Modeling Structural Change in Money Demand Using a Fourier-Series Approximation

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Abstract

The paper develops a simple method that can be used to test for a time-varying intercept and to approximate its form. The test is solidly grounded in asymptotic theory and has good small-sample properties. The methodology is based on the fact that a Fourier approximation can capture the variation in any absolutely integrable function of time. As such, it is possible to use successive applications of the test to 'back-out" the form of the time-varying intercept. We illustrate the methodology using an extended example concerning the demand for money.

Keywords: Structural break, Fourier Approximation, Money Demand

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1. Introduction

Consider an economic time-series model such that:

$$y_t = \alpha_t + \beta x_t + \varepsilon_t \qquad (1)$$

where: α_t is the time-varying intercept, x_t is a vector containing exogenous explanatory variables and/or lagged values of y_t and ε_t is an *i.i.d* disturbance term that is uncorrelated with any of the series contained in x_t. The notation in (1) is designed to emphasize the fact that the intercept term is a function of time. Although it is possible to allow the value of β to be time-varying, in order to highlight the effects of structural change, we focus only on the case in which the intercept changes over time. If the functional form of α_t is known, the series can be estimated, hypotheses can be tested and conditional forecasts of the various values of $\{y_{t+j}\}$ can be made. In practice, the key problems are that the econometrician may not be sure if there is parameter instability and, if such instability exists, what form it is likely to take.

The time-series literature does address the first problem in great detail. In addition to the standard Chow (1960) test, Brown, Durbin and Evans (1975) CUSUM test and Hausman (1978) test, survey articles by Rosenberg (1973) and Chow (1984) discuss numerous tests designed to detect structural change. More recently, Andrews (1993) and Andrews and Ploberger (1994) have shown how to determine if there is a one-time change in a parameter when the change point is unknown, Hansen (1992) has considered parameter instability in regressions containing $I(1)$ variables, Lin and Terasvirta (1994) show how to test for multiple breaks, and Tan and Ashley (1999) formulated a test for frequency dependence in regression parameters.

The second problem is more difficult to address since there are many potential ways to model a changing intercept when the functional form of α_t is unknown. For example, it is possible to include dummy variables to capture seasonal effects or the influence of one or more structural breaks. Similarly, the inclusion of additional explanatory variables may capture the underlying reason for the change in the intercept. Another track is to let the data determine the functional form of α_t. For example, the local-level model described in Harvey (1989) uses the Kalman Filter to estimate α_t as an autoregressive (or unit-root) process. Similarly, the time-varying intercept may be estimated using a Markov-switching process, flexible least squares or a threshold process.

If there is little *a priori* information concerning the actual form of α_t, any estimated model likely to perform poorly since it is difficult to discriminate among alternative specifications using the standard diagnostic tools. As noted by Clements and Hendry (1998, pp. 168-9), parameter change appears in many guises and can cause significant forecast error when models are used in practice. They also go on to add that it can be difficult to distinguish model misspecification from the problem of non-constant parameters. One purpose of this paper is demonstrate how these difficulties may be alleviated by the use of a test for a time-varying intercept which may simultaneously be used as modelling strategy to 'back-out" the form of the time-varying intercept. The other is to apply the methodology to the demand for money. It is particularly interesting that the time-varying intercept suggests that money demand was never a stable function of the price level, real income and the short-term interest rate.

The test for a time-varying intercept is based on the fact that a Fourier approximation can capture the variation in any absolutely-integrable function of time. Our proposed methodology represents α_t by a Fourier approximation so that the issue becomes one of deciding which frequencies to include in the approximation. The test was first introduced by Davies (1987), who derived the asymptotic critical values necessary to determine the statistical significance of a frequency component in a regression equation. However, Davies (1987) did not apply the test to any particular data set nor did he consider the possibility of successive applications of the test. If the frequencies used in the Fourier approximation were known, the test is similar to that of Farley and Hinich (1970, 1975) who consider a model with parameter trend. It is also analogous to that of Tan and Ashley (1999) if their frequency band is restricted to a single frequency. If there is a single structural break, the test works nearly as well as the Andrews and Ploberger (AP) optimal test. In the presence of more than one break, the test can have substantially more power than the AP test.

There are many tests for parameter instability and it is not the intention of this paper merely to present the empirical properties of yet another. Instead the methodology presented here is intended to be most helpful when it is not clear how to model the time-varying intercept. The novel feature of this approach is that it uses the time-varying intercept as a modelling device to capture the form of any potential structural breaks and hence lessen the influence of model misspecification on the estimated equations. The method is similar to that of Ludlow and Enders (2000) who use Monte Carlo critical values as a means to back-out a time-varying autoregressive coefficient. By contrast, the strategy used in this paper is solidly based on asymptotic theory.

The rest of the paper is structured as follows. Section 2 describes the test procedure of using a Fourier approximation to test for time-variation in the intercept term of a regression equation. Section 3 shows that it is possible to use successive applications of the test to 'back-out" the form of the time-varying intercept. Hence, the modeling strategy is to continue to apply the test until all significant frequencies have been eliminated from the regression residuals. Since the sum of the selected Fourier terms is a universal approximator, this sum yields an approximation of the intercept. In Section 4, apply the methodology to the demand for money (as measured by M3. In essence, we back-out the form of the so-called "missing money." There is the strong suggestion that the missing money has the same form as the major stock market indices. Conclusions and limitations of our work are discussed in the final section.

2. The Davies Test as a Fourier Approximation

If α_t is an absolutely integrable function, for any desired level of accuracy, it is possible to write:[1]

$$\alpha_t = A_0 + \sum_{k=1}^{s} \left[A_k \sin \frac{2\pi k}{T} \bullet t + B_k \cos \frac{2\pi k}{T} \bullet t \right] \quad ; \quad s \leq T/2 \tag{2}$$

where: s refers to the number of frequencies contained in the process generating α_t, k represents a particular frequency and T is the number of usable observations.

