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Gerald HL Cheang and Carl Chiarella

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Abstract

We consider a market consisting of multiple assets under jump-diffusion dynamics with European style options written on these assets. It is well-known that such markets are incomplete in the Harrison and Pliska sense. We derive a pricing relation by adopting a Radon-Nikodým derivative based on the exponential martingale of a correlated Brownian motion process and a multivariate compound Poisson process. The parameters in the Radon-Nikodým derivative define a family of equivalent martingale measures in the model, and we derive the corresponding integro-partial differential equation for the option price. We also derive the pricing relation by setting up a hedge portfolio containing an appropriate number of options to "complete" the market. The market prices of jump-risks are priced in the hedge portfolio and we relate these to the choice of the parameters in the Radon-Nikodým derivative used in the alternative derivation of the integro-partial differential equation.

Key words: Incomplete markets, Equivalent martingale measure, Compound Poisson processes, Radon-Nikodým derivative, Multi-asset options, Integro-partial differential equation.
JEL Classification: C00, G12, G13

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1 Introduction

The problem of pricing options on an underlying that is subject not only to diffusion risk, but also to jump risk was initiated in a classical paper by Merton (1976) who extended the celebrated Black and Scholes (1973) option pricing model to consider option pricing based on a stock following jump-diffusion dynamics. Merton (1976) considered a constant arrival intensity, log normally distributed jump sizes, set the market price of jump risk to zero and obtained a Poisson weighted sum of Black-Scholes type formulae. He also considered the same hedge portfolio used by Black and Scholes, namely one consisting of a position in the stock, the option and the risk-free asset only. In this case a perfect hedge does not exist and hedging was achieved by Merton by averaging out idiosyncratic risk. However, this leaves the market price of jump risk unpriced, and also the distribution of the jump components remain unchanged. Further extensions to the Merton (1976) model include those by Anderson (1984) and Aase (1988). However these authors also make assumptions that amount to leaving the jump risk unpriced. Furthermore these later derivations do not appeal to the traditional hedging argument but rather appeal directly to the risk-neutral valuation principle and change of measure arguments.

Whilst there are now a number of derivations of option pricing under jump diffusion dynamics (see below) these are usually derived either within a dynamic general equilibrium framework, or using risk-neutral valuation and change of measure arguments. The way in which the traditional economic hedging argument carries over to the jump-diffusion situation has not been treated very much in the literature, so the link between this approach and the approach based on change of measure arguments remains obscure.

In this paper we consider a market consisting of multiple assets driven by jump-diffusion dynamics and European style options written on these assets. We derive the option pricing relation by the risk-neutral valuation/change of measure approach, which may be considered a generalization of the approach of Aase (1988). We then show how to extend the traditional hedging argument to price options under jump-diffusion dynamics. We do this by adding to

the hedge portfolio an appropriate number of options of different maturities. As we allow the jump sizes to be drawn from some continuous distribution it is only possible to hedge perfectly for a particular set of draws from the jump size distribution. Thus we can only hedge in some approximate manner, and here we choose to eliminate the jump risks by averaging over the distribution of the jump sizes so that the portfolio is hedged against jump risks "on average". We show that the resulting pricing relationship is equivalent to that obtained under the earlier derivation if the market prices of jump risk are chosen in an appropriate manner.

The market we consider is incomplete in the Harrison and Pliska (1981) sense. As we consider the market prices of jump risks, then there are many equivalent martingale measures to choose from, each yielding different distributions for the jump components. For a single stock market, one could for example, apply a local risk-minimizing trading strategy in the manner of Schweizer (1991), Colwell and Elliott (1993), or a minimum entropy martingale measure approach in the manner of Miyahara (2001). Jeanblanc-Piqué and Pontier (1990) applied a general equilibrium model to the problem and used two assets driven by the same Wiener and Poisson noise factors. Jarrow and Madan (1995) included additional traded assets in order to hedge away the jump-risk in interest rate term-structure-related securities. Mercurio and Runggaldier (1993), and Runggaldier (2003), suggested that other assets driven by the same Wiener and Poisson noise factors as the stock be included in the hedge portfolio. Jarrow and Madan (1999), and Björk et al. (1997) considered infinite asset cases that allow for measure-valued trading portfolios, the latter also in the case of bond market structure. However, we do not take such an approach in this paper.

For our market consisting of multiple but finite number of assets, we adopt a Radon-Nikodým derivative based on the exponential of a correlated Brownian motion process and a multivariate compound Poisson process. The parameters in the Radon-Nikodým derivative define a family of equivalent martingale measures in the model, from which a suitable choice is made. The approach of selecting an equivalent martingale measure based on the selection of suitable parameters in a Radon-Nikodým derivative is not new. Gerber and Shiu (1994)

applied a similar approach using Radon-Nikodým derivatives based on Esscher transforms of Lévy processes for single stock markets under complete market conditions. The parameters in our Radon-Nikodým derivatives are based on the local characteristics of the various independent Poisson measures used, and many pairs of choices are possible. After selecting particular values of the parameters of the Radon-Nikodým derivative that determine the new local characteristics of the Poisson measures under the measure transformation, we derive an integro-partial differential equation for the option price where there is a change in the distributions of the jump components from that of the original physical measure. As stated above we also set up a hedge portfolio as another method of deriving the same integro-partial differential equation. In our case, a certain number of options of different maturities are required to "complete" the market. The jump risks are priced in our portfolio and we relate the market prices of these jump risks to the choice of the parameters in the Radon-Nikodým derivative used in the alternative derivation of the integro-partial differential equation based on the measure change approach.

The paper unfolds as follows. Section 2 lays out the framework of the model and states the main result for the pricing of options under jump-diffusion dynamics. Section 3 outlines the change of measure result for jump-diffusion processes. Section 4 discusses the market prices of Wiener and jump risks and the role these play in the change of measure to a martingale measure. Section 5 then uses the change of measure result to derive the option pricing relationship. Section 6 considers a hedge portfolio that includes an appropriate number of options of different maturities and shows how the portfolio may be made riskless on average. It is then shown that the resulting pricing relation is equivalent to the one obtained in Section 5 if market prices of jump risk are chosen appropriately. Section 7 concludes.

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2 Preliminaries and the Extended Merton's Model for Multiple Assets

We consider an economy consisting of d stocks with the ith stock paying dividends at the continuously compounded rate $\xi_{i,t}$. The dynamics of each stock are exposed to both diffusion and jumps components. There are two jump components, the first is a unique jump component that corresponds to idiosyncratic shocks to the individual stock price, and the second a common jump component with correlated relative jump sizes that arrive at the same time. The jumps that arrive at the same time for all the stocks can be interpreted as macroeconomic shocks. The dynamics for the stock prices can be written as (5) below. However, in order to understand the details of the model, we first introduce some necessary mathematical notation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a probability measure space and assume that the filtration $\{\mathcal{F}_t\}$ is well-defined to contain all the basic processes necessary in our multi-stock market model. These processes include a d-dimensional Wiener process with correlated components denoted by $W_t = (W_{1,t}, W_{2,t}, \ldots, W_{d,t})^\top$. The Wiener components are correlated with

$$dW_{i,t}dW_{j,t} = \begin{cases} \rho_{ij}dt, & i \neq j \\ dt, & i = j \end{cases}$$

where $\rho_{ij} = \rho_{ji}$ may be time-varying and the correlation matrix at time $t = u$ is

$$\Sigma_u = \begin{pmatrix} 1 & & \rho_{ij} \\ \vdots & \ddots & \vdots \\ \rho_{ji} & & 1 \end{pmatrix},$$

with possibly time-varying entries in the off-diagonals. In order to exclude some trivial cases in our model, we also require that $|\rho_{ij}| < 1$ and that Σ_t is Lebesgue-almost-everywhere t invertible.

