Pricing Interest Rate Exotics in Multi-Factor Gaussian Interest Rate Models

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Abstract

For many interest rate exotic options, for example options on the slope of the yield curve or American featured options, a one factor assumption for term structure evolution is inappropriate. These options derive their value from changes in the slope or curvature of the yield curve and hence are more realistically priced with multiple factor models. However, efficient construction of short rate trees becomes computationally intractable as we increase the number of factors and in particular as we move to non-Markovian models.

In this paper we describe a general framework for pricing a wide range of interest rate exotic options under a very general family of multi-factor Gaussian interest rate models. Our framework is based on a computationally efficient implementation of Monte Carlo integration utilising analytical approximations as control variates. These techniques extend the analysis of Clewlow, Pang, and Strickland [1997] for pricing interest rate caps and swaptions.

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1 Introduction

This paper develops a general framework for the efficient valuation of exotic interest rate derivatives including captions/floortions, yield spread options, average rate caps, barrier swaptions and American swaptions for a wide family of interest rate models. The models belong to the multi-factor Gaussian class of Heath, Jarrow, and Morton (HJM) [1992] models - their only required inputs are the initial yield curve and the volatility structures for pure discount bond price returns. The techniques developed do not require any restrictions on the form of the volatility functions.

In section 2 we briefly review the general class of models with which we work and the new general computational framework is described in Section 3. In Section 4 we show how the standard market instruments of caps and swaptions can be efficiently priced. The formula for caps, which has been previously described by Brace and Musiela [1995], is analytical in the sense of the Black-Scholes formula. We also show that swaption pricing in multi-factor Gaussian models reduces to an integration with dimension equal to the number of coupon payments due to the underlying bond. Although a similar result has been previously obtained by Brace and Musiela [1995], our extension is the incorporation of an extremely efficient control variate, the efficiency that we achieve suggests we can achieve realistic model calibration. In Section 5 we illustrate the general framework with examples of pricing a wide range of interest rate exotics including path-dependent options; yield spread options, captions, compound swaptions, average rate caps, barrier swaptions and American swaptions.

As far as we are aware, the problems with implementing multi-factor models has restricted the application of the HJM model, until now, to standard instruments and this exposition is the first to propose efficient implementation techniques for exotic options pricing[1]. Our approach is based on a computationally efficient implementation of Monte Carlo integration utilising eigen analysis of the covariance matrix of the pure discount bonds underlying the derivatives and secondly on the use of a general approach to obtaining analytical approximations which can be used as control variates. We also show how American style interest rate derivatives can be efficiently handled in this framework. We obtain an accurate lower bound on the price by supplementing the Monte Carlo integration with Markovian short rate trees in order to generate an approximate early exercise strategy. This method has the advantage that it handles high dimensional problems very well. We compare this method to recent work on the application of Monte Carlo methods for pricing American options. In Section 6 we calibrate a two factor model to a set of money market data and present computational evidence which indicates the viability of our methods, concentrating on accuracy and convergence. Section 7 provides a summary and conclusions.

2 The Model

Heath, Jarrow, and Morton [1992] extended the early term structure consistent framework of Ho and Lee [1986] by proposing the following stochastic differential equation (SDE) for the evolution of the instantaneous forward rate curve:

$$df(t,T) = a(t,T)dt + \sum_{i=1}^{n} s_i(t,T, f(t,T))dz_i(t) \tag{1}$$

where $f(0,t)$ is the initially observed forward curve and dz_i are independent Wiener processes[2]. Equation (1) is the most general formulation of the HJM approach with *n* sources

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1 Brace and Musiela [1995] deal with caps and swaption pricing, Carverhill and Pang [1995] look at pricing discount bond and coupon bond options, and Amin and Morton [1994] price Eurodollar futures options.

2 Note that throughout this paper we work entirely in the standard risk neutral measure with the usual assumptions (see Heath, Jarrow and Morton [1992] and Carverhill [1995]).

of randomness and with the volatilities of forward rates allowed to be dependent on the level of the forward rates. As shown by HJM the drift rate $\alpha(t,T)$ is determined by no arbitrage;

$$\alpha(t,T) = \sum_{i=1}^{n} \left\{ \sigma_i(t,T,f(t,T)) \left[\int_t^T \sigma_i(t,u,f(t,u))du \right] \right\} \tag{2}$$

and hence the forward rate process is completely specified by the volatility functions. In the following analysis we concentrate on Gaussian versions of the approach, dropping the dependence of the volatility function on the rate levels. Although the original formulation of the HJM approach is in terms of forward rates the model can be equivalently restated in terms of pure discount bond prices. Under this formulation (see for example Carverhill [1995]) bond price returns satisfy the stochastic differential equation;

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sum_{i=1}^{n} v_i(t,T)dz_i(t) \tag{3}$$

where $P(0,T)$ is the initially observed discount function and represents the price of a bond which matures with value 1, with certainty, at time *T*. The *T*-maturity bond price return volatility is related to the forward rate volatility function via;

$$\nu_i(t,T) = -\int_t^T \sigma_i(t,u)du \tag{4}$$

The process for bond price returns is more intuitive than the forward rate process; because the bonds are traded assets the drift in a risk-neutral world is simply the instantaneous rate of interest $r(t)$ or short rate. For the rest of this paper we will work with this formulation as implementation is easier and more intuitive. Another reformulation of the HJM model is in terms of the implied process for the short rate. It can be shown (Carverhill [1995]) that the short rate, $r(t)$, given by $f(t,t)$, satisfies the following SDE;

$$dr(t) = \left[\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^{n} \left\{ \int_0^t v_i(u,t) \frac{\partial^2 v_i(u,t)}{\partial t^2} + \frac{\partial v_i(u,t)^2}{\partial t} du + \int_0^t \frac{\partial^2 v_i(u,t)}{\partial t^2} dz_i(u) \right\} \right] dt + \sum_{i=1}^{n} \left. \frac{\partial v_i(u,t)}{\partial t} \right|_{u=t} dz_i(t) \tag{5}$$

The last component of the drift term, involving the integral with respect to the Brownian motions, indicates that the short rate is non-Markovian in general formulations of the HJM model.

3 The General Framework

In our general framework we work in terms of the prices of pure discount bonds which, under the risk-neutral measure, satisfy the stochastic differential equation (3) and can be written in integral form as:

$$P(T,s) = P(t,s)\exp\left(\int_t^T r(u)du\right)Y(t,T,s) \tag{6}$$

where $Y(t,T,s) = \exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_t^T v_i(u,s)^2 du + \int_t^T v_i(u,s)dz_i(u)\right\}\right]$ is an exponential martingale. Therefore, any set of pure discount bonds is jointly lognormally distributed at any set of future dates or alternatively the natural logarithms of pure discount bond prices are jointly normally distributed.

We consider any interest rate contingent claim to be a series of possibly contingent cashflows $C_k(s_k,\Theta,\{c_l P(s_k,s_l)\})$ occurring on dates s_k, k = 1, ..., m and depending on discount bonds with maturity dates also on the set of dates s_k, k = 1, ..., m with face values c_k, k = 1, ..., m and also on parameter vector Θ. The price of this contingent claim is given by

$$C(t,\Theta,\{c_k\},\{s_k\}) = \mathrm{E}_t\left[\sum_{k=1}^{m} P(t,s_k)Y(t,s_k,s_k)C_k\left(s_k,\Theta,\{c_l P(s_k,s_l)\}\right)\right] \tag{7}$$

where we have used $P(T,T) = 1 = P(t,T)\exp\left(\int_t^T r(u)du\right)Y(t,T,T)$. For clarity and without loss of generality we separate contingent claim cashflow dates (i.e. option expiration dates) from underlying pure discount bond maturity dates and label them with T. For example the price of a European coupon bond call option at time t with strike price K, maturity date T, on a bond with cashflows c_k at dates s_k represented by $Call_{CB}(t,K,T,\{c_k\},\{s_k\})$ is given by

$$
\begin{aligned}
&Call_{CB}(t,K,T,\{c_k\},\{s_k\})= \\
&\mathrm{E}_t\left[P(t,T)Y(t,T,T)\max\left(0,\sum_{k=1}^{m}c_k P(T,s_k)-K\right)\right]
\end{aligned}
\qquad (8)
$$

Note that although the dimensionality of the continuum of pure discount bond prices may be arbitrarily high, depending on the form of the volatility functions, the problem represented by equation (8) has a maximum dimensionality of m. That is the expectation is taken with respect to the m-dimensional distribution of the m pure discount bonds underlying the coupon bond.

