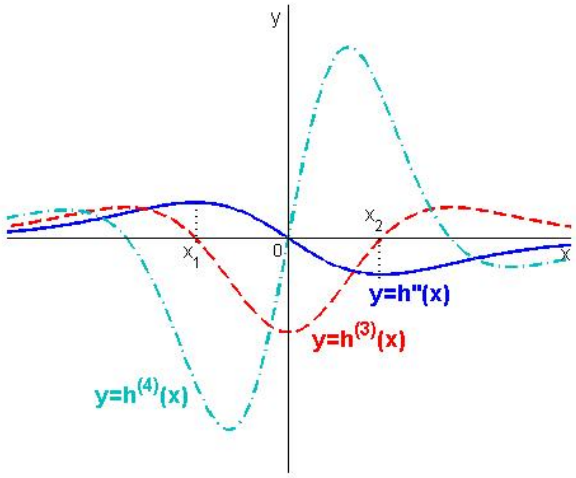


Figure 15: Plots of functions $h''(x)$, $h^{(3)}(x)$ and $h^{(4)}(x)$.

When $b < 1$, we have $1-h'(\psi) > 0$ for any ψ and there is no singularity in $(-\infty, +\infty)$. The only singular points are $\pm\infty$. Based on the theory of the classification of singular boundaries[10], the boundaries $\pm\infty$ are the repulsively natural boundaries. Furthermore, we obtain the following result.

Theorem 4.4 *When $b < 1$, the SDE (4.12) uniquely generates a local smooth RDS Ψ which is global (in the sense of Theorem 4.1). In addition, there exists a unique stationary density $\mathbb{p}$ for Ψ, where*

$$\mathbb{p}(\psi) = N\frac{(1-h'(\psi))}{a\sigma}\exp\left(\int_0^{\psi} -\frac{2y(1-h'(y))}{a\sigma^2}dy\right) \tag{4.15}$$

and N is the normalisation constant.

Note that when ψ satisfies

$$h''(\psi) + \frac{2\psi(1-h'(\psi))^2}{a\sigma^2} = 0, \tag{4.16}$$

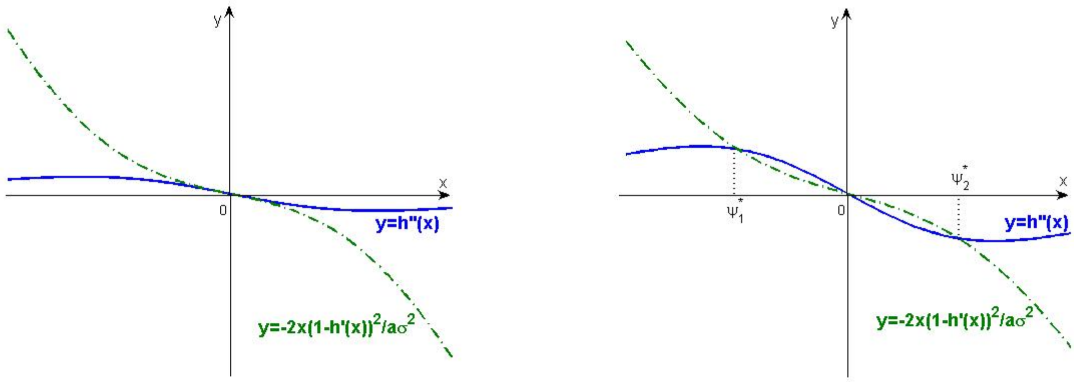
the stationary density $\mathbb{p}(\cdot)$ attains its extremum. This, together with the assumptions (2.4) and (4.14), leads to the following result on the P-bifurcation.

[10]We refer to Lin and Cai (2004) for more information about the theory of the classification of various singular boundaries, including entrance, regular, and (attractively and repulsively) natural boundaries used in our discussion.

Theorem 4.5 (P-Bifurcation) *Let* $b_p = 1 - \sqrt{-\frac{h^{(3)}(0)a\sigma^2}{2}}$.

