ASSET PRICING, VOLATILITY AND MARKET BEHAVIOUR: A MARKET FRACTION APPROACH

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ABSTRACT. Motivated by recent development in structural agent models on asset pricing, explanation power and calibration issue of those models, this paper presents a simple market fraction model of two types of traders—fundamentalists and trend followers—under a market maker scenario. It is found that asset prices, wealth dynamics and market behaviour are characterised by the dynamics of the underlying deterministic system. The model is able to explain various market behaviour, and generate some of the stylized facts. By introducing two measures on wealth dynamics, we are able to show the limitations of profitability and rationality of different trading strategies. Six significant autocorrelation coefficients (ACs) patterns are characterized by different types of bifurcation of the underlying deterministic system. In particular, an oscillating and decaying AC pattern with positive ACs for even lags and negative for odd lags can only be generated when the market is dominated by the fundamentalists (that is when the parameters are near the flip bifurcation boundary), and a positive decaying AC patterns with long memory can only be generated when the market is dominated by the trend followers with high decay memory (that is when the parameters are near the Hopf bifurcation boundary). The results show a promising power of stability analysis and bifurcation theory in explaining and calibrating asset price and wealth dynamics, market behaviour, and generating various econometric properties of financial data.

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1. INTRODUCTION

A great deal of well established economic and finance theory is based on the assumptions of investor homogeneity and the efficient market hypothesis. However, there is a growing dissatisfaction with (i) models of asset price dynamics based on the representative agent paradigm, as expressed for example by Kirman (1992), and (ii) the extreme informational assumptions of rational expectations. As a result, the literature has seen a rapidly increasing number of structural agent models to characterise the dynamics of financial asset prices resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices in recent years (e.g. Arthur et al (1997), Brock and Hommes (1997, 2002), Brock and LeBaron (1996), Bullard and Duffy (1999), Chen and Yeh (1997, 2002), Chiarella (1992), Chiarella et al (2001), Chiarella and He (2001, 2002, 2003*b*), Dacorogna et al (1995), Day and Huang (1990), De Long et al (1990), Farmer and Joshi (2002), Frankel and Froot (1987), Gaunersdorfer (2000), Hommes (2001, 2002), Iori (2001), LeBaron (2000, 2001, 2002), LeBaron et al (1999), Lux (1995, 1997, 1998) and Lux and Marchesi (1999)). This heterogeneous agent literature can be classified as either theoretical or computational oriented in general. Most of them is computational oriented (e.g. Arthur et al (1997), Brock and LeBaron (1996), Bullard and Duffy (1999), Chen and Yeh (1997, 2002), Dacorogna et al (1995), Iori (2001), LeBaron (2000, 2001, 2002), LeBaron et al (1999), Lux (1995, 1997, 1998)), including Santa Fe artificial stock market of Arthur et al (1997), Farmer and Joshi (2002) and LeBaron (2000, 2001, 2002), and the stochastic multi-agent models of Lux (1995, 1997, 1998) and Lux and Marchesi (1999). Computational agent-based modeling is a promising way to study the stock market as a complex adaptive system of many groups of heterogeneous traders learning about the relations between prices and market information. Recent progress in computing technology has made possible a more ambitious vehicle to construct and simulate the stock market. It attacks the problem of very complex heterogeneity which leaves the boundary of what can be handled analytically and has shown that many econometric properties (stylized facts) of financial time series, including volatility clustering, excess kurtosis, bubbles and crashes, unit roots, and many others, can be replicated.

Although the computational oriented approach provides useful insight and intuition, being able to generate rich dynamics is only minor part of complex adaptive system. A disadvantage of this approach is that it is not always clear what effect exactly causes certain simulation outcomes. Among these are the question of what type of trading strategies should be used to generate certain stylized facts. As pointed out by LeBaron (2000), '*researchers...may be frustrated by the fact that results are sensitive to the actual details of trading...Validation remains a critical issue if artificial financial markets are going to prove successful in helping to explain the dynamics of real market.*' This clearly indicates the weakness of this approach.

Theoretical oriented approach approximates complicated computer models by simple nonlinear dynamical system (e.g. Brock and Hommes (1997), Chiarella (1992), Chiarella et al (2001), Chiarella and He (2001, 2002, 2003*b*), Day and Huang (1990), Gaunersdorfer (2000), Hommes (2001, 2002), 2002)). It has been widely accepted in finance and economics that simple deterministic nonlinear system can exhibit very rich dynamics, including stability and various bifurcation routes to complex and even chaotical behaviour. Recently, Brock and Hommes (1997, 1998) have proposed simple *Adaptive Belief System* to model economic and financial markets, where agents base decisions upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. A key aspect of these models is that they exhibit expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. The resulting dynamical system is nonlinear and, as Brock and Hommes (1998) show, capable of generating the entire "zoo" of complex behaviour from local stability to high order cycles and even chaos as various key parameters of the model change. Brock and Hommes's framework has been extended further in Gaunersdorfer (2000) and Chiarella and He (2001, 2002, 2003*b*) to incorporate heterogeneous variance, risk and learning under both Walrasian auctioneer and market maker scenarios. By using both bifurcation theory and numerical analysis, it is found that the relative risk attitudes, different learning mechanism and different market clearing seniors affect asset pricing dynamics in a very complicated way. It has been shown (e.g. Hommes (2002)) that such simple nonlinear adaptive models are capable of explaining important stylized facts, including fat tails, clustered volatility and long memory, of real financial series. The most important advantage of the theoretical oriented approach is that theoretical analysis of stylized simple evolutionary adaptive system and its numerical analysis may contribute in providing insight into the connection between individual and market behaviour, in particular, the important question whether asset prices in real markets are driven only by news or, at least in part, driven by market psychology.

There are two goals this literature is trying to achieve, first, to replicate the econometric properties and stylized facts of financial time series, and second to explain various market behaviour. Both theoretical and computational approaches have shown some promising results in achieving these two goals, however, based on the following discussion, we are still have some distance to achieving these goals.

It is well known that most of the stylized facts can be observed only for high frequency data, such as weekly, daily or intraday data, not for low frequency data, such as monthly and yearly data. Most of existing literatures (e.g. Arthur et al (1997), Brock and Hommes (1997), Chen and Yeh (2002), Chiarella et al (2001), Chiarella and He (2002, 2003*b*), Iori (2001), LeBaron (2002), LeBaron et al (1999), Levy et al (1994)) that are capable of generating realistic market price, market behaviour, and stylized facts uses risk-free rate of $r_f = 0.10$[1]per trading period when an optimal portfolio of risk and risk-free assets is constructed. The risk-free rate plays a crucial role for traders to optimally determine their demand on the risky asset. For model calibration, this is not calibrated to any sort of actual data and no where is trading period specified as to what the time period is. This level of risk-free rate is too high to characterize the stylized facts of high frequency financial time series and those models have limitation for further reduction of the risk-free rate to a reasonable level[2], for example, to an annual rate of $r_a = 5 - 10\%$ that leads to $r_f = r_a/250$ for daily trading period[3]. As pointed out by LeBaron (2002), *'This (unrealistic trading period) is fine for early qualitative comparisons with stylized facts, but it is a problem for quantitative calibration to actual time series'*.

Another problem is related to the explanation power of various models to financial market behaviour. As mentioned early, the theoretical oriented approach is more capable in this respect than the computational oriented approach. As an adaptive evolution system of heterogeneous agents, the asset pricing and agents' behaviour can be modeled as simple nonlinear dynamics system, whose underlying deterministic system can be studied by using stability analysis and bifurcation theory (e.g. Brock and Hommes (1997), Chiarella (1992), Chiarella et al (2001), Chiarella and He (2001, 2002, 2003*b*), Day and Huang (1990), Gaunersdorfer (2000), Hommes (2001, 2002)). Following the studies on Brock and Hommes' framework, it is very interesting to find that many important elements, such as adaption, evolution, heterogeneity, and even learning, can be incorporated into the theoretical analysis framework and many rich and complicated dynamics developed from this framework lead some insight into the understanding and explaining the market behaviour. However, because of the complicity of the dynamics and the calibration problem mentioned early, the explanation power of this framework has not been fully developed yet. In particular, the questions that how market fractions of heterogeneous traders influence the asset prices, volatility and market behaviour, and how various types bifurcation are connected to certain patterns on prices and returns, have not been studied in deep. Those are some of issues this paper is trying to address.

It is well known from both empirical (e.g. Taylor and Allen (1992)) and theoretical (e.g. Brock and Hommes (1997)) studies that market fraction plays an important role in financial markets. Empirical evidence from Taylor and Allen (1992) suggests that at least 90% of traders place some weight on technical analysis at one or more

[1]Apart from the trading 'day' risk-free rate $r_f = 0.01$ in Gaunersdorfer (2000) and LeBaron (2001), which corresponds to an unreasonable annual risk-free rate of 250%, and $r_f = 0.0004$ in Hommes (2002), which corresponds to a reasonable annual risk-free rate of 10%.

[2]As risk-free rate of trading period decreases, demand on the risky asset increases and hence stock prices increases. Consequently, the price of the risky asset become rather larger numbers resulting sometimes in break-down in theoretic analysis and overflows in numerical simulations.

[3]In this paper, we follow a convention of 250 trading days a year.

time horizons. In particular, traders rely more on technical, as opposed to the fundamentalist, analysis at shorter horizons. As the length of time horizons increase, more traders rely on the fundamentalist, rather than technical, analysis. In addition, there is certain proportion of traders do not change their strategies over all time horizons. Theoretical study from Brock and Hommes (1997) shows that, when different groups of traders having different expectations about future prices and dividends, such as fundamentalists and chartists, compete between trading strategies and choose their strategy according to an evolutionary 'fitness measure', the corresponding deterministic system exhibits coexistence of a stable steady state and a stable limit cycle. When buffeted with dynamic noise, irregular switching occurs between close to the fundamental steady state fluctuations, when the market is dominated by fundamentalists, and periodic fluctuation when the market is dominated by the chartists. The adaptive switching mechanism proposed by Brock and Hommes (1997) is one of very important elements and it is based on certain 'fitness function' and discrete choice probability. However, because of the amplifying effect of the exponential function used in the discrete choice probability, the market fractions become very sensitive to price changes and the fitness functions, leading the market fractions to be switched significantly even when the prices change insignificantly. This amplifying effect becomes even more significantly when the model is calibrated to market date (such as when a realistic risk-free rate is used for daily trading period). Therefore, it is not very clear how the market fractions do actually influence the market price and this issue is partially addressed in this paper. More realistically, we propose a market fraction model that there are some market fractions of heterogeneous traders are fixed while the rest are switching based on some fitness functions and the adaptive switching mechanism introduced in Brock and Hommes (1997). To have a clear picture how market fraction influence market price and market behaviour, we consider in this paper a simple case where all market fractions are fixed.

This paper takes the model of Chiarella and He (2003*b*) by using a market-maker, instead of Walrasian auctioneer, scenario as the mechanism generating the market clearing price and investigates a simple model for the price dynamics involving only three types of participants in the asset market: two groups of traders—fundamentalists (also called informed traders) and trend followers (also called less informed traders or chartists)—and a market-maker. Both fundamentalists and trend followers' demands are determined by maximising their expected (exponential) utility function[4] of wealth one period ahead. The fundamentalists are assumed to adjust their expected price towards the fundamental price. The trend followers are assumed to extrapolate the latest observed price change over a long-run sample mean price which follows some learning

[4]The problem of exponential utility function is that traders' optimal demand of risk asset is independent of their wealth level, even though wealth is shifting among traders. In actual markets it is clear that wealthier traders will have a greater impact on price. Therefore further extension to power or logarithm utility function so that the wealth effect can be incorporated is necessary, see Chiarella and He (2001) for related discussion.

processes, such as a geometric decay (learning) process used in this paper[5]. The market maker at the beginning of each trading period announces a price and then receives all the buy and sell orders for the risky asset in that time period formed by agents on the basis of the announced price. The market-maker hence determines the excess demand and then takes an off-setting long or short position in the risky asset so as to clear the market. The market-maker announces the price for the next trading interval as a function of the excess demand in the current period. Whilst this scenario is still highly stylised it does bring the analysis closer to the functioning of real markets than does the Walrasian scenario. As pointed out by O'Hara (1995) that there is only one market in which market clearing prices are arrived at via the Walrasian auctioneer scenario. O'Hara (1995) also highlights the inadequacy of assuming Walrasian type of market clearing mechanism. Related literature on the behavior of the market maker in securities markets can also be found in Garman (1976), Stoll (1978), Beja and Goldman (1980), Ho and Stoll (1981), Peck (1990), Day and Huang (1990), Chiarella (1992), Sethi (1996) and Farmer and Joshi (2002). The market maker in this paper still remains highly stylised in that he or she does not change behavior, irrespective of the size of his or her long or short position. We assume the market maker is risk neutral, setting the price in response to excess demands from the traders, without worrying about accumulated inventory. The market framework and price formation mechanism are similar to that of Kyle (1985) and Farmer and Joshi (2002).

By using the market fraction approach, this paper seeks to determine how market price and behaviour are determined and influenced by these three groups of traders. In particular, we examine how the market fraction, the speed of price adjustment from the market maker, the speed of the expected price adjustment from the fundamentalists towards the fundamental price, and the memory decay rate and extrapolation of the trend followers affect asset prices, volatility and market behaviour. In particular, we are trying to establish a connection between various types of bifurcation of the underlying deterministic system and various econometric properties of time series generated from the corresponding stochastic system, such as excess volatility, volatility clustering, normality of return distributions, autocorrelation patterns etc.

Another central issue is the rationality and survivability of *irrational* speculators in the market. Market psychology and investors sentiment have been viewed by new classical economists as irrational and hence are inconsistent with the rational expectation hypothesis. For example, Friedman (1953) argued that irrational speculative traders would be driven out of the market by rational traders and prices would then be driven back to fundamental prices. However, it is shown (e.g. Brock and Hommes (1997)) that this need not be the case and that simple, technical strategies may survive evolutionary competition, even in the long run. This issue is also addressed from the point of market fraction. Although traders' wealth does not enter their optimal demand, wealth

[5]Learning on variance literature.

is shifting among traders. By introducing two relative wealth proportion measures, the wealth dynamics is also examined. When the fundamentalists become more informed about the fundamental price, it is shown that the fundamentalists accumulate more significant wealth proportion in long run, compared with the trend followers. This indicates that, in a long-run, trend followers have incentive to switch to the fundamentalists. However, given that traders are bounded rational, information is not costless, and market may not always be efficient, trend followers may survive over a long run and this is demonstrated in this paper.

The plan of the paper is as follows. Section 2 outlines a market fraction model of heterogeneous agents with the market clearing price set by a market maker, introduces the expectations function and learning mechanisms of the fundamentalists and trend followers, and derives a complete market fraction model. Sections 2 and 3 examine two special cases of the complete model by assuming that there is only one type of traders in the market. It becomes clear that the dynamics and statistic properties of these two special cases have a close connection to the dynamics and time series properties of the complete model, which is examined in Section 5. In all these sections, the local stability and bifurcation of the fundamental steady state is studied analytically and numerical analysis is then conducted to examine statistical properties. Section 6 concludes and all proofs are included in the Appendix.

2. Market Fraction and a Market-Maker Model

This section sets up a standard discounted value asset pricing model with heterogeneous agents, which is closely related to the framework of Brock and Hommes (1997, 1998) and Chiarella and He (2002). However, the market clearing price is arrived at via a market maker scenario in line with Chiarella and He (2003*b*) rather than the Walrasian scenario. We focus on the simple case in which there are three classes of participants in the asset market: two groups of traders, fundamentalists and trend followers, and a market maker, which are described in detail in the following discussion.

2.1. **Market Fraction and Market Clearing Price under a Market Maker.**

Consider an asset pricing model with one risky asset and one risk free asset. It is assumed that the risk free asset is perfectly elastically supplied at gross return of $R = 1 + r/K$, where r stands for a constant risk-free rate per annual and K stands for the frequency of trading period per year. Typically, $K = 1, 12, 52$ and 250 for trading period of year, month, week and day, respectively. To calibrate the stylized facts observed from daily price movement in financial market, we select $K = 250$ in our following discussion.

Let P_t be the price (ex dividend) per share of the risky asset at time t and $\{D_t\}$ be the stochastic dividend process of the risky asset. Then the wealth of a typical investor at $t + 1$ is given by

$$W_{t+1} = RW_t + [P_{t+1} + D_{t+1} - RP_t]z_t, \tag{2.1}$$

where W_t is investor's wealth at time t and z_t is the number of shares of the risky asset purchased by the investor at t. Denote by $F_t = \{P_t, P_{t-1}, \cdots; D_t, D_{t-1}, \cdots\}$ the common information set formed at time t. We assume that, apart from the common information set, the fundamentalists have 'superior' information on the fundamental price. Let $E_{h,t}$ and $V_{h,t}$ be the "beliefs" of type h traders about the conditional expectation and variance of quantities at $t+1$. Denote by r_{t+1} and R_{t+1} the return and the excess capital gain on the risky asset, respectively, at $t+1$, that is

$$r_{t+1} = \frac{P_{t+1} + D_{t+1} - RP_t}{P_t}, \qquad R_{t+1} = P_{t+1} + D_{t+1} - R\,P_t. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$\begin{aligned} E_{h,t}(W_{t+1}) &= RW_t + E_{h,t}(R_{t+1})z_{h,t}, \\ V_{h,t}(W_{t+1}) &= z_{h,t}^2 V_{h,t}(R_{t+1}), \end{aligned} \tag{2.3}$$

where $z_{h,t}$ is the demand by agent h for the risky asset.

Assume each type, say type h, of traders is an expected utility maximizer with exponential utility function, but having different attitudes towards risk, characterized by the risk aversion coefficient, a_h. That is $U_h(W) = -exp(-a_h W)$. Then, for type h of traders, the demand $z_{h,t}$ on the risky asset is given by

$$z_{h,t} = \frac{E_{h,t}(R_{t+1})}{a_h V_{h,t}(R_{t+1})}. \tag{2.4}$$

Let N be the total number of traders, among which, there are N_1 fundamentalists, classed as type 1 traders, and N_2 trend followers, classed as type 2 traders. Then the market fraction of traders are defined by

$$n_1 = \frac{N_1}{N}, \qquad n_2 = \frac{N_2}{N}. \tag{2.5}$$

We assume that the market fraction (n_1, n_2) is fixed. Denote $m = n_1 - n_2$. Then

$$n_1 = \frac{1+m}{2}, \qquad n_2 = \frac{1-m}{2}. \tag{2.6}$$

Obviously, $m \in [-1, 1]$ and $m = 1, -1$ corresponds to the case when all the traders are fundamentalists and trend followers, respectively. Assume zero supply of outside shares. Then the excess demand $z_{e,t}$ is given by

$$z_{e,t} \equiv n_1 z_{1,t} + n_2 z_{2,t}, \tag{2.7}$$

or (using (2.3) and (2.7))

$$z_{e,t} = \frac{1+m}{2} \frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + \frac{1-m}{2} \frac{E_{2,t}[R_{t+1}]}{a_2 V_{2,t}[R_{t+1}]}. \tag{2.8}$$

To complete the model the price changes must be made explicit. The role of the market maker is to take a long (when $z_{e,t} < 0$) or short (when $z_{e,t} > 0$) position so as to clear the market. At the end of period t, after the market maker has carried out all transactions, he or she adjusts the price for the next period in the direction of the observed excess demand. Using μ to denote the corresponding speed of price adjustment for each period and $\tilde{\epsilon}_t$ be an IID normally distributed random variable that captures a random demand noise process for unexpected news about fundamentals or noise created by "noise traders" with $\tilde{\epsilon}_t \sim N(0, \sigma_\epsilon^2)$, then the price adjustment equation would be given by

$$P_{t+1} = P_t + \mu z_{e,t} + \tilde{\epsilon}_t$$

which, by using (2.8), becomes

$$P_{t+1} = P_t + \frac{\mu}{2}\left[(1+m)\frac{E_{1,t}[R_{t+1}]}{a_1 V_{1,t}[R_{t+1}]} + (1-m)\frac{E_{2,t}[R_{t+1}]}{a_1 V_{2,t}[R_{t+1}]}\right] + \tilde{\epsilon}_t. \tag{2.9}$$

It should be pointed out that the market maker behavior in this model is highly stylised. For instance, the inventory of the market maker built up as a result of the accumulation of various long and short positions is not considered. This could affect his or her behavior, e.g. the market maker price setting role in (2.9) could be a function of the inventory. Allowing μ to be a function of inventory would be one way to model such behavior. Such considerations are left to future research. Future research should also seek to explore the microfoundations of the coefficient μ. In the present paper it is best thought of as a market friction, and an aim of our analysis is to understand how this friction affects the market dynamics.