The natural logarithms of this set of pure discount bonds has an $m*m$ covariance matrix Σ where the (k,j)-th element is represented by;

$$\Sigma_{kj} = Cov\left[\ln\left(P(T,s_k)\right),\ln\left(P(T,s_j)\right)\right] = \sum_{i=1}^{n}\left\{\int_t^T [v_i(u,s_k)v_i(u,s_j)]du\right\} \qquad (9)$$

The covariance matrix Σ can be represented by an orthogonal set of m eigenvectors $\underline{w}_i$ and associated m eigenvalues λ_i such that

$$\Sigma = \Gamma\Lambda\Gamma'$$

where $\Gamma = \begin{vmatrix} w_{11} & w_{21} & \dots & w_{m1} \\ w_{12} & w_{22} & \dots & w_{m2} \\ \dots & \dots & \dots & \dots \\ w_{1m} & w_{2m} & \dots & w_{mm} \end{vmatrix}$, $\Lambda = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_m \end{vmatrix}$ and where the prime denotes transpose. The columns of Γ are the eigenvectors.

We now have a very natural Monte Carlo integration approximation of equation (8). Let M be the number of samples or simulations and ε_i, $i = 1,\dots,m$ be independent standard normal random numbers. Therefore, we have;

$$Call_{CB}(t,K,T,\{c_k\},\{s_k\})=$$
$$\frac{1}{M}\sum_{j=1}^{M}\left[P(t,T)Y_j(t,T,T)\max\left(0,\sum_{k=1}^{m}c_kP(t,s_k)Y_j(t,T,s_k)-K\right)\right] \qquad (10)$$

where $Y_j(t,T,s_k)=\exp\left[-\frac{1}{2}\sum_{i=1}^{m}\left\{w_{ik}{}^2\lambda_i\right\}+\sum_{i=1}^{m}\left\{w_{ik}\sqrt{\lambda_i}\,\varepsilon_i\right\}\right]$

We also have a convenient way to trade-off speed and accuracy by truncation of the eigenvector set to a sub-set of the more important (measured by the size of the eigenvalues) eigenvectors. Although the integral implied by the expectation in equation (8) (or equivalently the equation (10)) has a notional dimensionality of *m*. The true dimensionality will be determined partly by how far the model is from being fully Markovian and partly by the structure of the contingent claim. We conjecture (and our numerical results support this) that when this model is calibrated to market volatility data (i.e. caps and swaptions) it will always be close to Markovian. Regarding the structure of the contingent claim we find for all the examples we consider in this paper that the dimensionality of the integral can be well approximated by the first three eigenvectors. Furthermore the primary eigenvector accounts on average for 99% of the variance. This observation allows us the formulate a general method for constructing control variates. We approximate the covariance matrix with the primary eigenvector, this effectively gives us a single factor model under which we can obtain a quasi-analytical solution (in the sense of the Black-Scholes formula) for any contingent claim with a single contingent cashflow date. Claims with two contingent cashflow dates can also usually be handled (we demonstrate this with the example of a compound option).

Finally we have a natural, one-factor, Markovian approximation to the model corresponding to the Hull and White extended Vasicek model (Hull and White [1990]). This is particularly useful for the pricing of American featured options that we discuss in section 5.

4 Caps and Swaptions Pricing

In this section we briefly review the pricing of two of the market standard interest rate derivatives - interest rate caps and European swaptions. The efficient pricing of these instruments was described in Clewlow, Pang and Strickland [1997].

Consider an *s*-year cap with reset period $\Delta\tau$ and a cap rate of r_C on an underlying principal of *L*, where the capped rate is the $\Delta\tau$ period spot rate (or LIBOR rate). The cap can be interpreted as a portfolio of 'caplets' covering each of the $s/\Delta\tau$ -1 capping periods. If $R(t, \Delta\tau)$ is the $\Delta\tau$-period spot rate at time *t* for the period between *t* and $t+\Delta\tau$ then the payoff at time $t+(k+1)\Delta\tau$ to the *k*'th caplet (k=1,...,$s/\Delta\tau$ -1) is given by

$$\max\left(0, R(t+k\Delta t, \Delta t) - r_c\right)\Delta t L \qquad (11)$$

Now we can write the spot rate $R(t,\Delta t)$, in terms of the price of a pure discount bond;

$$R(t,\Delta t) = \frac{1}{\Delta t}\left[\frac{1}{P(t,t+\Delta t)} - 1\right] \qquad (12)$$

The payoff in equation (11) can be written as (see Hull [1989] or Strickland [1996]);

$$\max\left(0,1 - \frac{1+r_c\Delta t}{1+R(t+k\Delta t, \Delta t)\Delta t}\right)L \qquad (13)$$

or in terms of the pure discount bond price

$$\max\left(0,1-\left(1+r_c\Delta t\right)P(t+k\Delta t, t+(k+1)\Delta t)\right)L \qquad (14)$$

Therefore a caplet is equivalent to a put option with a strike price of 1 and maturity date at the beginning of the capping period on a discount bond with face value $\left(1+r_c\Delta t\right)$ and maturity date at the end of the capping period. A cap is therefore a portfolio of put options on discount bonds and a floor is a portfolio of call options on discount bonds.

A number of authors have shown (see for example Brace and Musiela [1995], Clewlow *et al* [1997]) that the price of a call option at date *t* with strike price *K* and maturity date *T* on a pure discount bond with maturity date *s* in this framework is given by

$$c(t,K,T,s) = P(t,s)N(h) - KP(t,T)N(h-w) \qquad (15)$$

where

$$h = \frac{\ln\left(\dfrac{P(t,s)}{P(t,T)K}\right) + \dfrac{1}{2}w}{\sqrt{w}} \qquad \text{and} \qquad w = \sum_{i=1}^{n}\left\{\int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du\right\}$$

The implication of equation (15) is that for multi-factor Gaussian versions of the HJM model, evaluating European options on pure discount bonds is as straightforward as using the Black-Scholes equation. The calculation requires only univariate integrations involving the volatilities of the discount bonds maturing at the time of the option and the bond underlying the option. This integration will usually be analytical and if not standard numerical subroutines can be used to efficiently perform the numerical integration (see for example Press *et al*. [1992]).

It is straightforward to show that European payer (receiver) swaptions can be priced as put (call) options on coupon bonds where the strike price of the option is set equal to the principal underlying the swap and the coupon is set equal to the swap rate (see for example Brace and Musiela [1995] or Strickland [1996]). Clewlow et al [1997] showed that, since a coupon bond is a portfolio of discount bonds in which the cashflow associated with the longest maturity discount bond (maturing at s_m) is much larger than the other cashflows, the coupon bond is close to lognormally distributed and highly correlated with the longest dated discount bond. They use this information to derive an analytical approximation for the coupon bond option in the form of a discount bond option with the same maturity date *T*, strike price *K'*, and face value of the discount bond *L*, where

$$L = \sqrt{\frac{Var\left[CB(T)\right]}{E\left[P(T,s_m)^2\right] - E\left[P(T,s_m)\right]^2}} \qquad (16)$$

and

$$K' = K - \left(E_t\,[CB(T)] - LE\left[P(T, s_m)\right]\right) \qquad (17)$$

where *CB*(*T*) is the price of the coupon bond at date *T*. This discount bond option can also be used as a control variate for efficient and accurate Monte Carlo valuation of the coupon bond option.