(1) When $h^{(3)}(0) > -\frac{2}{a\sigma^2}$ *and* $0 < b < b_p$*, the stationary density* $\mathbb{p}(\cdot)$ *has a unique extreme point* $\psi = 0$*, at which* $\mathbb{p}(\cdot)$ *attains its maximum.*

(2) When $\max\{b_p, 0\} < b < 1$*, the stationary density* $\mathbb{p}(\cdot)$ *has three extreme points* ψ_1^**,* 0 *and* ψ_2^* *satisfying* $\psi_1^* < 0 < \psi_2^*$*. In addition, the stationary density* $\mathbb{p}(\cdot)$ *attains its minimum value at* $\psi = 0$ *and the maximum values at* $\psi = \psi_1^*$ *and* ψ_2^**.*



(a) $h^{(3)}(0) > -\frac{2}{a\sigma^2}$ and $b < b_p$ (b) $\max\{b_p, 0\} < b < 1$

Figure 16: The determination of the extreme points of the stationary density $\mathbb{p}(\cdot)$

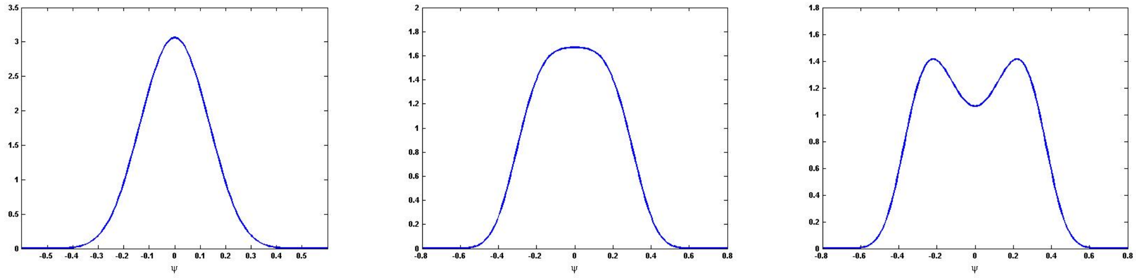
Theorem 4.5 indicates that, as the chartists place more and more weight on the recent price changes, the number of the extreme points of the stationary density $\mathbb{p}(\cdot)$ in (4.15) changes from one to three as the parameter b changes (but $b < 1$). This is shown in Fig. 16. This means that, a moderate increase in activity (such that $b < 1$) of the chartists when they are weighting recent price changes very heavily results in a large deviation of their estimate of the price trend ψ from its mean value, which is illustrated by the changes from unimodal to bimodal distribution in the upper panel of Fig. 17. The changes in distributions are further illustrated by the underlying time series for ψ in the bottom panel of Fig. 17. However, unlike the case of $\tau > 0$, such a P-bifurcation is inconsistent with the D-bifurcation, which does not occur based on the following theorem.

Theorem 4.6 (No D-Bifurcation) *For* $b < 1$*, let* $\mathbb{p}(\psi)$ *be as defined in Theorem 4.4 and consider* $\wp(\psi) = \mathbb{p}(\psi)d\psi$*. Then*

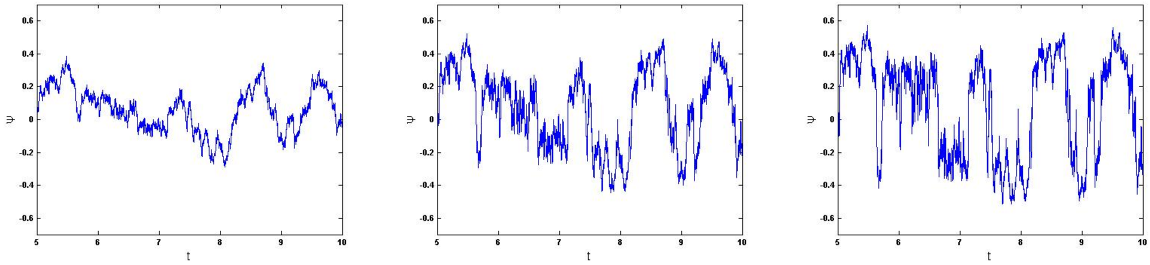
(i) the Lyapunov exponents of $\wp$ *satisfies*

$$\lambda = -2 \int_{\mathbb{R}} (\frac{m(\psi)}{\sigma(\psi)})^2 \mathbb{p}(\psi) d\psi < 0, \tag{4.17}$$

and hence the RDS Ψ *generated by* (4.12) *is always exponentially stable under* $\wp$*.*



(a) $b < b_p$, density (b) $b \lessapprox b_p$, density (c) $b_p \lessapprox b < 1$, density



(d) $b < b_p$, time series (e) $b \lessapprox b_p$, time series (f) $b_p \lessapprox b < 1$, time series

Figure 17: When $\tau = 0$ and $h^{(3)}(0) > -\frac{2}{a\sigma^2}$, a P-bifurcation occurs at $b = b_p$. The top panel shows different typical stationary densities $\mathbb{p}$ of ψ with different parameter ranges and the bottom panel shows the underlying time series.