2.2. **Heterogeneous Trading Strategies.** Based on the nature of asymmetric information among traders, we now formulate two most popular trading strategies for two types of traders—fundamentalists and trend followers.

2.2.1. *Fundamentalists.* It is assumed that the fundamental traders have some 'superior' information on the *fundamental value (or price)* P_t^* of the risky asset and they also realise the existence of non-fundamental traders, such as trend followers introduced in the following discussion. For various reasons, such as the existence of non-fundamental traders, or less confident about the fundamental price, they believe that the stock price may be driven away from the fundamental price in the short run, but it will eventually converge to the fundamental value. More precisely, for the fundamental traders, their conditional mean and variance are assumed to follow

$$E_{1,t}(P_{t+1}) = P_{t+1}^* + \alpha(P_t - P_{t+1}^*), \tag{2.10}$$

$$V_{1,t}(P_{t+1}) = \sigma_1^2, \tag{2.11}$$

where P_t^* denotes the fundamental price, constant $\alpha \in [0, 1]$ measures the speed of price adjustment toward the fundamental price and, σ_1^2 stands for a constant variance on the price. In terms of parameter α, two special cases are particular interesting.

- If $\alpha = 0$, then it follows from (2.10) that

$$E_{1,t}(P_{t+1}) = P_{t+1}^*.$$

  In this case, the fundamental traders adjust their expected price at next period instantaneously to the fundamental value.

- If $\alpha = 1$, then, from (2.10),

$$E_{1,t}(P_{t+1}) = P_t,$$

which corresponds to a naive expectation—today's price is the best forecast for tomorrow's price.

In general, the fundamental traders believe that markets are efficient and prices converge to the fundamental price. An increase in α may indicate a less confidence of the fundamental traders on the convergence of the asset price to the fundamental price, leading to a slow adjustment of their expected price towards the fundamental price.

2.2.2. *Trend followers.* Apart from the fundamental traders, we assume that there is another group of traders—the trend followers who extrapolate the latest observed price change over a long-run sample mean price. More precisely, their estimates on the conditional mean and variance are assumed to follow

$$E_{2,t}(P_{t+1}) = P_t + \gamma(P_t - u_t), \tag{2.12}$$

$$V_{2,t}(P_{t+1}) = \sigma_1^2 + b_2 v_t, \tag{2.13}$$

where $\gamma, b_2 \geq 0$ are constants, and u_t and v_t are sample mean and variance, respectively, which follow some learning processes. Parameter γ measures the extrapolation rate and high values of γ correspond to strong extrapolation from the trend followers. The coefficient b_2 measures the influence of the sample variance on the conditional variance estimated by the trend followers. It is in general assumed that $b_2 > 0$, indicating a believe of the trend followers on more volatile price movement. In terms of the sample mean u_t and variance v_t, it is assumed in this paper that

$$u_t = \delta u_{t-1} + (1-\delta)P_t, \tag{2.14}$$

$$v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2. \tag{2.15}$$

This process on sample mean and variance can be treated as a limiting process of *geometric decay process* when the memory lag length tends to infinity[6]. Parameter δ measures the geometric decay rate. For $\delta = 0$, the sample mean $u_t = P_t$, which is the latest observed price, while $\delta = 0.95$ gives a half life of 2.5 weeks, while $\delta = 0.999$ gives a half life of about 2.7 years. The selection of this process is intend to capture various aspects of asset price dynamics, such as asset volatility, certain pattern of autocorrelation coefficients and long memory.

2.2.3. *Demand Functions of Fundamentalists and Trend Followers.* Regarding the dividend process D_t, we assume $D_t \sim N(\bar{D}, \sigma_D^2)$. It follows from $D_t = (r/K)P_t$ that the long-run fundamental price $\bar{P} = (K/r)\bar{D}$, where r is the annual risk-free rate.

[6]For related studies on heterogeneous learning in general and asset pricing models with heterogeneous agents who's conditional mean and variance follow various learning processes, we refer to Chiarella and He (2002, 2003*a*, 2003*b*)

Let $\sigma\bar{P}$ be the annual volatility of P_t, then the trading period variances of price and dividend can be estimated as[7]

$$\sigma_1^2 = (\bar{P}\sigma)^2/K, \qquad \sigma_D^2 = r^2\sigma_1^2. \tag{2.16}$$

Throughout the paper, we choose

$$\bar{P} = \$100, \quad r = 5\% \text{ per annual}, \quad \sigma = 20\% \text{ per annual}, \quad K = 250. \tag{2.17}$$

Correspondingly, $R = 1 + 0.05/250 = 1.0002, \sigma_1^2 = (100 \times 0.2)^2/250 = 8/5$ and $\sigma_D^2 = 1/250$.

Based on assumptions (2.10)-(2.11),

$$\begin{aligned} E_{1,t}(R_{t+1}) &= E_{1,t}[P_{t+1} + D_{t+1} - R\,P_t] \\ &= P_{t+1}^* + \alpha(P_t - P_{t+1}^*) + \bar{D} - R\,P_t \\ &= (\alpha - 1)(P_t - P_{t+1}^*) - (R-1)(P_t - \bar{P}), \\ V_{1,t}(R_{t+1}) &= (1 + r^2)\sigma_1^2 \qquad \text{using (2.16)}, \end{aligned}$$

and hence the optimal demand for the fundamentalist is given by

$$z_{1,t} = \frac{1}{a_1(1+r^2)\sigma_1^2}[(\alpha - 1)(P_t - P_{t+1}^*) - (R-1)(P_t - \bar{P})]. \tag{2.18}$$

In particular, when $P_t^* = \bar{P}$,

$$z_{1,t} = \frac{(\alpha - R)(P_t - \bar{P})}{a_1(1+r^2)\sigma_1^2}. \tag{2.19}$$

Similarly, from (2.12) and (2.13),

$$\begin{aligned} E_{2,t}(R_{t+1}) &= E_{2,t}(P_{t+1} + D_{t+1} - R\,P_t) \\ &= P_t + \gamma(P_t - u_t) + \bar{D} - R\,P_t \\ &= \gamma(P_t - u_t) - (R-1)(P_t - \bar{P}) \quad (\text{using} \quad \bar{D} = (R-1)\bar{P}), \\ V_{2,t}(R_{t+1}) &= \sigma_1^2(1 + r^2 + b\,v_t), \end{aligned}$$

where $b = b_2/\sigma_1^2$. Hence the optimal demand of the trend followers is given by

$$z_{2,t} = \frac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + r^2 + b\,v_t)}. \tag{2.20}$$

[7]Let $\bar{D}_t$ and $\bar{\sigma}_D^2$ be the annual dividend and variance. Then it follows from $\bar{D}_t = rP_t$ and $\bar{\sigma}_D^2 = r^2(\bar{P}\sigma)^2$ that $\sigma_D^2 = \bar{\sigma}_D^2/K = r^2(\bar{P}\sigma)^2/K = r^2\sigma_1^2$.

2.3. **Complete Model.** To sum up, the price dynamics under a market maker is determined by the following 3-dimensional difference system

$$\begin{cases} P_{t+1} = P_t + \mu z_{e,t} + \tilde{\epsilon}_t, \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2, \end{cases} \tag{2.21}$$

where

$$\begin{cases} z_{e,t} = \dfrac{1+m}{2} z_{1,t} + \dfrac{1-m}{2} z_{2,t}, \\ z_{1,t} = \dfrac{1}{a_1(1+r^2)\sigma_1^2}[(\alpha-1)(P_t - P_{t+1}^*) - (R-1)(P_t - \bar{P})], \\ z_{2,t} = \dfrac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2\sigma_1^2(1 + r^2 + b\, v_t)}. \end{cases}$$

Because of the nonlinear demand function $z_{2,t}$ of the trend followers, the stochastic nature of the fundamental price P_t^* and the demand shock $\tilde{\epsilon}_t$, system (2.21) is a 3-dimensionally nonlinear stochastic difference system. In terms of the fundamental prices, we assume that P_t^* follows random walk

$$P_{t+1}^* = P_t^* + \tilde{\epsilon}_t^*, \qquad \tilde{\epsilon}_t^* \sim N(0, \sigma_1^2). \tag{2.22}$$

To understand the combined effect of different types of traders on the asset price dynamics, in the following sections, two special versions of the model—the market maker model with either fundamentalists or trend followers only—are analysed first (in Sections 3 and 4) and the complete market fractions model (2.21) with both fundamentalists and trend followers is then studied in Section 5. Various aspects of the model, including asset pricing and wealth dynamics, market dominance of one type of traders over the other, market behaviour, and econometric properties of the return series, are discussed.

It has been widely accepted that stability and bifurcation theory is a powerful tool in the study of asset-pricing dynamics (see, for example, Brock and Hommes (1997, 1998) and Chiarella and He (2002, 2003*b*)). However, how the stability and various types of bifurcation of the underlying deterministic system affect the nature of the stochastic system, including stationarity, distribution and statistic properties of the stochastic system, is not very clear at the current research stage, although the techniques discussed in Arnold (1998) may be useful in this regard. In this paper, we consider first the corresponding deterministic version of various model by assuming that the fundamental price is given by its long-run value $P_t^* = \bar{P}$ and there is no demand shocks $\tilde{\epsilon}_t = 0$. It becomes clear from the following discussion that understanding of the dynamics of the underlying deterministic system, including stability and bifurcation, plays an important role on the stochastic behaviour of system (2.21). It is shown that various parameters, including market fraction m, speeds of adjustment of the fundamentalists α and the market maker μ, extrapolation rate γ and memory decay rate δ of the trend followers, play different roles on the price dynamics and market behaviour. By using numerical simulation approach, the asset price dynamics and market behaviour of the stochastic system are analysed.

2.4. **Fundamental Steady State.** When the long run fundamental price is a constant, the following result on the existence and uniqueness of steady state of the corresponding deterministic system is obtained.

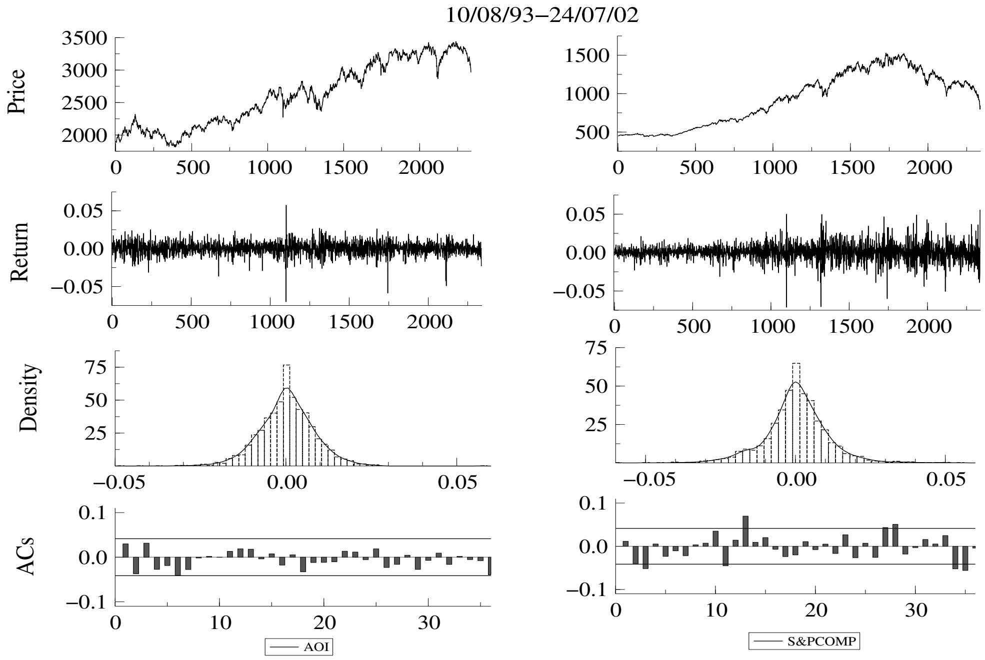
Proposition 2.1. *Assume*

$$P_t^* = \bar{P}, \qquad \sigma_\epsilon = 0. \tag{2.23}$$

Then $(P_t, u_t, v_t) = (\bar{P}, \bar{P}, 0)$ *is the unique steady state of system (2.21).*

Proof. See Appendix A.1. □

Proposition 2.1 shows that, when the fundamental price is a constant, there is a unique steady state of the system with $P_t = \bar{P}$. This steady state is therefore called *fundamental steady state*. Its stability and bifurcation plays a crucial role for understanding the dynamics of the stochastic system.



(a) (b)

FIGURE 2.1. Time series on prices and returns and density distributions and autocorrelation coefficients (ACs) of the return for S&P500 (a) and AOI (b) from Aug. 10, 1993 to July 24, 2002.

2.5. **Econometric Properties of Financial Time Series and Stylized Facts.** As a benchmark for econometric properties of financial times, we include time series plots on prices and returns for both S&P500 and AOI from Aug. 10, 1993 to July 24, 2002 in Fig.2.1. The corresponding density distributions, autocorrelation coefficients (ACs) and statistics of the returns are also illustrated in Fig.2.1 and Table 2.1.

Index	Mean	Median	Max.	Min.	Std. Dev.	Skew.	Kurt.	Jarque-Bera
S&P500	0.000194	0.0000433	0.057361	-0.070024	0.0083	-0.504638	8.215453	2746.706
AOI	0.000269	0.000106	0.055732	-0.071127	0.010613	-0.23127	7.263339	1789.96

TABLE 2.1. Statistics of returns series of for S&P500 and AOI from Aug. 10, 1993 to July 24, 2002.

A comprehensive discussion of stylized facts characterizing financial time series is given by Pagan (1996). They include excess volatility (relative to the dividends and underlying cash flows), volatility clustering, skewness, excess kurtosis, etc. Given the simplicity of the market fraction model established in this paper, we are not able to calibrate all those stylized facts. Instead, the aim of this paper is to establish a connection on price dynamics between the stochastic model and its underlying deterministic model, a relation between traders behaviour and market behaviour, and to have a theoretical understanding how different groups of traders influence the overall market behaviour. This effort is necessary, not only to replicate and calibrate financial time series, but also to understand real market.

3. A MARKET MAKER MODEL OF FUNDAMENTALISTS

This section is devoted to a special case of the complete model when $m = 1$, that is when all the traders are fundamentalists. In this case, system (2.21) is reduced to the following 1-dimensional difference system

$$P_{t+1} = P_t - \mu \frac{(R-1)(P_t - \bar{P}) + (1-\alpha)(P_t - P^*_{t+1})}{a_1(1+r^2)\sigma_1^2} + \tilde{\epsilon}_t. \tag{3.1}$$

Furthermore, under assumption (2.23), the underlying deterministic system of the stochastic system (3.1) is given by

$$P_{t+1} = P_t - \mu \frac{(R-\alpha)(P_t - \bar{P})}{a_1(1+r^2)\sigma_1^2}. \tag{3.2}$$

3.1. **Stability and Bifurcation Analysis.** For the deterministic system (3.2), the stability and bifurcation of the fundamental price $\bar{P}$ is obtained in the following Proposition 3.1.

Proposition 3.1. *Under assumption (2.23), if all the traders are fundamentalists, then the fundamental price $\bar{P}$ of (3.2) is globally asymptotically stable if and only if*

$$0 < \mu < \mu_{0,1} \equiv \frac{2a_1(1+r^2)\sigma_1^2}{(R-\alpha)}. \tag{3.3}$$

In addition, $\mu = \mu_{0,1}$ *leads to a flip bifurcation with* $\lambda = -1$*, where*

$$\lambda = 1 - \mu \frac{R - \alpha}{a_1(1 + r^2)\sigma_1^2}. \tag{3.4}$$

Proof. See Appendix A.2. □

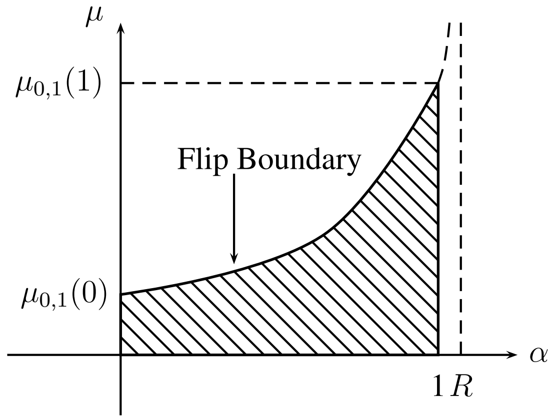


FIGURE 3.1. Stability region and bifurcation boundary for $m = 1$.

The stability region of the fundamental price $\bar{P}$ is plotted in (α, μ) plane in Fig. 3.1. One can see that $\mu_{0,1}$ is an increasing function of α and

$$\mu_{0,1} = \begin{cases} \mu_{0,1}(0) = \frac{2a_1(1+r^2)\sigma_1^2}{R} & \text{for} \quad \alpha = 0; \\ \mu_{0,1}(1) = \frac{2a_1(1+r^2)\sigma_1^2}{(R-1)} & \text{for} \quad \alpha = 1. \end{cases}$$

Proposition 3.1 leads to the following implications on price dynamics and market behaviour.

- An increase of α corresponds a slow price adjustment of the fundamentalists towards the fundamental price. For $\mu \in (\mu_{0,1}(0), \mu_{0,1}(1))$, as the fundamentalists adjust the price toward the fundamental price slowly, the fundamental price is stabilised.
- The region of the speed of price adjustment μ from the market maker to maintain the stability of the fundamental price is enlarged as α increases, that is as the fundamentalists adjust their expected price towards the fundamental price slowly. However, over-reactions from the market maker (in terms of $\mu > \mu_{0,1}$) or the fundamentalists (in terms of small α) can push prices to be exploded. In other word, when there is no trend followers in the market and all the traders are fundamentalists, the stability of the fundamental price is maintained when the market maker and fundamentalist are under-reaction and an over-reaction from either the market maker or the fundamentalists can push the price to explode.

- The stability boundary $\mu_{0,1}$ increases as the fundamentalists become more risk averse.

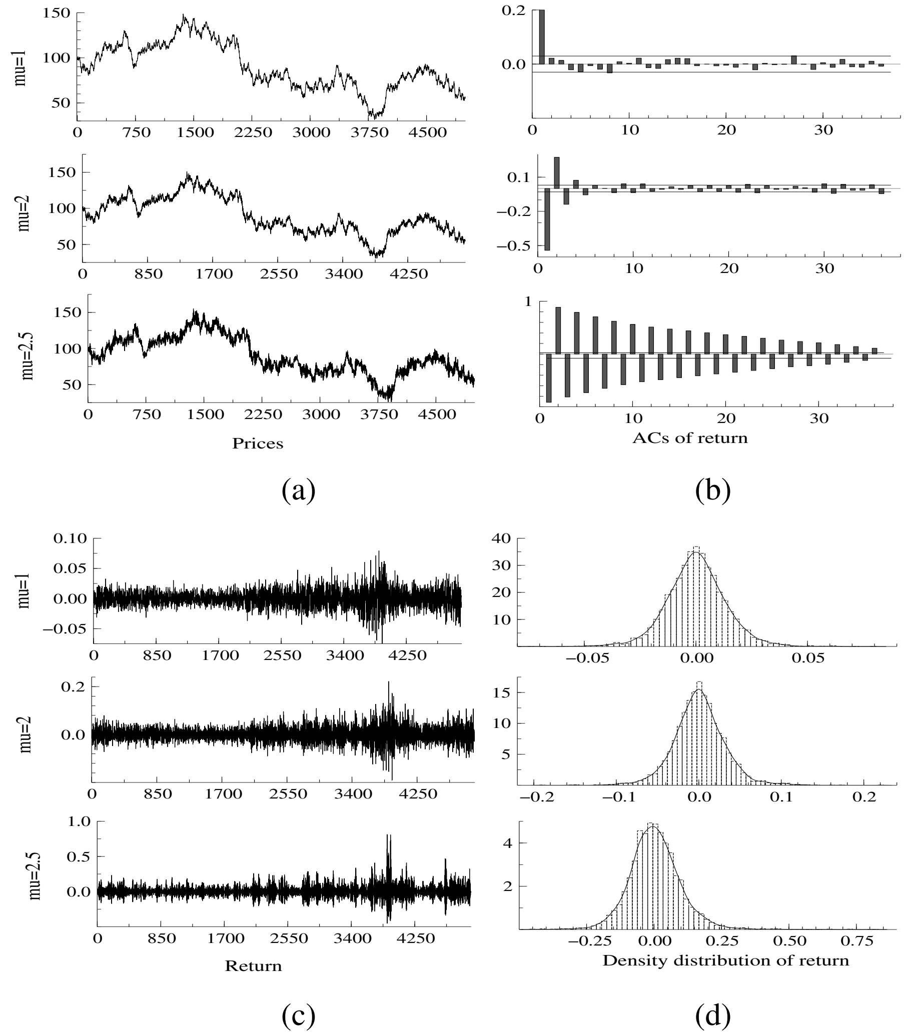


FIGURE 3.2. Time series on prices (a), returns (c), and autocorrelation coefficients (ACs) (b) and distribution densities (d) of the returns for fixed $\alpha = 0$ and $\mu = 1, 2, 2.5$.