Table 1 shows that the approximation is very accurate. The table gives prices of European at-the-money and 5% in- and out-of the money options with various maturities exercising into a new 5 year swap. The maturities of the options are 6 months, 1.0. and 1.5 years. The first row for each maturity is the analytical approximation, with the second and third rows the Monte Carlo value (with 1000 simulations) and the standard error of the Monte Carlo estimate.

Table 1: Swaption Prices Obtained via Monte Carlo Simulation and the Approximation

		Moneyness		
		0.95	1.00	1.05
Option Maturity	0.5	0.0937	0.0077	0.0000
		0.0937	0.0077	0.0000
		0.0000	0.0000	0.0000
	1	0.0906	0.0108	0.0000
		0.0906	0.0107	0.0000
		0.0000	0.0000	0.0000
	1.5	0.0875	0.0133	0.0001
		0.0875	0.0131	0.0001
		0.0000	0.0000	0.0000

Carverhill and Pang [1995] also look at the pricing of coupon bond options showing how carefully chosen Martingale Variance Reduction variates can be used to value the option. However, we can interpret their variance reduction scheme as a static delta hedge whereas the methodology we employ here is equivalent to a continuously rebalanced delta hedge and the variance reduction we obtain is therefore superior.

5 Pricing Exotic Interest Rate Derivatives

In this section we illustrate the efficiency and flexibility of our general framework by applying it to the valuation of yield spread options, captions, compound swaptions, average rate caps, barrier swaptions and American swaptions.

5.1 Yield Spread Options

Let $yso(t,K,T,s_1,s_2)$ represent the price at time t of a yield spread option with strike price K that matures at time T. The payoff to the option is dependent on spot rates maturing at dates s_1 and s_2, with $t < T < s_1 < s_2$, and is given by;

$$yso(T,K,T,s_1,s_2) = \max\left(0, R(T,s_1 - T) - R(T,s_2 - T) - K\right) \tag{18}$$

Thus the payoff is the difference, if positive, between the spread between two yields and a fixed strike price, K.

The price of the option with payoff given by equation (18) under the risk neutral measure is;

$$\begin{aligned} &yso(t,K,T,s_1,s_2) = \\ &\mathrm{E}_t\left[P(t,T)Y(t,T,T)\max(0, R(T,s_1 - T) - R(T,s_2 - T) - K)\right] \end{aligned} \tag{19}$$

Writing the yields in terms of pure discount bond prices we obtain;

$$\begin{aligned} &yso(t,K,T,s_1,s_2) \\ &= \mathrm{E}_t\left[\begin{aligned} &P(t,T)Y(t,T,T) \\ &\max\left(0, \left(\frac{1}{(s_1 - T)}\left(\frac{1}{P(T,s_1)} - 1\right)\right) - \left(\frac{1}{(s_2 - T)}\left(\frac{1}{P(T,s_2)} - 1\right)\right) - K\right) \end{aligned}\right] \end{aligned} \tag{20}$$

By changing numeraire to the pure discount bond which matures at T (the forward measure) we obtain;

$$yso(t,K,T,s_1,s_2) = P(t,T)$$

$$\mathrm{E}_t^T\left[\max\left(0,\left(\frac{1}{(s_1-T)}\left(\frac{1}{\frac{P(T,s_1)}{P(T,T)}}-1\right)\right)-\left(\frac{1}{(s_2-T)}\left(\frac{1}{\frac{P(T,s_2)}{P(T,T)}}-1\right)\right)-K\right)\right] \tag{21}$$

where $E_t^T[]$ indicates expectation with respect to the $P(t,T)$-numeraire and where

$$\frac{P(T,s_k)}{P(T,T)} = \frac{P(t,s_k)}{P(t,T)} \exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_t^T\left(v_i(u,s_k)-v_i(u,T)\right)^2 du + \int_t^T (v_i(u,s_k)-v_i(u,T))dz_i(u)\right\}\right] \tag{22}$$

Rearranging equation (21) we obtain

$$yso(t,K,T,s_1,s_2) = P(t,T)$$

$$E_t^{\,T}\left[\max\left(0,\left(\frac{1}{s_1-T}\left(\frac{P(T,T)}{P(T,s_1)}\right)\right)-\left(\frac{1}{s_2-T}\left(\frac{P(T,T)}{P(T,s_2)}\right)\right)-\left(K+\frac{1}{s_1-T}-\frac{1}{s_2-T}\right)\right)\right] \tag{23}$$

It follows easily from equation (22) that

$$\ln\left(\frac{1}{s-T}\frac{P(T,T)}{P(T,s)}\right) \sim N\left(\begin{array}{c}\ln\left(\frac{P(t,T)}{P(t,s)}\right)+\frac{1}{2}\left[\sum_{i=1}^{n}-\frac{1}{2}\int_t^T\left(v_i(u,s)-v_i(u,T)\right)^2 du\right]-\ln(s-T); \\ \sum_{i=1}^{n}\int_t^T\left(v_i(u,s)-v_i(u,T)\right)^2 du\end{array}\right) \tag{24}$$

Therefore the price of a yield spread option is equivalent to the price of a spread option on two lognormally distributed variables. This equivalence makes it possible to use the method of Ravindran (1993) to derive a quasi-closed form solution for the price of a yield spread option[3]. Let

[3] The authors would like to acknowledge the assistance of Palle Broman, an MSc in Economics and Finance 1997 student at Warwick Business School, in verifying this result.

$$X = \left(\frac{1}{s_1 - T} \left(\frac{P(T,T)}{P(T,s_1)} \right) \right) \qquad Y = \left(\frac{1}{s_2 - T} \left(\frac{P(T,T)}{P(T,s_2)} \right) \right) \qquad \hat{K} = \left(K + \frac{1}{s_1 - T} - \frac{1}{s_2 - T} \right)$$

and let E_X denotes the expectation over *X*. Therefore we can write equation (23) as:

$$yso(t,K,T,s_1,s_2) = P(t,T) E^T_{X,Y} \left[\max\left(0, X - Y - \hat{K}\right) \right] \tag{25}$$

By using the following property of conditional expectations we obtain:

$$yso(t,K,T,s_1,s_2) = P(t,T) E^T_Y \left[E^T_X \left[\max\left(0, X - Y - \hat{K}\right) \middle| Y \right] \right] \tag{26}$$

It can be shown that the conditional distribution of *X* given *Y* is again lognormal, therefore let

$$\ln X \middle| \ln Y \sim N\left(\bar{\mu}, \bar{\sigma}^2\right)$$

where

$$\bar{\mu} = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}\left(\ln Y - \mu_2\right), \; \bar{\sigma}^2 = \sigma_1^2\left(1 - \rho^2\right),$$

$$\mu_i = \ln\left(\frac{P(t,T)}{P(t,s_i)} \right) + \frac{1}{2}\left[\sum_{i=1}^n - \frac{1}{2}\int_t^T \left(v_i(u,s_i) - v_i(u,T)\right)^2 du \right] - \ln(s_i - T)$$

$$s_i^2 = \sum_{i=1}^n \int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du$$

with

$$r = \frac{\operatorname{cov}[\ln X, \ln Y]}{s_1 s_2} = \frac{\sum_{i=1}^n \left\{ \int_t^T \left(v_i(u,s_1) - v_i(u,T)\right)\left(v_i(u,s_2) - v_i(u,T)\right) du \right\}}{\sqrt{\sum_{i=1}^n \int_t^T \left(v_i(u,s_1) - v_i(u,T)\right)^2 du} \sqrt{\sum_{i=1}^n \int_t^T \left(v_i(u,s_2) - v_i(u,T)\right)^2 du}}$$

The inner expectation of equation (26) can be evaluated analytically, yielding

$$yso(t,K,T,s_1,s_2) = P(t,T)E_Y^T\left[\exp\left(\overline{m}+\frac{\overline{s}^2}{2}\right)N(h)-\left(Y+\hat{K}\right)N\left(h-\overline{s}\right)\right] \qquad (27)$$

where $h = \dfrac{\overline{m} - \ln\left(\hat{K}+Y\right)+\overline{s}^2}{\overline{s}}$

The price of the yield spread option can therefore be evaluated efficiently by a one-dimensional numerical integration of the expression within the expectation in equation (27) over the distribution of Y. We present numerical results for this in the following section.