(ii) the $\mathbb{p}$-invariant forward Markov measure $\mathbb{\mu}_\omega := \lim_{t\to\infty} \Psi(-t,\omega)^{-1}\mathbb{p}$ is a Dirac measure, $\mathbb{\mu}_\omega = \delta_{\mathbb{x}(\omega)}$, and $A(\omega) := \{\mathbb{x}(\omega)\}$ is the random attractor of Ψ in $\mathbb{R}$ in a universe of sets containing all bounded deterministic sets.

(iii) the RDS Ψ has no other invariant measure besides $\mathbb{\mu}_\omega$.

(iv) for any initial measure $\mathbb{\nu} \in \mathcal{P}r(\mathbb{R})$ and $\mathbb{\nu}_t(\cdot) := \int_{\mathbb{R}} P(t,\psi,\cdot)\mathbb{\nu}(d\psi)$,

$$\lim_{t\to\infty} \mathbb{\nu}_t = \mathbb{p} = E\delta_{\mathbb{x}} = \mathbb{P}(\mathbb{x} \in \cdot).$$

When $b < 1$, Theorem 4.6 implies that the RDS Ψ has no D-bifurcation since the measure $\mathbb{\mu}_\omega = \delta_{a(\omega)}$ corresponding to $\mathbb{p}$ is the unique invariant measure of Ψ and it always has a negative Lyapunov exponent. This result is similar to that of the corresponding deterministic model (that is when $b < 1$, the unique steady state $(F^*, 0)$ is stable). However, the appearance of the P-bifurcation discussed above cannot be inferred from any information in the corresponding deterministic case. This result illustrates that, from the P-bifurcation point of view, the stochastic dynamical system can be very different from that of the underlying deterministic system.

When $b = 1$, that is $h'(0) = 1$, the drift term $\mathbb{M}(0)$ in (4.13) becomes unbounded and further $\psi = 0$ is a regular boundary. With the increase of b to $b > 1$, by the assumptions (2.4), there exist $\psi_{1b} < 0$ and $\psi_{2b} > 0$ satisfying $h'(\psi_{1b}) = h'(\psi_{2b}) = 1$, both of which are also regular boundaries. The appearance of the regular boundary at $b = 1$ corresponds to the occurrence of a singularity induced bifurcation in the corresponding deterministic case. In fact, at the same time, the stochastic model may share some of the features of the deterministic model. For example, when $b > 1$, we know from the previous section that the deterministic dynamics exhibit fast motion in the ψ-direction and slow motion in the p-direction. For the stochastic model, we observe a similar behaviour, as shown in Fig. 18. However, once a regular boundary appears, it renders the stationary solution nonunique[11] without further stipulation on the behaviour of (4.13) (or (4.12)) (see Feller (1952)). This causes some difficulties in obtaining analytical properties of the stationary densities of the stochastic system, as was possible in the case of $b < 1$. However, from a practical point of view, this additional freedom may be a blessing since it may permit certain realistic behaviour to be incorporated into the context of the financial model, but we leave this consideration to future research.

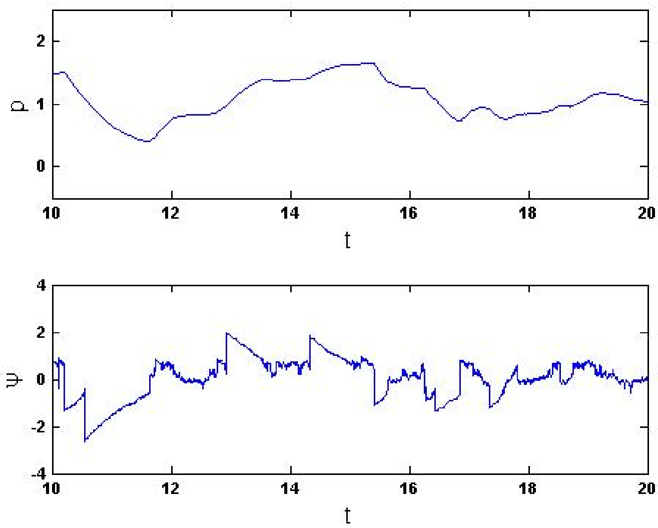


Figure 18: Fast motion in ψ and slow motion in p in the limit of $\tau \to 0^+$ for $b > 1$