3.2. **Time Series Analysis.** We now examine the time series properties for the stochastic system (3.1), in which the fundamental price P_t^* follows a random walk process (2.22). In the following simulations[8], apart from the parameters selected in (2.17),

[8]For all simulations in this paper, we choose a sample size of 5,000, which corresponds to about 20 years' daily data. For comparison purpose, a same set of random number generators are selected for all simulations. All the statistics presented in this paper are based on sample data between 500 and 5,000.

we choose $a_1 = 0.8$ and $b = 1$. With the selected parameters, for the deterministic system, the flip bifurcation value of μ for $\alpha = 0$ is given by $\mu_{0,1}(0) = 2.5589$. When the fundamental price P_t^* follows the random walk, it is found from numerical simulations that prices are exploded for the parameters of α and μ which are either near the flip bifurcation boundary or outside the stability region indicated in Fig. 3.1.

3.2.1. *Dynamics of the speed of price adjustment from the market maker—μ.* The dynamics of μ is illustrated by choosing $\mu = 1, 2, 2.5$ and fixed $\alpha = 0$ and numerical simulations show a similar feature for different $\alpha \in [0, 1]$. For fixed $\alpha = 0$, Fig. 3.2 shows the times series plots of prices (a) and returns (c), the corresponding autocorrelation coefficients (ACs) (c), and density distributions (d) of the return series for different values of $\mu = 1, 2, 2.5(< \mu_{0,1}(0) = 2.5589)$. For those combinations, the long-run constant fundamental price $\bar{P} = \$100$ of the corresponding deterministic system is stable. However, the price of the stochastic system displays different dynamics for different values of μ. In Fig. 3.2(a), the prices generated from the model are plotted against the fundamental prices. For $\mu = 1$, price follows the fundamental price closely. As μ increases, prices deviate from the fundamental price and become more volatile, as indicated by the return series in (c) and the density distributions of the returns in (d). This is further confirmed by the systematical increase in the standard deviations and the significant skewness and kurtosis of the statistical results, as μ increases, in the upper panel in Table 3.1.

(α, μ)	Mean	Median	Max.	Min.	Std. Dev.	Skew.	Kurt.	Jarque-Bera
(0, 1)	0.000221	0.000064	0.079361	-0.074376	0.01354	0.058219	4.960488	723.3618
(0, 2)	0.00062	0.000084	0.221923	-0.190233	0.031506	0.180158	5.546204	1240.21
(0, 2.5)	0.005071	-0.000335	0.813137	-0.451884	0.10111	0.883403	8.724681	6731.542
(0,2)	0.00062	0.000084	0.221923	-0.190233	0.031506	0.180158	5.546204	1240.21
(0.5, 2)	0.000221	0.000064	0.079334	-0.074351	0.013538	0.058189	4.958836	722.1447
(0.9, 2)	0.000139	0.000133	0.021129	-0.020756	0.004782	0.004617	4.025611	197.287
(1, 2)	0.000221	0.000231	0.002386	-0.001841	0.00063	-0.058484	2.934318	3.374964

TABLE 3.1. Statistics of return series of the market maker model for $m = 1$ and various combinations of (α, μ) for which the fundamental price of the underlying deterministic system is stable.

The ACs of the returns in Fig. 3.2(b) have clear patterns as μ increases and these patterns can be explained by the underlying dynamics of the deterministic system. In fact, the stochastic system (3.1) can be rewritten as follows:

$$P_{t+1} - \bar{P} = \lambda[P_t - \bar{P}] + \tilde{\nu}_t, \tag{3.5}$$

where

$$\tilde{\nu}_t = \frac{\mu(1-\alpha)}{a_1(1+r^2)\sigma_1^2}(P_{t+1}^* - \bar{P}) + \tilde{\epsilon}_t.$$

It follows that $\tilde{\nu}_t$ is a white noise process with mean of 0. Therefore, under the stability condition (3.3), the price process (3.5) is a stationary process and the autocorrelation coefficients of the price series are determined by the value of λ. The upper panel in

(α, μ)	(0, 0)	(0, 1)	(0, 2)	(0, 2.5)
λ	1	0.22	-0.56	-0.95
(α, μ)	(0, 2)	(0.5, 2)	(0.9, 2)	(1, 2)
λ	-0.56	0.22	0.84	0.99

TABLE 3.2. The values of λ for various combinations of (α, μ) of the market maker model with $m = 1$.

Table 3.2 lists values of λ for various combinations of (α, μ). Note that, for fixed $\alpha = 0$, λ decreases from 1 to -1 as μ increases. For those combinations, the long-run constant fundamental price $\bar{P} = \$100$ is stable. However, the ACs of the returns of the stochastic system are closely related to the underlying dynamics of the deterministic system.

- For $\mu = 1$, it follows from $\lambda = 0.22$ and (3.5) that the market maker under-adjusts the market price. As a result of instantaneous price adjustment of the fundamentalists towards the fundamental price, price series is positively correlated at the first lag and less correlated for all other lags. Consequently, the return series is positively correlated at the first lag and less significant for all other lags. In general, when $\lambda > 0$ is small, such positive ACs may become less significant for lag $L \geq 2$. This observation underlies the numerical simulation result for the stochastic system (3.1) in the first plot in Fig. 3.2(b) which shows a significant AC of the return series for $L = 1$ (AC(1)=0.214) and less significant ACs for $L \geq 2$.
- For $\mu = 2$, $\lambda = -0.56$ implies that the market price is over-adjusted by the market maker. As a result of the instantaneous price adjustment from the fundamentalist, price is negatively correlated for even lags and positively correlated for odd lags, which in turn lead to positive (negative) ACs on the return series for even (odd) lags. Because of $|\lambda| < 1$, the ACs of the absolute return decrease and ACs become less significant for high lags. This underlies the significant AC pattern for the first few lags (AC(1)=-0.542, AC(2)=0.274, AC(3)=-0.139, AC(4)=0.073) and less significant ACs for all other lags in the second plot in Fig. 3.2(b).
- For $\mu = 2.5$, $\lambda = -0.95$. This is the result of over-reaction from both the market maker and the fundamentalists. Because λ is close to -1, a strong oscillated autocorrelation patterns in both price and return are expected. This underlies the strong AC pattern for the return series across all the lags in the third plot in Fig. 3.2(b).

3.2.2. *Dynamics of the speed of the expected price adjustment of the fundamentalists towards the fundamental price—α.* To illustrate the time series properties of dynamics generated from α, we choose fixed $\mu = 2(< \mu_{0,1}(0))$. Fig. 3.3 shows the times series plots of prices (a) and returns (c), the corresponding autocorrelation coefficients (b) and density distributions (d) of the return series for $\alpha = 0, 0.5, 0.9, 1$, respectively. In terms of the price series, for $\alpha = 1$, price converge to the fundamental price $\bar{P} = \$100$. For $\alpha = 0.9$, price follows the fundamental price closely. As the speed of the price adjustment of the fundamentalists towards the fundamental prices is increases, that is as α decreases, prices deviate from the fundamental price and become more volatile, as indicated by the return series in Fig. 3.3(c) and their density distributions in Fig. 3.3(d). This is also confirmed by the systematical increase in the standard deviations and the significant skewness and kurtosis of the statistical results, as α decreases, in the lower panel in Table 3.1.

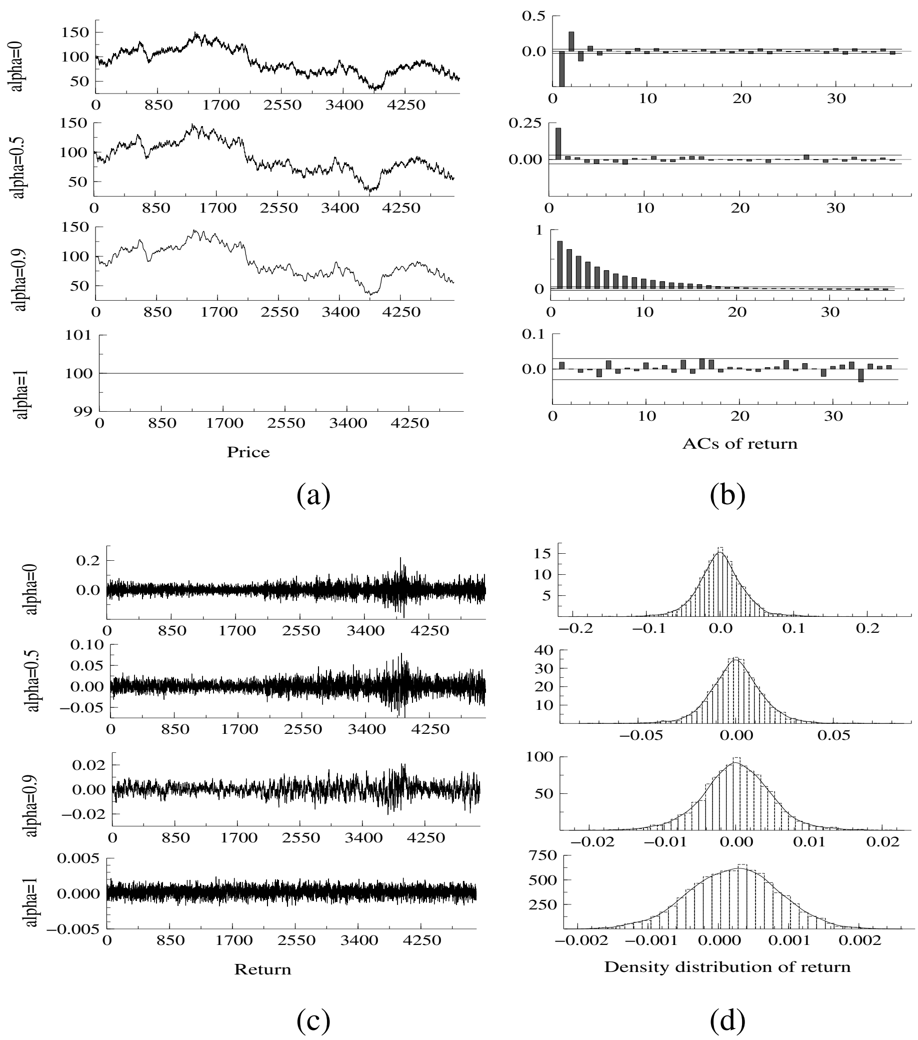


FIGURE 3.3. Time series on prices (a), returns (c), and autocorrelation coefficients (ACs) (b) and density distributions (d) of the returns for fixed $\mu = 2$ and $\alpha = 0, 0.5, 0.9, 1$.

The lower panel in Table 3.2 lists values of λ for various combinations of (α, μ). Note that, for fixed $\mu = 2$, λ increases from -1 to 1 as α increases. For those combinations, the long-run constant fundamental price $\bar{P} = \$100$ of the underlying deterministic system is stable. However, for fixed $\mu = 2$, the ACs of the returns in Fig. 3.3(b) have clear patterns as α increases and these patterns are connected to the underlying dynamics of the corresponding deterministic system.

- For $\alpha = 1$, it follows from $\lambda = 0.99$ that the price series has a close to unit root. Consequently, the ACs of the return series become less significantly for all lags. This underlies the numerical simulation result for the stochastic system (3.1) in the fourth plot in Fig. 3.3(b) which shows no significant pattern for the ACs of the return series across all the lags.
- For $\alpha = 0.9$, it follows from $\lambda = 0.84$ and (3.5) that the price series and hence the return series, are positively correlated across all the lags and ACs for the return are decreases as lag increases. This underlies the strong pattern for the ACs of the return series across all the lags for the stochastic system (3.1) in the third plot in Fig. 3.3(b).
- For $\alpha = 0.5$, $\lambda = 0.22$ implies significant ACs on the return series for the first few lags in the second plot in Fig. 3.3(b).
- For $\alpha = 0$, $\lambda = -0.56$ implies an oscillated patterns on the ACs of the return series in the first plot in Fig. 3.3(b).

3.2.3. *Overall Features.* To sum up, for the market maker model with fundamentalists only, the following features on the price dynamics and time series properties have been obtained:

- In terms of significant AC patterns of the return series, the following four patterns can be generated:
  - [AC-A] $AC(1) > 0$ is significant and $AC(i), i \geq 2$, are not significant, as indicated by the first AC pattern in Fig.3.2(b);
  - [AC-B] $AC(i)$ are significantly negative (positive) for odd (even) small lags i, as indicated by the second AC pattern in Fig.3.2(b);
  - [AC-C] $AC(i)$ are significantly negative (positive), but decreasing for all lags, as indicated by the third AC pattern in Fig.3.2(b);
  - [AC-D] $AC(i) > 0$ are significantly positive and decay for all lags, as indicated by the third AC pattern in Fig.3.3(b).
- When the fundamentalists adjust their expected price towards the fundamental price instantaneously, under-reaction from the market maker leads the market price moving closely towards to the fundamental price and hence the return has the pattern [AC-A]. As the speed of price adjust from the market maker increases, market price starts to be over adjusted towards the fundamental price, leading the return pattern [AC-B]. As the speed increases further, market price

volatility increases, reinforcing the AC patterns for returns to have pattern [AC-C].

- For fixed speed of price adjustment from the market maker, if the fundamentalists do not adjust their expected price towards the fundamental price (i.e. $\alpha = 1$), the return series have no significant ACs for all lags. As the speed of the expected adjustment of the fundamentalists towards the fundamental price is low (say $\alpha = 0.9$), the market price moves closely to the fundamental price, leading the returns have pattern [AC-D]. As the speed increases, the strong positive pattern [AC-D] is degenerated to pattern [AC-C]. As the speed increases further (when α is close to 0), over-reaction from the fundamentalists make the market price more volatile, leading return pattern [AC-B].
- In terms of AC patterns for the returns, [AC-B] can only be generated when both the market maker and the fundamentalists are over-reaction, and [AC-D] can only be generated when the fundamentalists are under-reaction.
- Normality of the return series, in terms of the first four moments of the return distribution, increases as both the market maker and the fundamentalists adjust their price and expected price, respectively, slowly (that is as (α, μ) moves away from the bifurcation boundaries).
- Excess volatility and strong ACs patterns occur as (α, μ) move towards the bifurcation boundaries and the patterns for the ACs are closed related to the underlying dynamics of the deterministic system. Near the flip bifurcation, significant oscillating pattern [AC-C] occurs. As parameters (α, μ) move away from the bifurcation boundary, patterns [AC-B], [AC-A], and [AC-D] appear in turn. The significant level of those AC patterns increases as the parameters move close to the boundary.

4. A Market Maker Model of Trend Followers

We now consider another special case of the complete model when $m = -1$, that is all the traders are trend followers. In this case, system (2.21) is reduced to the following 3-dimensional difference system

$$\begin{cases} P_{t+1} = P_t + \mu \dfrac{\gamma(P_t - u_t) - (R-1)(P_t - \bar{P})}{a_2(1 + r^2 + bv_t)\sigma_1^2} + \tilde{\epsilon}_t \\ u_t = \delta u_{t-1} + (1-\delta)P_t, \\ v_t = \delta v_{t-1} + \delta(1-\delta)(P_t - u_{t-1})^2. \end{cases} \tag{4.1}$$

In particular, when the memory decay rate $\delta = 0$, $E_{2,t}(P_{t+1}) = P_t$ and system (4.1) is reduced to the following 1-dimensional system

$$P_{t+1} = P_t - \mu \frac{(R-1)}{a_2(1+r^2)\sigma_1^2}(P_t - \bar{P}) + \tilde{\epsilon}_t. \tag{4.2}$$

4.1. **Stability and Bifurcation Analysis.** First of all, the stability of the fundamental steady state of the underlying deterministic system and its bifurcation is summarised in Proposition 4.1.

Proposition 4.1. *Under assumption (2.23), if all the traders are trend followers (that is $m = -1$), then*

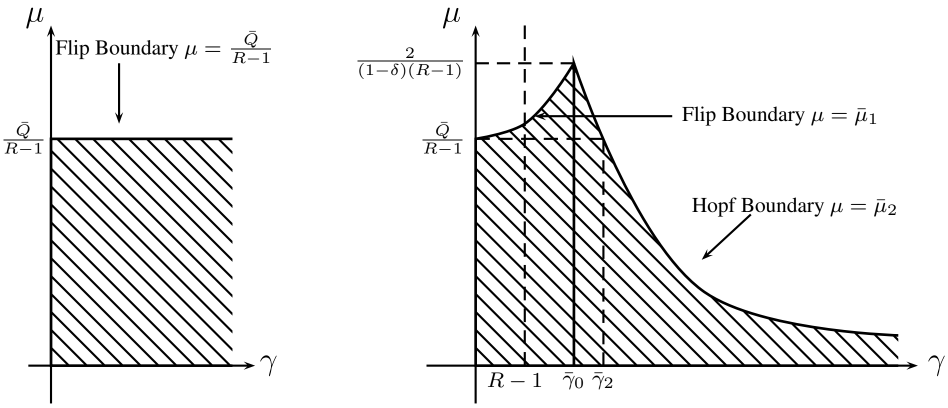
(1) *for $\delta = 0$, the fundamental steady state is globally asymptotically stable if and only if*
$$0 < \mu < \frac{Q}{R-1},$$
where $Q = 2a_2(1 + r^2)\sigma_1^2$. In addition, a flip bifurcation occurs along the boundary $\mu = Q/(R-1)$;

(2) *for $\delta \in (0, 1)$, the fundamental steady state is stable for*
$$0 < \mu < \begin{cases} \bar{\mu}_1 & 0 \le \gamma \le \bar{\gamma}_0 \\ \bar{\mu}_2, & \bar{\gamma}_0 \le \gamma, \end{cases}$$
where
$$\bar{\mu}_1 = \frac{Q}{(R-1) - \gamma 2\delta/(1+\delta)},$$
$$\bar{\mu}_2 = \frac{(1-\delta)Q}{2\delta[\gamma - (R-1)]},$$
$$\bar{\gamma}_0 = (R-1)\frac{(1+\delta)^2}{4\delta}.$$
In addition, a flip bifurcation occurs along the boundary $\mu = \bar{\mu}_1$ for $0 < \gamma \le \bar{\gamma}_0$ and a Hopf bifurcation occurs along the boundary $\mu = \bar{\mu}_2$ for $\gamma \ge \bar{\gamma}_0$.

Proof. See Appendix A.3. □

The local stability regions and bifurcation boundaries are indicated in Fig. 4.1 (a) for $\delta = 0$ and (b) for $\delta \in (0, 1)$, where $\bar{\gamma}_2 = (1+\delta)(R-1)/(2\delta)$ is obtained by letting $\bar{\mu}_2 = Q/(R-1)$. Given that $R = 1 + r/K$ is very close to 1, the value of μ along the flip boundary is very high and $\bar{\gamma}_o$ is very close to 0. This implies that, for $\delta = 0$, the fundamental price is stable for a wide range of μ, while for $\delta \in (0, 1)$, the two bifurcation boundaries degenerate to the Hopf bifurcation boundary. More precisely, based on Proposition 4.1, we obtain the following implications.

- When the memory decay rate $\delta = 0$, the trend followers use the naive expectation $E_{2,t}(P_{t+1}) = P_t$. In this case, the fundamental price becomes stable (exploded) when the market maker under-react (over-react) to the demand from the trend followers in the sense that $\mu < Q/(R-1)$ ($\mu \ge Q/(R-1)$). However, the critical value $Q/(R-1)$ for the market maker can be very high, implying that the fundamental price is stable in most of case.



