The Reduction of Forward Rate Dependent Volatility HJM Models to Markovian Form: Pricing European Bond Options

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Abstract:

We consider a single factor Heath-Jarrow-Morton model with a forward rate volatility function depending upon a function of time to maturity, the instantaneous spot rate of interest and a forward rate to a fixed maturity. With this specification the stochastic dynamics determining the prices of interest rate derivatives may be reduced to Markovian form. Furthermore, the evolution of the forward rate curve is completely determined by the two rates specified in the volatility function and it is thus possible to obtain a closed form expression for bond prices. The prices of bond options are determined by a partial differential equation involving two spatial variables. We discuss the evaluation of European bond options in this framework by use of the ADI method.

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1. Introduction

Approaches to modelling the term structure of interest rates in continuous time may be broadly described in terms of either the equilibrium approach or the no-arbitrage approach even though some early models include concepts from both approaches.

The equilibrium approach is now based primarily on the work of Cox, Ingersoll and Ross (1985). This approach begins with a description of the underlying economy. First, assumptions are made about the stochastic evolution of exogenous factors or state variables in the economy and about the preferences of individual investors. Then a stochastic intertemporal optimising framework is employed to endogenously derive the interest rate and the price of contingent claims.

The no-arbitrage approach starts with assumptions about the stochastic evolution of one or more underlying factors, usually interest rates. Bond prices are assumed to be functions of these driving stochastic processes. The prices of contingent claims are then derived by imposing the condition that there are no arbitrage opportunities in the economy between bonds of different maturity. Vasicek's (1977) model is an example of a single-factor no-arbitrage model in which the whole term structure depends on a single stochastic variable or factor, in this case the short interest rate. It employs an arbitrage argument in relation to the expected return on bonds of differing maturity. The Brennan and Schwartz (1979) approach is based on two factors, the shortest and the longest maturity yields and also uses a form of arbitrage argument in its approach.

The early no-arbitrage models contained terms related to the market price of interest rate risk, which is dependent upon unobservable investor preferences. These models also had the shortcoming of not being consistent with the currently observed yields. A major advance in the modelling of the term structure of interest rates was made by Ho and Lee (1986) who, in a binomial framework using no-arbitrage arguments, derived a model which does not depend upon the market price of interest rate risk. This feature was achieved by imposing the condition that the model is compatible with the currently observed yield curve, which has impounded in it investor preferences and the market price of interest rate risk. This approach was generalised to the continuous time framework by Heath, Jarrow and Morton (HJM, 1992) which now provides the most general methodology for term structure of interest rate models. The HJM approach uses as the driving stochastic dynamic variable forward rates whose evolution is dependent on a specified volatility function.

The crucial input to the HJM model is the volatility function of the forward interest rate and the most important modelling decision is the form of this volatility function. Rebonato (1996, page 316) makes the point that there is not “the HJM” model but a class of HJM models each characterised by a specification of this volatility function. Ritchken and Sankarasubramanian (1995a) also investigate the possible effect of this specification on estimated option prices. Hull and White (1990) develop a methodology to eliminate the market price of risk by imposing consistency with the currently observed yield curve. Pelsser (1996) indicates a more systematic procedure which works well for a certain class of models. It is now understood that these approaches can be seen to be equivalent to assuming certain forms for the forward rate volatility in the HJM framework. See Bhar and Chiarella (1997a), Chiarella and El-Hassan (1996) and Chiarella and Kwon (1999a).

In the arbitrage-free approach of HJM the dependence of the drift on the volatility function is explicitly recognised. A consequence of this is that the spot rate process is history dependent and results, under the most general specification, in a non-Markovian structure. Several authors have investigated the conditions under which the spot rate process may be made Markovian. Such studies include, for example, Cheyette (1992), Carverhill (1994), Ritchken and Sankarasubramanian (1995b), Jeffrey (1995) and Bhar and Chiarella (1997a). Although these studies differ in details of their implementation, the general nature of the volatility specification is given by,

$$\sigma(t,T,r(t)) = [a_0 + a_1(T-t) + .. + a_n(T-t)^n]\, h(r(t))\, e^{-\lambda(T-t)}, \quad 0 \le \mathrm{t} \le \mathrm{T} \qquad (1)$$

where *h(r(t))* is a function of the spot interest rate, and the a_i and λ are constant coefficients. The effect of such a specification is that the system becomes Markovian but with respect to an augmented set of state variables. Bhar and Chiarella (1997a) demonstrate, using the general specification as in equation (1), the number of additional state variables needed and the precise nature of these subsidiary variables. They also express such a higher dimensional system in state space form, which can then be estimated using the extended Kalman filter algorithm. Inui and Kijima (1998) consider multiple noise terms with each volatility function being of the form (1), with $a_1 = \ldots = a_n = 0$ and λ possibly being time dependent in a particular way.

In this paper we extend Bhar and Chiarella (1997a) to further generalise the form of the volatility function to include forward interest rates. In particular we work with the specification,

$$\sigma(t,T,r(t), f(t,\tau)) = g\big[r(t), f(t,\tau)\big]\, e^{-\lambda(T-t)}, \quad 0 \le t \le \tau < T \qquad (2)$$

and determine the additional state variables necessary to make the system Markovian although with a higher dimension. Here *g(r(t), f(t,õ))* is a function of the spot interest rate, *r(t)*, and of the forward interest rate, *f(t,õ)* of a fixed maturity τ. For example, *f(t,õ)* could be some long forward rate. The intuition behind such a specification is that not only the spot interest rate but also a fixed maturity forward interest rate influence the evolution of the term structure. The particular forward rate to be used may depend on the application under consideration. This approach may be considered to be equivalent in some sense, within the HJM framework, to the Brennan and Schwartz (1979) model where a short rate and a long rate are used to explain the evolution of the term structure.

The cited earlier works obtain Markovian representations by an expansion of the state space as we have already mentioned. However the additional state variables have never been given any economic interpretation. A distinguishing feature of our analysis is that we demonstrate how to express the additional state variable in terms of *r(t)* and *f(t,õ)*. Thus the Markovian representation which initially appears as a three dimensional stochastic dynamic system can be reduced to a two dimensional one. Furthermore, it turns out that the evolution of the forward rate curve is determined entirely by the evolution of *r(t)* and *f(t,õ)*.

In the next section we describe the basic HJM framework and show how under the volatility specification of equation (2) the driving non-Markovian stochastic dynamics may be reduced to Markovian form by the introduction of an additional state variable. The stochastic dynamics are then being driven by a three-dimensional Markovian system. In section 3, we show the additional state variable may be expressed in terms of the rates $r(t)$ and $f(t,\tau)$, thus reducing the driving stochastic dynamics to a two-dimensional Markovian system. In section 4 we show how an explicit expression for bond prices may be obtained. We also derive the underlying pricing partial differential operator which turns out to involve two spatial variables. In section 5 we discuss the pricing of bond options within the framework we have adopted. In section 6 we discuss the use of the alternating direction implicit (ADI) finite difference scheme to evaluate European bond options. Section 7 gives some numerical results for the evaluation of European bond options using the ADI method for a range of different discretisations. Section 8 concludes. Many of the technical derivations are gathered in a set of appendices.