5 Conclusion

In this paper, within the framework of the heterogeneous agent paradigm, we have extended the basic deterministic model of speculative price dynamics studied by Beja and Goldman (1980) and Chiarella (1992) to a stochastic model for the market price, which can be characterised by the invariant measures of a random dynamical system. In contrast to the indirect approach widely used in recent literature, we use a direct approach via stochastic bifurcation analysis to examine the market impact of speculative

[11]Note that in the deterministic case, the occurrence of the singularity induced bifurcation corresponds to the appearance of a singular point where the Implicit Function Theorem does not hold, that is to say that the solution of the implicit function can be nonunique.

behaviour. By using D(dynamical)- bifurcation analysis, we examine the quantitative changes of the stable invariant measures. By using P(Phenomenological)- bifurcation analysis, we examine the qualitative changes of the stationary measures. The difference between P-bifurcation and D-bifurcation lies in the fact that the P-bifurcation approach focuses on the stationary distribution and is, in general, not related to path-wise stability, while the D-bifurcation approach does focus on path-wise stability and is based on the invariant measure and the multiplicative ergodic theorem. However, the P-bifurcation has the advantage of allowing us to visualise the changes of the stationary density functions. Quantitative changes under the D-bifurcation can help us to obtain a better view on the qualitative changes under the P-bifurcation, but the combined analysis using both D- and P-bifurcations certainly gives us a relatively complete picture of the stochastic behaviour of the model.

For the simple stochastic financial market model studied here, when the time lag (τ) used by the chartists to form their expected price trends is not zero, we show that the market equilibrium price can be characterised by a unique and stable invariant measure when the activity of the speculators is below a certain critical value. If this threshold is surpassed, the market equilibrium can be characterised by a new stable invariant measure while the original invariant measure becomes unstable. In addition, the corresponding stationary measure displays a significant qualitative change near the threshold value. Below the threshold, the joint densities display one peak and the marginal densities are unimodal. Above the threshold, the joint densities have crater-like shapes and the marginal densities are bimodal. This indicates that the changes in the stationary distributions lead to a bimodal logarithm price distribution and fat tails. This qualitative change near the threshold is further confirmed by using a stochastic approximation method. We show that, near the steady state of the underlying deterministic model, the approximation of the stochastic model shares the corresponding Hopf bifurcation dynamics of the deterministic model. When the time lag (τ) used by the chartists to form their expected price changes approaches zero, so that the chartists are placing greater and greater weight on recent price changes, then the system can have singular points. In this case, the stochastic model displays very different dynamics from those of the underlying deterministic model. In particular, the fundamental noise can destabilise the market equilibrium and result in a change of the stationary distribution through a P-bifurcation of the stochastic model, while the corresponding deterministic model displays no bifurcation. In addition, the consistency between D- and P-bifurcations for $\tau > 0$ breaks down for $\tau \to 0^+$. Our results demonstrate the important connection, but also the significant differences, of the dynamics between deterministic and stochastic models.

Economically, our analysis indicates the significant role that the chartists play in the financial market model. At one level, this phenomenon is very intuitive, but a rigorous analysis of the stochastic behaviour underlying this intuition using the tools and concepts of stochastic bifurcation theory is our main contribution in this paper.

In order to bring out the basic phenomena associated with stochastic bifurcations we have focused on a highly simplified model of interacting heterogeneous agents in a financial market. The model could be embellished in a number of ways in future research, in particular; introducing other types of randomness such as market noise, since it is known from empirical literature (see for example the work of Shiller (1981) that fundamental noise is not sufficient to yield the type of volatility observed in financial markets; allowing for switching of strategies according to some fitness measure as proposed by Brock and Hommes (1997, 1998); and analysing in closer detail the stochastic dynamics in the case when the time lag of chartists in their estimate of the trend tends to zero.

Appendix

Proof of Theorem 3.2: The results can be obtained by using the Singularity Induced Bifurcation Theorem in Venkatasubramanian, Schättler and Zaborszky (1995). ∎

Proof of Theorem 3.3: This is a direct corollary of the main theorem in Yang, Tang and Du (2003). ∎

Proof of Theorem 4.1: (i) This follows from the general existence, uniqueness Theorem 2.3.36 given in Arnold (1998).