We also have $d+1$ independent Poisson arrival processes adapted to the filtration $\mathcal{F}_t$ denoted by $N_{0,t}$, $N_{1,t}$ and so on until $N_{d,t}$, each with possibly non-homogeneous arrival intensities (with respect to the measure $\mathbb{P}$) denoted by $\lambda_i \equiv \lambda_{i,t}$ for $i = 0, \ldots, d$. Following the notation

in Runggaldier (2003), if $(T_{i,n}, Y_{i,n})$ are independent point processes and the marks $Y_{i,n}$ form a sequence of random variables taking values in a set A_i, then for any set $G \in \sigma(A_i) \subset \mathcal{F}$, the arrival process $N_{i,t}$ is associated with the counting process

$$N_{i,t}(G) \equiv \sum_{n \geq 1} \mathbb{1}_{\{T_{i,n} \leq t\}} \mathbb{1}_{\{Y_{i,n} \in G\}}.$$

We shall assume that $Y_{i,n} \in \mathbb{R}$ and $Y_{0,n} \in \mathbb{R}^d$. The associated counting measure is

$$(1) \qquad p_i((0,t], G) = N_{i,t}(G), \quad G \in \sigma(A_i).$$

This measure allows us to obtain more concise expressions for the marked point process terms via integrals of the form

$$(2) \qquad \int_0^t \int_{A_i} H(s, y_i) p_i(ds, dy_i) \equiv \sum_{n \geq 1} H(T_{i,n}, Y_{i,n}) \mathbb{1}_{\{T_{i,n} \leq t\}} \equiv \sum_{n=1}^{N_{i,t}} H(T_{i,n}, Y_{i,n}),$$

where $H(t, \cdot)$ is $\mathcal{F}_t$-predictable. We assume for simplicity that the intensity takes the form

$$\lambda_{i,t}(dy) = \lambda_{i,t} m_{i,t}(dy_i),$$

where $\lambda_{i,t}$ is non-negative while $m_{i,t}(dy_i)$ is a probability measure on A_i and the $Y_{i,n}$ are independently and identically distributed (and independent from the intensity $\lambda_{i,t}$), so that one has

$$(3) \qquad \mathbb{E}_{\mathbb{P}}\left[\int_0^\infty \int_{A_i} H(s, y_i) p_i(ds, dy_i)\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^\infty \int_{A_i} H(s, y_i) \lambda_{i,t} m_{i,t}(dy_i) ds\right].$$

The pair $(\lambda_{i,t}, m_{i,t}(dy_i))$ is called the $(\mathbb{P}, \mathcal{F}_t)$-local characteristic of $p_i(ds, dy_i)$ and

$$(4) \qquad \hat{p}_i(dt, dy_i) = p_i(dt, dy_i) - \lambda_{i,t} m_{i,t}(dy_i) dt$$

is the corresponding compensated Poisson measure.

Given all of the above notation we can write the dynamics of the d stock prices as

$$\frac{dS_{i,t}}{S_{i,t-}} = \mu_i dt + \sigma_i dW_{i,t} + \int_{A_i} (Z_i(t, y_i) - 1) \hat{p}_i(dt, dy_i)$$

$$(5) \qquad\qquad\qquad + \int_{A_0} (Z_0^{(i)}(t, y_0) - 1) \hat{p}_0(dt, dy_0)$$

for $i = 1, 2, \cdots, d$. For the ith stock, its unique relative jump size, should a jump occur at time t is $Z_i(t, y_i) - 1$, where $Z_i(t, y_i) > 0$ and $\mathcal{F}_t$-predictable, so that

$$(6) \qquad \int_{A_i} (Z_i(t, y_i) - 1)p(dt, dy_i) = (Z_i(t, y_i) - 1)dN_{i,t},$$

and its average jump size increment (taken in the measure $\mathbb{P}$) is

$$(7) \qquad \kappa_{i,t} = \mathbb{E}_{\mathbb{P}}[Z_i - 1] = \int_{A_i} (Z_i(t, y_i) - 1)m_{i,t}(dy_i).$$

The relative jump size of its common jump component is $Z_0^{(i)}(t, y_0) - 1$, where $Z_0^{(i)}(t, y_0) > 0$ and $\mathcal{F}_t$-predictable, so that

$$(8) \qquad \int_{A_0} (Z_0^{(i)}(t, y_0) - 1)p(dt, dy_i) = (Z_0^{(i)}(t, y_0) - 1)dN_{0,t},$$

and its average jump size increment is

$$(9) \qquad \kappa_{0,t}^{(i)} = \mathbb{E}_{\mathbb{P}}[Z_0^{(i)} - 1] = \int_{A_0} (Z_0^{(i)}(t, y_0) - 1)m_{0,t}(dy_0).$$

Since the marks for the jumps $(Z_0^{(1)}, Z_0^{(2)}, \ldots, Z_0^{(d)})$ are drawn from the probability density function $m_{0,t}(dy_0)$, the jumps $(Z_0^{(1)}, Z_0^{(2)}, \ldots, Z_0^{(d)})$ are allowed to be correlated and thus allow for the macroeconomic shocks across all stock prices to be correlated.

Where no confusion arises, we drop the time subscript t for brevity and refer to (7) and (9) as κ_i and $\kappa_0^{(i)}$ respectively. Similarly for other time dependent parameters in the model of this paper, we drop the time subscript t for brevity if necessary. Thus we can rewrite (5) in the form involving the jump component compensators as

$$\frac{dS_{i,t}}{S_{i,t-}} = (\mu_i - \lambda_i \kappa_i - \lambda_0 \kappa_0^{(i)})dt + \sigma_i dW_{i,t}$$

$$(10) \qquad \qquad + (Z_i(t, y_i) - 1)dN_{i,t} + (Z_0^{(i)}(t, y_0) - 1)dN_{0,t}$$

for $i = 1, 2, \cdots, d$. Integrating (10), the stock prices are given by

$$S_{i,t} = S_{i,0} \exp\left[\int_0^t \left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i \kappa_i - \lambda_0 \kappa_0^{(i)}\right) du + \int_0^t \sigma_i dW_{i,u}\right]$$

$$(11) \qquad \qquad \times \prod_{n=1}^{N_{i,t}} Z_i(T_{i,n}, Y_{i,n}) \prod_{k=1}^{N_{0,t}} Z_0^{(i)}(T_{0,k}, Y_{0,k}).$$

7

We state a key result concerning the pricing of options underlying jump-diffusion processes. Note that throughout this paper, we assume for simplicity that the options have non path-dependent final payoffs $X_T(S_{1,T}, \ldots, S_{d,T})$. In other words, we are not considering path-dependent options, such as barrier options for instance.

Theorem 2.1. *Let $X_t(S_{1,t}, \cdots, S_{d,t})$ be a non-path dependent option written on the d stocks, with return dynamics given by (5) in the market measure $\mathbb{P}$, and the final non path-dependent payoff specified by $X_T(S_{1,T} \cdots, S_{d,T})$. Then the option price process satisfies the integro-partial differential equation*

$$
\frac{\partial X_t(S_{1,t-}, \cdots, S_{d,t-})}{\partial t}
$$
$$
+ \sum_{i=1}^{d} \left(r_t - \xi_{i,t} - \tilde{\lambda}_{i,t} \int_{A_i} [Z_i(t, y_i) - 1] \tilde{m}_{i,t}(dy_i) \right.
$$
$$
\left. - \tilde{\lambda}_{0,t} \int_{A_0} [Z_0^{(i)}(t, y_0) - 1] \tilde{m}_{0,t}(dy_0) \right) S_{i,t-} \frac{\partial X_t(S_{1,t-}, \cdots, S_{d,t-})}{\partial S_i}
$$
$$
+ \mathcal{L} X_t(S_{1,t-}, \cdots, S_{d,t-})
$$
$$
+ \sum_{i=1}^{d} \tilde{\lambda}_{i,t} \int_{A_i} [X_t(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_t(S_{1,t-}, \cdots, S_{d,t-})] \tilde{m}_{i,t}(dy_i)
$$
$$
+ \tilde{\lambda}_{0,t} \int_{A_0} [X_t(S_{1,t-} Z_0^{(1)}, \cdots, S_{d,t-} Z_0^{(d)}) - X_t(S_{1,t-}, \cdots, S_{d,t-})] \tilde{m}_{0,t}(dy)
$$
$$
(12) \qquad = r_t X_t(S_{1,t-}, \cdots, S_{d,t-}),
$$

where the operator $\mathcal{L}$ is defined as

$$
(13) \qquad \mathcal{L} \equiv \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \rho_{ij} \sigma_i \sigma_j S_{i,t-} S_{j,t-} \frac{\partial^2}{\partial S_i \partial S_j},
$$

and the Poisson counting measure $p_i(dt, dy_i)$ has $(\mathbb{Q}, \mathcal{F}_t)$-local characteristics $(\tilde{\lambda}_{i,t}, \tilde{m}_{i,t}(dy_i))$ for some equivalent martingale measure $\mathbb{Q}$. The local characteristics are given by

$$
\tilde{\lambda}_{i,t} = \psi_{i,t} \lambda_{i,t} \text{ and } \tilde{m}_{i,t}(dy_i) = h_{i,t}(y_i) m_{i,t}(dy_i),
$$

for some $\mathcal{F}_t$-predictable and positive $\psi_{i,t}$ and $h_{i,t}$, for $i = 0, 1, \cdots, d$, where the $(\mathbb{P}, \mathcal{F}_t)$-local characteristics of $p_i(dt, dy_i)$ are given by $(\lambda_{i,t}, m_{i,t}(dy_i))$ in the market measure $\mathbb{P}$.