2. The Forward Rate Process

2.1 The HJM Framework

We recall that the starting point of the one factor HJM (1992) model of the term structure of interest rates is the stochastic integral equation for the forward rate

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T,.)du + \int_0^t \sigma(u,T,.)dW(u), \quad (0 \le t \le T). \tag{3}$$

Here *f(t,T)* is the forward rate at time t applicable to time T(>t). The noise terms dW(u) are the increments of a standard Wiener process generated by a probability measure Q. These are the shock terms arising from the underlying factor. The functions $\alpha(u,T,.)$ and $\sigma(u,T,.)$ are respectively the instantaneous drift and volatility functions associated with the noise term. The third argument (.) indicates possible dependence on other state variables, typically such dependence could be on $f(t,T)$ itself or on $r(t)$, the instantaneous spot rate of interest both at time *t*.

HJM show that the absence of riskless arbitrage opportunities implies that the drift term cannot be chosen arbitrarily but rather will be a function of the volatility function, $\sigma(.)$, and the market price of interest rate risk, $\phi(t)$. This relationship is given by,

$$\alpha(t,T) = -\sigma(t,T,\cdot)\left[\phi(t) - \int_t^T \sigma(t,s,\cdot)ds\right]. \tag{4}$$

Furthermore, by an application of Girsanov's theorem the dependence on the market price of interest rate risk can be absorbed into an equivalent probability measure. The stochastic integral equation for the forward rate then becomes,

$$f(t,T) = f(0,T) + \int_0^t \sigma(u,T,\cdot)\int_u^T \sigma(u,s,\cdot)ds\,du + \int_0^t \sigma(u,T,\cdot)d\tilde{W}(u). \tag{5}$$

Here, $\tilde{W}(u)$ is the Wiener process generated by an equivalent martingale probability measure $\tilde{Q}$. The probability measures Q and $\tilde{Q}$ are related, via the Radon-Nikodym derivative as explained in HJM.

From equation (5) we obtain the stochastic integral equation for the spot interest rate as $r(t) \equiv f(t,t)$ which under $\tilde{Q}$ is given by,

$$r(t) = f(0,t) + \int_0^t \sigma(u,t,\cdot)\int_u^t \sigma(u,s,\cdot)ds\,du + \int_0^t \sigma(u,t,\cdot)d\tilde{W}(u)\,. \tag{6}$$

As a stochastic differential equation (6) becomes,

$$dr(t) = \left[f_2(0,t) + \frac{\partial}{\partial t}\int_0^t \sigma(u,t,\cdot)\int_u^t \sigma(u,s,\cdot)ds\,du + \int_0^t \sigma_2(u,t,\cdot)d\tilde{W}(u)\right]dt + \sigma(t,t,\cdot)d\tilde{W}(t)\,, \tag{7}$$

where $f_2(0,t)$ denotes the partial derivative of $f(0,t)$ with respect to the second argument. The non-Markovian nature of the stochastic dynamics of the system that we are considering stems from the integral terms of the drift coefficient. These integral terms depend on the entire history of the process up to time *t*.

Our aim now is to express the equations (5) and (6) under the volatility specification of equation (2) as a Markovian system of stochastic differential equations (SDE). With such a volatility specification we can rewrite the stochastic integral equations (5) and (6) as

$$\begin{aligned} f(t,T) = f(0,T) &+ \int_0^t \sigma(u,T,r(u), f(u,\tau))\int_u^T \sigma(u,s,r(u), f(u,\tau))ds\,du \\ &+ \int_0^t \sigma(u,T,r(u), f(u,\tau))d\tilde{W}(u)\,, \end{aligned} \tag{8}$$

and,

$$\begin{aligned} r(t) = f(0,t) &+ \int_0^t \sigma(u,t,r(u), f(u,\tau))\int_u^t \sigma(u,s,r(u), f(u,\tau))ds\,du \\ &+ \int_0^t \sigma(u,t,r(u), f(u,\tau))d\tilde{W}(u)\,. \end{aligned} \tag{9}$$

2.2 Reduction to Markovian Form

In order to express the stochastic integral equations (8) and (9) as stochastic differential equations, we need the stochastic differentials of the second and the third terms of these equations. These are evaluated in Appendix A, and for (8) the resulting SDE is,

$$df(t,T) = \left[\sigma(t,T,r(t), f(t,\tau))^2 \frac{\left(e^{\lambda(T-t)} - 1\right)}{\lambda}\right]dt + \sigma(t,T,r(t), f(t,\tau))d\tilde{W}(t). \tag{10}$$

Similarly, the SDE expression for the stochastic integral equation (9) is given by

$$dr(t) = \left[f_2(0,t) + \int_0^t \left\{ -\lambda\sigma(u,t,r(u), f(u,\tau)) \int_u^t \sigma(u,s,r(u), f(u,\tau))ds + \right.\right.$$

$$\left.\left. \sigma(u,t,r(u), f(u,\tau))^2 \right\} du - \lambda \int_0^t \sigma(u,t,r(u), f(u,\tau))d\tilde{W}(u) \right] dt \; + \qquad (11)$$

$$\sigma(t,t,r(t), f(t,\tau))d\tilde{W}(t).$$

The calculations of the differentials required to obtain the second and the third terms on the right hand side of equation (11) are given in Appendix B. We note from equation (9) that

$$\int_0^t \sigma(u,t,r(u), f(u,\tau)) \int_u^t \sigma(u,s,r(u), f(u,\tau))dsdu \qquad (12)$$

$$+ \int_0^t \sigma(u,t,r(u), f(u,\tau))d\tilde{W}(u) = r(t) - f(0,t).$$

Then the stochastic differential equation (11) may be simplified to

$$dr(t) = \left[f_2(0,t) + \lambda f(0,t) + \psi(t) - \lambda r(t) \right] dt + \sigma(t,t,r(t) f(t,\tau))d\tilde{W}(t), \qquad (13)$$

where we define the subsidiary variable

$$\text{Þ}(t) = \int_0^t \sigma(u,t,r(u), f(u,\tau))^2 du. \qquad (14)$$

It is a straightforward matter to show (see appendix C) that $\psi(t)$ satisfies the stochastic differential equation

$$d\psi(t) = [\sigma(t,t,r(t), f(t,\tau))^2 - 2\lambda\psi(t)]dt\,. \qquad (15)$$

We have now reduced the non-Markovian stochastic dynamics to a three dimensional Markovian stochastic dynamical system consisting of the stochastic differential equation for the discrete forward rate, namely,

$$df(t,\tau) = \sigma(t,\tau,r(t), f(t,\tau))^2 \frac{[e^{\lambda(\tau - t)} - 1]}{\lambda} dt + \sigma(t,\tau,r(t), f(t,\tau))d\tilde{W}(t)\,, \qquad (16)$$

and the two stochastic differential equations (13) and (15) for *r(t)* and $\psi(t)$ respectively.

Finally we recall that the dynamics of the forward rate to any maturity T , $f(t,T)$ is given by equation (8) and so is determined once $r(t)$ and $f(t,\tau)$ are determined. These latter quantities are driven by the three SDEs (13), (15), (16) which together form the Markovian representation.

The price of any derivative instrument would then have to depend on $r(t)$ and $f(t,\tau)$. Thus a bond of maturity T would have a price at time t denoted by, $P(t,T,r(t),f(t,\tau))$, and this price is also driven by the three-dimensional Markovian SDE system referred to above.

3. Interpreting the Subsidiary Variable Þ(t)

The subsidiary variable $\psi(t)$ defined in equation (14) plays a central role in allowing us to transform the original non-Markovian dynamics to Markovian form. Similar subsidiary variables appear in the reduction to Markovian forms of Cheyette (1992), Ritchken and Sankarasubramanian (1995b), Bhar and Chiarella (1997a), Inui and Kijima (1998) and Chiarella and Kwon (1999).