(ii) Since the Wiener process $\{W_t\}$ is scalar, the SDE system (4.2) has a path-wise interpretation (see Sussmann (1978), Theorem 8 and Section 7). We prove its strictly forward completeness by transforming it into an equivalent non-autonomous ODE via the transformation

$$\Phi_t = \phi_t - \frac{a\sigma}{\tau} W_t.$$

The result is

$$\begin{cases} \dot{\psi}_t = \Phi_t + \dfrac{a\sigma}{\tau} W_t, \\ \dot{\Phi}_t = \dfrac{1}{\tau}\Big[\big(\varsigma + h'(\psi_t) - b\big)\big(\Phi_t + \dfrac{a\sigma}{\tau} W_t\big) - a\psi_t\Big]. \end{cases} \tag{A.1}$$

Define the function $V = \psi^2 + \Phi^2$ and apply the chain rule to V. Then

$$\begin{aligned} dV &= 2\psi d\psi + 2\Phi d\Phi \\ &= 2\psi\Phi dt + \frac{2a}{\tau}\sigma W \psi dt + 2\Phi\frac{1}{\tau}\big[\varsigma + h'(\psi) - b\big](\Phi + \frac{a\sigma}{\tau}W)dt - \frac{2a}{\tau}\psi\Phi dt \\ &\le (1 + \frac{a}{\tau})(\psi^2 + \Phi^2)dt + \frac{a\sigma}{\tau}W(\psi^2 + 1)dt + M_0(2\Phi^2 + \frac{a\sigma}{\tau}W\Phi^2 + \frac{a\sigma}{\tau}W)dt \\ &\le M_1 V dt + M_2 dt, \end{aligned}$$

where $M_0 = \big(a\tau + 1 + \max_{\psi} |h'(\psi)|\big)/\tau$, $\beta = \sup_{s\in[0,T]} |W_s|$, $M_1 = 1 + a/\tau + \max\big\{a\sigma\beta/\tau, M_0(2 + a\sigma\beta/\tau)\big\}$ and $M_2 = \big(a\sigma\beta + M_0 a\sigma\beta\big)/\tau$. By the Gronwall lemma,

$$V_t \le K(T) < \infty, \qquad \text{for all} \qquad 0 \le t \le T.$$

Hence the solution of (4.2) exists on any interval $[0, T]$ and so its maximal interval of existence is $\mathbb{R}_+$.

Now the maximal non-explosive solution is defined for any $W \in C^0$ without any exceptional set. Therefore the solution is strictly forward complete.

Similarly, we also use the time reversed equations to get the strictly backward completeness of the solution of (4.2). ■

Proof of Theorem 4.2: The generator L of (4.2) is not elliptic because the diffusion matrix

$$\mathbb{g}(\mathbf{x})\mathbb{g}^\top(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 0 & a^2\sigma^2/\tau^2 \end{pmatrix}$$

has rank 1. The Lie bracket $[f, g]$ can be easily calculated to be

$$[\mathbb{f}, \mathbb{g}](\mathbf{x}) = \begin{pmatrix} a\sigma/\tau \\ a\sigma[\varsigma + h'(\psi) - b]/\tau^2 \end{pmatrix}, \qquad \text{where} \qquad \mathbf{x} = (\psi, \phi)^\top.$$

Hence for $\mathbf{x} \in \mathbb{R}^2$, the vectors $\mathbb{g}(\mathbf{x})$ and $[\mathbb{f}, \mathbb{g}](\mathbf{x})$ are clearly linearly independent, implying $dim\mathcal{LA}(\mathbb{f}, \mathbb{g})(\mathbf{x}) = 2$ for all $\mathbf{x} \in \mathbb{R}^2$. By the Hörmander Theorem, full rank of the Lie algebra ensures the hypoellipticity of L and L^*. Hence solutions v of $L^*v = q$ are smooth whenever q is smooth. ■

Proof of Theorem 4.3: The proof is based on the theory of the classification of singular boundaries in Lin and Cai (2004).