The key point is that the behavior of any deterministic sequence can be readily captured by a sinusoidal function even though the sequence in question is not periodic. As such, the intercept may be represented by a deterministic time-dependent coefficient model without first specifying the nature of the nonlinearity. The nature of the approximation is such that the case of a constant (possibly zero) intercept emerges as the special case in which all values of A_k and B_k are equal to zero. Thus, instead of positing a specific model, the specification problem is transformed into one of selecting the proper frequencies to include in (2).

For any particular frequency k the issue is to obtain the critical values for the null hypothesis $A_k = B_k = 0$. If the frequency were known, it would be possible to test this null hypothesis using a standard F-statistic. One could simply construct the variables $\sin(2\pi kt/T)$ and $\cos(2\pi kt/T)$ and perform the estimation using OLS. Unfortunately, the issue is complicated by the fact that the relevant frequencies are unknown and each is a nuisance parameter present only under the alternative hypothesis. Moreover, it is not practical include all possible frequencies. However, if α_t is not constant, there will be at least one frequency present in (2). To restate the issue, if it is found that any one frequency belongs in the equation for α_t, the intercept cannot be constant. Hence, a test for the presence of a Fourier component in (2) can be used to detect and model any change in intercept term of a regression. Such changes could result from any number of factors including structural breaks, seasonality of an unknown form and/or an omitted variable from a regression equation.

Davies (1987) considers the case in which a time-series $\{y_t\}$ consists of independent normal random variables with a known constant variance.[2] Suppose that we want to estimate y_t using only a single frequency:

$$y_t = A \sin \frac{2\pi k}{T} \bullet t + B \cos \frac{2\pi k}{T} \bullet t + \varepsilon_t \qquad (3)$$

Since the test statistic for $A = B = 0$ involves a nuisance parameter k that is unidentified under the null hypothesis, it is not possible to rely on standard asymptotic theory to obtain an appropriate test statistic. Instead, if $S(k)$ is the test statistic in question, Davies uses the supremum:

$$M = sup\{S(k): L \leq k \leq U\} \qquad (4)$$

where: $[\ L, U\ ]$ is the range of possible values of k.

To use the Davies test, demean and standardize the $\{y_t\}$ sequence to have a unit-variance and call the resultant sequence $\{x_t\}$. Reparameterize (3) such that:

$$E_{t-1}(x_t) = a_1\sin[\ (\ t - 0.5T - 0.5)\theta\ ] + b_1\cos[\ (\ t - 0.5T - 0.5\ )\theta\ ] \qquad (5)$$

In (5), the values of $\{x_t\}$ are zero-mean, unit-variance *i.i.d* normally distributed random variables with a period of oscillation equal to $2\pi/k$ (since $\theta = 2\pi k/T$). For the possible values of of θ in the range $[\,L, U\,]$ where $0 \le L < U \le \pi$, construct:

$$S(\boldsymbol{q}) = \left[\sum_{t=1}^{T} x_t \sin[(t - 0.5T - 0.5)\boldsymbol{q}]\right]^2 / v_1 + \left[\sum_{t=1}^{T} x_t \cos[(t - 0.5T - 0.5)\boldsymbol{q}]\right]^2 / v_2 \tag{6}$$

where: $v_1 = 0.5T - 0.5\sin(T\theta)/\sin(\theta)$ and $v_2 = 0.5T + 0.5\sin(T\theta)/\sin(\theta)$

Davies shows that:

$$prob\ [\ sup\ \{\ S(\theta): L \le \theta \le U\ \} > u\ ] \tag{7}$$

can be approximated by:[3]

$$Tu^{0.5}e^{-0.5u}(\ U - L\ )/(24\pi)^{0.5} + e^{-0.5u} \tag{8}$$

In essence, the Davies test is equivalent to estimating (3) for each possible frequency in the interval $U - L$. The frequency providing the smallest residual sum of squares is called k^* and the Fourier coefficients associated with that frequency are called A^* and B^*. However, the test of the null hypothesis $A^* = B^* = 0$ is performed using (6) instead of a traditional F-test. As an aside, notice that it is not necessary to estimate (3) to perform the test—$S(\theta)$ can be constructed for each potential value of θ and the value yielding the largest $S(\theta)$ is used to perform the test. However, it is necessary to estimate (3) to obtain the values A^* and B^*. Also note that, in principle, k and θ are continuous variables. As such, to perform the test, it is necessary to subdivide the interval $U - L$ into discrete parts.[4]

For example, suppose that we estimate (1) under the assumption of a constant value for α_t and want to determine whether there is a time-varying intercept. To perform the Davies test, standardize the regression residuals such that they have a unit variance—thus, these residuals become our $\{x_t\}$ sequence. Also suppose that we construct $S(\theta)$ as in equation (6) using all frequencies over the interval [0, 2] using steps of θ/256. Given the range of k, the values of U and L are 0.126 and 0, respectively ($U = 2*\pi*2/100 = 0.126$). Suppose that the largest calculated value of $S(\theta) = 10.0$. If there are 100 observations, equation (7) indicates that the critical values of $sup\ S(\theta)$ are 9.37, 10.88 and 12.86 at the 0.05, 0.025 and 0.01 significance levels, respectively. Hence, it would be possible to reject the null hypothesis $A^* = B^* = 0$ at the 5%, but not the 2.5% significance level. Thus, at the 5% level it is possible to maintain that the intercept term has the form:

$$\alpha_t = A_0 + A^* \sin \frac{2\pi k^*}{T} \bullet t + B^* \cos \frac{2\pi k^*}{T} \bullet t \tag{9}$$