The interest rate spread options which are typically traded involve the spread between a short maturity yield and a long maturity swap rate. We define the price of a yield-swap-rate spread option under the risk-neutral measure as

$$\begin{aligned}&ySRso\left(t,K,T,\{s_k\}\right)=\\&\mathrm{E}_t\left[P(t,T)Y(t,T,T)\max(0,R(T,s_1-T)-SR(T,s_m)-K)\right]\end{aligned} \qquad (28)$$

where $SR(T,s_m) = \dfrac{1-P(T,s_m)}{\sum_{k=1}^{m} P(T,s_k)}$

This option fits into the general framework described in Section 3 and can therefore be efficiently priced by Monte Carlo integration. The analytical result for the yield spread option can be used as a control variate where necessary, however the numerical results in section 5 indicate that this is not usually necessary.

5.2 Captions

A caption is a European option on a cap. Let T be the maturity date of the European option with strike price K. Upon exercise the underlying will be a s year cap with reset period $\Delta\tau$

and with cap rate r_c. Using the results of Section 3 we interpret the underlying cap as a portfolio of European discount bond put options and hence the caption as an option on a portfolio of discount bond options. Therefore the price of a caption at time *t* is given within the general framework of Section 3 by;

$$caption(t, K, T, s, r_c, \Delta t) = E_t\left[\begin{array}{l} P(t,T)Y(t,T,T) \\ \max\left(0, (1 + r_c\Delta t)\sum_{k=1}^{s/\Delta t - 1} p\left(T, \frac{1}{1 + r_c\Delta t}, T + k\Delta t, T + (k+1)\Delta t\right) - K\right) \end{array} \right] \quad (29)$$

Since the underlying cap can be priced analytically as a portfolio of pure discount bond options the price of the caption can be obtained by Monte Carlo integration of the payoff of the caption at date *T* over the distribution of the underlying pure discount bonds at this date. In order to apply a good control variate for this option we use a one factor approximation to the covariance structure and apply Gaussian Quadrature directly on the payoff.

5.3 Compound Swaptions

A European compound swaption is a European option on a European swaption where as before we interpret the swaption as a European coupon bond option. Let *T* represent the maturity date of the compound option having strike price *K*. Upon exercise the underlying will be an option maturing at time T_u and strike price K_u on a coupon bond with cashflows $\{c_k\}$ at times $\{s_k\}$. If the option is a call option on a receiver swaption then the time *t* price in our general framework is given by;

$$\begin{array}{l} compound\_swaption(t, K, T, K_u, T_u, \{c_k\}, \{s_k\}) = \\ E_t\left[P(t,T)Y(t,T,T)\max\left(0, Call_{CB}(T, K_u, T_u, \{c_k\}, \{s_k\}) - K\right)\right] \end{array} \quad (30)$$

where $Call_{CB}(T, K_u, T_u, \{c_k\}, \{s_k\})$ is defined by equation (7). Instead of using Monte Carlo integration to evaluate the underlying option price we approximate this option value with the analytical approximation of Section 4. In order to apply a good control variate for this option we again use a one factor approximation and apply Gaussian Quadrature directly on the payoff of the compound option.

5.4 Average Rate Caps

We consider only fixed strike Asian call options but the method generalises easily for floating strike versions and for put options. We define the payoff of an average rate caplet to be determined by the average level of the rate to be capped (e.g. three month LIBOR) observed at a set of predetermined dates over the capping period. Specifically, the payoff of the caplet for the period from T to $T+\Delta t$ is set by the average level of the Δt period yield observed at dates $s_k = T-\Delta t + k\Delta t$, $k=1,\ldots,m$ between $T-\Delta t$ and T where $\Delta t = \frac{\Delta t}{m}$. The payoff at time $T+\Delta t$ is therefore;

$$ARcaplet(T+\Delta t) = \max(0, A(T) - r_c)\Delta t L \qquad (31)$$

where $A(T) = \frac{1}{m}\sum_{k=1}^{m} R(s_k, \Delta t)$ and L is the principal amount of the caplet.

Therefore the price of the average rate caplet at time t in terms of the prices of pure discount bonds is given by

$$ARcaplet(t) = \mathrm{E}_t\left[P(t,T+\Delta t)Y(t,T+\Delta t,T+\Delta t)\max\left(0,\frac{1}{m}\sum_{k=1}^{m}\frac{1}{\Delta t}\left(\frac{1}{P(s_k,s_k+\Delta t)}-1\right)-r_c\right)\right]\Delta t L \qquad (32)$$

If the averaging only takes into account the last observed LIBOR rate (i.e. at date T) then the average rate caplet is the same as the standard caplet and so this becomes the natural control variate.

5.5 Barrier Swaptions

Without loss of generality we concentrate on a down-and-out receiver swaption. We define a down-and-out swaption as a standard swaption except that if the underlying swap rate is below the predetermined barrier level H on any of the swap reset dates $s_k = t + k\Delta t$, $k=1,...,m$ upto the maturity date of the swaption $T = s_{m'} < s_m$ the swaption expires worthless. The payoff to the down-and-out receiver swaption is therefore

$$DAORswaption(T) = \max\left(0, \sum_{k=m'+1}^{m} K\Delta t\, P(T,s_k) + P(T,s_m) - 1\right) 1_{SR(s_k)>H;k=1,...,m'} \quad (33)$$

where K is the strike rate of the swaption, $SR(s_k)$ is the par swap rate at the reset date s_k and $1_{condition}$ is the indicator function which takes the value 1 if the *condition* is true and zero otherwise. Therefore the value of the barrier swaption is

$$DAORswaption(t) =$$
$$\mathrm{E}_t\left[P(t,T)Y(t,T,T)\max\left(0, \sum_{k=m'+1}^{m} K\Delta t\, P(T,s_k) + P(T,s_m) - 1\right) 1_{SR(s_k)>H;k=1,...,m'}\right] \quad (34)$$

The natural control variate fo this instrument is the standard European swaption for which we have the analytical approximation given in Section 4.

5.6 American Swaptions

We conclude this section by proposing a method for obtaining an accurate approximation for the value of American style options. It has long been viewed that one of the drawbacks with using Monte Carlo simulation for derivative pricing is that it is difficult to value American options (see for example the comments of Hull [1997] page 364 and Campbell, Lo, and MacKinley [1997] page 390). The problem arises because simulation methods generate trajectories of state variables forward in time, whereas a backward dynamic programming approach is required to efficiently determine optimal decisions for pricing American options. We show in this section that by supplementing the Monte Carlo integration of previous sections with Markovian short rate trees, in order to generate an early exercise strategy, we can efficiently obtain approximations for prices of American style derivatives. This methodology is an alternative to recent work in the area and which can very easily be applied to multi-factor problems. Clewlow and Carverhill [1991] and Tilley [1993] represent one of the earliest attempts of extending simulation for pricing American options, bundling paths into groups according to the underlying asset price and applying dynamic programming using these paths to estimate a single continuation value for all paths within a bundle[4]. These methods, however, are difficult to extend to multiple factors[5]. Broadie and Glasserman [1997b] propose a method based on simulated trees which generates both a lower and upper bound, both convergent and asymptotically unbiased as the computational effort increases. The main drawbacks from the perspective of this chapter is that the work is exponential in the number of early exercise opportunities (resulting from the fact that each node generates its own independent subtree) and is again problematic to extend to high dimensions. Broadie and Glasserman [1997a] develop a 'stochastic mesh' whereby all the information from one time step is used to estimate the prices at the previous time step. Although this interlocking nature complicates the development of the algorithm, the advantage is that the work is linear in the number of early exercise opportunities. Although the authors claim that the method is viable, results from the tables in their paper suggest that, with the dimensions typical in the analysis of this chapter, computation times can take "hours" (Broadie and Glasserman [1997a], comments on Table 4).