First, note that (4.9) has two boundaries $r_l = 0$ and $r_r = +\infty$. The drift and diffusion coefficients corresponding to (4.9) are respectively

$$\mathbf{M}(r) = \mathscr{U}(r) + \frac{a^2\sigma^2}{2\tau^2 r}, \qquad \varrho^2(r) = \frac{a^2\sigma^2}{\tau^2}.$$

It is easy to see that the left boundary $r_l = 0$ is a singular boundary of the second kind, that is $\mathbf{M}(r_l) = \infty$. In addition,

$$\varrho^2(r) = O|r - r_l|^0, \quad \mathbf{M}(r) = O|r - r_l|^{-1} \quad \text{as} \quad r \to r_l^+;$$

so the diffusion and drift exponents of r_l are given by, respectively,

$$\mathbb{\alpha}_l = 0, \qquad \mathbb{\beta}_l = 1,$$

and the character value of r_l is

$$\mathbb{c}_l = \lim_{r \to r_l^+} \frac{2\mathbf{M}(r)(r - r_l)^{\mathbb{\beta}_l - \mathbb{\alpha}_l}}{\varrho^2(r)} = 1.$$

These quantities indicate that the left-hand boundary $r_l = 0$ is an entrance.

Similarly, the right-hand boundary $r_r = +\infty$ is a singular boundary of the second kind at infinity, i.e. $\mathbf{M}(+\infty) = \infty$. In this case, one can obtain the result that the diffusion and drift exponents of r_r are given by, respectively

$$\alpha_r = 0, \qquad \beta_r = 1.$$

Note that

$$b - \varsigma = 1 + a\tau > 0, \qquad \lim_{x\to\infty} h'(x) = 0,$$

so $\mathbf{M}(+\infty) < 0$. Therefore, the right-hand boundary $r_r = +\infty$ is repulsively natural. Since each of the two boundaries is either an entrance or repulsively natural, then a nontrivial stationary solution exists in $[0, \infty)$, which has the form

$$\mathbb{p}(r) = Cr \exp\{\frac{2\tau^2}{a^2\sigma^2}\int_0^r \mathscr{U}(s)ds\}.$$

Note that $\mathbb{p}(r) > 0$ for $r \in (0, \infty)$, so the stationary probability is unique, see Kliemann (1987). ■

Proof of Theorem 4.4: The existence and uniqueness of the solution of (4.12) follows from Theorem 2.3.36 in Arnold (1998).

For the global property, based on Theorem 2.3.38 in Arnold (1998), we only need to prove that the singular boundaries $\pm\infty$ are natural. Based on Lin and Cai (2004), it's not difficult to obtain the result that the boundaries $\pm\infty$ are singular boundaries of the second kind at infinity, that is $|\mathbb{M}(\pm\infty)| = \infty$ and the diffusion exponents and drift exponents of $\pm\infty$ are respectively

$$\alpha_{\pm\infty} = 0 \quad \text{and} \quad \beta_{\pm\infty} = 1.$$

In addition, $\mathbb{M}(\pm\infty) \lesseqgtr 0$. Therefore, $\pm\infty$ are repulsively natural. Similarly consider the backward SDE

$$d\psi = \frac{a\psi}{1 - h'(\psi)}dt - \frac{a\sigma}{1 - h'(\psi)} \circ dW. \tag{A.2}$$

The boundaries $\pm\infty$ are attractively natural for (A.2). Therefore, the RDS Ψ is strictly forward and backward complete.

In addition, for the forward SDE (4.12), $\pm\infty$ are repulsively natural, so there exists a nontrivial stationary solution in $(-\infty, +\infty)$. In fact, from the Fokker-Planck Equation, we know that the stationary probability is

$$\mathbb{p}(\psi) = N\frac{(1 - h'(\psi))}{a\sigma}\exp(\int_0^\psi -\frac{2y(1 - h'(y))}{a\sigma^2}dy),$$

where N is a normalisation constant. Note that $\mathbb{p}(\psi) > 0$ for all ψ, so the stationary probability density is unique. ■

Proof of Theorem 4.5: It is obvious that $\psi = 0$ is one of the solutions to (4.16) for any of the cases considered. In the following, we only consider the case of $(-\infty, 0)$, for the interval $(0, +\infty)$, a similar result can be easily demonstrated.