Davies provides a small Monte Carlo experiment designed to illustrate the power of the test. Using $L = 0$ and $U = \pi$ and various values for the sample size T, Davies generated 4000 series using the data generating process:

$$y_t = \begin{cases} a + bt + \varepsilon_t & t < \theta \\ a + bt + \xi(t - \theta) + \varepsilon_t & t \geq \theta \end{cases} \tag{10}$$

Hence, the deterministic portion of the $\{y_t\}$ sequence is a linear trend with a permanent break in the intercept and slope occurring at time period θ. For each series, $sup\ S(\theta)$ was calculated using subintervals of θ equal to π/128 and π/256. Four conclusions emerged from this study of the power of the test. *First*, the power of the test increases in the sample size T. *Second*, the power of the test seems to be moderately robust to non-normality. *Third*, if the frequency is not an integer, the use of integer frequencies entails a loss of power. *Fourth*, if the frequency k is an integer, the power of the discrete form of the test exceeds that of the test using fractional frequencies.

It should be clear that the search interval for $\theta = [\, 0 \leq L < U \leq \pi]$ is at the discretion of the researcher. As can be seen from equation (8), increasing the size of $U - L$ increases the probability of any given value of u. Thus, unnecessarily expanding the size of the interval will reduce the power of the test. If we are considering a small number of structural breaks, it makes sense to use a small value of U since a structural break is a ‘low frequency’ event. Similarly, in using the Fourier approximation to capture seasonal changes in the mean, a period of exactly one year is appropriate.

It is well known that the most powerful test for a one-time change in the mean is that of Andrews and Ploberger (1994). In order to illustrate the relative power of the Davies test, we used the data generating process:

$$y_t = \alpha_t + \beta x_t + \varepsilon_t, \qquad t = 1,\ldots,60 \tag{11}$$

where x_t and $\varepsilon_t \sim N(0,1)$, â = 1 and:

$$\boldsymbol{a}_t = \begin{cases} 0, & \forall\, t \leq 40 \\ \boldsymbol{d}, & \forall\, t > 40 \end{cases} \tag{12}$$

We considered values of k in the range [*0, 1*] in order to allow for the possibility of an infrequent change in the mean. After all, a frequency greater than one is not likely to replicate a single break. Table 1 shows the power of the AP (1994) and the Davies (1987) tests for different break sizes $\boldsymbol{d}$.

Of course, if it is known that there cannot be more than a single break in the intercept, the AP test is preferable to the Davies test. However, the Davies test does perform almost as well as the optimal test for a single break. As we show in the next section, the test can have far more power then the Andrews and Ploberger test if there is more than one break. We also show that the estimated coefficients and frequency (frequencies) mimic structural breaks extremely well.

3. Using a Fourier Approximation for Modeling Structural Breaks

Figure 1 illustrates the simple fact that use of a single frequency in a Fourier approximation can approximate a wide variety of functional forms. The solid line in each of the four panels represents a sequence that we approximate using a single frequency. We let the four panels depict sharp breaks since the smooth Fourier approximation has the most difficulty in mimicking a sharp break. Consider panel *a* in which the solid line represents a one-time change in the level of a series. Although the single frequency $k = 0.1953$ (so that $\theta = 0.01226$) approximates the sequence imperfectly, the approximation $\alpha_t = 2.4 - 0.705 sin(0.01226\ t) - 1.82 cos(0.01226\ t)$ does capture the fact that the sequence increases over time. In panel *b* there are two breaks in the series. In this case, the approximation $\alpha_t = 0.642 - 0.105 sin(0.586\ t) - 0.375 cos(0.586\ t)$, so that $k = \theta T/2\pi = 0.037$, captures the overall tendency of the series to increase. The solid line in panel *c* depicts a sequence with a temporary change in the level while the solid line in panel *d* depicts a "seasonal" sequence that is low in periods 1 – 25 and 51 – 75 and high in periods 26 – 50 and 76 – 100. Again, the approximations using a single frequency do reasonably well. It is interesting that the frequency used for the approximation in panel *d* is exactly 2.0 since there are two regular changes in the level of the sequence.

The point is that all of these sequences can be approximated by a single frequency of rather low order. We performed a second Monte Carlo experiment to validate the notion that a Fourier approximation can be especially useful to mimic a sequence with multiple breaks. As such, we modified the data generating process in (12) to have a second structural break:

$$\boldsymbol{a}_t = \begin{cases} 0, & t \leq 20 \\ \boldsymbol{d}, & 20 < t \leq 40 \\ 0, & t > 40 \end{cases} \tag{13}$$

As shown in Table 2, the Davies test still possesses reasonably high power, while the AP test has much weaker power compared to its power against a one time structural break.[5]

To this point, we have considered the use of only one frequency as a test for parameter instability. However, it is possible to use successive applications of the test to 'back-out' the form of the intercept term. The solid line in Figure 2 shows a sequence that we might want to approximate. If we approximate this sequence with a single frequency (k = 1.171) , we obtain

$$\alpha_t = -2.56 + 6.38 sin(0.0736\, t) - 1.65 cos(0.0736\, t)$$

and depict this as the dashed line labeled "1 Frequency." Next, we applied the Davies test to the difference y_t - α_t and obtained a second significant frequency component. The relevant approximation is now

$$\alpha_t = -2.56 + 6.38 sin(0.0736\, t) - 1.65 cos(0.0736\, t) + 2.49 sin(0.172\, t) + 2.74 cos(0.172\, t)$$

and is depicted by the line labeled "2 Frequencies" in Figure 2. Although this procedure can be repeated until the Davies test indicates that no additional frequency components are statistically significant, it is already clear that the simple addition of a second frequency provides a marked improvement in the performance of the approximation. The contention now is that the use of this approximation in a regression model with unknown structural breaks will lead to improved inference and to a better model.