The integro-partial differential equation (12) can be derived by both a martingale approach as well as a hedge portfolio approach. The proof of Theorem 2.1 is found in Sections 5 and 6, where in Section 5, the integro-partial differential equation (12) is derived using a martingale approach and in Section 6, it is derived using a hedging argument. Theorem 2.1 also demonstrates that the option price is not necessarily unique since the martingale measure $\mathbb{Q}$ depends on the choice of $\psi_{i,t}$ and $h_{i,t}$. What is novel in this paper is that $d+2$ options are used to hedge away Wiener diffusion risks and jump-component risks "on average" and that by judicious choice of the market prices of jump risk the resulting pricing formula is the same as that obtained from the equivalent martingale measure approach.

As special cases, Cheang and Chiarella (2007) specialise the model of this paper to the one asset case and derive a pricing formula for a call option under similar assumptions to those used by Merton (1976), but make explicit the role of the market price of jump risk parameters. Formulae could also be similarly derived for the double exponential jump size distribution of Kou and Wang (2004).

3 Transformation of Measures

In the classical Black-Scholes model (Black and Scholes 1973), no arbitrage arguments lead to the pricing of options in the risk-neutral measure $\mathbb{Q}$. The expected value of the payoff of the option at maturity must be adjusted by a state-price density if the expectation is calculated using the historical measure $\mathbb{P}$. The state-price density is actually a Radon-Nikodým derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ adjusted by a suitable discounting factor.

When the underlying stock price dynamics are modelled by a jump-diffusion model, the market is incomplete in the Harrison and Pliska (1981) sense, and so there are many equivalent martingale measures. A particular martingale measure $\mathbb{Q}$ can be chosen by specifying the parameters in the Radon-Nikodým derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$. The following theorem illustrates the form of the Radon-Nikodým derivative that we will be applying in our model.

Theorem 3.1. *On a finite time interval $[0,T]$ let $p_i(dt, dy_i)$ be independent Poisson measures*

with $(\mathbb{P}, \mathcal{F}_t)$-local characteristics $(\lambda_{i,t}, m_{i,t}(dy_i))$, independent from the correlated Brownian motion components $W_t = (W_{1,t}, W_{2,t}, \ldots, W_{d,t})^\top$. Let $\psi_{i,t} \geq 0$ be $\mathcal{F}_t$-predictable and $h_{i,t} \geq 0$ an $\mathcal{F}_t$-predictable and $\sigma(A_i)$-measurable process, such that $\mathbb{P}$-almost surely and for all $t \in [0, T]$, they satisfy

$$\int_0^t \psi_{i,u} \lambda_{i,u} du < \infty, \ \text{and} \ \int_{A_i} h_{i,u}(y_i) m_{i,u}(dy_i) = 1, \quad \text{for } i = 0, 1, \cdots, d.$$

Let $\theta_t = (\theta_{1,t}, \theta_{2,t}, \ldots, \theta_{d,t})^\top$ be square integrable $\mathcal{F}_t$-predictable processes. Then

$$
\begin{aligned}
\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t &= \exp\left[-\int_0^t (\Sigma_u^{-1} \theta_u)^\top dW_u - \frac{1}{2} \int_0^t \theta_u^\top \Sigma_u^{-1} \theta_u du \right] \\
&\quad \times \prod_{i=1}^d \left\{ \exp\left[\int_0^t \int_{A_i} (1 - \psi_{i,u} h_{i,u}(y_i)) \lambda_{i,u} m_{i,u}(dy_i) du \right] \prod_{n=1}^{N_{i,t}} \psi_{i,T_n} h_{i,T_n}(Y_{i,n}) \right\} \\
(14) &\quad \times \exp\left[\int_0^t \int_{A_0} (1 - \psi_{0,u} h_{0,u}(y_0)) \lambda_{0,u} m_{0,u}(dy_0) du \right] \prod_{n=1}^{N_{0,t}} \psi_{0,T_n} h_{0,T_n}(Y_{0,n})
\end{aligned}
$$

is a Radon-Nikodým derivative process under which $p_i(dt, dy_i)$ has $(\mathbb{Q}, \mathcal{F}_t)$-local characteristics $(\tilde{\lambda}_{i,t}, \tilde{m}_{i,t}(dy_i))$ where

$$\tilde{\lambda}_{i,t} = \psi_{i,t} \lambda_{i,t} \ \text{and} \ \tilde{m}_{i,t}(dy_i) = h_{i,t}(y_i) m_{i,t}(dy_i),$$

for $i = 0, 1, \cdots, d$ and

$$d\widetilde{W}_{i,t} = \theta_{i,t} dt + dW_{i,t}$$

where $\widetilde{W}_{i,t}$ is a standard Brownian motion component under $\mathbb{Q}$ and $\widetilde{W}_t = (\widetilde{W}_{1,t}, \ldots, \widetilde{W}_{d,t})^\top$ has the same correlation structure as $W_t = (W_{1,t}, W_{2,t}, \ldots, W_{d,t})^\top$.

Proof. The Radon-Nikodým derivative (14) is a product of terms all of which are Radon-Nikodým derivatives, and independent of each other since in our model, the Brownian motion components are assumed to be independent from the jump components. The first term is the usual Radon-Nikodým derivative for correlated Brownian motion and the change in the distributions of the Brownian motion components follows from the results of the usual Girsanov's measure transformation for Brownian motion. The remaining terms, again independent of each other, are Radon-Nikodým derivatives of the marked point processes. Details

on Girsanov's measure transformation for Poisson measures and marked point processes can be found in Runggaldier (2003) or Cont and Tankov (2004). □

Aase (1988) gives a simplified version of Theorem 3.1 for one Wiener and one compound Poisson noise factor only, and the parameter θ there was set to be the usual Black-Scholes market price of Wiener risk $\frac{\mu-r}{\sigma}$ for the single stock case. Although Merton (1976) did not use a Radon-Nikodým derivative approach, his arguments based on systematic and idiosyncratic risks essentially also led to the same Black-Scholes market price of Wiener risk. In our application of Theorem 3.1, the distribution of the jump components can be changed under the transformation from the historical measure $\mathbb{P}$ to a martingale measure $\mathbb{Q}$ given a choice of $\psi_{i,t}$ and $h_{i,t}$. In Sections 4 and 5, we will see that the choices of $\psi_{i,t}$ and $h_{i,t}$ determine the prices of the jump-risks. The $\theta_{i,t}$, which relates to the market prices of risk of the ith Wiener component, is then determined by a martingale condition on the discounted stock yield processes following specific choices of $\psi_{i,t}$ and $h_{i,t}$. The multi-asset equivalent of Merton's (1976) model under which the distribution of the jump components does not change under the transformation corresponds to the particular choice of setting both $\psi_{i,t}$ and $h_{i,t}$ equal to one, leaving the jump risks unpriced.