It is clear from equation (14) that $\psi(t)$ may be interpreted as a variable summarising the path history of the forward rate volatility. Similar interpretations have been given by the other cited authors. However, it would perhaps be more satisfying to relate $\psi(t)$ to the market rates $r(t)$ and $f(t,\tau)$. Indeed it turns out that such a relationship does exist for the forward rate volatility function assumed in equation (2).

Proposition 1

The subsidiary integrated square variance quantity $\psi(t)$ defined in equation (14) is related to the rates $r(t)$ and $f(t,\tau)$ via

$$\psi(t)=\lambda\alpha(t,\tau)[r(t)-f(0,t)]-\lambda e^{-\lambda(t-\tau)}\alpha(t,\tau)[f(t,\tau)-f(0,\tau)] \qquad (17)$$

where $\alpha(t,\tau)\equiv e^{-\lambda t}/\left(e^{-\lambda\tau}-e^{-\lambda t}\right)$

Proof

See appendix D. ■

An important consequence of Proposition 1 is that it allows us to reduce by one the dimension of the stochastic dynamic system (13),(15),(16) to the two-dimensional one consisting of the stochastic differential equations (13) and (16) with $\psi(t)$ begin defined by equation (17).

This reduction in dimension is quite significant if we seek to solve for derivative prices in this framework by use of partial differential equations or lattice based methods as in section 6, since then we need only deal with two rather than three spatial variables in the partial differential operator. The reduction is less significant, though still useful, when using Monte Carlo simulation. This is so since Monte Carlo simulation requires the simulation of the one Wiener increment, $d\tilde{W}(t)$. The generation of *ψ(t)* by equation (17) rather than discretising equation (15) should lead to some computational efficiency.

A consequence of proposition 1 is that we are able to express the forward rate to any maturity *T* in terms of the two rates *r(t)* and *f(t,õ)*.

Proposition 2

The forward rate f(t,T) to any maturity T is given by

$$f(t,T) - f(0,T) = \frac{\alpha(t,\tau)}{\alpha(T,\tau)}\left[f(t,\tau) - f(0,\tau)\right] + \left(1 + \frac{\alpha(t,\tau)}{\alpha(T,\tau)}\right)\left[r(t) - f(0,t)\right].$$

Proof

See appendix E. ■

4. The Term Structure of Interest Rates

We recall the HJM approach of defining a money account,

$$A(t) = \exp\left(\int_0^t r(y)dy\right), \tag{18}$$

and showing that the relative bond price

$$Z(t,T) = \frac{P(t,T)}{A(t)}, \tag{19}$$

is a martingale, so that the bond price can be written[1]

$$P(t,T) = \tilde{E}_t\left(\frac{A(t)}{A(T)}\right) = \tilde{E}_t\left(\exp\left\{-\int_t^T r(y)dy\right\}\right). \tag{20}$$

Here, $\tilde{E}_t$ is the expectation taken with respect to the probability distribution generated by the stochastic differential system (13) and (16). We use

$$\pi\left(r(t^*), f(t^*,\tau) \middle| r(t), f(t,\tau)\right), \tag{21}$$

to denote the transition probability density function between t and t^* $(t \le t^*)$. This quantity satisfies the Kolmogorov backward partial differential equation (see Oksendal (1992) for a discussion of

[1] See HJM (1992) or, Bhar and Chiarella (1997a) for a simplified version.

the Kolmogorov backward equation for a multi-dimensional diffusion process), which for our case is given by,

$$K\pi + \frac{\partial \pi}{\partial t} = 0\,, \tag{22}$$

where the operator K is the infinitesimal generator of the diffusion process for $f(t,\tau), r(t)$ driven by the stochastic differential equations (13) and (16). It turns out that K is given by [2](see appendix F),

$$K\pi \equiv \sigma_1^2 \frac{\left(e^{\lambda(\tau-t)}-1\right)}{\lambda}\frac{\partial \pi}{\partial f} + \left[f_2(0,t) - \lambda f(0,t) + \psi - \lambda r\right]\frac{\partial \pi}{\partial r} + \frac{1}{2}\sigma_1^2\frac{\partial^2 \pi}{\partial f^2} + \frac{1}{2}\sigma_r^2\frac{\partial^2 \pi}{\partial r^2} + \sigma_1\sigma_r\frac{\partial^2 \pi}{\partial f \partial r}. \tag{23}$$

By application of the Feynman-Kac formula to equation (24) we find that the bond price $P(t,T,r,f)$ satisfies the partial differential equation,

$$KP - rP + \frac{\partial P}{\partial t} = 0\,, \tag{24}$$

subject to the terminal condition

$$P(T,T,r,f) = 1, \tag{25}$$

and the boundary conditions

$$P(t,T,\infty,f) = 0, \qquad (f \geq 0),$$
$$P(t,T,r,\infty) = 0, \qquad (r \geq 0).$$

The further boundary conditions $P(t,T,0,f)$ and $P(t,T,r,0)$ may be obtained by an extrapolation procedure to be discussed in section 6 below.

Note that in subsequent discussion we set

$$D(t) \equiv f_2(0,t) + \lambda f(0,t). \tag{26}$$

A consequence of proposition 2 is that it turns out to be possible to obtain an analytical expression for the bond price. In fact we may state the following proposition:-

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[2] In order to alleviate the notation in describing the operator K we set $\sigma_1(t) \equiv \sigma(t,\tau,r(t),f(t,\tau))$ and $\sigma_r(t) \equiv (t,t,r(t),f(t,\tau))$

Proposition 3

The price of bonds driven by the Markovian stochastic differential equation system (13) and (16) can be expressed as

$$P(t,T,r,f)=\frac{P(0,T)}{P(0,t)}\exp\left\{-\beta(t,T)\left[r(t)-f(0,t)\right]-\tfrac{1}{2}\beta(t,T)^2\psi(t)\right\}. \qquad (27)$$

where

$$\beta(t,T)=\frac{1}{\lambda}\left[1-e^{-\lambda(T-t)}\right],$$

and $\psi(t)$ is defined in equation (17).

Proof

See appendix G. ■

The bond pricing equation (31) has precisely the same form[3] as the one derived by Ritchken and Sankarasubramanian (1995a) who (in current notation) assumed a form for the volatility function in equation (2) which is independent of the forward rate f (t,τ). In fact the results in propositions 2 and 3 can be considerably generalised. Chiarella and Kwon (1999) have shown that (27) holds in precisely the same form even when the forward rate volatility depends on a set of discrete forward rates $f(t,\tau_1), f(t,\tau_2),\ldots, f(t,\tau_r)$ where $t \le \tau_1 < \tau_2 < \cdots < \tau_r \le T$. Of course, under these different specifications the history variable $\psi(t)$ will evolve differently but the functional relationship remains the same.

5. Pricing European Bond Options

Consider an option written on the bond of maturity T. We suppose the option matures at time $T_c(<T)$ and denote its price by $C(t, T, r, f)$. This price satisfies the partial differential equation

$$KC - rC + \frac{\partial C}{\partial t} = 0 \quad (0 \le t \le T_c). \qquad (28)$$

If we are dealing with a European call option with strike price E then the terminal condition for equation (32) is

$$C(T_c,T,r,f) = \left(P(T_c,T,r,f)-E\right)^+. \qquad (29)$$

The boundary conditions at infinity are

[3] It is of course possible to express the exponent in equation (27) as a linear combination of *[r(t) – f(0,t)]* and *[f(t,õ)-f(0,õ)]*. We give the form shown in the proposition in order to allow easier comparison with the work of Ritchken and Sankarasubramanian (1995a).

$$C(t,T,\infty,f)=0,\qquad f\geq 0,$$
$$C(t,T,r,\infty)=0,\qquad r\geq 0.$$

We recall that the bond prices at option maturity for any given values of $r(T_c),\ f(T_c,\tau)$ can be obtained directly from equation (27) without the need to solve the bond pricing partial differential equation (24).