4. Structural Breaks in the Demand for Money

As discussed in a number of survey articles, including those by Goldfeld (1976) and Judd and Scadding (1982), there is a vast literature indicating a breakdown in the simple money demand relationship. As such, it seemed reasonable to apply our methodology to see if it could facilitate the modeling of a notorious problem. Consequently we obtained quarterly values of the U.S. money supply as measured by M3, seasonally adjusted real and nominal gdp, and the 3-

month treasury bill rate for the period 1959:1 – 2001:1 from the website of the Federal Reserve Bank of St. Louis (www.stls.frb.org/index.html).[6] We constructed the price level as the ratio of nominal to real gdp. As shown in Table 3, augmented Dickey-Fuller tests including a time trend in the estimating equation indicated that the logarithms of M3 (m), real gdp (y) and the price level (p) do not act as trend stationary processes. In contrast, the 3-month T-bill rate (r) shows some evidence of stationarity.

4.1 The long-run model

We then estimated the simple money demand function (with t-statistics in parentheses):

$$\underset{}{m_t} = \underset{(-0.231)}{-0.106} + \underset{(29.89)}{1.07}p_t + \underset{(18.75)}{0.962}y_t + \underset{(6.22)}{0.011}r_t \tag{14}$$

aic = -98.14, bic = -85.62

Although the price and income elasticities are statistically significant and are of the correct sign and magnitude, there are some serious problems with the regression equation. In addition to the fact that the interest rate semi-elasticity of demand is positive, the residuals are not well-behaved. For example, the autocorrelations of the residuals are quite high:

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
0.96	0.90	0.83	0.75	0.68	0.60	0.53	0.46

The impression that (14) is not a cointegrating vector is confirmed by the Engle-Granger (1987) test. For the lag lengths selected by the aic and sbc, the t-statistics for the null hypothesis that the residual series is nonstationary are –2.94 and –2.75, respectively.

Of course, a structural break or a missing variable may be one reason that the residuals appear to be nonstationary. At this point, it is not our aim to determine whether the residuals pass a test for white-noise. Instead, we want to determine the most appropriate frequency to include in our Fourier approximation of the intercept term. We used the standardized residuals $\{x_t\}$ to construct the value $S(\theta)$ shown in (6) for each frequency in the interval [0, 8].[7] Since there are 169 observations, this is equivalent to searching over θ in the interval 0 to 0.297. The frequency yielding the largest value of $S(\theta)$ is such that $k^* = 2.49$ and an associated value of $sup\ S(\theta) = 53.72$. For $L = 0$ and $U = 0.297$, this value of $S(\theta)$ is significant at the 9.389×10^{-11} level.[8] Hence, there is at least one frequency present in the regression residuals. We then used this frequency k^* to estimate a money demand function in the form:

$$m = \alpha_t + \alpha_1 p + \alpha_2 y + \alpha_3 r \tag{15}$$

where: $\alpha_t = a_0 + A_1^* \sin[(t\text{-}0.5T\text{-}0.5)\theta] + B_1^* \cos[(t\text{-}0.5T\text{-}0.5)\theta]$

Table 4 reports these values along with the value of the *aic* and *bic* for the resulting regression. The resulting residuals from this equation were again standardized and the procedure was repeated. As shown in the second row of Table 4, the new value of $sup\ S(\theta)$ is 87.96 with a $k^* = 3.45$. We re-estimated the entire money demand equation including the two frequencies in α_t. We continued to repeat the process until we found no additional statistically significant frequencies in the regression residuals. Since the sixth iteration produces a value of $sup\ S(\theta)$ that was not significant at conventional levels, we retained only the results from the first five iterations. The final estimate of the money demand relationship is:

$$\begin{aligned} m_t = \alpha_t + &\underset{(30.14)}{1.07}p_t + \underset{(22.14)}{0.947}y_t - \underset{(-8.10)}{0.007}r_t \end{aligned} \tag{16}$$

where: $\alpha_t = a_0 + \sum_{i=1}^{5} \left[A_i^* \sin(2\pi k_i t / T) + B_i^* \cos(2\pi k_i t / T) \right]$

and: $a_0 = 0.138$ with a t-statistic of 0.358 and the A_i^* and B_i^* are given in Table 4.

The final model fits the data quite well. The *aic* and *bic* (incorporating the fact that two additional coefficients plus the frequency are estimated at each new iteration) steadily decline as the number of iterations increases through iteration 5. The key point, however, is that the residuals are well-behaved. The last column of the table shows the *t*-statistic for the Engle-Granger (1987) test that the residuals from the money demand function using the frequency components through iteration *i* are nonstationary.

As in (14), the price and income elasticities are of the correct magnitude. However, the interest rate semi-elasticity of demand for money now has the correct sign with a magnitude that is 8.1 times its standard error. Figure 3 provides a visual representation of α_t. The striking impression is that the demand for money generally rose from 1959 through 1987. At this point, the demand for money suddenly declined. The decline continued thorough 1995 and then resumed its upward movement.