(a) $\delta = 0$ (b) $\delta \in (0, 1)$

FIGURE 4.1. Stability region and bifurcation boundaries for the trend followers and market maker model with $\delta = 0$ (a) and $\delta \in (0, 1)$ (b).

- When $\delta \in (0, 1)$ is fixed and the trend followers extrapolate weakly (with $\gamma < \bar{\gamma}_2$), stability region for the market maker (in terms of μ) is enlarged comparing with the case of $\delta = 0$. However, as the trend followers extrapolate strongly (for $\gamma \geq \bar{\gamma}_2$), the stability region becomes smaller, implying a destabilising role of the trend followers. In addition, the fundamental price becomes unstable mainly through a Hopf bifurcation, implying that, near the bifurcation boundary, price either converges periodically to the fundamental price or oscillates regularly or irregularly (determined by the nature of the Hopf bifurcation).
- For fixed $\gamma > \bar{\gamma}_0$, $\bar{\mu}_2$ decreases as δ increases. This implies that the stability region for the market maker becomes smaller as the trend followers give more weight to the most recent prices. This observation also leads to the destabilising role of the geometric decay rate of the trend followers.

4.2. **Time Series Analysis.** In the following simulations, apart from the parameters in (2.17), we select a basic set of parameters, unless specified otherwise,

$$\delta = 0.5, \qquad \mu = 1.2, \qquad \gamma = 1, \qquad \sigma_{\tilde{\epsilon}} = 0.05.$$

4.2.1. *Price Dynamics Near the Hopf Boundary.* For $\gamma = 1$ fixed, the Hopf bifurcation values of $\bar{\mu}_2$ for different δ are listed in Table 4.1.

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\bar{\mu}_2$	11.551	5.133	2.994	1.925	1.283	0.855	0.550	0.320	0.142

TABLE 4.1. Hopf bifurcation values of $\bar{\mu}_2$ for different δ.

Fig. 4.2 illustrates the price dynamics for the parameter μ near the Hopf boundary for $\delta = 0.1, 0.5$ and 0.9 and Fig. 4.3 plots the corresponding return series and their ACs, where parameter q is an indicator—$q = 0, 1$ corresponds to $\sigma_\epsilon = 0, 0.05$, respectively.

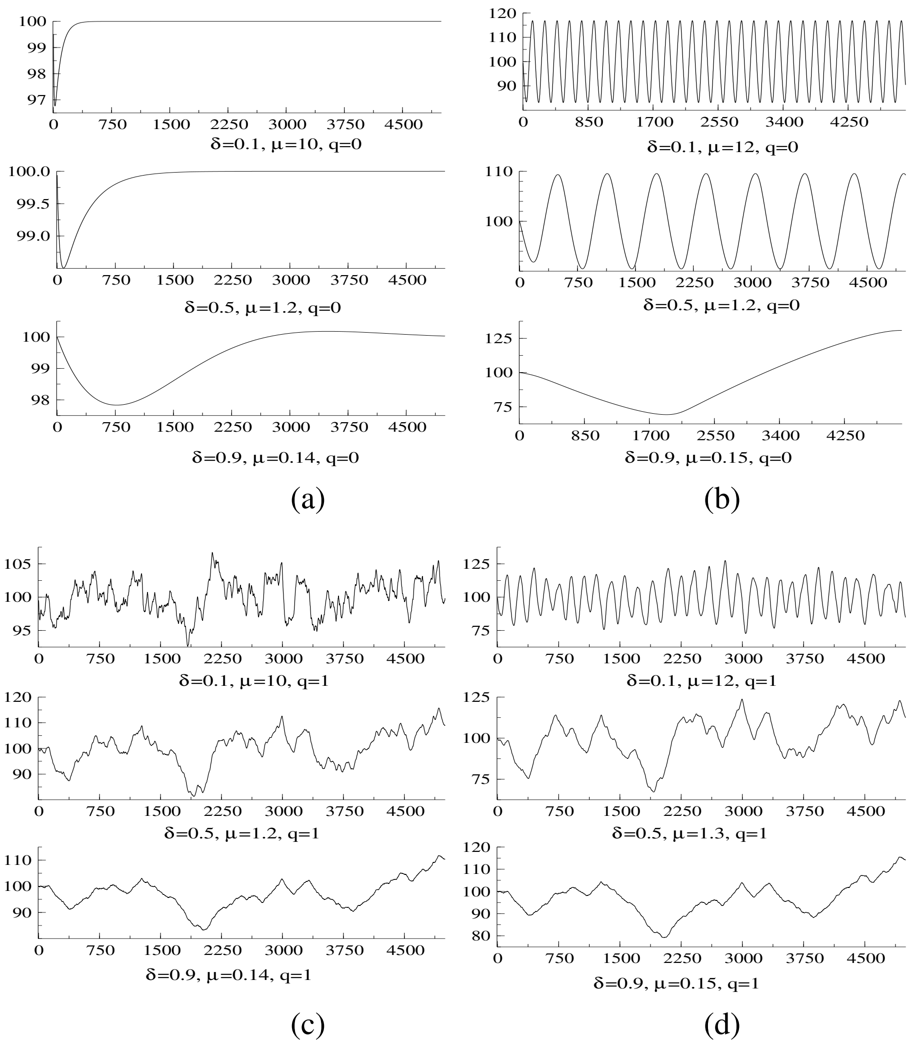


FIGURE 4.2. Time series of prices for the parameters of μ on both sides of the Hopf bifurcation boundary with $\delta = 0.1, 0.5, 0.9$. $q = 0$ corresponds to the case without demand shocks (a) and (b), and $q = 1$ corresponds to the case with demand shocks (c) and (d). Here $\gamma = 1$ is fixed.

Based on the simulations, we obtain the following observations.

- Without the demand shock, prices converge to the fundamental price $\bar{P} = \$100$ for the parameters inside the Hopf boundary and to periodic or quasi-periodic cycles for the parameters outside the boundary. However, along the Hopf bifurcation boundary, as δ increases, the speed of the convergence (when $\bar{P}$ is stable) and the periodicity of the oscillations (when $\bar{P}$ is unstable) decrease.

- With the demand shock, along the Hopf bifurcation boundary, the prices are dominated by the underlying price patterns without the shock. In addition, as δ increases, the volatility of the price is reduced and the normality (in terms of the first four moments) of the return distribution is improved on the both sides of the boundary, as indicated by the statistic results in Table 4.2.

(δ, μ)	Mean	Median	Max.	Min.	Std. Dev.	Skew.	Kurt.	Jarque-Bera
(0.1, 10)	0.000218	0.000221	0.004455	-0.003897	0.001164	-0.096387	2.984778	7.012829
(0.5, 1.2)	0.000251	0.000284	0.004296	-0.003749	0.001128	-0.072582	2.86204	7.521431
(0.9, 0.14)	0.000266	0.000289	0.003442	-0.00317	0.000877	-0.048872	2.990261	1.80956
(0.1, 12)	0.000212	0.000113	0.010933	-0.009671	0.004614	0.08355	1.870704	244.4105
(0.5, 1.3)	0.000266	0.000305	0.00524	-0.004602	0.001453	-0.062007	2.610581	31.32442
(0.9, 0.15)	0.000279	0.0003	0.003682	-0.00327	0.000903	-0.050813	3.02206	2.02816

TABLE 4.2. Statistics of return series of the market maker model for $m = -1$ and various combinations of (δ, μ).

- The ACs of the return series are significantly positive across all the lags for the parameters near the Hopf bifurcation boundary, and the significance is reduced as δ increases. This is partly determined by the Hopf bifurcation of the underlying deterministic system[9], partly influenced by an increase of the geometric decay rate[10].

4.2.2. *Dynamics of the Extrapolation Rate of the Trend Followers—γ.* For fixed $\delta = 0.5$ and $\mu = 2$, the fundamental price of the deterministic system is stable for $\gamma = 1$ and unstable for $\gamma = 1.2$. As the fundamental price becomes unstable, it bifurcates a periodic cycles, implied by the Hopf bifurcation near the boundary. Fig. 4.4 illustrates the time series of price with and without demand shock. For $\gamma = 1.2$, under the demand shock, the price pattern is influenced by the cyclical movement of the price without shock, although the periodicity of the price is broken down. This implies that a strong extrapolation from the trend followers can push the price away from the fundamental price, leading to cyclical price movement.

4.2.3. *Overall Features.* Further extensive simulations lead to the following observations on the overall features of the market maker model with trend followers only.

[9]When the underlying deterministic system displays a Hopf bifurcation, the linearised system at the fundamental steady state has a complex eigenvalues, implying a damped oscillation AC pattern. The frequency of these oscillation is determined by the geometric decay rate.

[10]As δ increases, the weights to the past prices increase and the frequency of the oscillation of ACs increases as well.

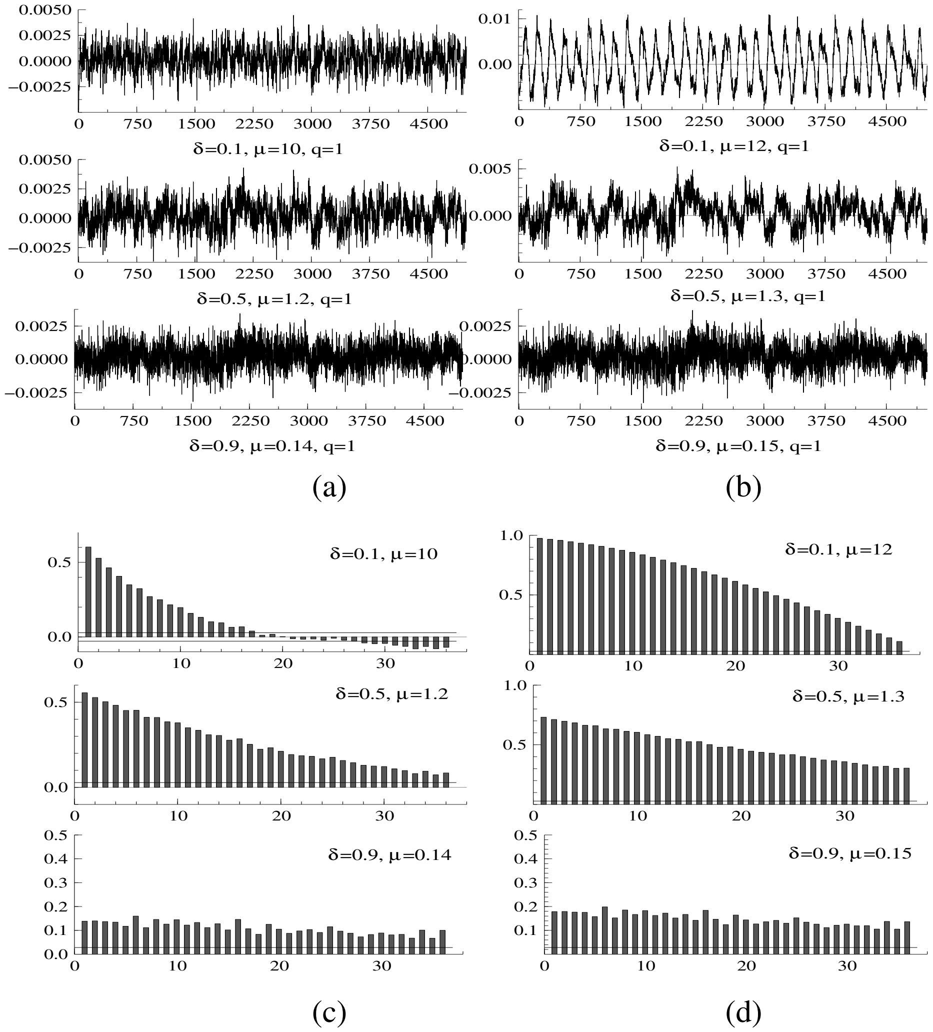


FIGURE 4.3. Time series (a) and (b) and ACs (c) and (d) of the return series with demand shocks for δ and μ on both sides of the Hopf bifurcation boundary.

- Near the Hopf bifurcation boundary, prices tend to have cyclical movement. The frequency of such cyclical price movement is reduced as either the market maker adjust market price to the demand quickly or the trend followers extrapolate the trend strongly.
- Different from the market maker model with the fundamentalists only (discussed in the previous section), the current model can generate only three significant AC patterns when the underlying deterministic system is stable: [AC-D] (defined in the previous section) and [AC-E], [AC-F] defined below:

[AC-E] $AC(i)$ are significant (at relative low level) across all lags, as indicated in the third AC pattern in Fig.4.3(c);

[AC-F] $AC(i)$ oscillate and decay with significantly positive for all small lags and less significantly negative for high lags, as indicated in the first AC pattern in Fig.4.3(c).

- As the result of trend chasing by the trend followers and reinforcing price adjustment by the market maker, price series and hence the return series tend to be positively correlated across all the lags.
- The significance and pattern of the ACs of the return series are mainly determined by the geometric decay rate. For small decay rate (such as $\delta = 0.1$), returns have pattern [AC-F], ACs oscillate and die out quickly. However, for high decay rate (such as $\delta = 0.9$), returns have pattern [AC-E], ACs may take long time to die out, a long memory feature generated from trend following strategy with high decay rate.
- Along the Hopf bifurcation boundary, returns have pattern [AC-F], [AC-D] and [AC-E], in turn, as the decay rate δ increases and the level of the significance can be reinforced by either quick price adjustment from the market maker or strong extrapolation from the trend followers. As parameters (γ, μ) move away (towards the inside of the stability region), those three AC patterns become less, and even not, significantly.

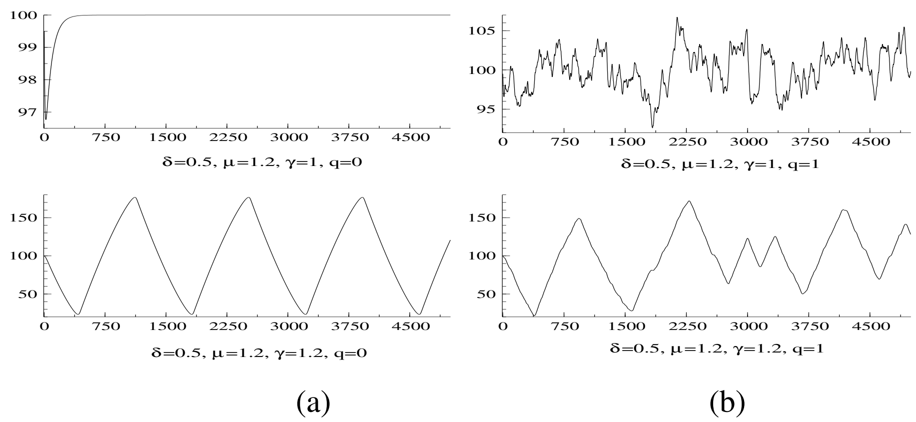


FIGURE 4.4. Price time series for $\gamma = 1$ and 1.2 without demand shock (a) and with shock (b).

5. DYNAMICS OF THE COMPLETE MARKET FRACTION MODEL

We now consider the complete market fraction model with both fundamentalists and trend followers by assuming $m \in (-1, 1)$. Let $a = a_2/a_1$ be the ratio of the absolute risk aversion coefficients. As in the previous sections, the stability and bifurcation of the fundamental steady state of the deterministic system are discussed first. To characterise the market dominance, wealth dynamics and asset pricing behaviour, we then introduce two relative wealth measures. Various aspects of time series properties and market behaviour are followed by using numerical analysis.

5.1. **Stability and Bifurcation Analysis.** It turns out that the stability and bifurcation of the fundamental steady state are different from the previous two models and they are determined jointly by the geometric decay rate and extrapolation rate of the trend followers, the speed of the price adjustment of the fundamentalists towards the fundamental steady state, and the speed of adjustment of the market maker towards the market aggregate demand.

Proposition 5.1. *Under assumption (2.23),*

(1) *for $\delta = 0$, the fundamental steady state is stable for $0 < \mu < \mu^*$, where*

$$\mu^* = \frac{2Q}{(R-1)(1-m) + a(R-\alpha)(1+m)}.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \mu^*$ with $\alpha \in [0,1]$;

(2) *for $\delta \in (0,1)$, the fundamental steady state is stable for*

$$0 < \mu < \begin{cases} \mu_1 & 0 \le \gamma \le \gamma_0 \\ \mu_2, & \gamma_0 \le \gamma, \end{cases}$$

where

$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2 - \gamma},$$
$$\mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma - \gamma_1},$$
$$\gamma_1 = (R-1) + a(R-\alpha)\frac{1+m}{1-m},$$
$$\gamma_0 = \frac{(1+\delta)^2}{4\delta}\gamma_1, \qquad \gamma_2 = \frac{1+\delta}{2\delta}\gamma_1.$$

In addition, a flip bifurcation occurs along the boundary $\mu = \mu_1$ for $0 < \gamma \le \gamma_0$ and a Hopf bifurcation occurs along the boundary $\mu = \mu_2$ for $\gamma \ge \gamma_0$.

Proof. See Appendix A.3. □

For $\delta = 0$, $E_{2,t}(P_{t+1}) = P_t$ corresponds a the naive expectation. In this case, the stability region of the fundamental steady state in (α, μ) plane have the same feature as in Fig. 3.1. The fundamental price becomes unstable through a flip bifurcation only. Therefore the model with the fundamentalists only discussed in section 3 can be treated as a degenerated case of the complete model with $\delta = 0$. However, for $\delta \in (0,1)$, the fundamental price becomes unstable through either flip or Hopf bifurcation.

The stability region and bifurcation boundaries for $\delta \in (0,1)$ are plotted in Fig.5.1, where

$$\bar{\mu}_0 = \frac{2}{1-\delta}\bar{\mu}, \qquad \bar{\mu} = \frac{2Q}{(R-1)(1-m) + a(R-\alpha)(1+m)}.$$

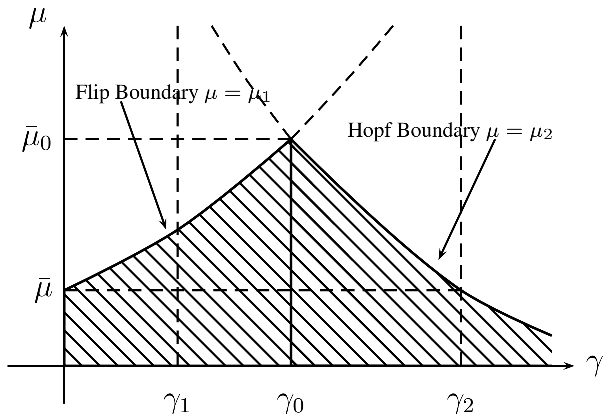


FIGURE 5.1. Stability region and bifurcation boundaries for $m \in (-1, 1)$ and $\delta \in (0, 1)$.

The implications of the stability conditions in Proposition 5.1 can be stated from different aspects of the parameters involved. For convenient, we denote

$$\Omega_F \equiv \{(\gamma, \mu) : 0 \le \gamma \le \gamma_0, 0 < \mu < \bar{\mu}_1\}, \qquad \Omega_T \equiv \{(\gamma, \mu) : \gamma \ge \gamma_0, 0 < \mu < \bar{\mu}_2\}$$

and call the region Ω_F as the stability region dominated by the fundamentalists and Ω_T as the stability region dominated by the trend followers.

5.1.1. *Implication of the Market Fraction—m.* It can be verified that, on the one hand, γ_1 and hence γ_0 and γ_2 increase as m increases. On the other hand, $\bar{\mu}$ and hence $\bar{\mu}_0$ increase for $a = a_2/a_1 < a^*$ and decrease for $a = a_2/a_1 > a^*$, where $a^* = (R-1)/(R-\alpha)$. Obviously $a^* \in (1 - 1/R, 1]$. These facts lead to the following observations.

- When the trend followers are less risk averse than the fundamentalists in the sense of $a_2 < a^* a_1$, the local stability region is enlarged (shift to the upper-right) as the fraction of the fundamental traders increases. The stability region is mainly bounded by the flip bifurcation boundary. Combinations of over-adjusted price from both fundamentalists and the market maker can lead the price to explode. However, the local stability region is also enlarged (shift to the upper-left) as the fraction of the trend followers increases and the stability region is mainly bounded by the Hopf bifurcation boundary. Combinations of over-adjusted price from the market maker and strong extrapolation from the trend followers can lead the price to oscillate regularly or irregularly. In both cases, the market maker can maintain the stability of the fundamental price over a large region of the price adjustment (in terms of μ).
- When the trend followers are more risk averse than the fundamentalists in the sense of $a_2 > a^* a_1$, the local stability region becomes smaller as the fraction of the fundamental traders increases (shift to the lower-right). The stability

region is mainly bounded by the flip bifurcation boundary. However, the local stability region becomes smaller too (shift to the left) as the fraction of the trend followers increases and the stability region is mainly bounded by the Hopf bifurcation boundary.