In order to obtain real time approximations we pursue an alternative strategy which involves numerically solving for an early exercise strategy in a low-dimension model similar to the higher dimensional model of interest. Perhaps the published material which is closest in spirit to ours can be illustrated by the papers of Fu and Hu [1995] and Li and Zhang [1996] who specify a parametric family to represent possible exercise regions[6].

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4 Barraquand and Martineau [1995] also propose a method that involves partitioning paths, but in the payoff space instead of the state space.

5 Raymar and Zwecher [1997] extend Barraquand and Martineau [1995] by basing the early exercise decision on a partition of two state variables.

6 Although some numerical tests have been performed on single asset problems, there has been little investigation of these methods on multiple state variable problems.

To illustrate our techniques we derive the price of a Bermudan swaption which exercises into an existing swap. This instrument, which is currently popular with practitioners, can be interpreted as an American (or Bermudan) coupon bond option. Consider an option with strike price *K* and maturity date *T* on a coupon bond with cashflows c_k on dates s_k, , $k = 1,...,m$ which can be exercised on any of the dates $s_k \leq s_{m'} = T$ (so called Bermudan style exercise). As before, in the case of a swaption the dates would be seperated by the swap reset period Δt and the cashflows would be equal to the product of the swap rate and the reset period.

Let *K*(t)* be the value of the coupon bond *CB(t)* above which it is optimal to exercise the call option early. The value of the Bermudan coupon bond option is given by

$$\begin{aligned} &C_{CB}(t,K,T,\{c_k\},\{s_k\}) = \\ &E_t\left[P(t,s^*)Y(t,s^*,s^*)\max\left(0,CB(s^*)-K\right)\middle| CB(s^*) > K^*(s^*), s^* \in [s_1,....,s_{m'}]\right] \end{aligned} \quad (35)$$

We obtain an approximate early exercise boundary by using a single factor Gaussian Markovian short rate approximation to our multi-factor non-Markovian model. Our approximating model is essentially the Hull and White Extended Vasicek model (Hull and White [1990]). For models of this class we can obtain the early exercise boundary by building a re-combining tree for the short rate, and determining the critical coupon bond prices at the early exercise dates via backwards induction in the tree. Our reason for choosing one factor rather than a two or higher factor Markovian short rate approximation is that we can obtain a very accurate early exercise boundary by increasing the number of time steps in the tree. This allows us to study the error introduced in the option value by the approximation without having to consider the discretisation error introduced by using a tree.

The stochastic differential equation for the short rate in our approximating model is

$$dr = [\theta(t) - \alpha r]dt + \sigma dz \quad (36)$$

The time-dependent function in the drift $q(t)$ allows the model to fit exactly the initial term structure of interest rates. The parameters α and σ describe the speed of mean reversion and volatility of the short rate respectively and determine the volatilities of forward rates, yields or pure discount bond prices. The volatility function of instantaneous forward rates is given by

$$s\,(t,s) = s\; e^{-a\,(s-t)} \qquad (37)$$

We 'best fit' in a least squares sense at the standard market maturities 0.25, 0.5, 1, 2, 3, 4, 5, 7, 10 years, the Hull-White volatility function to the total instantaneous forward rate volatility of our multi-factor model. A detailed numerical example is described in the following section where we also investigate the accuracy of this approximation technique.

Since a version of this work was first presented we have become aware of two authors who have suggested the same general framework[7]. Hull [1998] suggests that when valuing American options "one idea is to estimate the early exercise boundary using a Markov model and then use HJM in conjunction with Monte Carlo simulation" (page 8.18). Rebonato [1996] says that "... it might be very useful to use 'in parallel' a tree-based model which displays distributional assumptions as close as possible to the chosen HJM model. This 'germane' model mapped onto a recombining tree could provide an (almost) internally consistent approach to pricing and risk managing a complex options book" (page 332).

6 Calibration and Numerical Results

In this section we calibrate a two-factor version of our multi-factor model to market data and evaluate numerical results for the instruments discussed in Section 5. The only other published attempts of which we are aware to calibrate a model of this type to market prices of standard interest rate derivatives are Brace and Musiela [1995] and Brace, Gatarek and Musiela [1995].

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7 The idea behind pricing American options in this way was first presented at the September 1996 Financial Options Research Centre Annual Conference.

6.1 Calibration to Standard Instruments

Table 2 shows the market data used. This data consists of Eurodollar interest rates with maturities from one month out to twelve months, interest rate swap levels from two years out to fifteen years, with the volatility information consisting of standard Black volatilities for interest rate caps with maturities from one year out to ten years and interest rate swaptions with maturities from 3 months to 5 years on swaps with the standard maturities of 1, 2, 3, 4, 5, 7 and 10 years. The reset frequency for both the swaps and caps is semi-annual.

Table 2 : Money Market Data For Rates and Volatilities for 30 May 96

Eurodollar Rates				**Swap Rates**					
1m	**3m**	**6m**	**12m**	**2y**	**3y**	**5y**	**7y**	**10y**	**15y**
5.37	5.45	5.57	5.69	6.425	6.643	6.908	7.075	7.241	7.415

Interest Rate Caps Black Volatilities

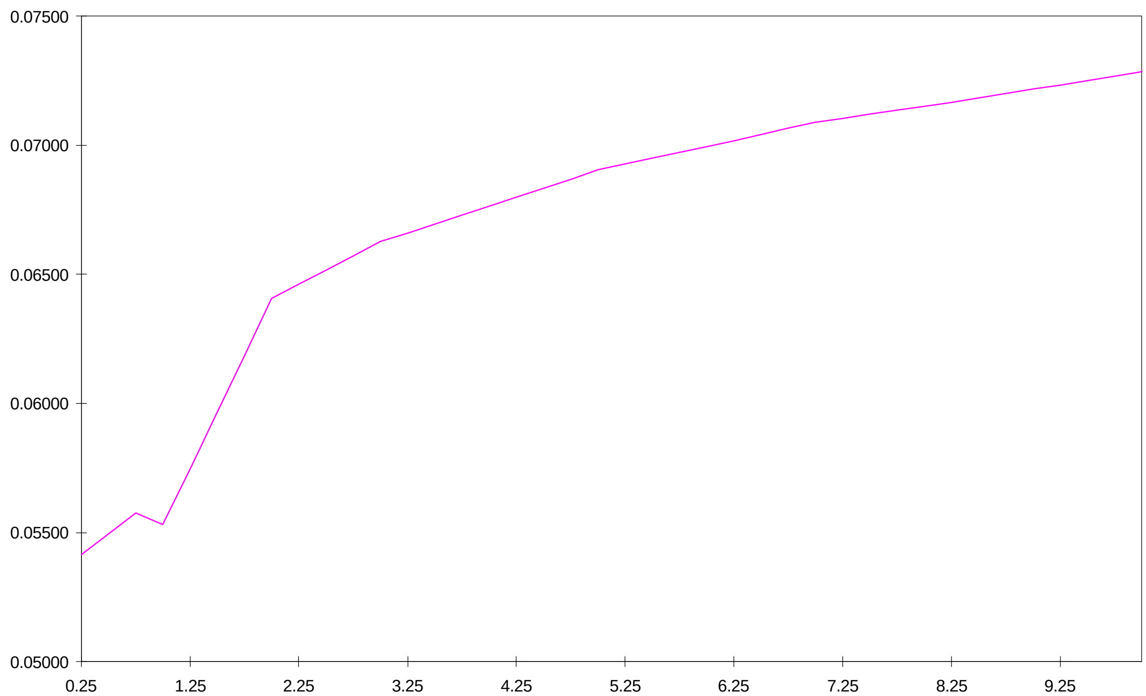
1y	**2y**	**3y**	**4y**	**5y**	**7y**	**10y**
15.5	19.0	19.5	19.2	19.0	18.0	17.0