4.2 The error-correction model

In the presence of α_t, the four variables appear to form a cointegrating relationship; as such, there exists an error-correction representation such that *lm*, *ly*, *lp* and *r* adjust to the discrepancy from the long-run equilibrium relationship. However, unlike a traditional error-correction model, adjustment will be nonlinear since the constant in the cointegrating vector is a function of time. As such, we estimated the following error-correcting model using the residuals from (16) as the error-correction term. Consider:

$$\Delta lm_t = \underset{(-5.74)}{-0.231} ec_{t-1} + \underset{(0.000)}{A_{11}(L)}\Delta lm_{t-1} + \underset{(0.031)}{A_{12}(L)}\Delta lp_{t-1} + \underset{(0.025)}{A_{13}(L)}\Delta ly_{t-1} + \underset{(0.082)}{A_{14}(L)}\Delta r_{t-1} \qquad (17)$$

$$\Delta lp_t = \underset{(3.72)}{0.082} ec_{t-1} + \underset{(0.281)}{A_{21}(L)}\Delta lm_{t-1} + \underset{(0.000)}{A_{22}(L)}\Delta lp_{t-1} + \underset{(0.317)}{A_{23}(L)}\Delta ly_{t-1} + \underset{(0.180)}{A_{24}(L)}\Delta r_{t-1} \qquad (18)$$

$$\Delta ly_t = 0.122 ec_{t-1} + A_{31}(L)\Delta lm_{t-1} + A_{32}(L)\Delta lp_{t-1} + A_{33}(L)\Delta ly_{t-1} + A_{34}(L)\Delta r_{t-1} \qquad (19)$$

$$\begin{array}{lllll} (0.176) & (0.405) & (0.515) & (0.081) & (0.003) \end{array}$$

$$\Delta r_t = 0.187 ec_{t\text{-}1} + A_{41}(L)\Delta lm_{t\text{-}1} + A_{42}(L)\Delta lp_{t\text{-}1} + A_{43}(L)\Delta ly_{t\text{-}1} + A_{44}(L)\Delta r_{t\text{-}1} \quad (20)$$
$$\begin{array}{lllll} (0.297) & (0.285) & (0.005) & (0.009) & (0.000) \end{array}$$

where: $ec_{t\text{-}1}$ = error-correction term (as measured by the residual from (6), $A_{ij}(L)$ = third-order polynomials in the lag operator L, parenthesis contain the t-statistic for the null hypothesis that the coefficient on the error-correction term is zero or the F-statistic for the null-hypothesis that all coefficients in $A_{ij}(L) = 0$, and constant terms in the intercepts are not reported.

Note that the money supply contracts and the price level increases in response to the previous period's deviation from the long-run equilibrium. However, income and the interest rate appear to be weakly exogenous.

4.3 The restricted model

Given that the income and price elasticities of the money demand function are so close to unity, we also investigated the restricted money demand equation:

$$lmp_t = -0.492 + 0.014 r_t \quad (21)$$
$$\quad (-41.52) \quad (7.79)$$

$aic = -74.51$ $bic = -71.38$

This regression suffered the same problems as the unconstrained form of the money demand function. After applying our methodology to the constrained money demand function we obtained:

$$mp_t = \alpha(t) - 0.004 r_t \quad (22)$$
$$\quad (-4.87)$$

$aic = -599.25$ $bic = -539.77$

where: mp = the logarithm of real money balanced divided by real *gdp* (i.e., $lm3 - lp - ly$) and α_t = has the same form as (16).

The time path of α_t is virtually identical to that shown in Figure 3. The error-correction model using the constrained form of the money-demand function is:

$$\Delta lmp_t = \underset{(-6.84)}{-0.394ec_{t-1}} + \underset{(0.000)}{A_{11}(L)\Delta lmp_{t-1}} + \underset{(0.000)}{A_{12}(L)\Delta r_{t-1}} \tag{23}$$

$$\Delta lr_t = \underset{(0.341)}{1.86ec_{t-1}} + \underset{(0.208)}{A_{11}(L)\Delta lmp_{t-1}} + \underset{(0.000)}{A_{12}(L)\Delta r_{t-1}} \tag{24}$$

where: ec_{t-1} = error-correction term (as measured by the residual from (22), $A_{ij}(L)$ = third-order polynomials in the lag operator L, parenthesis contain the t-statistic for the null hypothesis that the coefficient on the error-correction term is zero or the F-statistic for the null-hypothesis that all coefficients in $A_{ij}(L) = 0$, and intercepts are not reported.

4.4 Integer Frequencies

There are instances in which it might be desirable to limit the search to integer frequencies. For example, in using highly seasonal data, it seems natural to consider a frequency with a period of 4 quarters of 12 months. Integer frequencies guarantee that the initial and terminal values of α_t with be the same and integer frequency components also have the desirable statistical property that they are orthogonal to each other.[9] When only discrete frequencies are used, Davies (1987) shows that the critical values for $sup\ S(\theta)$ implied by (8) should be modified such that $prob\ [\ sup\ \{\ S(\theta): L \leq \theta \leq U\ \} > u\ ]$ is:

$$1 - (1 - e^{-0.5u})^{0.5T(U-L)/\pi} \tag{25}$$

In order to illustrate the use of integer frequencies and to compare the approximation to that using continuous frequencies, we re-estimated the money demand function using discrete frequencies in the interval [1, 8] so that θ ranges from 0.0372 to 0.2974 in steps of 0.0372.

The results from estimating the money demand function with integer frequencies are shown in Table 5. The form is the same as that in (16) except that discrete frequencies 1, 2, 3, 5 and 6 are used in the approximation for α_t. Although the fit (as measured by the *aic* and *bic*) is not as good as that using continuous frequencies, the Engle-Granger test strongly suggests that the residuals are stationary. The time-path of α_t using discrete frequencies is shown in Figure 4. The approximation seems to work quite well--in comparing Figures 3 and 4, notice that the two approximations are quite similar.