4 The Market Prices of Risk and Equivalent Martingale Measures

An equivalent martingale measure $\mathbb{Q}$ under which all the discounted stock yield processes $\left\{S_{i,t}e^{\int_0^t(\xi_{i,u}-r_u)du}\right\}$ are martingales can be obtained by specifying choices of $\psi_{i,t}$ and $h_{i,t}$ in the Radon-Nikodým derivative (14) prior to solving for the values of θ_i that appear in that formula. It is not the goal of this paper to discuss the various methods under which an equivalent martingale measure may be obtained, for example the minimal entropy martingale method of Miyahara (2001), the R-minimality method of Schweizer (1991) just to name a few. We simply assume that $\psi_{i,t}$ and $h_{i,t}$ have already been chosen or calibrated. Then from Theorem 3.1, the Poisson measure $p_i(dt, dy_i)$ has $(\mathbb{Q}, \mathcal{F}_t)$-local characteristics $(\tilde{\lambda}_{i,t}, \tilde{m}_{i,t}(dy_i))$

where $\tilde{\lambda}_{i,t} = \psi_{i,t}\lambda_{i,t}$ and $\tilde{m}_{i,t}(dy_i) = h_{i,t}(y_i)m_{i,t}(dy_i)$ for such a measure $\mathbb{Q}$ associated with such $\psi_{i,t}$ and $h_{i,t}$. Thus from (4) and (5), the stock price dynamics can be expressed as

$$
\begin{aligned}
\frac{dS_{i,t}}{S_{i,t-}} &= \mu_i dt + \sigma_i dW_{i,t} + \int_{A_i} (Z_i(t, y_i) - 1)\hat{q}_i(dt, dy_i) \\
&\quad + \lambda_{i,t} \int_{A_i} (Z_i(t, y_i) - 1)(\psi_{i,t}h_{i,t}(dy_i) - 1)m_{i,t}(dy_i)dt \\
&\quad + \int_{A_i} (Z_0^{(i)}(t, y_0) - 1)\hat{q}_0(dt, dy_0) \\
&\quad + \lambda_{0,t} \int_{A_0} (Z_0^{(i)}(t, y_0) - 1)(\psi_{0,t}h_{0,t}(dy_0) - 1)m_{0,t}(dy_0)dt,
\end{aligned}
$$

(15)

where,

$$
\hat{q}_i(dt, dy_i) = p_i(dt, dy_i) - \lambda_{i,t}\psi_{i,t}h_{i,t}(y_i)m_{i,t}(dy_i)dt = p_i(dt, dy_i) - \tilde{\lambda}_{i,t}\tilde{m}_{i,t}(dy_i)dt
$$

is the Poisson measure $p_i(dt, dy_i)$ compensated in the measure $\mathbb{Q}$. The application of Theorem 3.1 and the martingale condition allows us to express (15) as

$$
\begin{aligned}
\frac{dS_{i,t}}{S_{i,t-}} &= (r_t - \xi_{i,t})dt + \sigma_i d\widetilde{W}_{i,t} + \int_{A_i} (Z_i(t, y_i) - 1)\hat{q}_i(dt, dy_i) \\
&\quad + \int_{A_0} (Z_0^{(i)}(t, y_0) - 1)\hat{q}_0(dt, dy_0),
\end{aligned}
$$

(16)

if and only if there exists a correlated Brownian process $\widetilde{W}_t = (W_{1,t}, \ldots, W_{d,t})^\top$ in the measure $\mathbb{Q}$ where

$$
d\widetilde{W}_{i,t} = \theta_{i,t}dt + dW_{i,t}
$$

and

$$
\begin{aligned}
r_t + \sigma_i\theta_{i,t} + \lambda_{i,t} &\int_{A_i} [Z_i(t, y_i) - 1](1 - \psi_{i,t}h_{i,t}(y_i))m_{i,t}(dy_i) \\
&+ \lambda_{0,t} \int_{A_0} [Z_0^{(i)}(t, y_0) - 1](1 - \psi_{0,t}h_{0,t}(y_0))m_{0,t}(dy_0) \\
&= \mu_i + \xi_{i,t}
\end{aligned}
$$

(17)

for $i = 1, 2, \ldots, d$. The set of equations in (17) are the market price of risk equations for the d stocks and they relate the risk premia of the stocks to the risk premia due to the Wiener components (the second term on the left hand side of (17)), the idiosyncractic jump

component (the third term on the left hand side of (17)) and, the common jump component (the last term on the left hand side of (17)). Following Runggaldier (2003), we may interpret $(\psi_{i,t} h_{i,t}(y_i) - 1) m_{i,t}(dy_i)$ as the risk premium per unit jump volatility due to the ith jump component.

By rewriting the average jump size increment taken in the measure $\mathbb{Q}$ as

$$(18) \qquad \tilde{\kappa}_i \equiv \kappa_{i,t} = \mathbb{E}_{\mathbb{Q}}[Z_i - 1] = \int_{A_i} (Z_i(t, y_i) - 1) \tilde{m}_{i,t}(dy_i),$$

and the average relative jump size of its common jump component as

$$(19) \qquad \tilde{\kappa}_0^{(i)} \equiv \tilde{\kappa}_{0,t}^{(i)} = \mathbb{E}_{\mathbb{Q}}[Z_0^{(i)} - 1] = \int_{A_0} (Z_0^{(i)}(t, y_0) - 1) \tilde{m}_{0,t}(dy_0),$$

the market prices of risk equations (17) can be written as

$$(20) \qquad \theta_{i,t} = \frac{1}{\sigma_i} \left[\mu_i + \xi_{i,t} - r_t - (\lambda_{i,t} \kappa_{i,t} - \tilde{\lambda}_{i,t} \tilde{\kappa}_{i,t}) - (\lambda_{0,t} \kappa_{0,t}^{(i)} - \tilde{\lambda}_{0,t} \tilde{\kappa}_{0,t}^{(i)}) \right]$$

for $i = 1, 2, \ldots, d$, which express the market price of risk of each Wiener component as the risk premium of the respective stock less the jump-risks per unit volatility. From (16), the ith stock price can be expressed as

$$S_{i,t} = S_{i,0} \exp \left[\int_0^t \left(r_u - \xi_{i,u} - \frac{\sigma_i^2}{2} - \tilde{\lambda}_i \tilde{\kappa}_i - \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)} \right) du + \int_0^t \sigma_i d\widetilde{W}_{i,u} \right]$$

$$(21) \qquad \times \prod_{n=1}^{N_{i,t}} Z_i(T_{i,n}, Y_{i,n}) \prod_{k=1}^{N_{0,t}} Z_0^{(i)}(T_{0,k}, Y_{0,k}),$$

where the Poisson arrivals $N_{i,t}$ now have intensities $\tilde{\lambda}_i$ in the martingale measure $\mathbb{Q}$.

Three special cases arise depending on the choices of the parameters $\psi_{i,t}$ and $h_{i,t}$ in the Radon-Nikodým derivative (14). As discussed in Section 3, the setting of both $\psi_{i,t}$ and $h_{i,t}$ to one leaves the distribution of the jump sizes and the jump-arrival intensities unchanged under the transformation of measures and thus the jump-risks remain unpriced. This generalizes the situation considered in Merton (1976) to the multi-asset case. When all the $\psi_{i,t} = 1$ but at least one $h_{i,t} \neq 1$, then there are no changes to the jump-arrival intensities but there is a change to the distribution of at least one of the jump sizes. Lastly, when all the $h_{i,t} = 1$ but at least one $\psi_{i,t} \neq 1$, then there are no changes to the distribution of the jump sizes, but at least one of the jump-arrival intensities change.

5 An IPDE for Option Pricing

Let $X_t(S_{1,t}, S_{2,t}, \cdots, S_{d,t})$ be an option written on the d stocks, for which we adopt the short-form notation

$$X_t \equiv X_t(S_{1,t}, S_{2,t}, \cdots, S_{d,t}) \text{ and } X_{t-} \equiv X_t(S_{1,t-}, S_{2,t-}, \cdots, S_{d,t-}).$$

By the application Itô's Lemma for jump-diffusion processes to the option price X_t with (16) as the dynamics of the underlying stock prices, the dynamics of the option price are given by

$$
\begin{aligned}
dX_t &= \left[\frac{\partial X_{t-}}{\partial t} + \sum_{i=1}^d \left(r_t - \xi_{i,t} - \tilde{\lambda}_{i,t} \int_{A_i} [Z_i(t, y_i) - 1] \tilde{m}_{i,t}(dy_i) \right. \right. \\
&\qquad \left. \left. - \tilde{\lambda}_{0,t} \int_{A_0} [Z_0^{(i)}(t, y_0) - 1] \tilde{m}_{0,t}(dy_0) \right) S_{i,t-} \frac{\partial X_{t-}}{\partial S_i} + \mathcal{L} X_{t-} \right] dt \\
&\quad + \sum_{i=1}^d \sigma_i S_{i,t-} \frac{\partial X_{t-}}{\partial S_i} \widetilde{dW}_{i,t} \\
&\quad + \sum_{i=1}^d \tilde{\lambda}_{i,t} \int_{A_i} [X_t(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{t-}] \tilde{m}_{i,t}(dy_i) dt \\
&\quad + \sum_{i=1}^d \int_{A_i} [X_t(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{t-}] \hat{q}_i(dt, dy_i) \\
&\quad + \tilde{\lambda}_{0,t} \int_{A_0} [X_t(S_{1,t-} Z_0^{(1)}, \cdots, S_{d,t-} Z_0^{(d)}) - X_{t-}] \tilde{m}_{0,t}(dy_0) dt \\
&\quad + \int_{A_0} [X_t(S_{1,t-} Z_0^{(1)}, \cdots, S_{d,t-} Z_0^{(d)}) - X_{t-}] \hat{q}_0(dt, dy_0),
\end{aligned}
$$