Overall, the stability region of the fundamental price becomes large (small), in terms of (γ, μ), when the fundamentalists become more (less) risk averse. Despite different risk attitude of two types of traders, the stability and bifurcation of the complete model tend to have the feature of the model with the fundamentalists (trend followers) only discussed in section 3 (section 4) as the market fraction of the fundamentalists (trend followers) increases. The intuition of this observation is obvious.

5.1.2. *Implication of Speed of the Price Adjustment of the Fundamental Traders Towards the Fundamental Price—α.* Note that parameter α measures the speed of the price adjustment of the fundamentalists towards the fundamental price, $\alpha = 0$ corresponds to an instantaneous adjustment and $\alpha = 1$ corresponds to no adjustment. As α decreases (increases), $\bar{\mu}$ and hence $\bar{\mu}_0$ decrease (increase), and γ_1 and hence γ_0, γ_2 increase (decrease). Consequently, the flip (Hopf) bifurcation boundary of the local stability region in Fig. 5.1 is enlarged, leading to the following observations.

- As the fundamental traders decrease the speed of their expected price adjustment toward the long-run constant fundamental price $\bar{P}$ (that is, as α increases), the local stability region in Fig. 5.1 shifts to upper-left and the Hopf bifurcation boundary is enlarged. In this case, the market maker can maintain the stability of the fundamental price over a large (small) region of the price adjustment when the trend followers extrapolate weakly, i.e. $\gamma \leq \gamma_0$ (strongly, i.e. $\gamma \geq \gamma_0$).
- As the fundamental traders increase the speed of their expected price toward the long-run constant fundamental price $\bar{P}$ (that is, as α decreases), the local stability region in Fig. 5.1 shifts to low-right and the flip bifurcation boundary is enlarged.
- When trend followers extrapolate weakly ($\gamma \leq \gamma_o$), over-reaction (in terms of over-adjustment of the market price) from the market maker leads price to explode. However, when trend followers extrapolate strongly ($\gamma \geq \gamma_0$), such over-reaction from the market maker can make the price to fluctuate (either regularly or irregularly, depending on the nature of the Hopf bifurcation).

Overall, the stability and bifurcation of the complete model tend to have the feature of the model with the fundamentalists (trend followers) only discussed in section 3 (section 4) as the fundamentalists increase (decrease) the speed of the adjustment of their expected price towards the fundamental price, just like the effect of the market fraction m.

5.1.3. *Implication of the Geometric Decay Rate From the Trend Followers—δ.* Note that both γ_1 and $\bar{\mu}$ are independent of δ. However, as δ decreases, both γ_0 and γ_2 increase and $\bar{\mu}_0$ increases as well. In particular, as a limiting case, $\gamma_0, \gamma_2 \to +\infty$

as $\delta \to 0$, and the stability and bifurcation can then be characterised by the model with the fundamentalists only, discussed in section 3 and indicated by Fig.3.1. On the other hand, as δ increases, both γ_0 and γ_2 decrease, but $\bar{\mu}_0$ increases. In particular, as $\delta \to 1_-$, both γ_0 and γ_2 tend to γ_1 whilst $\bar{\mu}_0$ tends to infinity and the stability and bifurcation can then be characterised by the model with the trend followers only, discussed in section 4 and indicated by Fig.4.1. These analyses lead to the following observations.

- The less geometric decay rate the trend followers have for the past prices, the less influence of their extrapolation on the stability of the fundamental price. The price either converges to the fundamental price or explodes.
- As the trend followers increase their geometric decay rate on the past prices, their influence on the price dynamics becomes significantly. When they extrapolate strongly, the price either converge to the fundamental price (when μ is small) or fluctuate regularly or irregularly (when μ is large). In particular, for γ near γ_1, the market maker can maintain the stability of the fundamental price for a wilder range of price adjustment (not that $\bar{\mu}_0 \to +\infty$ as $\delta \to 1$).

Overall, in terms of the local stability and bifurcation of the fundamental steady state, a high (low) geometric decay rate has a similar effect as either high (low) market fraction of the trend followers or low (high) speed of the price adjustment of the fundamentalists towards the fundamental price.

5.1.4. *Summary.* Based on the above analysis, a broad picture on the features of the stability and bifurcation of the deterministic system can be summarised as following.

- The model with fundamentalists (trend followers) only discussed in the previous section 3 (4) can be treated as a degenerate case of the complete model with either (i) a low (high) geometric decay rate—δ close 0 (1)—of the trend followers or (ii) a high (low) speed of the price adjustment—α close to 0 (1)—of the fundamentalists.
- The stability region and bifurcation boundary are mainly determined by the degenerated model with fundamentalists (trend followers) only in section 3 (section 4) as either (i), or (ii), or (iii) a high market fraction of the fundamentalists (trend followers).
- Give that the stability region Ω_F (Ω_T) has a flip (Hopf) bifurcation boundary, instability of the fundamental steady state through the bifurcation boundary of Ω_F (Ω_T) leads the price to explode (oscillate).

Given the fact that the complete model is jointly characterised by the model with either fundamentalists and trend followers only, returns are expected to have all those features described in the previous sections, including the six significant AC patterns. Depending on the dominance of either type of traders, some patterns are expected to dominant others, as shown in the following discussion.

5.2. **Wealth Dynamics and Relative Wealth Measures.** Traders' wealth in general follow some growing processes. To be able to measure the wealth dynamics among different trading strategies, to examine the market dominance and behaviour, we introduce the following two relative wealth measures. The first measures the absolute level of the wealth proportion of representative agent from each type, called the *absolute wealth proportion* for short, defined by

$$w_{1,t} = \frac{W_{1,t}}{W_{1,t} + W_{2,t}}, \qquad w_{2,t} = \frac{W_{2,t}}{W_{2,t} + W_{2,t}}. \tag{5.1}$$

The second measures the overall market level of the wealth proportion, called the *market wealth proportion* for short, which is defined as market fraction weighted average of the absolute level of the wealth proportion,

$$\begin{cases} \bar{w}_{1,t} = \dfrac{(1+m)W_{1,t}}{(1+m)W_{1,t} + (1-m)W_{2,t}}, \\ \bar{w}_{2,t} = \dfrac{(1-m)W_{2,t}}{(1+m)W_{1,t} + (1-m)W_{2,t}}. \end{cases} \tag{5.2}$$

Let

$$V_{1,t} = 1/W_{1,t}, \qquad V_{2,t} = 1/W_{2,t}.$$

Then it follows from (2.1) that

$$\begin{cases} V_{1,t+1} = \dfrac{V_{1,t}}{R + R_{t+1} z_{1,t} V_{1,t}}, \\ V_{2,t+1} = \dfrac{V_{2,t}}{R + R_{t+1} z_{1,t} V_{1,t}}. \end{cases}$$

Note that

$$\frac{V_{1,t}}{V_{1,t} + V_{2,t}} = \frac{1/W_{1,t}}{1/W_{1,t} + 1/W_{2,t}} = \frac{W_{2,t}}{W_{1,t} + W_{2,t}},$$
$$\frac{V_{2,t}}{V_{1,t} + V_{2,t}} = \frac{1/W_{2,t}}{1/W_{1,t} + 1/W_{2,t}} = \frac{W_{1,t}}{W_{1,t} + W_{2,t}}$$

and therefore

$$w_{1,t} = \frac{V_{2,t}}{V_{1,t} + V_{2,t}}, \qquad w_{2,t} = \frac{V_{1,t}}{V_{1,t} + V_{2,t}} \tag{5.3}$$

and

$$\begin{cases} \bar{w}_{1,t} = \dfrac{(1+m)V_{2,t}}{(1+m)V_{2,t} + (1-m)V_{1,t}}, \\ \bar{w}_{2,t} = \dfrac{(1-m)V_{1,t}}{(1+m)V_{2,t} + (1-m)V_{1,t}}. \end{cases} \tag{5.4}$$

By using auxiliary functions $(V_{1,t}, V_{2,t})$ and numerical simulations, we are able to study the wealth dynamics of the fundamentalists and trend followers.

5.3. **Time Series Analysis.** In the following simulations, apart from the parameters in (2.17), we also select $a_1 = a_2 = 0.8$. To characterise various aspects of the model, we first examine an extreme case $\alpha = 0$ where the fundamentalists adjust their expected price towards the fundamental price instantaneously. The case $\alpha \in (0, 1)$ is then discussed to see its effect one the dynamics. The model is analysed from different aspect through different parameters.

5.3.1. *Dynamics of the Market Fraction* m. We first examine the dynamics generated from the market fraction m. In this subsection, we select

$$\alpha = 0, \qquad \delta = 0.85, \qquad \gamma = 2.1, \qquad \mu = 0.43, \qquad w_{1,0} = 0.5.$$

For the selected parameters, Table 5.1 lists the bifurcation values of μ and types of bifurcation for a set of market fraction parameters. For fixed $\mu = 0.43$, the fundamental steady state is stable for $m \geq \bar{m}$ and unstable for $m < \bar{m}$ with $\bar{m} \in (0, 0.05)$. For given $\mu > 0$, one can see that the market fraction m plays a stabilising role as the market fraction of the fundamentalist increases.

Market Fraction (m)	-0.95	-0.5	-0.2	0	0.3	0.35
Hopf Bif. Value ($\bar{\mu}_2$)	0.1119	0.1709	0.2633	0.4118	2.6703	31.0201
Market Fraction (m)	0.4	0.5	0.6	0.7	0.8	0.95
Flip Bif. Value ($\bar{\mu}_1$)	21.1607	9.5844	6.1952	4.5768	3.6288	2.7686

TABLE 5.1. Bifurcation values of μ for various market fraction m.

In Figures 5.2-5.3, we choose $m = -0.95, -0.5, 0, 0.5$, respectively. In terms of the market fraction, they correspond to market fraction of the fundamentalists $m_F = 2.5\%, 25\%, 50\%, 75\%$, respectively. For the corresponding deterministic system, the fundamental steady state is not stable for $m = -0.95$, $-0.5, 0$ and stable for $m = 0.5$.

<u>*The Market Prices.*</u> The stochastic fundament price (generated from the random walk process (2.22)) and two market price series of the model, corresponding to either the constant or the stochastic fundamental price, are plotted and compared in Fig.5.2(a)-(b).

- When the underlying fundamental price is constant $\bar{P} = \$100$, the market price of the complete model converges to the fundamental price $\bar{P} = \$100$ for $m_F = 75\%$ and displays regular oscillation around the constant fundamental price $\bar{P} = \$100$ for $m_F = 2.5\%, 25\%, 50\%$. In addition, as the market fraction of the fundamentalists increases, the frequency of the oscillation increases but the dispersion of the oscillation from the fundamental price $\bar{P}$ decreases.
- When the underlying fundamental price follows the random walk process, for $m_F = 2.5\%$, the price is dominated by the large irregular swings generated by the trend followers. However, convergence of the price to the fundamental price is improved as the market fraction of the fundamentalists increases. Such

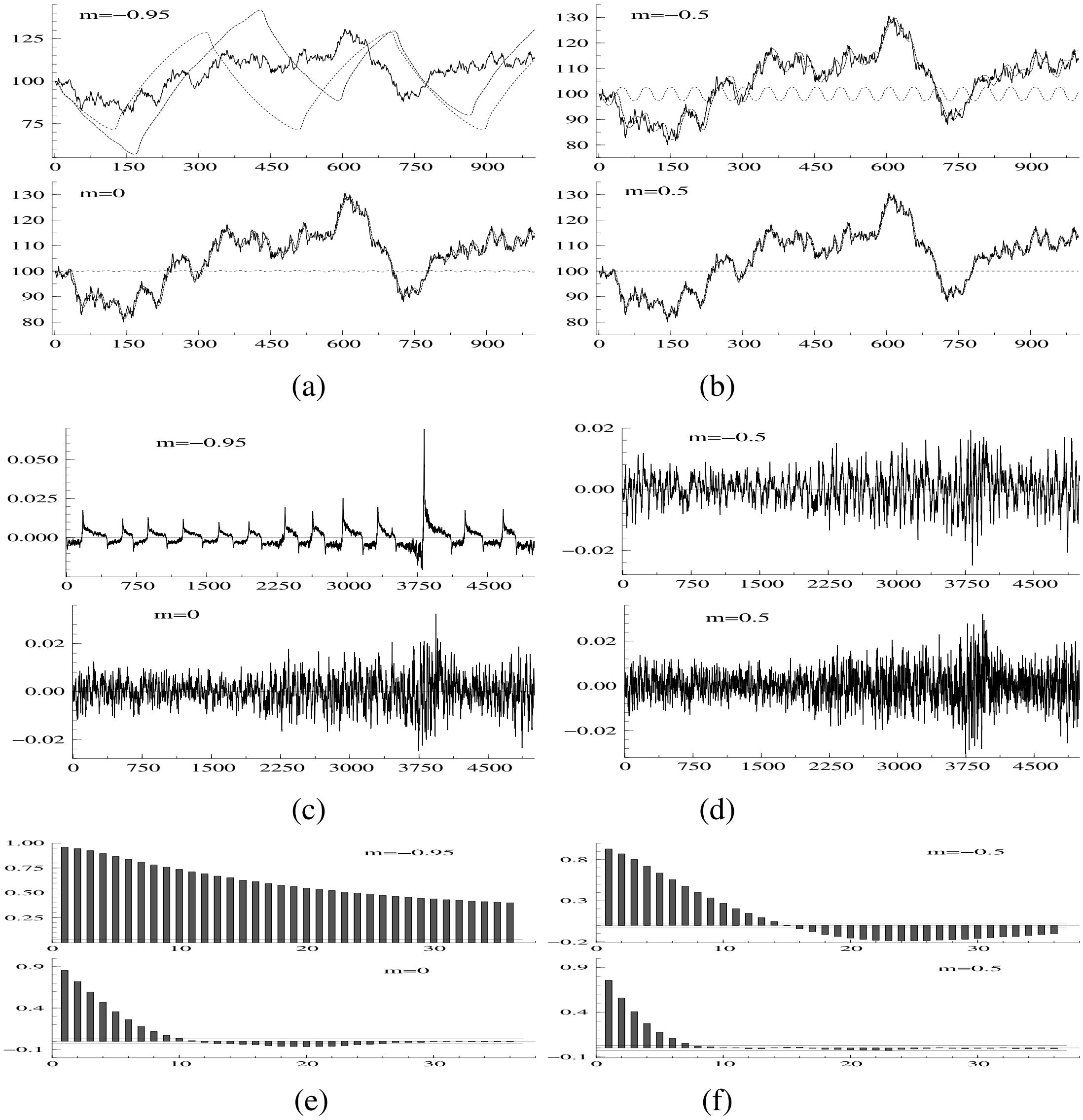


FIGURE 5.2. Effect of m: (a)-(b) Comparison of stochastic fundamental price and the prices generated from the complete model with underlying fundamental price is either constant or follows the stochastic fundamental price for different $m = -0.95, -0.5, 0, 0.5$; (c)-(d): The corresponding return series with underlying stochastic fundamental price; (e)-(f): ACs of the return series.

convergence may be slow down as the speed of the price adjustment of the fundamentalist towards the fundamental price, as discussed late.

Returns and ACs. The corresponding return series and ACs of the returns when the fundamental price follows the random walk process are plotted in Fig.5.2(c)-(d) and (e)-(f), respectively. The AC patterns of the returns are clearly related and influenced by the six AC patterns of the model with either fundamentalists or trend followers only discussed in sections 3-4.

- When there is a very low market fraction of the fundamentalists, such as $m_F = 2.5\%$ (for $m = -0.95$), the market is dominated by the trend followers. Returns have pattern [AC-D]: ACs are significantly positively correlated across all lags when the geometric decay rate is high (such as $\delta = 0.85$ used here). The significant level of the ACs can be increased as the result of strong extrapolations from the trend followers. Consequently, the market price oscillate regularly away from the fundamental price.
- As the market fraction of the fundamentalists increases (from $m_F = 2.5\%$ to either $m_F = 25\%$ or $m_F = 50\%$), the market price trend is partly corrected towards the fundamental price by the fundamentalists' instantaneous adjustment. Consequently, returns have AC pattern [AC-F]: ACs oscillate, indicated by the ACs for either $m_F = 25\%$ or 50% in Fig. 5.2(e)-(f). The level of the significance is reduced as m_F increases.
- As the market fraction of the fundamentalists increases further so that there is enough market fraction of the fundamentalists to dominate the market, the market price follow the fundamental price closely and returns have AC pattern [AC-D], as the case when $m_F = 75\%$.

The above analysis illustrates the important role of the market fraction and explanation power of the model in asset price dynamics and market behaviour. When the market is dominated by the trend followers, asset prices are driven far away from the fundamental price and returns are characterized by AC pattern [AC-D] with very significant ACs; when the market is dominated by the fundamentalists, asset prices follows closely the fundamental price and returns are characterized by AC pattern [AC-D] with less significant ACs; when two groups of traders are balanced in some way, asset prices are driven away, but not significantly, from the fundamental price and returns have AC pattern [AC-F].

Wealth Proportions. The wealth dynamics of the fundamentalists among the two types of traders is illustrated in Fig. 5.3, where the wealth proportions of the fundamentalists are plotted.

- When the underlying fundamental price is the constant $\bar{P}$, the absolute wealth proportion of the fundamentalists stay below 50%, as shown in Fig. 5.3(a). For $m_F = 2.5\%$, the wealth proportion is dropped from 50% to about 40% quickly when the market price is driven down below the fundamental price. The most part of the 10% loss may be quickly recovered when the market price moves back upwards to the fundamental price. Hence the wealth proportion of the fundamentalists oscillates, but over the time, follows a downward trend. It is very interesting to notice that their wealth proportion always stays

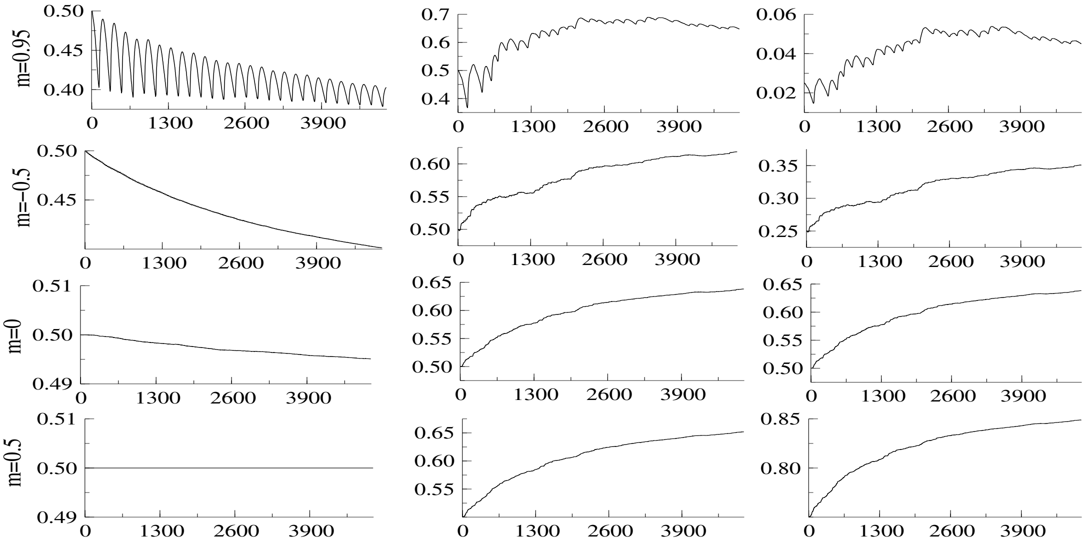


FIGURE 5.3. Wealth dynamics of the market fraction—the absolute level of wealth proportions of the fundamentalists with (a) constant, (b) stochastic fundamental price; the market fraction weighted wealth proportion of the fundamentalists with underlying stochastic fundamental price. Here the initial absolute wealth proportion $w_{1,0} = 0.5$.

above at about 35% over the 20 years period. This situation may not be improved by increasing their fraction, as indicated by the plot for $m_F = 25\%$ (for $m = -0.5$) in Fig. 5.3(a). In this case, the wealth proportion of the fundamentalists monotonically decreases from 50% to about 30% over the time period. This is clearly indicated by the small regular oscillating trend on the price. Since the dispersion of the price from the constant fundamental price is small, the demand for the fundamentalists is also small and hence the profit opportunities for the fundamentalists when the prices revert to the fundamental price diminishes. However, when the market fraction of the fundamentalists increases further (to $m_F = 50\%$ or 75%), their wealth proportions stay at about 50%, their initial level.