4.5 Missing Variables

As suggested by Clements and Hendry (1998), a specification error resulting from an omitted variable can manifest itself in parameter instability. One major advantage of 'backing-out' the form of α_t is that it might help to suggest the missing variable responsible for parameter instability. In terms of our money demand analysis, the inclusion of a variable having the time profile exhibited in Figure 3 (or Figure 4) might eliminate the parameter instability. To demonstrate the point, we included a time trend in the demand for money function such that:

$$\alpha_t = a_0 + b_0 t + (a_1 + b_1 t)d_1 + (a_2 + b_2 t)d_2 \tag{26}$$

where: $d_1 = 1$ for $1982{:}2 < t \leq 1995{:}2$ and 0 otherwise

$d_2 = 1$ for $t > 1995{:}2$ and 0 otherwise

Thus, instead of using our Fourier approximation, we represent α_t by a linear trend with breaks in the intercept and slope coefficients occurring at the time periods suggested by Figure 3. The estimated money demand function is:

$$m_t = \alpha_t + 0.822 p_t + 0.609 y_t - 0.004 r_t \qquad (27)$$
$$\qquad (19.16) \quad (6.59) \quad (-4.14)$$

$$\alpha_t = 2.21 + 0.008t + (\ 1.64 - 0.014t\ ) d_1 + (-0.953 + 0.004t\ ) d_2$$
$$(3.38) \quad (6.22) \quad (21.02)\ (-22.15) \qquad (-9.82) \quad (6.28)$$

aic = -465.27 bic = -437.10

The Engle-Granger test indicates that the residuals from (27) are stationary: with four lags in the augmented form of the test, the t-statistic on the lagged level of the residuals is –4.79. As measured by the *aic* and *bic,* this form of the money demand function does not fit the data quite as well as those using the Fourier approximation. Moreover, the price and income elasticities have been shifted downward.

Although the Fourier approximations have better overall properties than (27), we used a trend-line containing two breaks for illustrative purposes only. The point is that a Fourier approximation can be used to 'back-out' the time-varying intercept. As such, the visual depiction of the time-varying intercept can be suggestive of a missing explanatory variable. Of course, in addition to a broken trend-line, there are other candidate variables. Figure 3 suggests that the large decline in wealth following Black Monday in October of 1987 might have been responsible for the decline in money demand. As stock prices recovered, the demand for M3 seemed to have resumed its upward trend.

5. Conclusion

In the paper we developed a simple method that can be used to test for a time-varying intercept and to approximate its form. The procedure is solidly grounded in asymptotic theory and has good small-sample properties. The method uses a Fourier approximation to capture any variation in the intercept term. As such, the issue becomes one of deciding which frequencies to include in the approximation. The test for a structural break works nearly as well as the Andrews and Ploberger (1994) optimal test if there is one break and can have substantially more power in the presence of multiple breaks. Perhaps, the most important point is that successive applications of the test can be used to 'back-out" the form of the time-varying intercept.

We explored the nature of the approximation using an extended example concerning the demand for M3. Using quarterly U.S. data over the 1959:1 – 2001:1 period, we confirmed the standard result that the demand for money is not a stable linear function of real income, the price level and a short-term interest rate. The incorporation of the time-varying intercept resulting from the Fourier approximation does result in a stable money demand function. Moreover, the magnitudes of the coefficients are quite plausible and all are significant at conventional levels. The form of the intercept term suggests a fairly steady growth rate in the demand for M3 until late-1987. At that point, there was a sharp and sustained drop in demand. Money demand continued to decline until mid-1995 and then resumed its upward trend. The implied error-correction model appears to be reasonable in that money and the price level (but neither income nor the interest rate) adjust to eliminate any discrepancy in money demand.

There are a number of important limitations of the methodology. First, in a regression analysis, a structural break may affect the slope coefficients as well as the intercept. Our methodology forces the effects of the structural change to manifest itself only in the intercept term. A related point is that the alternative hypothesis in the test is that the residuals are not white-noise. It is quite possible that the methodology captures any number of departures from white-noise and places them in the intercept term. Third, we have not addressed the issue of out-of-sample forecasting. Although the Fourier approximation has very good in-sample properties, it is not clear how to extend the intercept term beyond the observed data. Our preference is to use an average of the last few values of α_t for out-of-sample forecasts. However, there are a number of other possibilities that are equally plausible. Anyone who has read the paper to this point can certainly add to the list of limitations. Nevertheless, we believe that the methodology explored in this paper can be useful for modeling in the presence of structural change.

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Table 1: Power of the Andrews-Ploberger and Davies Tests with One Break

Andrews	d **= 0**	d **= 0.5**	d **= 1**
1%	0.008	0.115	0.652
5%	0.043	0.274	0.825
10%	0.094	0.399	0.896
Davies	d **= 0**	d **= 0.5**	d **= 1**
1%	0.007	0.105	0.585
5%	0.047	0.290	0.794
10%	0.096	0.409	0.891
Table 1: Reports size (d = 0) and power statistics for Andrews and Davies test applied to the process in (11 and 12). Significance evaluated by means of bootstrap.			

Table 2: Power of the Andrews-Ploberger and Davies Tests with Two Breaks

Andrews	d **= 0**	d **= 0.5**	d **= 1**
1%	0.008	0.026	0.103
5%	0.043	0.103	0.294
10%	0.094	0.185	0.443
Davies	d **= 0**	d **= 0.5**	d **= 1**
1%	0.007	0.074	0.444
5%	0.047	0.213	0.671
10%	0.096	0.335	0.772
Table 2: Reports size (d = 0) and power statistics for Andrews and Davies test applied to the process in (13). Significance evaluated by means of bootstrap.			

Table 3: Results of the Dickey-Fuller Tests

Variable	Lags	t-statistic
Δm	1	-1.37
Δy	1	0.41
Δp	3	-1.85
Δr	5	-2.88

Table 4: Results of the Successive Iterations

Iteration	*sup S*(q)	k_i*	*aic*	*bic*	A_i*	B_i*	*t*
1	53.72	2.49	-169.7	-147.8	-0.025	-0.035	-3.38
2	87.96	3.45	-303.1	-271.8	0.016	-0.032	-4.33
3	45.59	1.08	-507.8	-467.1	0.047	-0.064	-5.32
4	50.57	4.69	-595.7	-545.6	-0.012	-0.016	-6.33
5	16.35	5.95	-617.6	-558.2	0.008	0.002	-6.43
6	5.32	*na*	*na*	*na*	*na*	*na*	*na*

NOTE: The critical values for *sup* $S(\theta)$ are 10.58, 12.09, 13.59 and 15.55 at the 10%, 5%, 2.5% and 1% significance levels, respectively.