(22)

where the operator $\mathcal{L}$ has been defined in (13). We require the discounted option price process $\left\{ X_t e^{- \int_0^t r_u du} \right\}$ to be a martingale under the martingale measure $\mathbb{Q}$. From (22), the

stochastic differential satisfied by this quantity is

$$
\begin{aligned}
d(X_t e^{-\int_0^t r_u du}) \;=\;& e^{-\int_0^t r_u du}\left[\frac{\partial X_{t-}}{\partial t} - r_t X_{t-} + \sum_{i=1}^d \left(r_t - \xi_{i,t} - \tilde{\lambda}_{i,t}\int_{A_i}[Z_i(t,y_i)-1]\tilde{m}_{i,t}(dy_i)\right.\right.\\
& \left.\left. - \tilde{\lambda}_{0,t}\int_{A_0}[Z_0^{(i)}(t,y_0)-1]\tilde{m}_{0,t}(dy_0)\right)S_{i,t-}\frac{\partial X_{t-}}{\partial S_i} + \mathcal{L}X_{t-}\right]dt\\
& + \sum_{i=1}^d \sigma_i e^{-\int_0^t r_u du} S_{i,t-}\frac{\partial X_{t-}}{\partial S_i}d\widetilde{W}_{i,t}\\
& + \sum_{i=1}^d \tilde{\lambda}_{i,t}e^{-\int_0^t r_u du}\int_{A_i}[X_t(S_{1,t-},\cdots,S_{i,t-}Z_i,\cdots,S_{d,t-}) - X_{t-}]\tilde{m}_{i,t}(dy_i)dt\\
& + \sum_{i=1}^d e^{-\int_0^t r_u du}\int_{A_i}[X_t(S_{1,t-},\cdots,S_{i,t-}Z_i,\cdots,S_{d,t-}) - X_{t-}]\hat{q}_i(dt,dy_i)\\
& + \tilde{\lambda}_{0,t}e^{-\int_0^t r_u du}\int_{A_0}[X_t(S_{1,t-}Z_0^{(1)},\cdots,S_{d,t-}Z_0^{(d)}) - X_{t-}]\tilde{m}_{0,t}(dy_0)dt\\
\text{(23)}\qquad & + e^{-\int_0^t r_u du}\int_{A_0}[X_t(S_{1,t-}Z_0^{(1)},\cdots,S_{d,t-}Z_0^{(d)}) - X_{t-}]\hat{q}_0(dt,dy_0),
\end{aligned}
$$

In (23) the coefficient of dt must be zero so that the martingale condition on the discounted option price is satisfied. Hence we obtain the integro-partial differential equation (12), subject to the terminal condition $X_T(S_{1,T},\cdots,S_{d,T})$ being the non-negative valued final payoff of the option. For example, a basket call option with strike K has the terminal payoff $X_T(S_{1,T},\cdots,S_{d,T}) = (\sum_{i=1}^d S_{i,T} - K)^+$. From the Feynmann-Kac Theorem for jump-diffusion processes, the solution for the option price in the form of a conditional expectation is

$$
\text{(24)}\qquad X_t(S_{1,t},\cdots,S_{d,t}) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}X_T(S_{1,T},\cdots,S_{d,T})\,\Big|\,\mathcal{F}_t\right].
$$

Using the Radon-Nikodým derivative (14) from Theorem 3.1 with some particular choices of $\psi_{i,t}$ and $h_{i,t}$ and the martingale condition on the choice of θ_i, the option price (24) is equivalent to

$$
\text{(25)}\qquad X_t(S_{1,t},\cdots,S_{d,t}) = \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_t\right)^{-1}\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_T e^{-\int_t^T r(u)du}X_T(S_{1,T},\cdots,S_{d,T})\,\Big|\,\mathcal{F}_t\right].
$$

6   The Hedging Portfolio

Now in order to obtain some economic intuition behind the derivation of the integro-partial differential equation (12) we here derive it by use of a hedging argument. Jarrow and Madan (1999) suggested that market completion may require dynamic trading in more than one European option. Runggaldier (2003) considered an option written on a stock following a jump-diffusion process, with a fixed jump size. He found that a hedge portfolio consisting of the underlying asset and another asset driven by the same noise process (such as a derivative) was sufficient to create a risk-free portfolio; one asset to hedge away the Wiener diffusion risk, the other to hedge away the jump-risk. Here we note the suggestion by Jarrow and Madan, and extend the approach of Runggaldier and set up a hedge portfolio consisting of d stocks and $d+2$ (European) options of different maturities [1], of which the d stocks and one option are needed to hedge away the d Wiener diffusion risks and the remaining $d+1$ options to hedge away the $d+1$ jump-risks, for a given jump-size. We use the $d+1$ options on the d stocks for the remaining $d+1$ assets in the portfolio since these options are driven by the same Wiener and Poisson noise factors as the d stocks.

Let the options be $X_{1,t}, \cdots, X_{d+2,t}$, where $X_{1,t}$ is the option in which we are originally interested and $X_{2,t}, \cdots, X_{d+2,t}$ are other options on the same d stocks that are traded in the market with different maturities. Let the corresponding maturities be $T_1, \ldots, T_{d+2}$ with each $T_i \leq T$, that is, all the options have finite maturities. Although not expressed explicitly, the price of option $X_{j,t}$ is also function of its maturity time T_j and once selected into the portfolio, option maturities do not change over time. The portfolio selected is

$$(26) \qquad \Pi_t = \sum_{i=1}^{d} H_i S_{i,t} + \sum_{j=1}^{d+2} Q_j X_{j,t},$$

where H_i are $\mathcal{F}_t$-predictable positions held in the stock, and Q_j are $\mathcal{F}_t$-predictable positions held in the respective options. In the interests of brevity, we drop the subscripts t in all the parameters, asset holding positions and prices where there is no confusion, although these

---

[1]Technically, we could have also chosen $d+2$ options of the same class on the same d stocks with the same maturity but with different strikes.

quantities are always functions of time and time-varying.

The dynamics of the option price $X_{j,t}$ obtained after application of Ito's lemma for jump-diffusion processes may be conveniently written (similar in form to (10)) as

$$\frac{dX_{j,t}}{X_{j,t}} = \left(\mu_{X_j} - \sum_{i=1}^{d} \lambda_i \kappa_{X_j}^{(i)} - \lambda_0 \kappa_{X_j}^{(0)} \right) dt + \sum_{i=1}^{d} \sigma_{X_j}^{(i)} dW_{i,t}$$

(27)
$$+ \sum_{i=1}^{d} (Z_{X_j}^{(i)} - 1) dN_{i,t} + (Z_{X,j}^{(0)} - 1) dN_{0,t},$$

where the drift of the option is defined as

$$\mu_{X_j} = \frac{1}{X_{j,t-}} \left[\frac{\partial X_{j,t-}}{\partial t} + \sum_{i=1}^{d} (\mu_i - \lambda_i \kappa_i - \lambda_0 \kappa_0^{(i)}) S_{i,t-} \frac{\partial X_{j,t-}}{\partial S_i} \right.$$

$$\left. + \mathcal{L} X_{j,t-} + \sum_{i=1}^{d} \lambda_i \kappa_{X_j}^{(i)} X_{j,t-} + \lambda_0 \kappa_{X_j}^{(0)} X_{j,t-} \right],$$

with the option price increments due to the ith idiosyncratic jump component given by

$$(Z_{X_j}^{(i)} - 1) X_{j,t-} dN_{i,t} \equiv \int_{A_i} [X_{j,t}(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{j,t-}] p_i(dt, dy_i)$$

$$\equiv [X_{j,t}(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{j,t-}] dN_{i,t},$$

and the expected option price increment due to the ith idiosyncratic jump component is

$$X_{j,t-} \kappa_{X_j}^{(i)} \equiv \mathbb{E}_{\mathbb{P}} [X_{j,t}(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{j,t-}]$$

$$\equiv \int_{A_i} [X_{j,t}(S_{1,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{j,t-}] m_{i,t}(dy_i).$$