In section 6 we discuss the solution of the partial differential equation (31) by means of the alternating directions implicit (ADI) method.

An alternative approach to pricing the European option is to use the result (also derived by HJM) that

$$C(t,T,r,f)=\tilde{E}_t\left[e^{-\int_t^{T_c} r(y)dy}\left(P(T_c,T,r(T_c),\ f(T_c,\tau))-E\right)^+\right]. \tag{30}$$

The expectation in (30) could be approximated by simulating an appropriate number of times the stochastic differential equation system (13) and (16) from t to T_c.

6. Numerical solution with the ADI Method

For the purposes of the numerical results reported in this paper we have taken the volatility function

$$\sigma(t,T,r(t),\ f(t,\tau))=\left(\alpha_0+\alpha_r r(t)+\alpha_f\ f(t,\tau)\right)^{\gamma} e^{-\lambda(T-t)}, \tag{31}$$

where $\alpha_0,\alpha_r,\alpha_f\geq 0$ and $0<\gamma<1$. The latter restriction on γ should avoid the potential for exploding forward rates pointed out by HJM (1992) in the case γ=1 and α_r=0. However this point requires further investigation.

Also, we take $f(0,t)=\beta_0+\beta_1(1-e^{-\eta t})$. Given the definition of $\sigma\left(t,T,r(t),\ f(t,\tau)\right)$ in equation (35) it turns out that (see footnote 2)

$$\sigma_r=\left(\alpha_0+\alpha_r r(t)+\alpha_f\ f(t,\tau)\right)^{\gamma},$$

and

$$\sigma_1=e^{-\lambda(\tau-t)}\sigma_r.$$

The partial differential equations (24) and (28) are parabolic and involve two spatial variables, namely $r(t)$ and $f(t,\tau)$. A range of numerical techniques have been used in the physical sciences to successfully solve numerically parabolic partial differential equations with two and three spatial dimensions; we refer the reader to Lapidus and Pinder (1982) for a good discussion. The most commonly used methods are the ADI (Alternating Direction Implicit) method and the Crank-Nicholson scheme. Here we outline use of the ADI scheme to solve equations (24) and

(28).

We have from proposition 2 an exact solution for the bond price. We use this exact solution to test the appropriateness and accuracy of the discretisation procedure we have used in setting up the ADI method. This is important since for two and three dimensional partial differential equations it is much more difficult to make strong statements about the stability of the discretisation schemes we employ. This is particularly so for partial differential operator of the type with which we are dealing, which have time varying coefficients. Furthermore there are many possible ways to handle the cross derivative terms and there is little theory to guide us as to which is more appropriate for any given problem. For these reasons we have chosen to first test the discretisation on the bond pricing partial differential equation which has a known solution. The numerical results below indicate that for this problem the discretisation employed yields a satisfactory accuracy.

It is therefore with some confidence that we then go on to apply the same discretisation procedure to the option pricing problem, which has the same partial differential operator but different boundary conditions.

Let $\bar{t} = T - t,$ then the partial differential operator governing both equations (24) and (28) can be written as (here V equals P or C)

$$\frac{\partial V}{\partial \bar{t}} = \mu_r \frac{\partial V}{\partial r} + \mu_1 \frac{\partial V}{\partial f} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 V}{\partial f^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \sigma_1\sigma_r \frac{\partial^2 V}{\partial f \partial r} - rV \tag{32}$$

where we have set

$$\mu_r = D(T - \bar{t}) + \psi - \lambda r \text{ , and } \mu_1 = \sigma_1^2 \frac{e^{\lambda(\tau - T + \bar{t})} - 1}{\lambda} . \tag{33}$$

Equation (32) is to be solved subject to the appropriate initial and boundary conditions.

In particular, for the solution of the bond pricing problem (V = P), these conditions are

$$\begin{aligned} &P(0, T, r, f) = 1, \\ &P(T - \bar{t}, T, \infty, f) = 0, \qquad f \geq 0, \ \ 0 \leq \bar{t} \leq T, \\ &P(T - \bar{t}, T, r, \infty) = 0, \qquad r \geq 0, \ \ 0 \leq \bar{t} \leq T, \end{aligned} \tag{34}$$

On the other hand, for the option pricing equation (V = C) these conditions become

$$C(T_c, T, r, f) = \left(P(T_c, T, r, f) - E\right)^+ . \tag{35}$$

$$C(T-\bar{t},T,\infty,f)=0, \qquad f\geq 0,$$
$$C(T-\bar{t},T,r,\infty)=0, \qquad r\geq 0.$$

In solving both the bond and the option pricing equation by finite difference methods, we require values for P and C at the boundaries r = 0 and f = 0. These are not specified a priori and we have determined them by extrapolating from the three neighbouring solution points.

Assume that $\Delta r, \Delta f$ and Δt denote the step sizes in the directions of *r, f* and *t* then the bond price at the time step n and the point (i, j) is written as

$$P_{i,j}^{n}=P(n\Delta t,T,i\Delta r, j\Delta f),$$

$i=0,1,\cdots,I$, $j=0,1,\cdots J$ and $n=0,1,\cdots N$. We define a 2-step ADI (Alternating Direction Implicit) scheme. The essence of this scheme is that the price $P_{i,j}^{n+1}$ at time step $(n+1)$ is obtained from the price $P_{i,j}^{n}$ at time step n in a two-step process. First $P_{i,j}^{n+\frac{1}{2}}$ is calculated by holding *j* (ie. *f*) fixed and varying *i* (ie. *r*). Then $P_{i,j}^{n+1}$ is calculated by holding *i* fixed and varying *j*. In this way, the problem in two spatial variables is reduced to a sequence of two one-variable problems. Thus we first calculate

$$P_{i,j}^{n+\frac{1}{2}}+\left\{\frac{1}{\theta}-\frac{\Delta t}{4(\Delta r)^2}\left(\sigma_r^{n+\frac{1}{2}}\right)^2\right\}\left(P_{i+1,j}^{n+\frac{1}{2}}-2P_{i,j}^{n+\frac{1}{2}}+P_{i-1,j}^{n+\frac{1}{2}}\right)-\frac{\Delta t}{2\Delta r}\mu_r^{n+\frac{1}{2}}\left(P_{i+1,j}^{n+\frac{1}{2}}-P_{i-1,j}^{n+\frac{1}{2}}\right)=L_{ij}^{n}, \qquad (36)$$

for the set of bond prices at $t=(n+\frac{1}{2})\Delta t$ and then solve

$$P_{i,j}^{n+1}+\left\{\frac{1}{\theta}-\frac{\Delta t}{4(\Delta f)^2}\left(\sigma_1^{n+1}\right)^2\right\}\left(P_{i,j+1}^{n+1}-2P_{i,j}^{n+1}+P_{i,j-1}^{n+1}\right)-\frac{\Delta t}{2\Delta f}\mu^{n+1}\left(P_{i,j+1}^{n+1}-P_{i,j-1}^{n+1}\right)=L_{ij}^{n+\frac{1}{2}}, \qquad (37)$$

for the set of bond prices at $t=(n+1)\Delta t$.