- When the underlying fundamental price follows the random walk process, the absolute wealth proportions of the fundamentalists improves from 50% up to about 65% over the whole time period, although their wealth proportion may be dropped initially because of their lower market fraction ($m_F = 2.5\%$). This is clearly indicated by the price patterns in Fig. 5.2(a)-(b). Because of the randomness of the fundamental price, the market price trend generated from trend followers becomes less informed and lagged for the trend followers, which puts the fundamentalists in a more favourable position to be in the right price at the right time.

- The market wealth proportions of the fundamentalists are plotted in Fig.5.3(c), which clearly indicates that, on the one hand, the market wealth proportions of the fundamentalists can be very small when their market fraction is small (for $m_F = 2.5\%$), although their absolute wealth proportion is high. On the other hand, their market wealth proportions can be vary high when their market fraction is high.

The two wealth proportion measures clearly indicate the absolute and market wealth proportions of each group of traders. They confirm the market dominance and behaviour discussed in the previous subsection.

5.3.2. *Dynamics of the Price Adjustment From the Market Maker μ.* To see the effect of the price adjustment of the market maker on the price dynamics and market behaviour, we select in this subsection

$$\alpha = 0, \qquad m = 0, \qquad \delta = 0.85, \qquad \gamma = 0.3, \qquad w_{1,0} = 50\%$$

and let the fundamental price follow the same random walk process. With this selection, $m_F = 50\%$, the fundamental steady state of the deterministic system is stable for $\mu < \bar{\mu}_1^* = 7.082417$ and a flip bifurcation occurs for $\mu = \bar{\mu}_1^*$. To illustrate the effect of μ, in the following simulations, $\mu = 1, 3$ and 5 are selected. Obviously, the steady state of the deterministic system is stable for $\mu = 1, 3, 5 (< \bar{\mu}^*)$. Fig. 5.4 plots time series of prices, returns, wealth proportions, and the ACs of the returns for $\mu = 1, 3, 5$ in (a), (b) and (c), respectively.

Prices, Returns, and Volatilities. Because of the instantaneous price adjustment of the fundamentalists towards the fundamental price ($\alpha = 0$), 50% of the market fraction of each type of traders, and the stability of the fundamental steady state of the underlying deterministic system, the market price follows the fundamental price closely. However, as μ increases, that is as the speed of the price adjustment from the market maker increases, the market prices become more volatile. In other words, over-reaction from the market maker can generate excess volatility on prices, which is clearly indicated by both the price and return series in Fig. 5.4 and the statistical result in Table 5.2 on the return series. As μ increases, the volatility, skewness and kurtosis increase too.

μ	Mean	Median	Max.	Min.	Std. Dev.	Skew.	Kurt.	Jarque-Bera
1	0.000165	0.000166	0.040037	-0.042549	0.008408	0.01345	4.465331	402.8243
3	0.000327	0.000014	0.124849	-0.112888	0.019978	0.081569	5.068942	807.7675
5	0.003328	-0.000677	0.610357	-0.382382	0.080999	0.723064	8.096536	5263.537

TABLE 5.2. Statistics of return series for $\mu = 1, 3, 5$ with $\alpha = 0, \delta = 0.85, m = 0$.

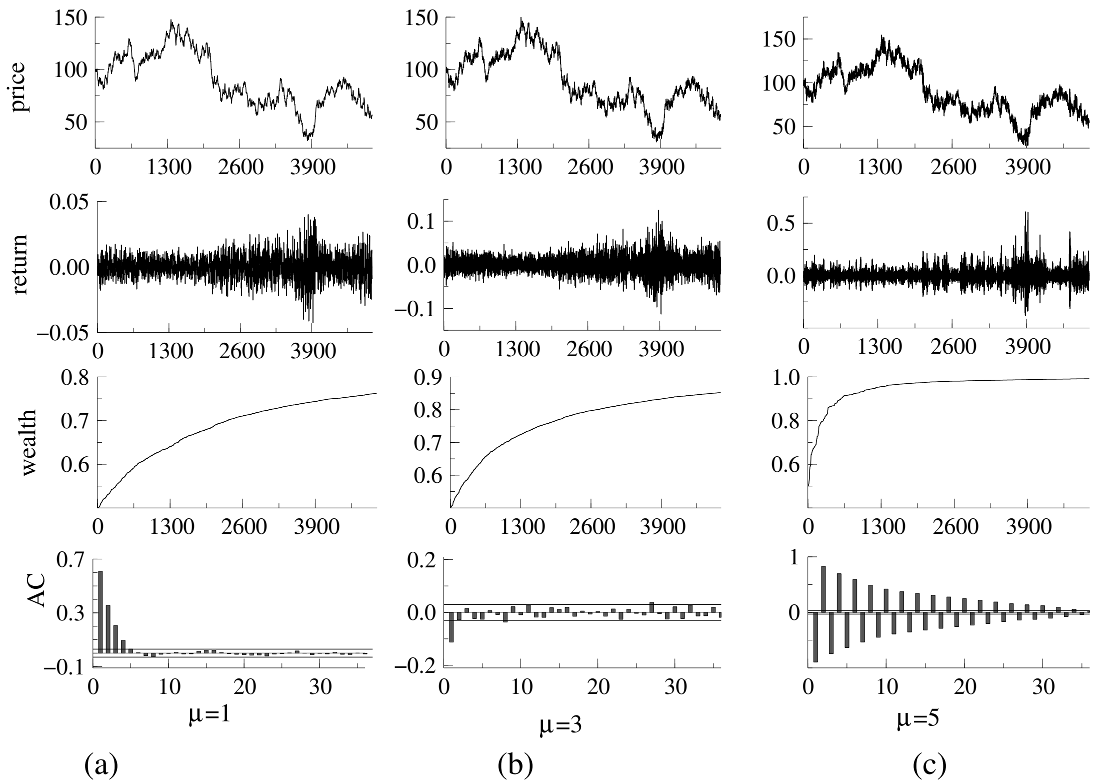
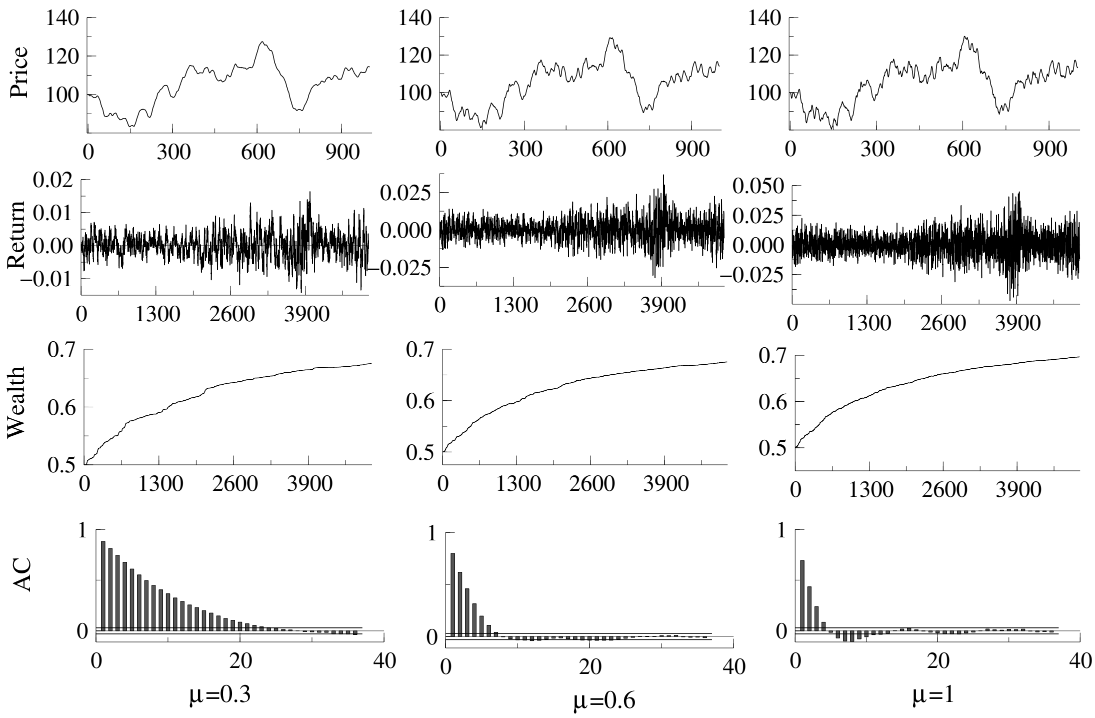


FIGURE 5.4. Effect of μ: Prices, returns, wealth proportions, and ACs of the returns for (a) $\mu = 1$, (b) $\mu = 3$, and (c) $\mu = 5$. Here $m = 0, \delta = 0.85, \gamma = 0.3$.

ACs of the Returns. The selected parameters are located in Ω_F, the stability region dominated by the fundamentalists. When the market maker under-reacts to the demand, which mainly comes from the fundamentalists, it may need few trading periods for the market price move towards the fundamental price. Hence the market prices are positively correlated and such price correlation becomes less significant as the price is getting closer and closer to the fundamental price (and hence the demand from the fundamentalists becomes less and less). Consequently, ACs of the return are positively significant over the first few lags, but diminish very quickly as time goes. This underlies the AC pattern [AC-F], illustrated in Fig. 5.4(a) for $\mu = 1$. As μ increases, under-reaction from the market maker is correct partially and the positive AC pattern of the return becomes less significant, such as pattern [AC-B] in Fig. 5.4 (b) for $\mu = 3$, and even disappears. However, as μ increases further, the market maker over-reacts to the demand (mainly from the fundamentalists) and price is over-shotted. Because of the instantaneous price adjustment from the fundamentalists towards the fundamental price, the market price is negatively correlated and the return has pattern [AC-C] illustrated by the AC patterns in Fig. 5.4 (c) for $\mu = 5$.

Wealth Proportions. Since $m = 0$, both the absolute and market wealth proportions are the same. Fig. 5.4 illustrates the wealth proportions of the fundamentalists. Because of the dominance of the fundamentalists, as illustrated in Fig. 5.4, their wealth proportions increase from the initial level of 50% up to 76.2% for $\mu = 1$, 85.2% for $\mu = 3$, and 99.2% for $\mu = 5$ over the whole time period and even become more significantly as the market maker over-react more and more. Therefore, in terms of the relative wealth in the market, the trend following strategy becomes less and less popular, although their market fraction is still at 50%.



(a) (b) (c)

FIGURE 5.5. Effect of μ: Prices, returns, wealth proportions, and ACs of the returns for (a) $\mu = 0.3$, (b) $\mu = 0.6$, and (c) $\mu = 1$. Here $m = 0, \delta = 0.85, \gamma = 2.1$.

Remark. To see the effect of μ on the price dynamics and market behaviour over the stability region Ω_T where the market is dominated by the trend followers, we select $\gamma = 2.1$, instead of $\gamma = 0.3$ in the above analysis. In this case, the fundamental steady state of the deterministic system is stable for $\mu < \bar{\mu}_2^* = 0.411872$ (see also Table 5.1) and $\mu = \bar{\mu}_2^*$ leads to a Hopf bifurcation. Fig. 5.5 illustrates the price and wealth dynamics for $\mu = 0.3, 0.6$ and 1. In general, it shows a similar feature to the model of the trend followers only discussed in section 4. Prices become more volatile as μ increases while the returns have pattern [AC-F]. The prices and hence returns are positive correlated, but because of the existence of the fundamentalists, the level of the positive ACs of the returns is reduced significantly as lag increases. In addition, as the result of the over-reaction from the market maker, the demand from fundamentalists increases, and hence some negative ACs for the market prices and returns become significant, as indicated in Fig. 5.5 (b) for $\mu = 0.6$ and (c) for $\mu = 1$. Because of the randomness of the fundamental price, the trend generated from the trend followers becomes less informed and hence the wealth proportion of the fundamentalists is increased from their initial level of 50% to about 68-70%. It is however interesting to notice that, unlike the previous case, the wealth proportion of the fundamentalists does not improved significantly as μ increases.

5.3.3. *Dynamics of the Geometric Decay Rate of the Trend Followers δ.* To see the effect of the geometric decay rate of the trend followers on the price dynamics, we choose in this subsection

$$\alpha = 0, \qquad m = 0, \qquad \gamma = 2, \qquad \mu = 1, \qquad w_{1,0} = 0.5$$

and let $q = 0$ be the case when the underlying fundamental price is the constant $\bar{P} = \$100$ and $q = 1$ be the case when the underlying fundamental price follows the same random walk process. Based on the selected parameters, for the deterministic system, there exists $\delta^* \in (0.7, 0.8)$ such that the fundamental steady state is stable for $\delta < \delta^*$ and $\delta = \delta^*$ leads to a Hopf bifurcation. To see the dynamics of the complete model near the bifurcation value, we select $\delta = 0.7$ and 0.8 in the following simulations and the corresponding results are shown in Figs. 5.6-5.7.

<u>*Price and Wealth Dynamics.*</u> When the underlying fundamental price is the constant $\bar{P} = \$100$, that is when $q = 0$, the market prices converge to the fundamental price for $\delta = 0.7$ and oscillate periodically near the fundamental price for $\delta = 0.8$. However, when the fundamental price follows the random walk process, the trend generated by the trend followers become less informed and hence the prices for both $\delta = 0.7$ and 0.8 follow the fundamental price closely. This comparison is illustrated in Fig. 5.6 (a). For $m = 0$, both the absolute and market wealth proportions are the same. For $q = 0$, the wealth proportion of the fundamentalists stays at their initial wealth level for $\delta = 0.7$, but declines by about 2-3% over the whole time period for $\delta = 0.8$, shown in Fig. 5.6(b). However, for $q = 1$, their wealth proportion increases from the initial level of 50% to about 68-70% for both $q = 0.7$ and 0.8, shown in Fig. 5.6(c).

<u>*Returns and ACs.*</u> The corresponding return series and the distribution densities and ACs of the returns are plotted in Fig. 5.7. For $q = 0$ and $\delta = 0.7$, the return is normally distributed with no significant pattern on ACs, reflecting the underlying dividend

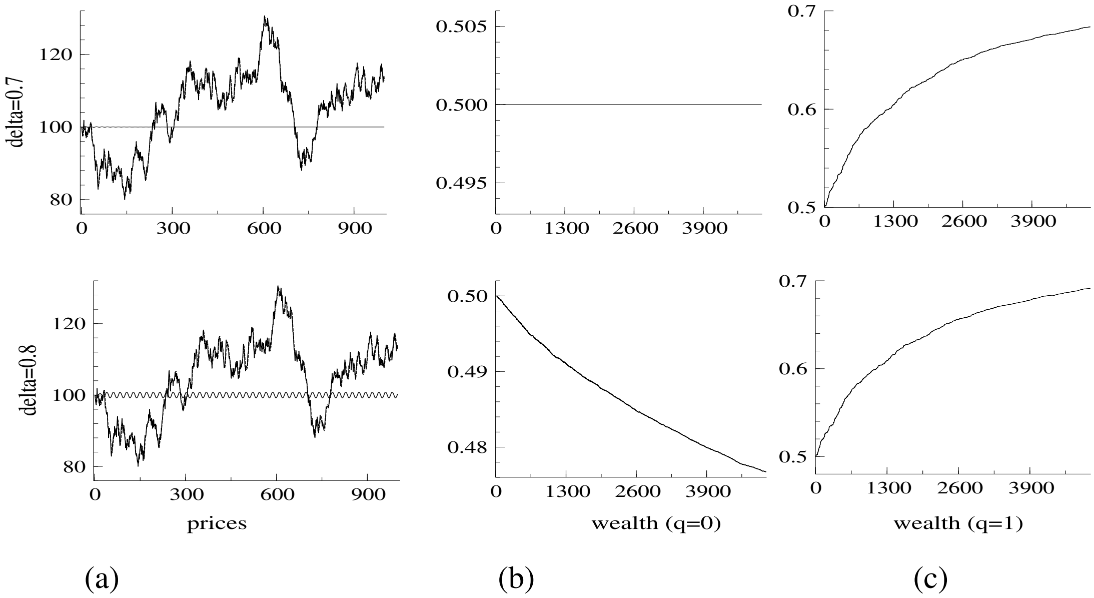


FIGURE 5.6. Effect of δ: Price series for both $q = 0$ and $q = 1$ (a) and Wealth proportions of the fundamentalists for $q = 0$ (b) and $q = 1$ (c) with $\delta = 0.7, 0.8$. Here $\alpha = 0, m = 0, \gamma = 2, \mu = 1, w_{1,0} = 0.5$.

process. For $q = 0$ and $\delta = 0.8$, the return displays a bi-mode distribution and strong oscillating AC pattern, reflecting the periodical fluctuation of the market price and the underlying dynamics of the deterministic system. For $q = 1$, the returns display excess volatility and volatility clustering for both $q = 0.7$ and 0.8. The ACs of the returns follow pattern [AC-F]: Acs are significantly positive for the first few lags and less significantly negative for the following lags, reflecting the geometric decay memory on the past prices from the trend followers and the market price reverting activity to the fundamental price from the fundamentalists.

5.3.4. *Dynamics of the Extrapolation Rate of the Trend Followers* γ. For $\alpha = 0, m = 0, \mu = 1, \delta = 0.85$, numerical simulations are conducted for $\gamma = 1, 2$ and 3. It is found that, as γ increases, the price and wealth dynamics and returns have a similar feature as the case when δ increases discussed in the previous subsection.

In the above analysis, we assume that $\alpha = 0$, that is the fundamentalists adjust their expected price instantaneously towards the fundamental price which follows the random walk process. As α increases, the speed of such adjustment from the fundamentalists is slowing down. This may partially reflect that the fundamentalists are less informed or confident on the fundamental price, partially because of the information cost. Consequently, the role of the fundamentalists on the market price becomes less dominated and less important, leading the returns and ACs to display features when the

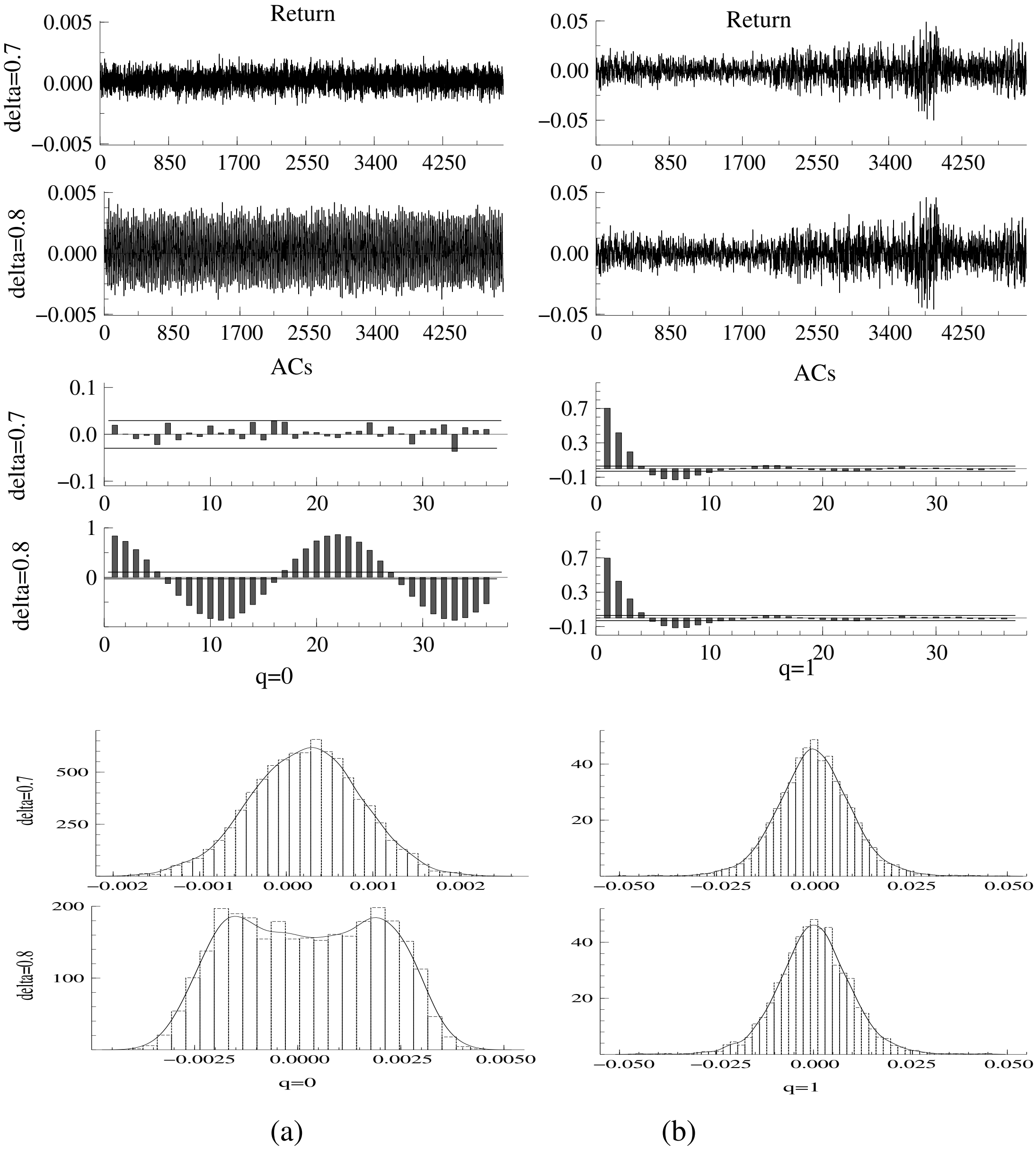


FIGURE 5.7. Effect of δ: Return series, distribution density and ACs of the returns for both $q = 0$ (a), and $q = 1$ (b), with $\delta = 0.7, 0.8$. Here $\alpha = 0, m = 0, \gamma = 2, \mu = 1, w_{1,0} = 0.5$.

market prices are more dominated by the trend followers. In the rest of the discussion, we demonstrate such changes by examining few cases when $\alpha \in (0, 1]$.