Let $S(\psi) = -\frac{2\psi(1-h'(\psi))^2}{a\sigma^2}$. By the assumptions (2.4) and (4.14), we know that $h''(\psi)$ is a concave function on $(x_1, 0)$ and $\lim_{\psi \to -\infty} h''(\psi) = \ell$ where $0 \le \ell < +\infty$. In addition,

$$\lim_{\psi \to -\infty} S(\psi) = +\infty, \quad S'(\psi) < 0 \quad \text{for} \quad \psi \in (-\infty, 0), \tag{A.3a}$$

$$S''(\psi) > 0 \quad \text{for} \quad \psi \in (x_1, 0). \tag{A.3b}$$

When

$$h^{(3)}(0) > -\frac{2}{a\sigma^2} \quad \text{and} \quad b < 1 - \sqrt{-\frac{h^{(3)}(0)a\sigma^2}{2}}, \tag{A.4}$$

then $S'(0) < h^{(3)}(0) < 0$. By the convexity and concavity of $S(\cdot)$ and $h''(\cdot)$ in $[x_1, 0)$ and monotonicity of h in $(-\infty, x_1)$, there is no solution of (4.16) on $(-\infty, 0)$. Then (4.16) has a unique solution $\psi = 0$ on $(-\infty, +\infty)$ when (A.4) holds. In addition, if (A.4) is satisfied, then

$$\mathbb{p}''(0) = \frac{N}{a\sigma}(-h^{(3)}(0) - \frac{2(1-b)^2}{a\sigma^2}) < 0.$$

So $\psi = 0$ is the maximum point of $\mathbb{p}(\cdot)$.

When $b > 1 - \sqrt{-h^{(3)}(0)a\sigma^2/2}$, then $S'(0) > h^{(3)}(0)$. Therefore, there exists a solution of (4.16) in $(-\infty, 0)$, denoted by ψ_1^*. To demonstrate the uniqueness of the solution of (4.16) in $(-\infty, 0)$, we consider the following two cases:

(i) If the solution ψ_1^* is on $[x_1, 0)$, by the concavity of $h''(\cdot)$ and convexity of $S(\cdot)$ in $[x_1, 0)$, there is a unique solution of (4.16) on $[x_1, 0)$. On the other hand, on $(-\infty, x_1)$, $h''(\cdot)$ is monotonically increasing and however $S(\cdot)$ is monotonically decreasing. Hence, there is no other solution on $(-\infty, 0)$.

(ii) When $\psi_1^* \in (-\infty, x_1)$, there is no solution of (4.16) on the interval $[x_1, 0)$, otherwise there is a contradiction with the case (i). With the monotonicity of $h''(\cdot)$ and $S(\cdot)$ on $(-\infty, x_1)$, there is only one solution ψ_1^* on $(-\infty, 0)$.

In addition, when $b > 1 - \sqrt{-h^{(3)}(0)a\sigma^2/2}$, we have $\mathbb{p}''(0) > 0$. So $\psi = 0$ is the minimum point of $\mathbb{p}(\cdot)$. We note that at ψ_1^*, it must be the case that

$$h^{(3)}(\psi)\Big|_{\psi_1^*} > (\frac{-2\psi(1-h'(\psi))^2}{a\sigma^2})'\Big|_{\psi_1^*}, \tag{A.5}$$

otherwise, by $\lim_{\psi \to -\infty} h''(\psi) = \ell(< +\infty)$ and $\lim_{\psi \to -\infty} S(\psi) = +\infty$, there must be another solution $\widetilde{\psi}(< \psi_1^*)$ of (4.16), which is a contradiction of the uniqueness of the solution of

(4.16) on $(-\infty, 0)$. Then, through (4.16) and (A.5)

$$\begin{aligned} \mathbb{p}''(\psi)\Big|_{\psi=\psi_1^*} &= \Big[-\frac{h^{(3)}(\psi)}{a\sigma} + \frac{6\psi h''(\psi)(1-h'(\psi))}{a^2\sigma^3} - \frac{2(1-h'(\psi))^2}{a^2\sigma^3} \\ &+ \frac{4\psi^2(1-h'(\psi))^3}{a^3\sigma^5}\Big] N\exp\Big(\int_0^{\psi} -\frac{2y(1-h'(y))}{a\sigma^2}dy\Big)\Big|_{\psi=\psi_1^*} < 0. \quad \text{(A.6)} \end{aligned}$$

Therefore, ψ_1^* is a maximum point of $\mathbb{p}(\cdot)$. ■

Proof of Theorem 4.6: Follows by the fact that $a\sigma/(1-h'(\psi)) > 0$ and Theorem 9.2.4 in Arnold (1998). ■

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