Interest Rate Swaptions Black Volatilities

Swaption Maturity	**Swap Maturity**						
	1y	**2y**	**3y**	**4y**	**5y**	**7y**	**10y**
3m	18.0	19.6	19.0	18.7	18.5	17.0	16.0
6m	18.2	18.7	18.5	18.5	18.1	16.5	15.5
1y	20.0	19.2	18.5	17.7	17.1	16.0	15.2
2y	19.2	18.2	17.5	16.7	16.2	15.3	14.5
3y	18.2	17.5	16.7	16.2	15.5	15.0	13.7
4y	17.5	16.7	16.2	15.5	15.2	14.2	13.3
5y	15.5	19.0	19.5	19.2	19.0	18.0	17.0

Figure 1 shows the resulting yield curve. The yield curve is upward sloping from about 5.5% for the short end to just over 7.0% for a maturity of ten years. In order to construct the yield curve we used the Eurodollar rates out to twelve months and then bootstrapped into the swap rates by using linear interpolation of the swap rates to obtain the swap rates for every intermediate reset date.

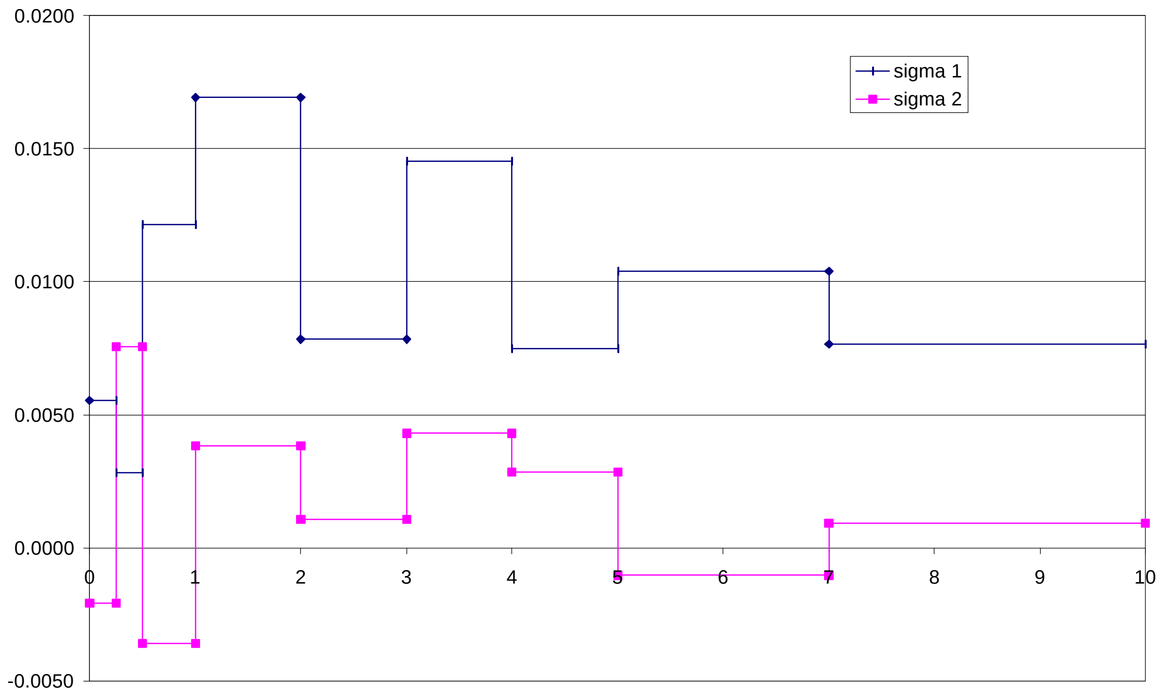
Figure 1 : Spot Rate Yield Curve for 30 May 96



The volatility functions are calibrated to the caps and swaptions data of Table 2 and for these examples are assumed to be time-homogeneous and piecewise flat. We assume that there are 9 segments determining the volatility structure and the maturities of these segments match the maturity structure of the instruments underlying the yield and volatility curves, i.e. 0 to 3 months, 3 to 6 months, 6 months to 1 year, 1 to 2 years, 2 to 3 years, 3 to 4 years, 4 to 5 years, 5 to 7 years, and 7 years onwards. The volatility functions are determined by minimising the sum of the squared differences between the market prices of the caps and swaptions and the prices of these instruments under the two factor model.

Figure 2 shows the resulting volatility functions. The first volatility function (sigma1) exhibits the peak at around 2 years which is commonly observed and then declines in an approximately exponential fashion before flattening out at longer maturities. The second volatility function (sigma2) shows a roughly linearly decreasing level of absolute volatility with increasing maturity but also exhibits a component which corresponds to a bending or twisting of the yield curve. These volatility structures broadly match the parallel shifts and twists in the yield curve which are typically found with principal components analysis.

Figure 2 : Calibrated Time-Homogeneous Piece Wise Flat Volatility Function for a Two-Factor Version of the General Model



6.2 Numerical Results

In this section we price the interest rate exotics described in Section 5 and illustrate the efficiency of our methodology. All option prices in this section are evaluated with antithetic control variates and the specific control variates described in section 5 for each instrument. All computation times are for a IBM compatible personal computer with a Pentium running at 200 MHz.

6.2.1 Yield Spread Options

Table 3 shows the prices of yield spread options with one year to maturity where the first rate has a maturity at the option maturity date of three months and second rate has a maturity of five years. A range of typical strike prices are shown. The number of simulations is 1,000. Prices are firstly shown under the assumption that the two rates are simple spot rates (the column labelled ‘yso’) using the quasi-analytical formula represented by equation (27). Secondly the prices are shown under the assumption that the short rate is a simple spot rate and the long rate is a swap rate using the Monte Carlo simulation of equation (28) with the simple spot rate spread option as a control variate (the column labelled 'ySRso'). Column 'se' is the standard error for the Monte Carlo simulation and the time taken in seconds to evaluate the prices are in the columns headed 't'. We see that for this instrument the standard error is small even though we are only performing 1000 simulations indicating the control variate works well. As we increase the strike price to 1% the value of the option decreases to zero indicating that the probability of the spread between the rates exceeding 1% is very small.

Table 3 : Prices of Yield Spread Options

Strike	yso	YSRso	se	t
-0.010	0.000291	0.003590	0.000051	0.06
-0.005	0.000063	0.001287	0.000049	0.06
0.000	0.000009	0.000305	0.000025	0.06
0.005	0.000001	0.000036	0.000007	0.06
0.010	0.000000	0.000001	0.000001	0.06

Table 4 shows the convergence of the price of the Yield-Swap Rate spread option as a function of the number of simulations 'nr' for the a strike of -1%. This shows that with 1000 simulations we are indeed getting a price accurate to the fourth decimal place.

Table 4 : Convergence of Yield Spread Option Prices

nr	ySRso	se	t
100	0.003423	0.000147	0.01
200	0.003362	0.000094	0.02
300	0.003400	0.000080	0.03
400	0.003418	0.000073	0.03
500	0.003479	0.000066	0.03
600	0.003537	0.000064	0.04
700	0.003552	0.000060	0.05
800	0.003579	0.000057	0.05
900	0.003600	0.000054	0.05
1000	0.003590	0.000051	0.06

6.2.2 Captions

Table 5 shows the prices of a one year option on a five year cap. The reset frequency for the underlying cap is assumed to be 6 months. The strike levels are chosen to be at-the-money forward, and 10% in and out of the money. The second column 'Approx' is the analytical approximation using Gaussian Quadrature. The Monte Carlo price ‘MC Price’ is evaluated using equation (29) with 10,000 simulations and the analytical approximation as a control variate. The analytical approximation can be seen to be very fast and accurate.