Here $(n=1,\cdots,N)$, $(i=1,\cdots,I-1)$ *and* $(j=1,\cdots,J-1)$. Furthermore,

$$L_{ij}^{n}=\left(1-r_i\Delta t\right)P_{ij}^{n}+\left[\frac{1}{\theta}+\frac{\Delta t}{4(\Delta r)^2}\left(\sigma_r^{n}\right)^2\right]\left(P_{i+1,j}^{n}-2P_{i,j}^{n}+P_{i-1,j}^{n}\right)+\frac{\Delta t}{2(\Delta f)^2}\left(\sigma_1^{n}\right)^2\left(P_{i,j+1}^{n}-2P_{i,j}^{n}+P_{i,j-1}^{n}\right)$$
$$+\frac{\Delta t}{4\Delta r\Delta f}\sigma_1^{n}\sigma_r^{n}\left(P_{i+1,j+1}^{n}-P_{i+1,j-1}^{n}-P_{i-1,j+1}^{n}+P_{i-1,j-1}^{n}\right),$$
$$L_{ij}^{n+\frac{1}{2}}=P_{i,j}^{n+\frac{1}{2}}+\left[\frac{1}{\theta}-\frac{\Delta t}{4(\Delta f)^2}\left(\sigma_1^{n}\right)^2\right]\left(P_{i,j+1}^{n}-2P_{i,j}^{n}+P_{i,j-1}^{n}\right).$$

and $\theta \geq 4$ or $\theta < 0$ is a tuning parameter whose choice is somewhat a matter of numerical experimentation but which improves the stability of the numerical scheme. Note that in the above equations, $\mu_r, \mu_1, \sigma_r, \sigma_1$ are all evaluated at the point ($i\Delta r, j\Delta f$) and the time indicated by the superscript. The details of setting of the foregoing discretisation scheme in a convenient matrix form are outlined in appendix H.

7. Numerical Results

As we stated in the previous section, it is much more difficult to establish stability results for parabolic partial differential equations with two spatial variables than for those with one spatial variable. We have therefore chosen to check the stability and accuracy of the particular discretisation used for the ADI method by using it to first solve the bond pricing partial differential equation for which we have an exact solution.

In table 1, we display the parameter set used in our numerical experiments.

α_0 =0.001 α_r = 0.04, γ = 0.5, λ = 0.2
β_0 = 0.04, β_1 = 0.04, η =0.05
T_B = 1 year, τ = 1 year

Table 1 **Parameter Set**

In table 2, we display the maximum error of the solution for the bond price incurred by using the ADI method with the discretisations indicated. The partial differential equation has been solved from a bond maturity of 6 months back to time 3 months. The maximum error and percentage error between the approximate and exact bond price (according to equation 31) on the grid at this point are what is recorded in table 2. We have considered 3 and 6 month maturity bonds on the basis that this is the time period over which we will be solving the partial differential equations for the option price. For fixed α_r we have considered a range of α_f. We see from table 2 that the discretisation used yielded at least two decimal accuracy.

In table 3 we display the values of European bond options calculated using both ADI method for a range of discretisations. We have taken that T_B=1 year as the maturity of the underlying bond and the strike price E=0.8. We have also taken $T_{max} = f_{max} = 3, r_{max} = 3$. Option values are calculated at the point of $r=f=\beta_0$ at $t=0$. Given the range of år, åf and N_t (number of time steps) used, the calculated option values generally seem accurate to three decimal places. A range of values for the parameter Ø were experimented with, the values reported gave the best results, though differences across a wide range of values were not great. The computational times were reasonable. However more work could be done in finding different discretisations which give faster convergence.

	Time	**Maximum Error**	**Maximum Error (%)**
α_f=0	3 months	0.00188599	0.240095
	6 months	0.00287108	0.462340
α_f=0.1α_r	3 months	0.00146113	0.189027
	6 months	0.00178074	0.285249
α_f=0.5α_r	3 months	0.000678249	0.111420
	6 months	0.000973979	0.218679
α_f=α_r	3 months	0.000689449	0.113260
	6 months	0.000915286	0.201260

Discretisation: Δt= 0.01, Δr=Δf=0.01.

Table 2 **Maximum Errors for Numerical Solution of the Bond Pricing PDE**

	Option Maturity Time	**Option Value**	**N_t**	**Δr=åf**	**θ**
α_f=0	3 months	0.1639	300	0.002	12
		0.1634	300	0.004	12
		0.1633	300	0.005	12
	6 months	0.1502	300	0.002	12
		0.1493	300	0.004	12
		0.1491	300	0.005	12
α_f=0.1α_r	3 months	0.1630	300	0.002	12
		0.1625	300	0.004	12
		0.1623	300	0.005	12
	6 months	0.1479	300	0.002	12
		0.1468	300	0.004	12
		0.1467	300	0.005	12
α_f=0.5α_r	3 months	0.1589	300	0.002	12
		0.1584	300	0.004	12
		0.1583	300	0.005	12
	6 months	0.1385	300	0.002	12
		0.1373	300	0.004	12
		0.1370	300	0.005	12
α_f=α_r	3 months	0.1530	300	0.002	12
		0.1524	300	0.004	12
		0.1523	300	0.005	12
	6 months	0.1264	300	0.002	12
		0.1261	300	0.004	12
		0.1256	300	0.005	12

Table 3 **European Bond Option Prices**

8. Conclusions

We have considered option pricing in the HJM framework with a forward rate volatility function depending upon time to maturity, the instantaneous spot rate of interest and a fixed forward rate. We demonstrated how this specification allowed a two-dimensional Markovian representation of the stochastic dynamics. It thus becomes possible to price interest rate derivatives using partial differential equation techniques. Despite the fact that we are dealing with a partial differential equation with two-spatial dimensions the approach is feasible as it is possible to obtain an explicit expression for the price of the underlying bond. Here we computed prices of European call bond options using the ADI finite difference method. We have given numerical results for European bond options showing the effect of the long forward rate on option values.

A number of directions for future research suggest themselves. First, alternative implementations of the ADI schemes should be considered with a view to improving speed of convergence; see Thomas (1995) for a detailed discussion of the ADI scheme. Second, Monte Carlo schemes could be applied. For instance, it should be possible to develop a very effective control variate using the known closed form solution when the forward rate volatility is a function of time to maturity only (see Brace and Musiela (1995) for such solutions). Third, the pricing of American options needs to be considered. Chiarella and El-Hassan (1999) have found the method of lines to be very effective in the case when the forward rate volatility of this paper is independent of the discrete forward rate. The same method should be applicable in the framework of this paper as the spatial dimensions of the partial differential equation remains the same. Fourth, it should be possible to develop computational methods based on a lattice technique similar to the one developed by Li, Ritchken and Sankarasubramanian (1995). They also considered a forward rate volatility specification similar to the one of this paper, but without dependence on the forward rate volatility. This lattice method provides the most natural generalisation of the popular binomial method to the two spatial dimensional structure of the model of this paper. Fifth, it is also necessary to undertake some empirical studies to determine when the types of volatility functions discussed in this paper fit market conditions. Some initial steps in this direction have been take by Bhar and Chiarella (1997b).