5.3.5. *Dynamics of the Extrapolation Rate γ When $\alpha = 1$.* We first examine another extreme case when $\alpha = 1$, that is when the fundamentalists do not adjust their expected price towards the fundamental price. For illustration, we select

$$\alpha = 1, \qquad m = 0, \qquad \mu = 1, \qquad \delta = 0.85, \qquad w_{1,0} = 0.5$$

and different $\gamma = 0.3, 0.4, 0.5$ and 0.6. For the select parameters, there exists a Hopf bifurcation value $\gamma^* \in (0.4, 0.5)$ such that the market price converges to the fundamental price for $\gamma = 0.3, 0.4$ and fluctuates around the fundamental price for $\gamma = 0.5, 0.6$. The price series and wealth proportion for both $q = 0$ and $q = 1$ are plotted in Fig. 5.8. Because of $\alpha = 1$, the random fundamental price has no influence on the market price. Therefore, the indicator parameter q stands for the demand shocks.

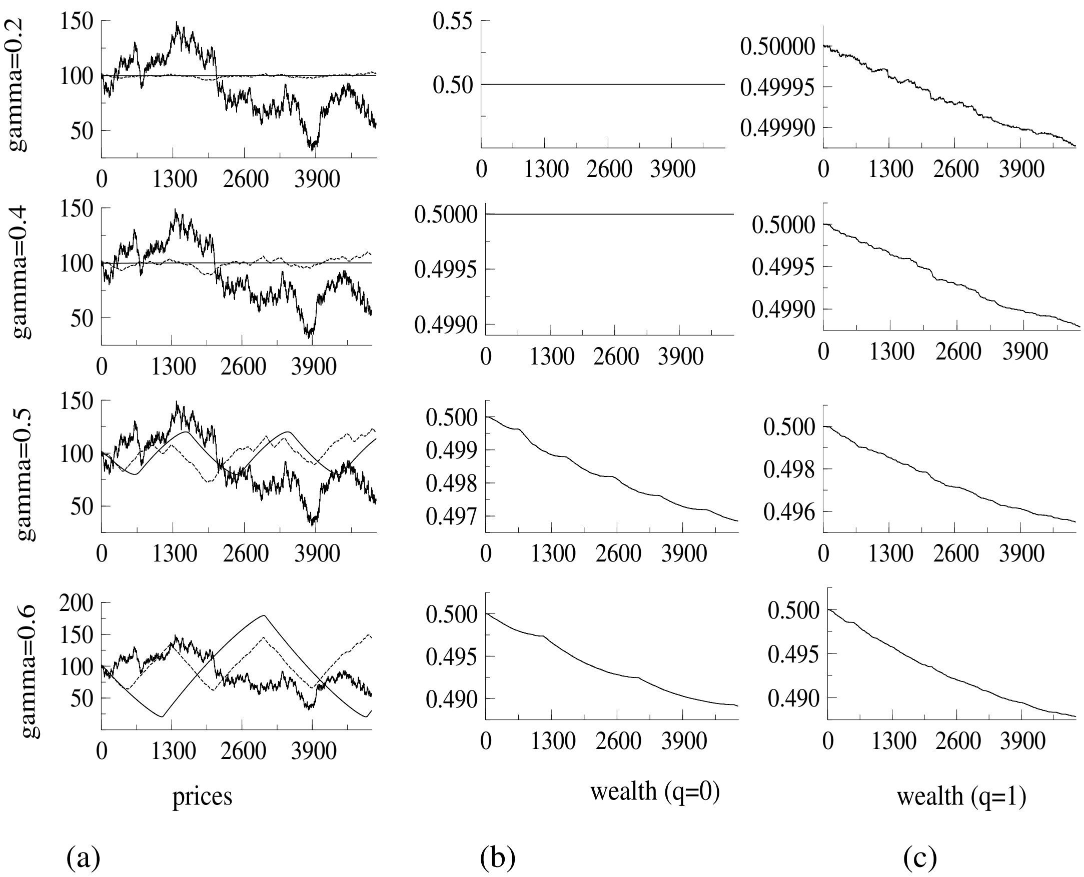


FIGURE 5.8. Effect of γ when $\alpha = 1$: Price series (a), wealth proportions for $q = 0$ (b) and $q = 1$ (c) with different values of $\gamma = 0.3, 0.4, 0.5, 0.6$.

Prices and Wealth Proportions. One can see from Fig. 5.8(a) that, when trend followers extrapolate weakly (with $\gamma = 0.3, 0.4$), the market price fluctuate around the constant fundamental price $\bar{P} = \$100$ and the wealth proportions stay at about the same level for both without ($q = 0$) and with ($q = 1$) demand shocks, although the wealth proportion of the fundamentalists is dropped slightly for $q = 1$. However, when trend followers extrapolate strongly (with $\gamma = 0.5, 0.6$), the market prices are driven away from both the constant and random fundamental prices. In addition, the dispersion and the frequency of oscillation of the market price increases as γ increases, shown in Fig. 5.8(a) for $\gamma = 0.6$. The wealth proportions of the fundamentalists decline by about 1-2% over the whole time period for both $q = 0$ and $q = 1$.

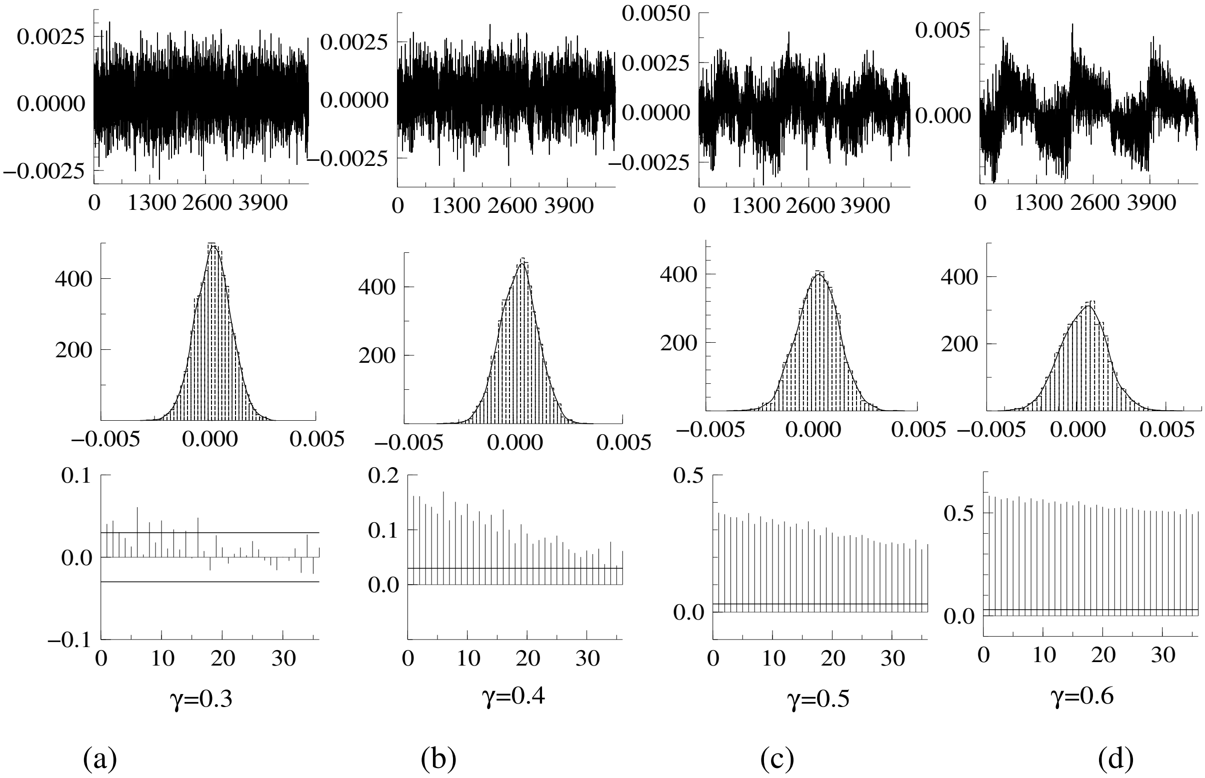


FIGURE 5.9. Effect of γ when $\alpha = 1$: Returns, distribution densities and ACs of the returns for $\gamma = 0.3, 0.4, 0.5, 0.6$ and $q = 1$.

Returns and Distribution Densities and ACs of the Returns. The corresponding returns and their distribution densities and ACs are plotted in Fig. 5.9 for $\gamma = 0.3, 0.4, 0.5, 0.6$. For $\gamma = 0.3$, the return has skewness of -0.022445, kurtosis of 2.9767, Jarque-Bera of 0.479305 and probability of normality 0.7869. The ACs of the return are not significant for all the lags. As γ increases, normality of the return distribution is destroyed, volatility clustering appears and ACs have pattern [AC-E]: ACs are significantly positive across all the lags.

Overall, when the fundamentalists do not adjust their expected towards the fundamental price (or have no information on the fundamental price), they simple become trend followers with no memory on the past price. In such case, the market prices become more volatile as the trend followers extrapolate strongly and returns have AC pattern [AC-E]. In terms of the wealth dynamics, the wealth proportions of the fundamentalists are reduced, but not significantly, over the whole time period.

5.3.6. *Dynamics of the Speed of Expected Price Adjustment of the Fundamentalists Towards the Fundamental Price α.* After examining the two extreme cases of $\alpha = 0, 1$, we now explore the price and wealth dynamics and market behaviour for $\alpha \in (0, 1)$. In this subsection, we select

$$m = 0, \qquad \gamma = 2, \qquad \delta = 0.85, \qquad \mu = 0.4, \qquad w_{1,0} = 0.5$$

and let q be the indicator parameter for the underlying fundamental price. To see the effect of α, we select $\alpha = 0, 0.5, 0.9, 0.995$ for our simulations in Fig. 5.10.

Note that a low (high) value of α corresponds to a high (low) speed of the price adjustment of the fundamentalists towards the fundamental price. One can see from Fig. 5.10 that, for $\alpha = 0$ and $q = 0$, the market price converges to the constant fundamental price and the wealth proportion does not change. However, the wealth proportion of the fundamentalists increases by about 17% for $q = 1$. For $\alpha = 0.5$, the wealth proportion of the fundamentalists is reduced by about 2-3% for $q = 0$, but increased by about 7% for $q = 1$. As the speed of the price adjustment of the fundamentalists towards the fundamental price decreases (that is as α increases), the market price are driven away from the fundamental price for both $q = 0$ and $q = 1$, as the case for $\alpha = 0.9, 0.995$ in Fig. 5.10. With slow adjustment of their expected price from the fundamentalists towards the fundamental price, the market prices are largely influenced by the oscillating trend generated by the trend followers. This results in declines of the wealth proportions of the fundamentalists by about 3-5% over the whole time period for both $q = 0$ and 1.

In terms of the return series, as the fundamentalists adjust their expected price towards the fundamental price slowly, they share the same feature as the case when the market is dominated by the trend followers (e.g. the dynamics of γ when $\alpha = 1$ and γ is high, discussed in the previous subsection). Because of the dominance of the trend followers, the returns tend to be positively correlated (but decreasing as lag increases) as α increases.

It is interesting to notice that, when the fundamentalists have less confidence on the convergence of the market price to the fundamental price, the market price can be

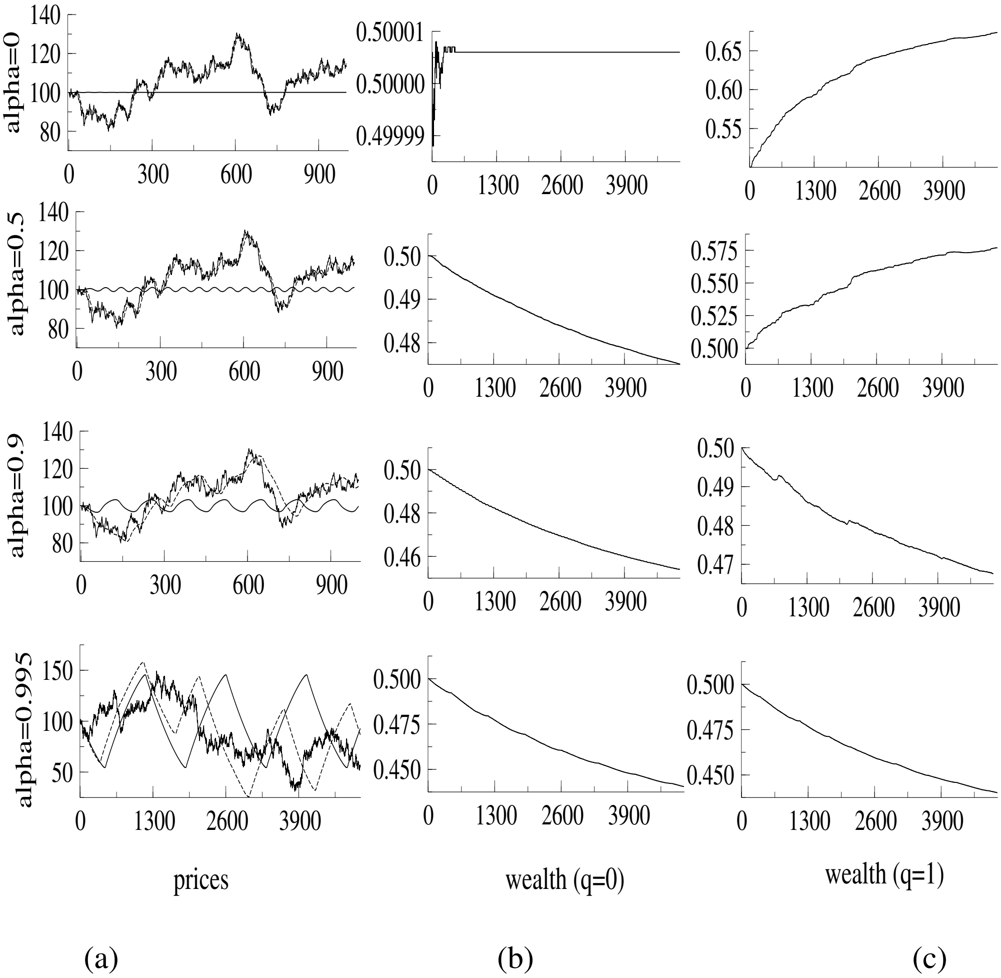


FIGURE 5.10. Effect of $\alpha \neq 1$: Price series, and wealth proportions of the fundamentalists for both $q = 0$ and $q = 1$ with different value of $\alpha = 0, 0.5, 0.9, 0.995$. Here $m = 0, \gamma = 2, \delta = 0.85, \mu = 0.4. w_{1,0} = 0.5$.

driven away by the trend followers and the wealth proportions of the trend followers increases. However, the level of the increase of their wealth proportions is relative low, unlike the case when the market is dominated by the fundamentalists, the wealth proportions of the fundamentalists increased significantly. This indicates that the profitability of the trend following strategy is limited and the trend followers may be driven out of the market over long time periods when the fundamentalists become more informed and more confident about the fundamental price. To be able to survive, the trend followers may have to adapt their beliefs from time to time, leading to future research on adaptive model. Of course, the existence of such fundamentalists having full information to know the fundamental price is also questionable.

5.3.7. *Dynamics of the Geometric Decay Rate* δ *When* $\alpha \in (0, 1)$. For $\alpha = 0$ and δ near the Hopf bifurcation boundary, we have seen from out previous discussion that the market prices follow the random fundamental price closely and the wealth proportions of the fundamentalists can be increased by about 20%. We now examine the case when $\alpha \neq 0$. In this subsection, we select

$$m = 0, \qquad \mu = 0.4, \qquad \alpha = 0.9, \qquad \gamma = 2, \qquad w_{1,0} = 0.5$$

and let $q = 0$ ($q = 1$) be the case when the underlying fundamental price is the constant (follows the same random walk process).

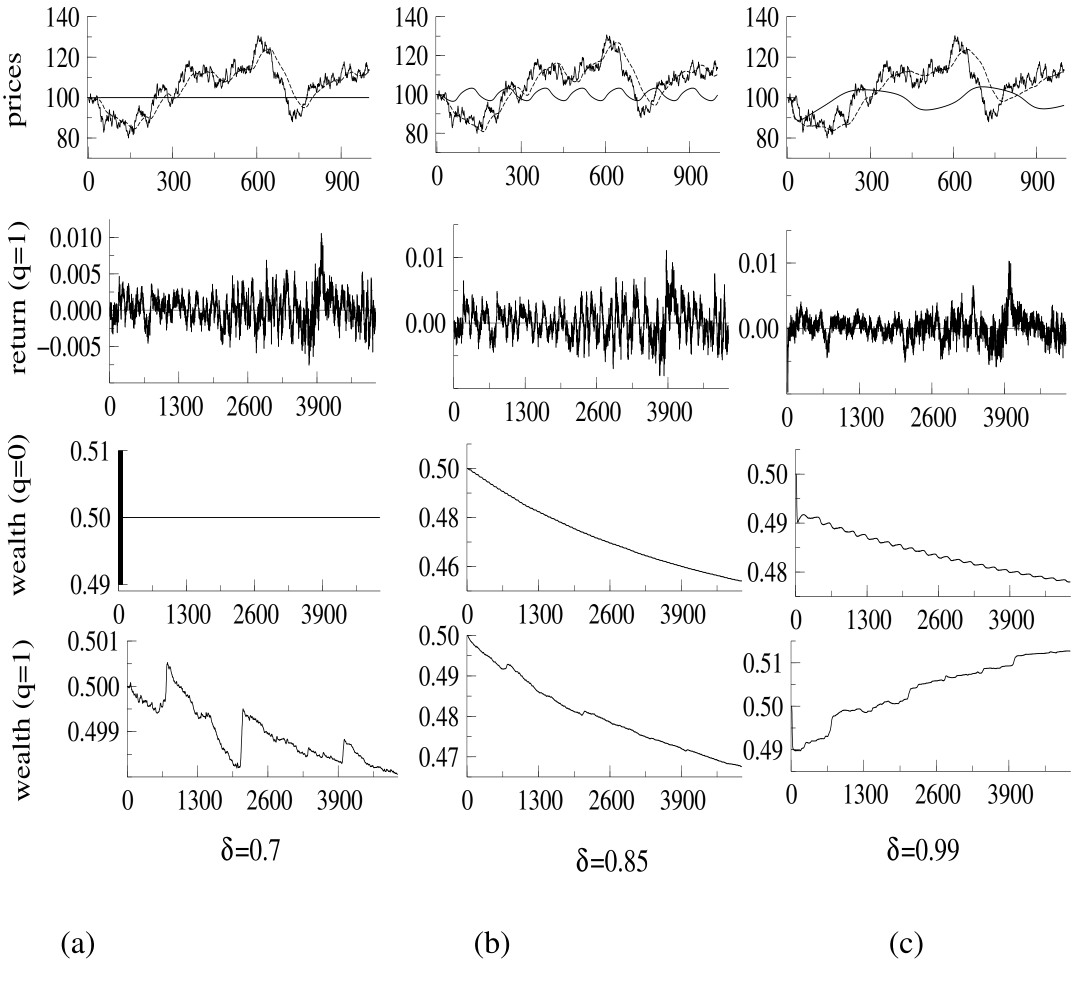


FIGURE 5.11. Effect of the decay rate δ: Price, return and wealth proportion series for $q = 0, 1$ and $\delta = 0.7, 0.85, 0.99$. Here $m = 0, \mu = 0.4, \alpha = 0.9, \gamma = 2, w_{1,0} = 0.5$.