Table 5 : Prices of a One Year Option on a Five Year Cap

K	Approx.	t	MC Price	Se	t
0.039001	0.022836	0.01	0. 022829	0.000018	1.05
0.043334	0.020953	0.01	0. 020944	0.000021	1.05
0.047668	0.019124	0.01	0. 019115	0.000025	1.05

Table 6 shows the convergence of the prices of the ‘at the money’ one year option into a five year cap with semi-annual resets. The standard error is reduced from 0.000072 for 1000 simulations to 0.000021 for 10,000 simulations.

Table 6 : Convergence of Caption Prices

nr	MC Price	se	t
1000	0.020991	0.000072	0.11
2000	0.020977	0.000049	0.22
3000	0.020943	0.000039	0.32
4000	0.020941	0.000033	0.43
5000	0.020944	0.000030	0.53
6000	0.020934	0.000027	0.63
7000	0.020935	0.000025	0.74
8000	0.020941	0.000024	0.84
9000	0.020934	0.000022	0.94
10000	0.020944	0.000021	1.05

6.2.3 Compound Swaptions

Table 7 shows the prices of compound swaptions evaluated using equation (30) where the first option has six months to maturity and is written on a swaption which itself has six months to maturity and has as its underlying a swap with an original maturity of five years. The strike prices are again chosen to reflect ‘at the money’ and 10% ‘in the money’ and ‘out of the money’.

Approx’ is the analytical approximation derived by reducing to a one-factor model and applying Gaussian Quadrature. The column headed ‘MC Price’ is the Monte Carlo simulated price, with the standard errors and calculation time also shown for 10000 simulations. The analytical approximartion can be seen to be both fast and accurate.

Table 7 : Prices of Compound Swaptions

K	Approx.	MC Price	se	t
0.000453	0.000345	0.000343	0.000003	70
0.000503	0.000333	0.000334	0.000003	70
0.000554	0.000324	0.000326	0.000003	70

Table 8 shows the convergence of compound swaption prices for the at-the-money option, increasing the number of simulations from 100 to 1,000. The standard error is seen to reduce from 0.000111 taking 1 seconds to 0.000035 taking 8 seconds.

Table 8 : Convergence of Compound Swaption Prices

nr	MC Price	Se	t
100	0.000455	0.000111	1
200	0.000353	0.000072	2
300	0.000352	0.000062	2
400	0.000371	0.000065	3
500	0.000372	0.000056	4
600	0.000385	0.000050	4
700	0.000378	0.000045	6
800	0.000378	0.000042	6
900	0.000358	0.000038	6
1000	0.000351	0.000035	8

6.2.4 Average Rate Caps

The average rate cap provides a good illustration of the efficiency gains we obtain by determination of the effective dimensionality of the problem via the eigen analysis of the pure discount bond covariance matrix. The size of the largest three eigenvalues for this case are 0.007951, 0.000095, and 0.000005 with the remaining eigenvalues being less than $3.0*10^{-6}$. This implies that we can reduce the dimension of the simulation and still achieve good results. Table 9 shows the prices of a fixed strike average rate caplets ('

between two and two and a half years for a decreasing number of retained eigenvectors (NOE) from the full set down to the primary eigenvector only. We assume that there are five equally spaced fixings for the average during this period which results in a full set of ten eigenvectors. The control variate is the standard caplet with two years to maturity.

Table 9 : Prices of Average Rate Caplets for Numbers of Factors

NOE	Arcaplet	se	t
10	0.009451	0.000005	0.6
9	0.009456	0.000005	0.4
8	0.009458	0.000005	0.4
7	0.009462	0.000005	0.4
6	0.009458	0.000005	0.3
5	0.009455	0.000005	0.3
4	0.009457	0.000005	0.3
3	0.009459	0.000005	0.2
2	0.009464	0.000003	0.2
1	0.009475	0.000003	0.1

Table 9 confirms the results of the eigenvalue analysis - significant time savings can be made by considering only the first 3 factors with only a small loss of accuracy. Table 10 gives a set of average rate caplet prices. Prices are based on 10,000 simulations and the table shows the prices of caplets under various strike prices from 4% to 6%. The standard errors are again seen to be very small with the simulations taking only 0.3 seconds.

Table 10 : Prices of Average Rate Caplets

K	Arcaplet	se	T
0.04	0.013159	0.000003	0.3
0.05	0.009459	0.000005	0.3
0.06	0.006222	0.000007	0.3

Table 11 shows the convergence of average rate caplet prices for simulations between 1,000 and 10,000 for the two year maturing caplet with a strike price of 6%. The results again indicate that we can achieve real-time pricing.

Table 11 : Convergence of Average Rate Caplet Prices

nr	ARcaplet	se	T
1000	0.009464	0.000016	0.1
2000	0.009449	0.000012	0.1
3000	0.009449	0.000010	0.1
4000	0.009453	0.000008	0.2
5000	0.009450	0.000007	0.2
6000	0.009449	0.000007	0.2
7000	0.009454	0.000006	0.3
8000	0.009452	0.000006	0.4
9000	0.009450	0.000005	0.4
10000	0.009450	0.000005	0.4

6.2.5 Barrier Swaptions

Table 12 shows the convergence of the down and out receiver swaption price DAORswaption' where the swaption has three years to maturity on an underlying swap with five years to maturity. The strike price of the swaption is 5%, the barrier is set at a level of 4.5%.

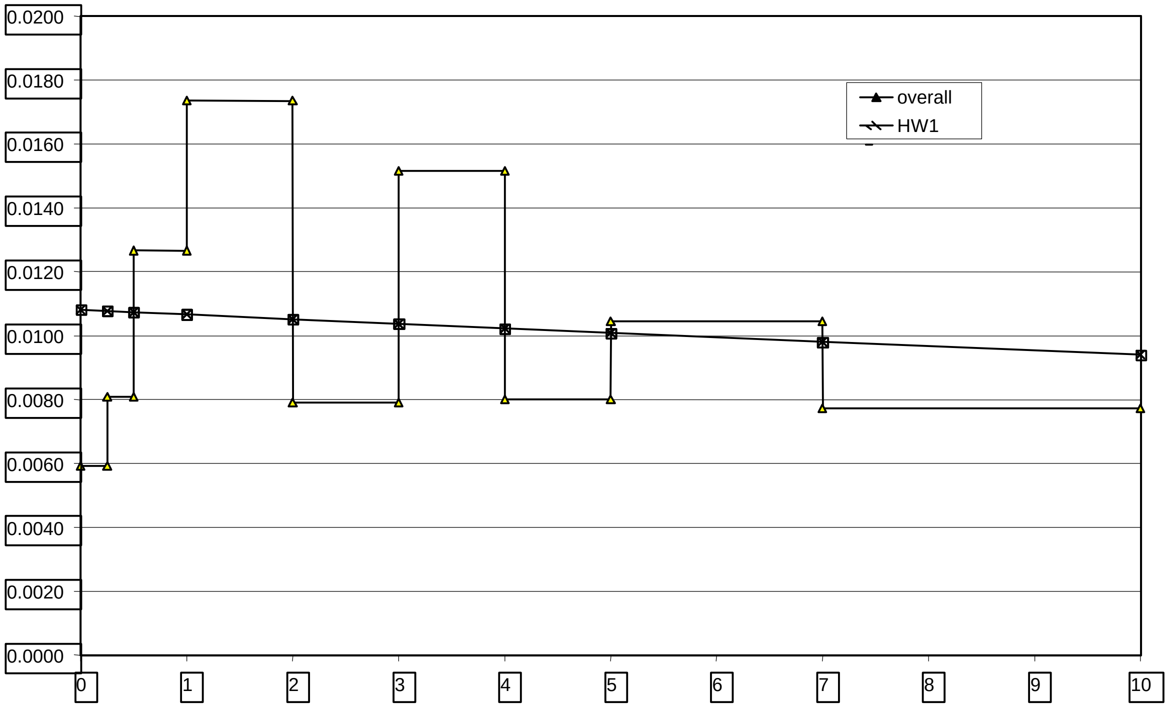
Table 12 : Convergence of Barrier Swaption Prices

nr	DAORswaption	se	t
1000	0.000107	0.000016	0.3
2000	0.000116	0.000012	0.4
3000	0.000115	0.000010	0.6
4000	0.000113	0.000008	0.8
5000	0.000112	0.000007	1.0
6000	0.000111	0.000007	1.1
7000	0.000110	0.000006	1.4
8000	0.000108	0.000006	1.6
9000	0.000110	0.000005	1.8
10000	0.000110	0.000005	2.0