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Appendix A (Derivation of equation 10)

To take the differential of the first integral term in equation (8), we need to evaluate,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t \sigma(u,T,r(u),\ f(u,\tau))\int_u^T \sigma(u,s,r(u),\ f(u,\tau))ds\,du \\
&= \sigma(t,T,r(t),\ f(t,\tau))\int_t^T \sigma(t,s,r(t),\ f(t,\tau))ds \\
&= \sigma(t,T,r(t),\ f(t,\tau))g(r(t),\ f(t,\tau))\int_t^T e^{-\lambda(s-t)}ds \\
&= \sigma(t,T,r(t),\ f(t,\tau))^2\,\frac{\left[e^{\lambda(T-t)}-1\right]}{\lambda}.
\end{aligned}
\tag{A1}
$$

The differential of the stochastic integral term is simply

$$
d\int_0^t \sigma(u,T,r(u),\ f(u,\tau))d\tilde{W}(u) = \sigma(t,T,r(t),\ f(t,\tau))d\tilde{W}(t). \tag{A2}
$$

With these two results we readily obtain the stochastic differential equation (10).

Appendix B (Derivation of equation 11)

Consider the time derivative of the first integral term of equation (9)

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t \sigma(u,t,r(u),\ f(u,\tau))\int_u^t \sigma(u,s,r(u),\ f(u,\tau))ds\,du \\
&= \int_0^t\left\{\sigma_2(u,t,r(u),\ f(u,\tau))\int_u^t \sigma(u,s,r(u),\ f(u,\tau))ds + \sigma(u,t,r(u),\ f(u,\tau))^2\right\}du\,.
\end{aligned}
\tag{B1}
$$

The notation σ_2 represents the partial derivative of σ with respect to its second argument. Given the expression for σ in equation (2), we can write,

$$
\sigma_2(u,t,r(u),\ f(u,\tau)) = -\lambda\sigma(u,t,r(u),\ f(u,\tau)). \tag{B2}
$$

Thus, the right hand side of equation (B1) becomes,

$$
\int_0^t\left\{-\lambda\sigma(u,t,r(u),\ f(u,\tau))\int_u^t \sigma(u,s,r(u),\ f(u,\tau))ds + \sigma(u,t,r(u),\ f(u,\tau))^2\right\}du\,. \tag{B3}
$$

Similarly for the stochastic term,

$$
\begin{aligned}
& d\int_0^t \sigma(u,t,r(u),f(u,\tau))d\tilde{W}(u) \\
& = \left[\int_0^t \sigma^{(2)}(u,t,r(u),f(u,\tau))d\tilde{W}(u)\right]dt + \sigma(t,t,r(t),f(t,\tau))d\tilde{W}(t) \\
& = \left[-\lambda\int_0^t \sigma(u,t,r(u),f(u,\tau))d\tilde{W}(u)\right]dt + \sigma(t,t,r(t),f(t,\tau))d\tilde{W}(t).
\end{aligned} \tag{B4}
$$

Using results (B3) and (B4) we readily obtain the stochastic differential equation (11).

Appendix C (Derivation of equation 15)

Taking in turn differentials of (11), (12), (13), making use of (B1) and (B3) we obtain the stochastic differential equations for ψ viz

$$
\begin{aligned}
d\psi(t) &= \left[\sigma(t,t,r(t),f(t,\tau))^2 + 2\int_0^t \sigma(u,t,r(u),f(u,\tau))\cdot\sigma_2(u,t,r(u),f(u,\tau))du\right]dt \\
&= \left[\sigma(t,t,r(t),f(t,\tau))^2 - 2\lambda\int_0^t \sigma(u,t,r(u),f(u,\tau))^2 du\right]dt \\
&= \left[\sigma(t,t,r(t),f(t,\tau))^2 - 2\lambda\psi(t)\right]dt.
\end{aligned} \tag{C1}
$$

Appendix D (Proof of Proposition 1)

Recall that $r(t)$ satisfies the stochastic integral equation (9) and $f(t,\tau)$, the stochastic integral equation (8) with T set equal to τ.

We have the forward rate volatility function

$$
\sigma(u,t,r(u),f(u,\tau)) = e^{-\lambda(t-u)}g[r(u),f(u,\tau)] \tag{D1}
$$

and set

$$\sigma*\left(u,t,r(u),\,f(u,\tau)\right)=\sigma\left(u,t,r(u),\,f(u,\tau)\right)\int_u^t\sigma\left(u,s,r(u),\,f(u,\tau)\right)ds$$

$$=e^{-\lambda(t-u)}g\left[r(u),\,f(u,\tau)\right]\int_u^t e^{-\lambda(s-u)}g\left[r(u),\,f(u,\tau)\right]ds$$

$$=g\left[r(u),\,f(u,\tau)\right]^2 e^{-\lambda(t-u)}\left[\frac{1-e^{-\lambda(t-u)}}{\lambda}\right]. \tag{D2}$$

Note that the second integral term in equation (9) can be written

$$\int_0^t\sigma*\left(u,t,r(u),\,f(u,\tau)\right)du$$

$$=\int_0^t g\left[r(u),\,f(u,\tau)\right]^2 e^{-\lambda(t-u)}\,\frac{\left[1-e^{-\lambda(t-u)}\right]}{\lambda}du$$

$$=\frac{e^{-\lambda t}}{\lambda}\int_0^t g\left[r(u),\,f(u,\tau)\right]^2 e^{\lambda u}du-\frac{e^{-2\lambda t}}{\lambda}\int_0^t g\left[r(u),\,f(u,\tau)\right]^2 e^{2\lambda u}du$$

$$\equiv\frac{e^{-\lambda t}}{\lambda}\mathrm{I}(t;\lambda)-\frac{e^{-2\lambda t}}{\lambda}\mathrm{I}(t;2\lambda). \tag{D3}$$

Similarly the second integral term in (8) can be written

$$\int_0^t\sigma*\left(u,\tau,r(u),\,f(u,\tau)\right)du$$

$$=\int_0^t g\left[r(u),\,f(u,\tau)\right]^2 e^{-\lambda(\tau-u)}\,\frac{\left[1-e^{-\lambda(\tau-u)}\right]}{\lambda}du$$

$$=\frac{e^{-\lambda\tau}}{\lambda}\mathrm{I}(t;\lambda)-\frac{e^{-2\lambda\tau}}{\lambda}\mathrm{I}(t;2\lambda). \tag{D4}$$

Next note that the second integral in equation (9) may be written as

$$\int_0^t\sigma\left(u,t,r(u),\,f(u,\tau)\right)d\widetilde{W}(u)=\int_0^t e^{-\lambda(t-u)}g\left[r(u),\,f(u,\tau)\right]d\widetilde{W}(u)$$

$$=e^{-\lambda t}\int_0^t e^{\lambda u}g\left[r(u),\,f(u,\tau)\right]d\widetilde{W}(u)$$

$$\equiv e^{-\lambda t}\mathrm{J}(t,\lambda). \tag{D5}$$

The second integral term in equation (8) may be similarly treated, so that

$$\int_0^t \sigma\big(u,\tau,r(u),f(u,\tau)\big)\,d\widetilde{W}(u) = \int_0^t e^{-\lambda(\tau-u)} g\big[r(u),f(u,\tau)\big]d\widetilde{W}(u)$$
$$= e^{-\lambda\tau}\int_0^t e^{\lambda u} g\big[r(u),f(u,\tau)\big]d\widetilde{W}(u)$$
$$\equiv e^{-\lambda\tau}\mathrm{J}(t,\lambda). \qquad \text{(D6)}$$