Fig. 5.11 plots the price, return and wealth proportion series for $q = 0, 1$ and $\delta = 0.7, 0.85, 0.99$.

For $q = 0$, the market price converges to the constant fundamental price for $\delta = 0.7$ and periodically oscillates around the constant fundamental price for $\delta = 0.85, 0.99$. Furthermore, as the result of high decay rate, the market price has long memory and hence the frequency of the oscillation decreases as δ increases. The wealth proportions of the fundamentalists is dropped by about 3-5% over the whole time period.

For $q = 1$, the market prices follow the random fundamental price approximatively when the geometric decay rate is low (e.g. for $\delta = 0.7$ and 0.8) and the wealth proportions of the fundamentalists declines slightly (by 0.1$\%$ for $\delta = 0.7$ and 3.5$\%$ for $\delta = 0.85$). However, as δ increases (to $\delta = 0.99$), the market price is driven away from the fundamental price and the fundamentalists benefit from the large oscillating trend, as the result the wealth proportion of the fundamentalists increases slightly (by about 1-2$\%$).

It is interesting to notice that, when $\delta = 0.85$, the dispersion of the trend generated by the trend followers is about the right size for the trend followers to generate a relative high wealth proportion. On the one hand, if such dispersion is not enough, the price trend generated by the trend followers becomes less informative, do not help the trend followers. On the other hand, if such dispersion is too large, the fundamentalists can benefit from the big swings with low frequency to accumulate more wealth proportion than the trend followers do.

Remarks: The effects of other parameters, such as the market fraction m and the speed of the price adjustment of the market maker μ when $\alpha \in (0, 1)$ are also examined and numerical simulations (not reported here) lead to the following broad features.

- *Dynamics of the market fraction* m *when* $\alpha = 0.9$. We select $\alpha = 0.9, \gamma = 2, \mu = 0.4, \delta = 0.85$ and different values of $m = 0.5, 0, -0.5$. For the corresponding deterministic system, the market price converges to the constant fundamental price for $m = 0.5$ and oscillates periodically around the constant fundamental price for $m = 0, -0.5$. When the underlying fundamental price follows the same random walk process, the market price is dominated by the price trend generated by the trend followers, leading volatility clustering and positively significant ACs on the returns. The absolute wealth proportions of the fundamentalists are dropped by about 2$\%$, 4$\%$ and 6$\%$ for $m = 0.5, 0$ and -0.5, respectively. Correspondingly, the market wealth proportions of the fundamentalists are dropped from their initial levels of 75$\%$, 50$\%$. 25$\%$ to 73.5$\%$, 46.7$\%$ and 21.2$\%$, respectively.
- *Dynamics of the speed of price adjustment of the market maker* μ *when* $\alpha = 0.9$. We select $\alpha = 0.9, \gamma = 2, m = 0, \delta = 0.85$ and different values of

$\mu = 1, 0.5, 0.1$. For the corresponding deterministic system, the market price converges to the constant fundamental price for $\mu = 0.1$ and oscillates periodically around the constant fundamental price for $\mu = 0.5, 1$. When the underlying fundamental price follows the same random walk process, the market prices become more volatile as the market maker adjusts the prices quickly. The returns have oscillating AC pattern [AC-F]: with positive ones for the small lags (between 1 and 18) and negative for high lags for $\mu = 1$, persistent and decay positive ACs for $\mu = 0.5$, and persistent (and not much decay) ACs for $\mu = 0.1$. For both $\mu = 1$ and 0.5, the wealth proportion of the fundamentalists declines by about 6% and 4% for $\mu = 1$ and 0.5, respectively. However, for $\mu = 0.1$, their wealth proportion increases by about 1.5%.

5.3.8. *Overall Features.* Based on the above analysis, one can see that the price and wealth dynamics of the stochastic system of the market fraction are closely underlined by the dynamics of the underlying deterministic system and the market behaviour is jointly determined by the fundamentalists, the trend followers and the market maker. When the market is dominated by either the fundamentalists or the trend followers, the price and wealth dynamics and the statistic properties of the return series can be characterized by the market maker model with one type of traders only, discussed in Sections 3 and 4. The main features of the complete model can be summarised as follows.

- With the rest of parameters fixed, the market is dominated by the fundamentalists (trend followers) under either one of the following cases, in particular, when parameters belong to the stability region Ω_F (Ω_T):
  - High (low) market fraction of the fundamentalists;
  - High (low) speed of the expected price adjustment from the fundamentalists towards the fundamental price;
  - Low (high) geometric decay rate from the trend followers;
  - Low (high) extrapolation from the trend followers.
- When the market is dominated by the fundamentalists, we obtain the following results.
  - The market prices tend to follow the fundamental price closely.
  - Returns tend to be less (more) volatile when the market marker adjust price slowly (quickly) and consequently, ACs have patterns [AC-A, D]([AC-B, C]).
  - Both the absolute and market wealth proportions of the fundamentalists increase significantly over the whole time period.

  In particular, when the fundamentalists become more informed and more confident about the convergence to the fundamental price, the wealth proportions of the trend followers can be reduced dramatically over the long-run. This partially supports the traditional view in economic and finance theory, for example

Friedman (1953), that irrational traders would be driven out of the market by the rational traders in long-run.

- When the market is dominated by the trend followers, the following results are obtained.
  - The market prices is driven away from the fundamental price, the dispersion and frequency of oscillation depend on the extrapolation rate and geometric decay rate from the trend followers. The oscillating frequency can be reduced by either quick price adjustment from the market maker or strong extrapolation from the trend followers.
  - Returns tend to be more volatile and ACs have patterns [AC-D, E, F] with long memory for high decay rate.
  - Both the absolute and market wealth proportions of the trend followers increase insignificantly when they extrapolate weakly with low geometric decay rate. However, their wealth proportions may even be reduced when the market prices have big swings with low frequent oscillations.

  In such case, the level of significance of ACs can be reinforced as either the trend followers extrapolate strongly or the market maker adjust the market price quickly.
- When two types of traders are balanced (for parameters near the jointed part of two stability regions Ω_F and Ω_T), the market is jointly determined by both types of traders.
  - The market prices have no regular cyclical trend and they tend to depart from the fundamental price from time to time. Both excess volatility and volatility clustering can be observed.
  - Returns tend to have AC pattern [AC-F]: ACs oscillate and decay with significantly positive for small lags and negative for high lags. The frequency of the oscillation of ACs tends to be high when the decay rate is low and to have a long memory when the decay rate is high.
  - Wealth proportions for the fundamentalists increase in most of cases.
- Under- and over-reaction from different types of traders lead to significant AC patterns.
  - Under-reaction from the market maker and weak extrapolation from the trend followers tend to generate AC pattern [AC-D]: ACs are positively correlated across all lags and decay as lag increases.
  - Over-reaction from the market maker or quick adjustment of their expected price from the fundamentalists towards the fundamental price tend to generate AC pattern [AC-F]. The frequency of oscillating AC is high (low) when the decay rate is low (high).
  - Trend following tend to generate AC patterns [AC-D, E].

  In general, ACs of returns tend to be positively correlated over short time periods and negatively correlated over long time periods, one of features found in empirical literature. This also reflects a popular view that the market price can be driven away from the fundamental price over a short-run, but converge to the fundamental price over long-run.

6. Conclusion

Motivated by recent development in structural agent models on asset pricing, oriented both computationally and theoretically, explanation power and calibration issue of those models, this paper presents a simple market fraction model of two types of traders—fundamentalists and trend followers—under a market maker scenario. It is found that asset prices, wealth dynamics and market behaviour are characterised by and related closely to the dynamics of the underlying deterministic system. The model is able to explain various market behaviour, and generate some of the stylized facts, such as excess volatility, volatility clustering, skewness and kurtosis. Excess volatility is created by various trading process, such as over-reaction from the market maker and quick adjustment of the expected price from the fundamentalists towards the fundamental price. In terms of the market prices, on the one hand, they tend to follow the fundamental price closely when the market is dominated by the fundamentalists, and on the other hand, they can be driven away from the fundamental price when the market is dominated by the trend followers.

Two measures on the wealth dynamics of different types of traders are introduced for the first time in this paper. They correctly reflect a connection between the market dominance and the wealth dynamics. On the one hand, when the market is dominated by the fundamentalists, both absolute and market wealth proportions of the fundamentalists increase significantly in long-run. On the other hand, when the market is dominated by the trend followers, the wealth proportions of the trend followers increase insignificantly in some cases (when the dispersion of the trend generated by the trend followers is small) and even decrease in other cases (when the dispersion of the trend from the fundamental price is large). It is interesting to notice that, when the fundamentalists are less confident on the convergence of the market price to the fundamental price, the market price can be driven away by the trend followers and the wealth proportions of the trend followers increases. However, the level of the increase of their wealth proportions is relative low, unlike the case when the market is dominated by the fundamentalists. When the fundamentalists become more informed and confident about the fundamental price, the wealth proportions of the trend followers can be reduced dramatically over the long-run. This partially suggests that, in long-run, the profitability of the trend following strategy is limited, and partially supports the traditional view in economic and finance theory (see Friedman (1953)) that irrational traders would be driven out of the market by the rational traders in long-run. Of course, the existence of such fundamentalists having full information to know the fundamental price is questionable. To be able to survive, the trend followers may have to adapt their beliefs from time to time. This leads to our future research on a more rational market fraction model we proposed in Section 1 that part of the market fractions are fixed and the rest part follows some evolutionary adaptive processes, inspirited by Brock and Hommes (1997, 1998).

When the underlying deterministic system is stable, different types of autocorrelation coefficients (ACs) patterns of returns can be generated and explained through different types of bifurcation. Typically, when the market is dominated by the fundamentalists, the stability region of the deterministic system is bounded by a flip type of bifurcation. Near the flip bifurcation boundary, four types of significant AC patterns [AC-A, B, C, D] (see the discussion in Section 3) can be generated. In particular, the oscillating AC patterns [AC-B, C] with negative ACs for odd lags and positive ACs for even lags can only be generated when both the market maker and the fundamentalists over-react and the positive decaying AC patterns [AC-A, D] can be generated when the market maker under-react and the fundamentalists over-react. When the market is dominated by the trend followers, the stability region of the deterministic system is bounded by a Hopf type of bifurcation. Near the Hopf bifurcation boundary, three types of significant AC patterns [AC-D, E, F] (see the discussion in Section 4) can be generated. In particular, the oscillating AC patterns [AC-F] with positive ACs for low lags and negative ACs for high lags can be generated when both the market maker and the trend followers over-react and the positive AC patterns [AC-E] with long memory can be generated when the trend followers extrapolate strongly using high geometric decay rate. In general, when the market is balanced by both fundamentalists and trend followers, there is no significant AC patterns when the parameters are far inside of the stable boundaries, however, significant AC pattern [AC-F] is presented when the parameters near the stability boundaries. Under AC pattern [AC-F], ACs are positively correlated over short time periods and negatively correlated over long time periods, one of features found in empirical literature. This also reflects a popular view that the market price can be driven away from the fundamental price over a short-run, but converge to the fundamental price over long-run. Such explicit relation between types of bifurcation and AC patterns of returns has not been seen in the literature.

The model established in this paper has shown a very promising power in explaining asset price and wealth dynamics, and market behaviour. It also demonstrates that the theoretical oriented approach (through stability analysis and bifurcation theory) of then underlying deterministic system can be used to characterize various features of the stochastic system, such as autocorrelation patterns of returns. Given the bounded rationality of agents and profitability of various trading strategies, it is very interesting to extend the current research to a adaptive market fraction model and we leave this to our future study.

APPENDIX A. PROOFS OF PROPOSITIONS

A.1. **Proof of Proposition 2.1.** Under assumption (2.23), the demand function for the fundamentalists becomes

$$z_{1,t} = \frac{(\alpha - R)(P_t - \bar{P})}{a_1(1+r^2)\sigma_1^2}.$$

Let $(P_t, u_t, v_t) = (P_0, u_0, v_0)$ be the steady state of the system. Then (P_0, u_0, v_0) satisfies

$$P_0 = P_0 + \frac{\mu}{2}\bigg[(1+m)\frac{(\alpha - R)(P_0 - \bar{P})}{a_1(1+r^2)\sigma_1^2} + (1-m)\frac{\gamma(P_0 - u_0) - (R-1)(P_0 - \bar{P})}{a_2\sigma_1^2(1+r^2+b\,v_0)}\bigg], \tag{A.1}$$

$$u_0 = \delta u_0 + (1-\delta)P_0, \tag{A.2}$$

$$v_0 = \delta v_0 + \delta(1-\delta)(P_0 - u_0)^2. \tag{A.3}$$

One can verify that $(P_0, u_0, v_0) = (\bar{P}, \bar{P}, 0)$ satisfies (A.1)-(A.3); that is the fundamental steady state is one of the steady state of the system (2.21). It follows from (A.2)-(A.3) and $\delta \in [0, 1)$ that $P_0 = u_0, v_0 = 0$. This together with (A.1) implies that $P_0 = \bar{P}$. In fact, if $P_0 \neq \bar{P}$, then (A.1) implies that

$$\frac{1+m}{a_1}(\alpha - R) + \frac{1-m}{a_2}(1-R) = 0. \tag{A.4}$$

However, since $\alpha \in [0, 1], R = 1 + r/K > 1$ and $m \in [-1, 1]$, equation (A.4) cannot be hold. Therefore the fundamental steady state is the unique steady state of the system.

A.2. **Proof of Proposition 3.1.** Under assumptions (2.23) and $m = 1$, equation (3.1) becomes

$$P_{t+1} = P_t - \mu\frac{(R-\alpha)(P_t - \bar{P})}{a_1(1+r^2)\sigma_1^2}, \tag{A.5}$$

which can be rewritten as

$$P_{t+1} - \bar{P} = \lambda[P_t - \bar{P}], \tag{A.6}$$

where

$$\lambda \equiv 1 - \mu\frac{R-\alpha}{a_1(1+r^2)\sigma_1^2}.$$

Obviously, from (A.6), the fundamental price $\bar{P}$ is globally asymptotically attractive if and only if $|\lambda| < 1$, which in turn is equivalent to $0 < \mu < \mu_o$.

A.3. **Proof of Propositions 4.1 and 5.1.** Under assumptions (2.23), system (2.21) is reduced to the following 3-dimensional difference deterministic system

$$\begin{cases} P_{t+1} = F_1(P_t, u_t, v_t), \\ u_{t+1} = F_2(P_t, u_t, v_t), \\ v_{t+1} = F_3(P_t, u_t, v_t), \end{cases} \tag{A.7}$$

where

$$\begin{aligned}
F_1(P,u,v) &= P + \frac{\mu}{2}\Big[(1+m)\frac{(\alpha-R)(P-\bar{P})}{a_1(1+r^2)\sigma_1^2} \\
&\qquad + (1-m)\frac{\gamma(P-u)-(R-1)(P-\bar{P})}{a_2\sigma_1^2(1+r^2+b\,v)}\Big],\\
F_2(P,u,v) &= \delta u + (1-\delta)F_1(P,u,v),\\
F_3(P,u,v) &= \delta v + \delta(1-\delta)(F_1-u)^2.
\end{aligned}$$

Denote

$$a = \frac{a_2}{a_1}, \qquad Q = 2a_2(1+r^2)\sigma_1^2.$$

At the fundamental steady state $(\bar{P},\bar{P},0)$,

$$\begin{aligned}
&\frac{\partial F_1}{\partial P} = A \equiv 1 + \frac{\mu}{Q}[(1+m)a(\alpha-R) + (1-m)(1+\gamma-R)],\\
&\frac{\partial F_1}{\partial u} = B \equiv -\frac{\mu\gamma(1-m)}{Q}, \qquad \frac{\partial F_1}{\partial v} = 0;\\
&\frac{\partial F_2}{\partial P} = (1-\delta)A, \qquad \frac{\partial F_2}{\partial u} = C \equiv \delta + (1-\delta)B, \qquad \frac{\partial F_2}{\partial v} = 0;\\
&\frac{\partial F_3}{\partial P} = \frac{\partial F_3}{\partial u} = \frac{\partial F_3}{\partial v} = 0.
\end{aligned}$$

Then the Jacobina matrix of the system at the fundamental steady state J is given by

$$J = \begin{pmatrix} A & B & 0 \\ (1-\delta)A & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

and hence the corresponding characteristic equation becomes

$$\lambda\Gamma(\lambda) = 0,$$

where

$$\Gamma(\lambda) = \lambda^2 - [A+\delta+(1-\delta)B]\lambda + \delta A.$$

It is well known that the fundamental steady state is stable if all three eigenvalues λ_i satisfy $|\lambda_i| < 1$ $(i=1,2,3)$, where $\lambda_3 = 0$ and $\lambda_{1,2}$ solve the equation $\Gamma(\lambda) = 0$.

For $\delta = 0$, $\Gamma(\lambda) = \lambda[\lambda - (A+B)]$. The first result of Proposition 4.1 is then follows from $-1 < \lambda = A+B < 1$ and $\lambda = -1$ when $A+B = 1$.

For $\delta \in (0,1)$, the fundamental steady state is stable if

(i). $\Gamma(1) > 0$;
(ii). $\Gamma(-1) > 0$;
(iii). $\delta A < 1$.

It can be verified that

(i). For $\alpha \in [0,1]$, $\Gamma(1) > 0$ holds;

(ii). $\Gamma(-1) > 0$ is equivalent to

$$\text{either} \quad \gamma \geq \gamma_2 \quad \text{or} \quad 0 < \gamma < \gamma_2 \quad \text{and} \quad 0 < \mu < \mu_1,$$

where

$$\gamma_2 = \frac{1+\delta}{2\delta}[(R-1) + a(R-\alpha)\frac{1+m}{1-m}],$$
$$\mu_1 = \frac{1+\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma_2 - \gamma}.$$

(iii). The condition $\delta A < 1$ is equivalent to

$$\text{either} \quad \gamma \leq \gamma_1 \quad \text{or} \quad \gamma > \gamma_1 \quad \text{and} \quad 0 < \mu < \mu_2,$$

where

$$\gamma_1 = (R-1) + a(R-\alpha)\frac{1+m}{1-m},$$
$$\mu_2 = \frac{1-\delta}{\delta}\frac{Q}{1-m}\frac{1}{\gamma - \gamma_1}.$$

Noting that, for $\delta \in (0,1)$, $\gamma_1 < \gamma_0 < \gamma_2$, where

$$\gamma_0 = \frac{(1+\delta)^2}{4\delta}\left[(R-1) + a(R-\alpha)\frac{1+m}{1-m}\right]$$

solves the equation $\mu_1 = \mu_2$. Also, μ_1 is an increasing function of γ for $\gamma < \gamma_2$ while μ_2 is a decreasing function of γ for $\gamma > \gamma_1$. Hence the two conditions for the stability are reduced to $0 < \mu < \mu_1$ for $0 \leq \gamma \leq \gamma_0$ and $0 \leq \mu \leq \mu_2$ for $\gamma > \gamma_0$. In addition, the two eigenvalues of $\Gamma(\lambda) = 0$ satisfy $\lambda_1 = -1$ and $\lambda_2 \in (-1, 1)$ when $\mu = \mu_1$ and $\lambda_{1,2}$ are complex numbers satisfying $|\lambda_{1,2}| < 1$ when $\mu = \mu_2$. Therefore, a flip bifurcation occurs along the boundary $\mu = \mu_1$ for $0 < \gamma \leq \gamma_0$ and a Hopf bifurcation occurs along the boundary $\mu = \mu_2$ for $\gamma \geq \gamma_0$.

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