6.2.6 Bermudan Swaption

The methodology for pricing options with early exercise features involves approximating the general *n*-factor non-Markovian Gaussian model with a single-factor Markovian short rate model which corresponds to the Hull-White extended Vasicek model. Figure 3 shows the overall instantaneous forward rate volatility function of our two-factor model and the least squared fitted Hull-White volatility function. The best fit values for the parameters describing the volatility function in the approximating model are $\alpha = 0.014069$ and $\sigma = 0.010815$.

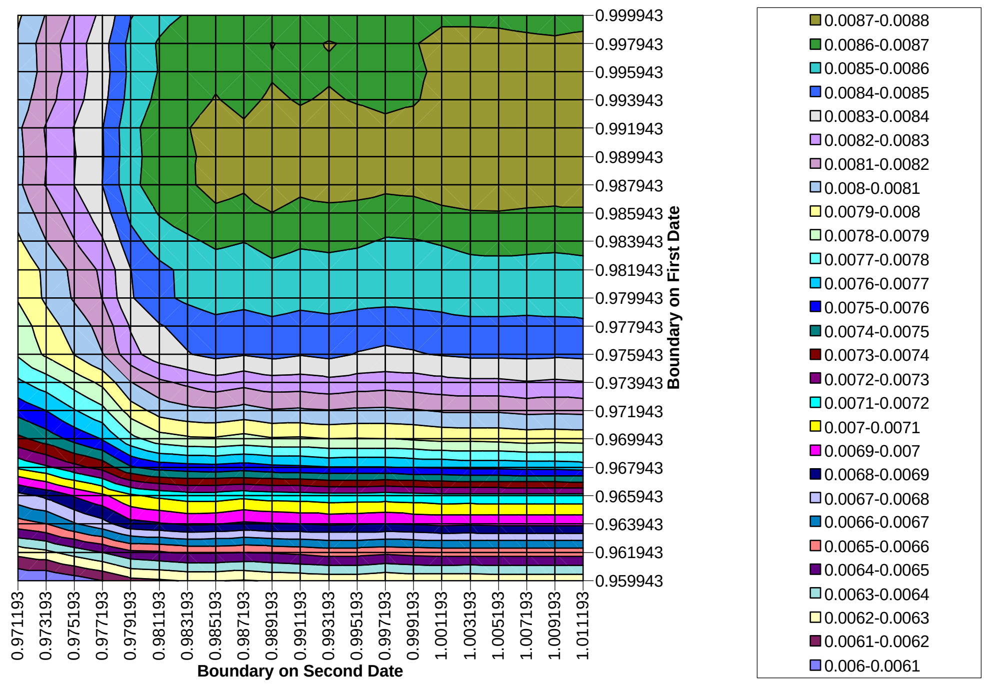
Figure 3 : Overall Two Factor Model Instantaneous Forward Rate Volatility Function and fitted Hull-White Volatility Function



We price a 1.5 year Bermudan option on a 3-year semi-annual 5% coupon bond with early exercise dates on the coupon dates (i.e. early exercise opportunities at 0.5 and 1.0 years). The second stage of the process is to determine the early exercise boundary points in the restricted model. The early exercise boundary points, in terms of the level of the underlying coupon bond above which it is optimal to exercise the option, are found to be 0.979640 at 0.5 years and 0.986887 at 1 year. The value of the European and Bermudan coupon bond option values derived from the short rate tree are 0.006680 and 0.006931 respectively. The corresponding values obtained via our 2 factor model using the European analytical approximation as a control variate and with 1000 simulations are 0.005456 and 0.008633 with standard errors of 0.000000 and 0.000054 respectively.

The robustness of the procedure we have outlined depends on the accuracy of the early exercise strategy derived from the single-factor Markovian short rate model. We can illustrate the accuracy of the early exercise strategy by evaluating the Bermudan option price for a range of values for the two points determining the early exercise boundary centred on the values obtained from the single-factor short rate model. Figure 4 shows the results of this procedure. The *y*- and *x*-axis are centred on the critical coupon bond prices for the first (0.979640) and second (0.986887) early exercise dates identified from the short rate tree, respectively. The true value for the Bermudan option should be the maximum value possible as a function of the early exercise boundary points. Figure 4 indicates that the approximate value obtained from the single-factor Markovian short rate model are very close to optimal because the maximum value for the option is close the value obtained from the approximate early exercise boundary. The percentage error in the early exercise premium is approximately 2%. Note that it would be straightforward to refine the early exercise boundary by optimisation since an individual evaluation of the Bermudan option price takes only about 0.1 seconds to obtain sufficient accuracy for the optimisation.

Figure 4 : Bermudan Swaption Prices for a Range of Values Determining the Early Exercise Boundary



7 Summary and Conclusions

In this paper we have decribed a framework for evaluating interest rate exotics under a very general class of multi-factor Gaussian non-Markovian interest rate models. We began by showing that, within our framework, and with the use of an innovative control variate, we can calibrate the model realistically to caps and swaption prices.

The second main part of this chapter is the first published attempt to answer the concerns of a number of authors as to the ability of multi-factor versions of HJM to efficiently price path-dependent claims. We have developed a general methodology for pricing interest rate exotics which is very flexible and, as examples we look at yield spread options (for which we have developed an analytical solution if the two rates are LIBOR rates), captions/floortions, average rate caps, barrier caps, compound swaptions, and barrier swaptions. The nature of the formulation means that we do not have to model every element of the observed term structure, only the pure discount bonds which correspond to underlying cashflows. Applying this to interest rate exotics yields valuation formulae in terms of expectations that are efficient to compute. A further contribution of this paper is that our general framework lends itself in a natural way to efficient sampling of the state space by truncation of the eigenvector representation of the covariance structure to a subset of the important factors. Finally this representation also lead to a general method of obtaining analytical approximations which can also be used as control variates for the simulation.

Although until recently it has been viewed that one of the drawbacks with using Monte-Carlo simulation for derivative pricing is that it is difficult to value American options, we show that by supplementing the Monte Carlo integration with Markovian short rate trees, to generate an early exercise strategy, we are able to obtain approximations to the prices of American style derivatives. We compared our approximation with recent research on Monte Carlo methods for pricing American options.

We finish the paper by calibrating a general two factor HJM model to a set of money market data - one of the first studies to look at multi-factor implied calibration to market data and the first to look at Eurodollar data - and produce numerical results showing the viability of the

Monte Carlo method and the computational efficiency for all instruments detailed in this paper. For all options, the methodology is computationally extremely efficient with many path dependent option prices calculated to within reasonable accuracy in a few seconds. Results concerning sampling the state space by truncation of the eigenvector set are also impressive. We find that the first 3 or 4 factors account for nearly all of the total variation and so by reducing the full dimensionality of the continuum of discount bond prices to these important factors we obtain an efficient approximation to the option value. Our results show that reduction from 10 factors to 3 only slightly reduces the accuracy of the approximation whilst more than halving the computation time.

The results we obtain for pricing a Bermudan swaption via simulation are similarly encouraging. We tentatively conclude that for typical shapes of volatility functions implied by market data, the non-Markovian character of the general model does not rule out approximating the early exercise strategy by using a Markovian restriction.

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