We may thus write the stochastic integral equations for $r(t),\ f(t,\tau)$ in terms of the integrals $I(t;\lambda),\ I(t;2\lambda)$ and $J(t;\lambda)$ as

$$r(t) = f(0,t) + \frac{e^{-\lambda t}}{\lambda}\mathrm{I}(t;\lambda) - \frac{e^{-2\lambda t}}{\lambda}\mathrm{I}(t;2\lambda) + e^{-\lambda t}\mathrm{J}(t;\lambda), \qquad \text{(D7)}$$

$$f(t,\tau) = f(0,\tau) + \frac{e^{-\lambda\tau}}{\lambda}\mathrm{I}(t;\lambda) - \frac{e^{-2\lambda\tau}}{\lambda}\mathrm{I}(t;2\lambda) + e^{-\lambda\tau}\mathrm{J}(t;\lambda). \qquad \text{(D8)}$$

We note that equations (D7) and (D8) can be re-expressed as

$$r(t) - f(0,t) + \frac{e^{-2\lambda t}}{\lambda}\mathrm{I}(t;2\lambda) = e^{-\lambda t}\left[\frac{\mathrm{I}(t;\lambda)}{\lambda} + \mathrm{J}(t;\lambda)\right], \qquad \text{(D9)}$$

$$f(t,\tau) - f(0,\tau) + \frac{e^{-2\lambda\tau}}{\lambda}\mathrm{I}(t;2\lambda) = e^{-\lambda\tau}\left[\frac{\mathrm{I}(t;\lambda)}{\lambda} + \mathrm{J}(t;\lambda)\right]. \qquad \text{(D10)}$$

We may combine (D9) and (D10) to express $\mathrm{I}(t;2\lambda)$ as a function of $r(t)$ and $f(t,\tau)$ ie.

$$\mathrm{I}(t;2\lambda) = \frac{\lambda e^{\lambda\tau}}{\left(e^{-\lambda t} - e^{-\lambda\tau}\right)}\big[f(t,\tau) - f(0,\tau)\big] - \frac{\lambda e^{\lambda t}}{\left(e^{-\lambda t} - e^{-\lambda\tau}\right)}\big[r(t) - f(0,t)\big] \qquad \text{(D11)}$$

Finally we note that

$$\psi(t) = \int_0^t \sigma\big(u,t,r(u),f(u,\tau)\big)^2 du$$
$$= \int_0^t e^{-2\lambda(t-u)} g\big[r(u),f(u,\tau)\big]^2 du$$
$$= e^{-2\lambda t}\int_0^t e^{2\lambda u} g\big[r(u),f(u,\tau)\big]^2 du$$
$$= e^{-2\lambda t}\mathrm{I}(t;2\lambda). \qquad \text{(D12)}$$

Thus from (D11) and (D12) we finally have

$$\psi(t) = \lambda\alpha(t,\tau)\big[r(t) - f(0,t)\big] - \lambda e^{-\lambda(t-\tau)}\alpha(t,\tau)\big[f(t,\tau) - f(0,\tau)\big]\ . \qquad \text{(D13)}$$

where we set

$$\alpha(t,\tau) \equiv e^{-\lambda t} / \left(e^{-\lambda\tau} - e^{-\lambda t}\right). \tag{D14}$$

Appendix E

It is readily verified that the manipulations that led to equation (D8) of appendix D are equally valid for õ set to a general maturity *T*. Thus equation (D11) holds for õ set to *T* ie.

$$\begin{aligned} I(t;2\lambda) &= \frac{\lambda e^{\lambda T}}{e^{-\lambda t} - e^{-\lambda T}}\left[f(t,T) - f(0,T)\right] - \frac{\lambda e^{\lambda t}}{e^{-\lambda t} - e^{-\lambda T}}\left[r(t) - f(0,t)\right] \\ &= e^{2\lambda t}\psi(t), \qquad \text{(By use of D12).} \end{aligned}$$

Substituting the expression for Þ*(t)* from (D13) and solving for *f(t,T)-f(0,T)* we obtain the expression given in proposition 2.

Appendix F (Details of the infinitesimal generator K)

We recall the following result from Oksendal (1992) (theorem 7.9) concerning the infinitesimal generator of an *n* dimensional Ito process.

Consider the *n*-dimensional Ito stochastic differential system

$$dX_i = a_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t), \tag{E1}$$

where $W_1(t),\ldots,W_n(t)$ are independent Wiener processes. Let σ denote the matrix whose elements are the σ_{ij} and define the matrix *S* as

$$S = \sigma\sigma^T. \tag{E2}$$

The infinitesimal generator *K* of the process *X* is given by

$$K = \sum_{i=j}^{n} a_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=j}^{n}\sum_{i=j}^{n} S_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{E3}$$

So that the Kolmogorov backward equation for the transition probability density function of the process generating *X* is given by

$$\frac{\partial \pi}{\partial t} + K\pi = 0. \tag{E4}$$

In our application we set

$$
\begin{aligned}
X_1 &\equiv f(t,\tau), \\
a_1 &\equiv \sigma(t,\tau,r(t),f(t,\tau))^2 \frac{[e^{\lambda(\tau-t)}-1]}{\lambda}, \\
\sigma_{11} &\equiv \sigma_1 \equiv \sigma(t,\tau,r(t)f(t,\tau)), \\
X_2 &\equiv r(t), \\
a_2 &\equiv f_2(0,t)-\lambda f(0,t)+\psi(t)-\lambda r(t), \\
\sigma_{21} &\equiv \sigma_r \equiv \sigma(t,t,r(t)f(t,\tau)).
\end{aligned}
\tag{E5}
$$

Thus the matrix S assumes the form

$$
S=\begin{pmatrix} \sigma_1{}^2 & \sigma_1\sigma_r \\ \sigma_1\sigma_r & \sigma_r{}^2 \end{pmatrix}. \tag{E6}
$$

Using the foregoing expression for S the expression for K in equation (27) is readily derived.

Appendix G (Proof of Proposition 3)

Using the relationship

$$
P(t,T)=\exp\left(-\int_t^T f(t,s)ds\right)
$$

and equation (8) for the forward rate $f(t,s)$ we obtain for the bond price the expression

$$
P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left[-\left\{\int_t^T \int_0^t \sigma^*(s,u,\cdot)\,ds\,du+\int_t^T \int_0^t \sigma(s,u,\cdot)\,d\widetilde{W}(s)\,du\right\}\right],
$$

where

$$
\begin{aligned}
\sigma(s,T,\cdot) &= e^{-\lambda(T-s)}\ \sigma\big(r(s),\ f(s,\tau)\big) \\
\sigma^*(s,T,\cdot) &= \sigma(s,T,\cdot)\int_s^T \sigma(s,u,\cdot)\,du \\
&= \sigma\big(r(s),\ f(s,\tau)\big)^2 e^{-\lambda(T-s)} \int_s^T e^{-\lambda(u-s)}du
\end{aligned}
$$