Similarly, the option price increment due to the common jump component is given by

$$(Z_{X_j}^{(0)} - 1) X_{j,t-} dN_{0,t} \equiv \int_{A_0} [X_{j,t}(S_{1,t-} Z_0^{(1)}, \cdots, S_{i,t-} Z_0^{(i)}, \cdots, S_{d,t-} Z_d^{(0)}) - X_{j,t-}] m_{0,t}(dy_0)$$

$$\equiv [X_{j,t}(S_{1,t-} Z_0^{(1)}, \cdots, S_{i,t-} Z_0^{(i)}, \cdots, S_{d,t-} Z_d^{(0)}) - X_{j,t-}] dN_{0,t},$$

and the expected option price increment due to the common jump component is

$$X_{j,t-} \kappa_{X_j}^{(0)} \equiv \mathbb{E}_{\mathbb{P}} \left[X_{j,t}(S_{1,t-} Z_0^{(1)}, \cdots, S_{i,t-} Z_0^{(i)}, \cdots, S_{d,t-} Z_d^{(0)}) - X_{j,t-} \right]$$

$$\equiv \int_{A_0} [X_{j,t}(S_{1,t-} Z_0^{(1)}, \cdots, S_{i,t-} Z_0^{(i)}, \cdots, S_{d,t-} Z_d^{(0)}) - X_{j,t-}] m_{0,t}(dy_0).$$

Finally the volatility of the option with respect to the ith Wiener component is

$$\sigma_{X_j}^{(i)} = \frac{\sigma_i S_{i,t-}}{X_{j,t-}} \frac{\partial X_{j,t-}}{\partial S_i}.$$

The infinitesimal change in the portfolio value $d\Pi_t$ over the time interval $[t, t+dt)$ evolves according to

$$(28) \qquad d\Pi_t = \sum_{i=1}^{d} H_i dS_{i,t} + \sum_{i=1}^{d} H_i S_{i,t-} \xi_i dt + \sum_{j=1}^{d+2} Q_j dX_j.$$

That is, it is the sum of the weighted change in the stock prices, the weighted sum of the dividends accrued, and the sum of the weighted change in the option prices. By (10) and (27), the change in the value of the portfolio $d\Pi_t$ is thus given by

$$
\begin{aligned}
d\Pi_t \;=\; & \left[\sum_{i=1}^{d} H_i (\mu_i + \xi_i - \lambda_i \kappa_i - \lambda_0 \kappa_0^{(i)}) S_{i,t-} \right. \\
& \left. + \sum_{j=1}^{d+2} Q_j X_{j,t-} \left(\mu_{X_j} - \sum_{i=1}^{d} \lambda_i \kappa_{X_j}^{(i)} - \lambda_0 \kappa_{X_j}^{(0)} \right) \right] dt \\
& + \sum_{i=1}^{d} \left[H_i S_{i,t-} \sigma_i + \sum_{j=1}^{d+2} Q_j \sigma_{X_j}^{(i)} X_{j,t-} \right] dW_{i,t} \\
& + \sum_{i=1}^{d} \left[H_i S_{i,t-} (Z_i - 1) + \sum_{j=1}^{d+2} Q_j X_{j,t-} \sum_{i=1}^{d} (Z_{X_j}^{(i)} - 1) \right] dN_{i,t} \\
(29) \qquad & + \left[\sum_{i=1}^{d} H_i S_{i,t-} (Z_0^{(i)} - 1) + \sum_{j=1}^{d+2} Q_j X_{j,t-} (Z_{X_j}^{(0)} - 1) \right] dN_{0,t}.
\end{aligned}
$$

The portfolio (29) is riskless over $[t, t+dt)$ when the weights satisfy the conditions

$$(30) \qquad H_i S_{i,t-} \sigma_i + \sum_{j=1}^{d+2} Q_j \sigma_{X_j}^{(i)} X_{j,t-} = 0, \qquad \text{for } i = 1, \cdots, d,$$

and

$$(31) \qquad H_i S_{i,t-} (Z_i - 1) + \sum_{j=1}^{d+2} Q_j X_{j,t-} (Z_{X_j}^{(i)} - 1) = 0, \qquad \text{for } i = 1, \cdots, d,$$

as well as

$$(32) \qquad \sum_{i=1}^{d} H_i S_{i,t-} (Z_0^{(i)} - 1) + \sum_{j=1}^{d+2} Q_j X_{j,t-} (Z_{X_j}^{(0)} - 1) = 0,$$

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for some particular values of the jump-sizes $Z_1, \ldots, Z_d$ and $Z_0^{(1)}, \ldots, Z_0^{(d)}$. The change in the value of portfolio (29), now riskless over $[t, t + dt)$, becomes

$$d\Pi_t = \left[\sum_{i=1}^{d} H_i(\mu_i + \xi_i - \lambda_i \kappa_i - \lambda_0 \kappa_0^{(i)}) S_{i,t-} \right.$$

(33)
$$\left. + \sum_{j=1}^{d+2} Q_j (\mu_{X_j} - \sum_{i=1}^{d} \lambda_i \kappa_{X_j}^{(i)} - \lambda_0 \kappa_{X_j}^{(0)}) X_{j,t-} \right] dt,$$

where the weights H_i and Q_j satisfy (30), (31) and (32).

At this point, it is already possible to compute the weights of the portfolio that would hedge it best given the particular jump-sizes $Z_1, \ldots, Z_d$ and $Z_0^{(1)}, \ldots, Z_0^{(d)}$. Consider the ratios of the pre-jump dollar values of the options $Q_j X_{j,t-}$ to the dollar value of the option $Q_1 X_{1,t-}$ and string them out along a vector

(34)
$$q = \begin{pmatrix} \frac{Q_2 X_{2,t-}}{Q_1 X_{1,t-}} \\ \frac{Q_3 X_{3,t-}}{Q_1 X_{1,t-}} \\ \vdots \\ \frac{Q_{d+2} X_{d+2,t-}}{Q_1 X_{1,t-}} \end{pmatrix}.$$

Similarly consider a vector z where the entries are

(35)
$$z = \begin{pmatrix} (Z_{X_1}^{(0)} - 1) - \sum_{i=1}^{d} \frac{\sigma_{X_1}^{(i)}}{\sigma_i} (Z_0^{(i)} - 1) \\ (Z_{X_1}^{(1)} - 1) - \frac{\sigma_{X_1}^{(1)}}{\sigma_1} (Z_1 - 1) \\ \vdots \\ (Z_{X_1}^{(d)} - 1) - \frac{\sigma_{X_1}^{(d)}}{\sigma_d} (Z_d - 1) \end{pmatrix},$$

and let the matrix A be a $(d+1) \times (d+1)$ matrix with entries

$$a_{0,j} = (Z_{X_j}^{(0)} - 1) - \sum_{i=1}^{d} \frac{\sigma_{X_j}^{(i)}}{\sigma_i} (Z_0^{(i)} - 1)$$

and

$$a_{i,j} = \frac{\sigma_{X_j}^{(i)}}{\sigma_i} (Z_i - 1) - (Z_{X_j}^{(i)} - 1),$$

for $i = 1, 2, \ldots, d$ and $j = 2, 3, \ldots, d + 2$. From (30), (31) and (32), it can be shown (see Appendix) that the ratios of the pre-jump dollar values of the options $Q_j X_{j,t-}$ to the dollar

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value of the option $Q_1 X_{1,t-}$ form a linear system

$$\text{(36)} \qquad\qquad \boldsymbol{Aq} = \boldsymbol{z}.$$

Since no two options are identical, no two columns of the matrix $\boldsymbol{A}$ are linearly dependent. The linear system (36) is invertible Lebesgue-almost-everywhere t (from Proposition A.1 in the Appendix), so that a unique solution for the vector $\boldsymbol{q}$ exists. Thus the solution for the option dollar amount ratios, that is, for the entries of $\boldsymbol{q}$ must be of the form

$$\text{(37)} \quad \frac{Q_j X_{j,t-}}{Q_1 X_{1,t-}} = \zeta_{j,0} \left[(Z_{X_1}^{(0)} - 1) - \sum_{i=1}^{d} \frac{\sigma_{X_1}^{(i)}}{\sigma_i} (Z_0^{(i)} - 1) \right] + \sum_{i=1}^{d} \zeta_{j,i} \left[(Z_{X_1}^{(i)} - 1) - \frac{\sigma_{X_1}^{(i)}}{\sigma_i} (Z_i - 1) \right],$$

for $j = 2, 3, \ldots, d + 2$, where the weights $\zeta_{j,i}$ are some polynomial functions of the entries in the matrix $\boldsymbol{A}$. Since option X_1 is the option of interest, we can always set $Q_1 = 1$ and compute the other weights Q_j from the solution to $\boldsymbol{q}$ in (37).