Table 5: The Approximation with Discrete Frequencies

Iteration	*sup S*(q)	k_i*	*aic*	*bic*	A_i*	B_i*	*t*
1	41.72	3	-146.2	-124.2	0.020	0.023	-3.76
2	47.38	2	-210.5	-179.2	0.060	-0.002	-4.13
3	52.47	1	-485.8	-445.1	0.014	-0.087	-4.98
4	58.77	5	-599.4	-549.4	-0.021	0.004	-6.49
5	13.04	6	-612.6	-553.2	0.006	-0.001	-7.02
6	4.62	8	*na*	*na*	*na*	*na*	*na*

NOTE: The critical values for *sup* $S(\theta)$ are 8.54, 9.98, 11.39 and 13.23 at the 10%, 5%, 2.5% and 1% significance levels, respectively.

Endnotes

[1] Let the function $\alpha(t)$ have the Fourier expansion:

$$\alpha(t) = \alpha_0 + \sum_{k=1}^{\infty}\left[A_k \sin\frac{2\pi k}{T}\bullet t + B_k \cos\frac{2\pi k}{T}\bullet t \right]$$

and define $F_s(t)$ to be the sum of the Fourier coefficients:

$$F_s(t) = \sum_{k=1}^{s}\left[A_k \sin\frac{2\pi k}{T}\bullet t + B_k \cos\frac{2\pi k}{T}\bullet t \right]$$

Then, for any arbitrary positive number h, there exists a number N such that:

$| \alpha(t) - F_s(t) | \leq h$ for all $s \geq N$.

[2] If T is large, the assumption of the known variance is overly strong; the asymptotic results go through using the estimated variance.

[3] Since the approximation works extremely well, even for a sample size of 16, we use only the approximate forms of the test statistic. Also note that θ need not be chosen such that k is an integer; in fact, below we illustrate that fractional values of k can provide good approximations to changes in the conditional mean of a series.

[4] We found that a direst estimate of the parameters A^*, B^* and k^* in (3) using non-linear least squares yields very poor estimates of k^*. Instead, estimate (2) using a using the frequency obtained for $sup\ S(\theta)$. Note that a direct grid search of (3) yields the same frequency as a grid search for $sup\ S(\theta)$.

[5] The Andrews-Ploberger test is only included for illustrative purposes--it is well known that it is not the optimal test for a double break.

[6] Almost identical results to those reported below hold if we use M2 instead of M3.

[7] We used a maximum value of $k = 8$ since we wanted to consider only ‘low frequency’ changes in the intercept. Also note that we searched at intervals of 1/512. The results turn out to be similar if we use integer frequencies.

[8] If we substitute $T = 169$, $k = 8$, $U = 2*\pi*k/169 = 0.297$, $L = 0$, and $u = 53.72$ into (8), we obtain 9.389×10^{-11}.

[9] It is well known that for integer values of k, the discrete frequency components in equation (2) form an orthogonal basis.