On the other hand, since the portfolio Π_t is now riskless over $[t, t + dt)$, it must grow at the risk-free rate to avoid arbitrage opportunities, so that

$$
\begin{aligned}
\text{(38)} \qquad d\Pi_t &= r_t \Pi_{t-} dt, \\
&= r_t \left[\sum_{i=1}^{d} H_i S_{i,t-} + \sum_{j=1}^{d+2} Q_j X_{j,t-} \right] dt.
\end{aligned}
$$

Equating (33) with (38), one obtains

$$
\begin{aligned}
\text{(39)} \qquad & \sum_{i=1}^{d} H_i S_{i,t-} (\mu_i + \xi_i - r_t - \lambda_i \kappa_i - \lambda_0 \kappa_0^{(i)}) \\
& + \sum_{j=1}^{d+2} Q_j X_{j,t-} \left(\mu_{X_j} - r_t - \sum_{i=1}^{d} \lambda_i \kappa_{X_j}^{(i)} - \lambda_0 \kappa_{X_j}^{(0)} \right) = 0.
\end{aligned}
$$

Recall the expression for the market price of Wiener risk $\theta_{i,t}$ in (20) for the Wiener components in Section 4, which is obtained after a choice of $\psi_{i,t}$ and $h_{i,t}$ is made in the Radon-Nikodým derivative (14). Recall also that the choice of $\psi_{i,t}$ and $h_{i,t}$ determine the martingale measure $\mathbb{Q}$ which is used in the martingale approach in the derivation of the integro-partial differential equation (12) in Section 2. The market price of Wiener risk $\theta_{i,t}$ in (20) depends

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only on the choice of $\psi_{i,t}$ and $h_{i,t}$, and then the integro-partial differential equation (12) is derived using the martingale approach. In what follows in the rest of this section, we complete the derivation of the integro-partial differential equation (12) using the hedging approach. Other than the use of the common market price of Wiener risk $\theta_{i,t}$, we stress that the derivation of the integro-partial differential equation (12) using the hedging approach is completely independent of the derivation using the martingale approach.

Multiply (30) by the market price of Wiener risk (20) and sum over the d stocks to obtain

$$
\sum_{i=1}^{d} H_i S_{i,t-} \left[\mu_i + \xi_i - r_t - (\lambda_i \kappa_i - \tilde{\lambda}_i \tilde{\kappa}_i) - (\lambda_0 \kappa_0^{(i)} - \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)}) \right]
$$
$$
+ \sum_{j=1}^{d+2} Q_j X_{j,t-} \sum_{i=1}^{d} \frac{\sigma_{X_j}^{(i)}}{\sigma_i} \left[\mu_i + \xi_i - r_t - (\lambda_i \kappa_i - \tilde{\lambda}_i \tilde{\kappa}_i) - (\lambda_0 \kappa_0^{(i)} - \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)}) \right]
$$
(40) $\qquad = \ 0.$

Also multiply (31) by $\lambda_i \varphi_i$ (for some positive and $\mathcal{F}_t$-predictable φ_i) and sum over the d stocks to obtain

$$
(41) \qquad \sum_{i=1}^{d} H_i S_{i,t-} \lambda_i (Z_i - 1)\varphi_i + \sum_{j=1}^{d+2} Q_j X_{j,t-} \sum_{i=1}^{d} \lambda_i (Z_{X_j}^{(i)} - 1)\varphi_i = 0.
$$

Similarly multiply (32) by $\lambda_0 \varphi_0$ (for some positive and $\mathcal{F}_t$-predictable φ_0) to obtain

$$
(42) \qquad \sum_{i=1}^{d} H_i S_{i,t-} \lambda_0 (Z_0^{(i)} - 1)\varphi_0 + \sum_{j=1}^{d+2} Q_j X_{j,t-} \lambda_0 (Z_{X_j}^{(0)} - 1)\varphi_0 = 0.
$$

The terms φ_i (for $i = 0, \ldots, d$) in (41) and (42) are chosen so that $\lambda_i \varphi_i$ (for $i = 0, \ldots, d$) decomposes as

$$
\lambda_i \varphi_i = \lambda_i \int_{A_i} \varphi_i(y_i) m_{i,t}(dy_i) \frac{\varphi_i(y_i)}{\int_{A_i} \varphi_i(y_i) m_{i,t}(dy_i)},
$$

where $\int_{A_i} \varphi_i(y_i) m_{i,t}(dy_i)$ is set equal to the particular $\psi_{i,t}$ and $\frac{\varphi_i(y_i)}{\int_{A_i} \varphi_i(y_i) m_{i,t}(dy_i)}$ equal to the particular $h_{i,t}(y_i)$ in the Radon-Nikodým derivative (14) in Theorem 3.1 that in turn induces the market price of Wiener risk $\theta_{i,t}$ in (20).

Rewrite (41) and (42) respectively as

$$
(43) \qquad \sum_{i=1}^{d} H_i S_{i,t-} \lambda_i \psi_{i,t} (Z_i - 1) h_{i,t}(y_i) + \sum_{j=1}^{d+2} Q_j X_{j,t-} \sum_{i=1}^{d} \lambda_i \psi_{i,t} (Z_{X_j}^{(i)} - 1) h_{i,t}(y_i) = 0
$$

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and

$$(44) \qquad \sum_{i=1}^{d} H_i S_{i,t-} \lambda_0 \psi_{0,t}(Z_0^{(i)} - 1) h_{0,t}(y_0) + \sum_{j=1}^{d+2} Q_j X_{j,t-} \lambda_0 \psi_{0,t}(Z_{X_j}^{(0)} - 1) h_{0,t}(y_0) = 0.$$

Now we integrate both sides of (43) over the regions $A_1 \times \cdots \times A_d$ with respect to the measures $m_{1,t}(dy_1) \cdots m_{d,t}(dy_d)$. This is analogous to taking expectations of both sides of (43) under the measure $\mathbb{P}$ restricted to the jump-sizes only, since the jump-sizes Z_i and $Z_{X_j}^{(i)}$ are by definition functions of the marks y_i. Define the expected option price increment due to the ith idiosyncratic jump component under the measure $\mathbb{Q}$, now the same equivalent martingale measure $\mathbb{Q}$ as used in the martingale approach in Section 5, by

$$\begin{aligned} X_{j,t-} \tilde{\kappa}_{X_j}^{(i)} &\equiv \mathbb{E}_{\mathbb{Q}} \left[X_{j,t}(S_{i,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{j,t-} \right] \\ &\equiv \int_{A_i} [X_t(S_{i,t-}, \cdots, S_{i,t-} Z_i, \cdots, S_{d,t-}) - X_{j,t-}] h_{i,t}(y_i) m_{i,t}(dy_i) \end{aligned}$$

and the expected option price increment due to the common jump component under the measure $\mathbb{Q}$ by

$$\begin{aligned} X_{j,t-} \tilde{\kappa}_{X_j}^{(0)} &\equiv \mathbb{E}_{\mathbb{Q}} \left[X_{j,t}(S_{i,t-} Z_0^{(1)}, \cdots, S_{i,t-} Z_0^{(i)}, \cdots, S_{d,t-} Z_d^{(0)}) - X_{j,t-} \right] \\ &\equiv \int_{A_0} [X_{j,t}(S_{i,t-} Z_0^{(1)}, \cdots, S_{i,t-} Z_0^{(i)}, \cdots, S_{d,t-} Z_d^{(0)}) - X_{j,t-}] h_{0,t}(y_0) m_{0,t}(dy_0). \end{aligned}$$

Hence (43) becomes

$$(45) \qquad \sum_{i=1}^{d} H_i S_{i,t-} \tilde{\lambda}_i \tilde{\kappa}_i + \sum_{j=1}^{d+2} Q_j X_{j,t-} \sum_{i=1}^{d} \tilde{\lambda}_i \tilde{\kappa}_{X_j}^{(i)} = 0,$$

by recalling the definitions of $\tilde{\lambda}_i$, $\tilde{\kappa}_i$ and $\tilde{\kappa}_{X_j}^{(i)}$. Similarly integrate both sides of (44) over the region A_0 with respect to the measures $m_{0,t}(dy_0)$ and recall the definitions of $\tilde{\lambda}_0$, $\tilde{\kappa}_0$ and $\tilde{\kappa}_{X_j}^{(0)}$ to obtain

$$(46) \qquad \sum_{i=1}^{d} H_i S_{i,t-} \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)} + \sum_{j=1}^{d+2} Q_j X_{j,t-} \tilde{\lambda}_0 \tilde{\kappa}_{X_j}^{(0)} = 0.$$

Finally subtract (40) from the sum of (39), (45) and (46) to obtain

$$(47) \qquad \begin{aligned} \sum_{j=1}^{d+2} Q_j X_{j,t-} &\left[\mu_{X_j} - r_t - \sum_{i=1}^{d} (\lambda_i \kappa_{X_j}^{(i)} - \tilde{\lambda}_i \tilde{\kappa}_{X_j}^{(0)}) - (\lambda_0 \kappa_{X_j}^{(0)} - \tilde{\lambda}_0 \tilde{\kappa}_{X_j}^{(0)}) \right. \\ &\left. - \sum_{i=1}^{d} \frac{\sigma_{X_j}^{(i)}}{\sigma_i} (\mu_i + \xi_i - r_t - \lambda_i \kappa_i + \tilde{\lambda}_i \tilde{\kappa}_i - \lambda_0 \kappa_0^{(i)} - \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)}) \right] = 0. \end{aligned}$$

In (47), the terms $Q_j X_{j,t-}$ are pre-jump dollar amounts of the options and can never be identically zero simultaneously (unless all the options have expired). Thus it follows that

$$
\mu_{X_j} - r_t - \sum_{i=1}^{d}(\lambda_i \kappa_{X_j}^{(i)} - \tilde{\lambda}_i \tilde{\kappa}_{X_j}^{(0)}) - (\lambda_0 \kappa_{X_j}^{(0)} - \tilde{\lambda}_0 \tilde{\kappa}_{X_j}^{(0)})
$$

(48)
$$
= \sum_{i=1}^{d} \frac{\sigma_{X_j}^{(i)}}{\sigma_i}(\mu_i + \xi_i - r_t - \lambda_i \kappa_i + \tilde{\lambda}_i \tilde{\kappa}_i - \lambda_0 \kappa_0^{(i)} - \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)}),
$$

for options $X_1, \ldots, X_{d+2}$. Since (48) holds for any of the options X_j whose maturities were arbitrarily chosen, it holds for any option X written on the same d stocks $S_1, \ldots, S_d$, that is

$$
\mu_X - r_t - \sum_{i=1}^{d}(\lambda_i \kappa_X^{(i)} - \tilde{\lambda}_i \tilde{\kappa}_X^{(0)}) - (\lambda_0 \kappa_X^{(0)} - \tilde{\lambda}_0 \tilde{\kappa}_X^{(0)})
$$

(49)
$$
= \sum_{i=1}^{d} \frac{\sigma_X^{(i)}}{\sigma_i}(\mu_i + \xi_i - r_t - \lambda_i \kappa_i + \tilde{\lambda}_i \tilde{\kappa}_i - \lambda_0 \kappa_0^{(i)} - \tilde{\lambda}_0 \tilde{\kappa}_0^{(i)}),
$$

where the risk premium of the option less jump-risk is expressed as the weighted sum of the market prices of risk of the Wiener components. Multiplying (49) by the pre-jump option price X_{t-} and the substitution of the equivalent expressions for all the drift, volatility and expected jump-size increment terms yields again the integro-partial differential equation (12).

In Section 5, the integro-partial differential equation for the option price is obtained after selecting a martingale measure based on the choice of the parameters in the Radon-Nikodým derivative in Theorem 3.1. In this section, the choice of our equivalent martingale measure corresponds to a perfect hedge only when the marks $y_0, y_1, \cdots, y_d$ take on some particular values that in turn induce particular values of the jump-sizes $Z_1, \ldots, Z_d$ and $Z_0^{(1)}, \ldots, Z_0^{(d)}$ that makes the portfolio (29) riskless over $[t, t + dt)$. Thus the distribution of the jump components (induced by the Poisson counting measures $p_i(dt, dy_i)$) may change, and we have seen that the market prices of jump-risk due to each jump component are present in our model in contrast to Merton's (1976) hedge portfolio with one option only.

7 Conclusion

In this paper we have extended the hedging argument in Merton (1976) to multi-asset markets where asset prices exhibit jump-diffusion dynamics, allowing in particular for both asset specific jumps as well as economy wide jumps. We first derive an integro-partial differential equation for the option price using the usual martingale conditions after the parameters of the Radon-Nikodým derivative have been selected, and the local characteristics of the Poisson measures driving the asset price return dynamics change accordingly under the measure transformation. We then derive an integro-partial differential equation for the option price by including more than one option written on the underlying assets in a hedge portfolio. The derivation of this integro-partial differential equation takes the market prices of jump-risks into account, hedges away the jump risks "on average" and provides financial economic interpretations for the steps in the derivation. The choice of parameters of the Radon-Nikodým derivative in the first derivation of the integro-partial differential equation via a martingale approach can be chosen so that we obtain the same integro-partial differential equation using the hedging argument. In doing so, we demonstrate how the market prices of jump-risks are priced in the hedge portfolio.

Our focus here has been on drawing out the relationship between the martingale approach and the hedging argument approach to option pricing under jump-diffusion dynamics. We have not derived any particular pricing formulae, though such could be derived by specifying the intensities for the Poisson arrival process and distributions for the jump sizes. The reader is referred to Cheang and Chiarella (2007) for a pricing formula for a call option, based on a single stock, that is a Poisson weighted average of a Black-Scholes type formula, but with jump-arrival intensities and jump-sizes distributions changed under the measure transformation (in contrast to Merton 1976).

Appendix

The weights of the portfolio (29), made riskless over the interval $[t, t + dt)$ for particular values of the jump-sizes $Z_1, \ldots, Z_d$ and $Z_0^{(1)}, \ldots, Z_0^{(d)}$, satisfy (30), (31) and (32). From (30), the pre-jump dollar amount of the ith stock is related to the pre-jump dollar amount of the $d + 2$ options by

$$
(50) \qquad H_i S_{i,t-} = -\sum_{j=1}^{d+2} Q_j \frac{\sigma_{X_j}^{(i)}}{\sigma_i} X_{j,t-}.
$$

Substitute (50) into (31) to get

$$
Q_1 X_{1,t-} \frac{\sigma_{X_1}^{(i)}}{\sigma_i} (Z_i - 1) + \sum_{j=2}^{d+2} Q_j X_{j,t-} \frac{\sigma_{X_j}^{(i)}}{\sigma_i} (Z_i - 1)
$$

$$
(51) \qquad = Q_1 X_{1,t-} (Z_{X_1}^{(i)} - 1) + \sum_{j=2}^{d+2} Q_j X_{j,t-} (Z_{X_j}^{(i)} - 1), \qquad \text{for } i = 1, \cdots, d.
$$

The substitution of (50) into (32) yields

$$
Q_1 X_{1,t-} (Z_{X_1}^{(0)} - 1) - \sum_{i=1}^{d} Q_1 X_{1,t-} \frac{\sigma_{X_1}^{(i)}}{\sigma_i} (Z_0^{(i)} - 1)
$$

$$
(52) \qquad = \sum_{j=2}^{d+2} Q_j X_{j,t-} \left[(Z_{X_j}^{(0)} - 1) - \sum_{i=1}^{d} \frac{\sigma_{X_j}^{(i)}}{\sigma_i} (Z_0^{(i)} - 1) \right].
$$

Divide both sides of (51) and (52) by the pre-jump dollar values $Q_1 X_{1,t-}$ of the first option. Then the ratios of the pre-jump dollar values of the options $Q_j X_{j,t-}$ to the dollar value of the option $Q_1 X_{1,t-}$ form the linear system (36).

In order to ensure the Lebesque-almost-everywhere t invertibility of the matrix $\boldsymbol{A}$ in the linear system (36), we have applied the following proposition from Björk et al. (1997), which we reproduce here for the convenience of the reader.

Proposition A.1. *Let $f_1, \ldots f_M$ be a set of real-valued functions such that*

    i. For each i, the function f_i is real-valued analytic.

    ii. The functions $f_1, \ldots f_M$ are linearly independent.

For each choice of reals $T_1, \ldots, T_M$, consider the matrix B defined by

(53)
$$B(T_1, \ldots, T_M) = \{f_i(T_j)\}_{i,j}.$$

Then, given any finite interval $[I_L, I_R]$ of a positive length, we can choose $T_1, \ldots, T_M$ in $[I_L, I_R]$ such that B is invertible. Furthermore, apart from a finite set of points, we can choose $T_1, \ldots, T_M$ arbitrarily in $[I_L, I_R]$ as long as they are distinct.

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