Figure 1: Four Fourier Approximations to Changes in the Mean

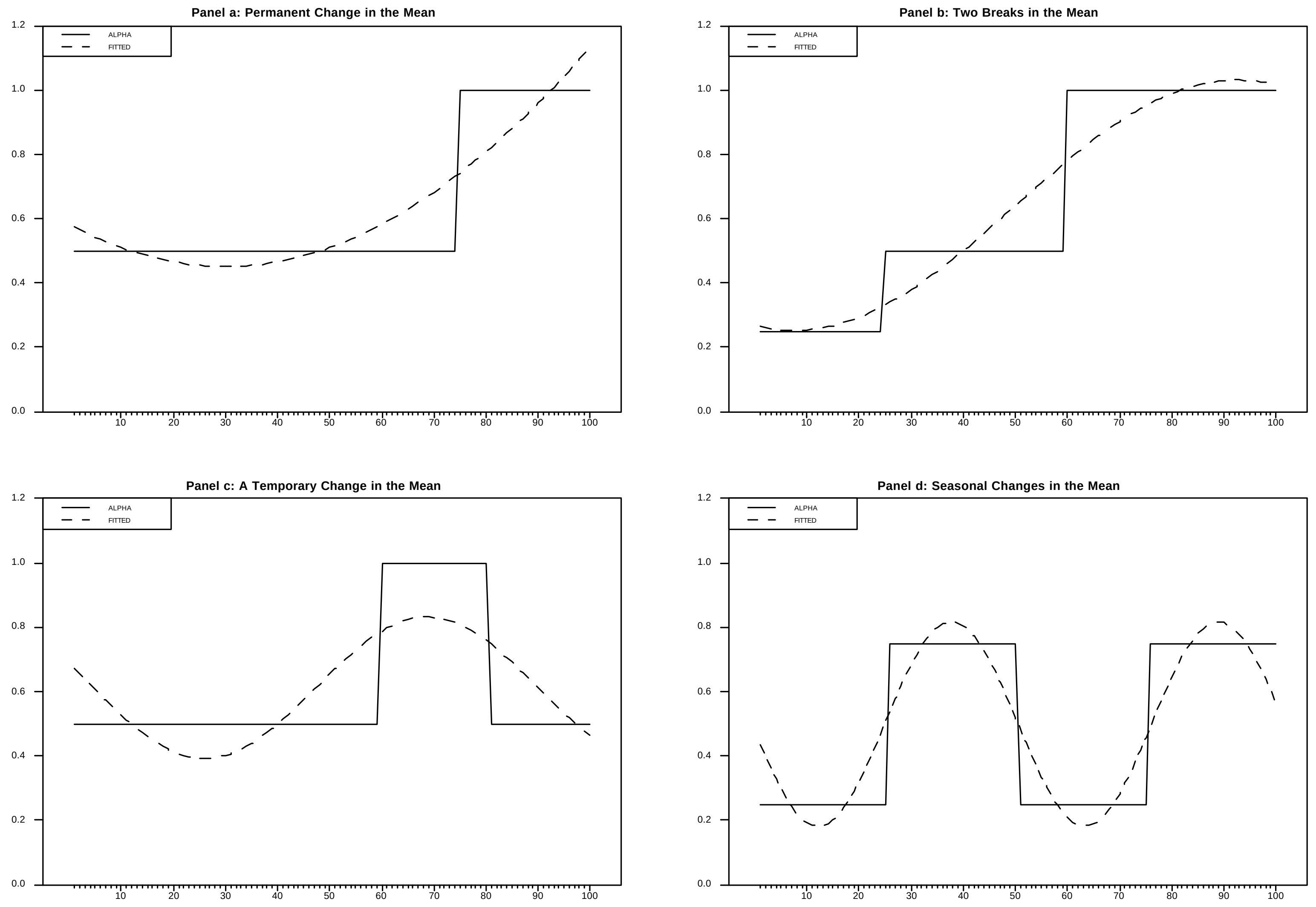

Figure 2: Increasing the Number of Freqencies

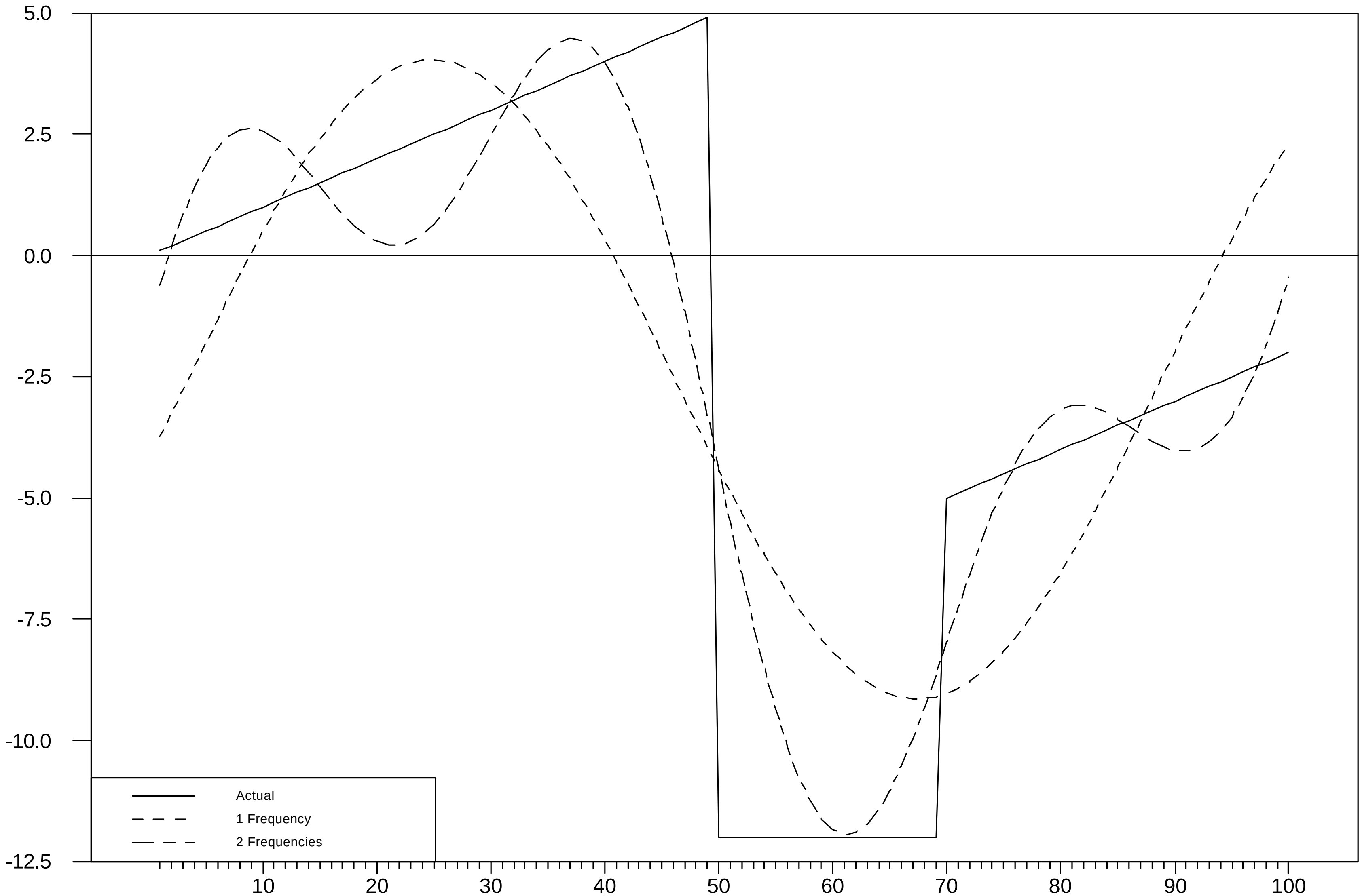

Figure 3: Intercept of the Demand for Money

(5 Frequencies)

Figure 4: Intercept of the Demand for Money

(Integer Frequencies)