Set

$$
\begin{aligned}
I &= \int_t^T \int_0^t \sigma^*(s,u,\cdot)\,ds\,du + \int_t^T \int_0^t \sigma(s,u,\cdot)\,d\tilde{W}(s)\,du \\
&\equiv I_1 + I_2 \\
&= \int_0^t \int_t^T \sigma^*(s,u,\cdot)\,du\,ds + \int_0^t \int_t^T \sigma(s,u,\cdot)\,du\,d\tilde{W}(s),
\end{aligned}
$$

where we have interchanged the order of integration to obtain the last equality. Next note that

$$
\begin{aligned}
\int_t^T \sigma^*(s,u,\cdot)du &= \sigma\big(r(s), f(s,\tau)\big)\int_t^T e^{-\lambda(u-s)} \int_s^u e^{-\lambda(y-s)}\sigma\big(r(s), f(s,\tau)\big)dy\,du \\
&= \sigma\big(r(s), f(s,\tau)\big)\int_t^T e^{-\lambda(u-s)} \Big\{\int_s^t e^{-\lambda(y-s)}\sigma\big(r(s), f(s,\tau)\big)dy \\
&\quad + \int_t^u e^{-\lambda(y-s)}\sigma\big(r(s), f(s,\tau)\big)dy \Big\}du \\
&= \sigma^2\big(r(s), f(s,\tau)\big)\int_t^T e^{-\lambda(u-s)}du \int_s^t e^{-\lambda(y-s)}dy \\
&\quad + \sigma^2\big(r(s), f(s,\tau)\big)\int_t^T e^{-\lambda(u-s)} \int_t^u e^{-\lambda(y-s)}dy\,du \\
&= \sigma^2\big(r(s), f(s,\tau)\big)e^{-\lambda(t-s)}\left(\int_t^T e^{-\lambda(u-t)}\,du\right)\int_s^t e^{-\lambda(y-s)}dy \\
&\quad + \sigma^2\big(r(s), f(s,\tau)\big)e^{-2\lambda(t-s)}\int_t^T e^{-\lambda(u-t)} \int_t^u e^{-\lambda(y-t)}dy\,du \\
&= \sigma^*(s,t,\cdot)\beta(t,T) + \sigma^2(s,t,\cdot)\alpha(t,T)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta(t,T) &= \int_t^T e^{-\lambda(u-t)}du = \tfrac{1}{\lambda}\left[1 - e^{-\lambda(T-t)}\right] \\
\alpha(t,T) &= \int_t^T e^{-\lambda(u-t)} \int_t^u e^{-\lambda(y-t)}dy\,du = \tfrac{1}{2}\beta(t,T)^2
\end{aligned}
$$

i.e. we have shown that

$$
\boxed{\int_t^T \sigma^*(s,u,\cdot)du = \beta(t,T)\,\sigma^*(s,t,\cdot) + \tfrac{1}{2}\beta(t,T)^2\,\sigma^2(s,t,\cdot)\,.}
$$

Next consider

$$
\begin{aligned}
\int_t^T \sigma(s,u,\cdot)du &= \int_t^T e^{-\lambda(u-s)}\,\sigma\big(r(s), f(s,\tau)\big)\,du \\
&= \sigma\big(r(s), f(s,\tau)\big)\,e^{-\lambda(t-s)}\left[\int_t^T e^{-\lambda(u-t)}\,du\right]
\end{aligned}
$$

i.e. we have shown that

$$
\boxed{\int_t^T \sigma(s,u,\cdot)du = \sigma(s,t,\cdot)\,\beta(t,T)\,.}
$$

Returning to the expressions for I_1, I_2 we can now write

$$I_1 = \int_0^t \left\{\beta(t,T)\,\sigma^*(s,t,\cdot) + \tfrac{1}{2}\beta(t,T)^2\sigma^2(s,t,\cdot)\right\} ds$$

and

$$I_2 = \int_0^t \beta(t,T)\,\sigma(s,t,\cdot)\,d\tilde{W}(s)$$

so that

$$I = \tfrac{1}{2}\beta^2(t,T)\int_0^t \sigma^2(s,t,\cdot)ds + \beta(t,T)\left\{\int_0^t \sigma^*(s,t,\cdot)ds + \int_0^t \sigma(s,t,\cdot)\,d\tilde{W}(s)\right\} .$$

However we note from equation (6), for the instantaneous spot rate $r(t)$, that

$$\int_0^t \sigma^*(s,t,\cdot)ds + \int_0^t \sigma(s,t,\cdot)\,d\tilde{W}(s) = r(t) - f(0,t) .$$

Hence

$$I = \tfrac{1}{2}\beta^2(t,T)\int_0^t \sigma^2(s,t,\cdot)ds + \beta(t,T)\left[r(t) - f(0,t)\right].$$

Recalling the definition of the subsidiary stochastic variable $\psi(t)$ we can finally write

$$I = \tfrac{1}{2}\beta^2(t,T)\psi(t) + \beta(t,T)\left[r(t) - f(0,t)\right].$$

Hence the expression for the bond price may be written as in proposition 3.

Appendix H (Matrix representation of the ADI scheme)

In order to set up equations (39)-(42) in matrix form, we need to define a number of vectors and matrices. First we define the coefficients

$$a_{1i}^j = \frac{1}{\theta} - \frac{\Delta t}{2\Delta r}\left[\frac{1}{2\Delta r}\left(\sigma_r^{n+\frac{1}{2}}\right)^2 - \mu_r^{n+\frac{1}{2}}\right], \quad a_{2i}^j = 1 - \frac{2}{\theta} + \frac{\Delta t}{2(\Delta r)^2}\left(\sigma_r^{n+\frac{1}{2}}\right)^2 ,$$

$$a_{3i}^j = \frac{1}{\theta} - \frac{\Delta t}{2\Delta r}\left[\frac{1}{2\Delta r}\left(\sigma_r^{n+\frac{1}{2}}\right)^2 + \mu_r^{n+\frac{1}{2}}\right],$$

$$c_{1j}^{i}=\frac{1}{\theta}-\frac{\Delta t}{2\Delta f}\left[\frac{1}{2\Delta f}\left(\sigma_{1}^{n+1}\right)^{2}-\sigma_{1}^{n+1}\right],\quad c_{2j}^{i}=1-\frac{2}{\theta}+\frac{\Delta t}{2(\Delta f)^{2}}\left(\sigma_{1}^{n+1}\right)^{2},$$
$$c_{3j}^{i}=\frac{1}{\theta}-\frac{\Delta t}{2\Delta f}\left[\frac{1}{2\Delta f}\left(\sigma_{1}^{n+1}\right)^{2}+\sigma_{1}^{n+1}\right]$$

and the vectors x_j, y_k and B_j, D_i given by

$$x_j=\left(P_{1,j}^{n+\frac{1}{2}},\cdots,P_{i-1,j}^{n+\frac{1}{2}}\right)^{T},\quad y_i=\left(P_{i,1}^{n+1},\cdots,P_{i,J-1}^{n+1}\right)^{T},$$
$$B_j=\left(L_{1,j}^{n}-a_{11}^{j}P_{0,j}^{n+\frac{1}{2}},L_{2,j}^{n},\cdots,L_{I-2,j}^{n},L_{I-1,j}^{n}-a_{3,i-1}^{j}P_{i,j}^{n+\frac{1}{2}}\right)^{T},$$
$$D_{ik}=\left(L_{i,1}^{n+\frac{1}{2}}-c_{11}^{i}P_{i,0}^{n+1},L_{i,2}^{n+\frac{1}{2}},\cdots,L_{i,J-2}^{n+\frac{1}{2}},L_{i,J-1}^{n+\frac{1}{2}}-c_{3,J-1}^{i}P_{i,J}^{n+1}\right)^{T}.$$

The matrix form of the above discretized equations can be written as:

(G1)

Ax = B, Cy = D

Where **A and C** are the block diagonal matrices

$$A=diag\left(A_1,A_2,\cdots,A_{J-1}\right)$$
$$C=diag\left(C_1,C_2,\cdots,C_{I-1}\right)$$

and we define the vectors

$$x=\left(x_1,\cdots,x_{J\text{-}1}\right)^{T}$$
$$B=\left(B_1,\cdots,B_{J\text{-}1}\right)^{T}$$
$$y=\left(y_1,\cdots,y_{I\text{-}1}\right)^{T}$$
$$D=\left(D_1,\cdots,D_{I\text{-}1}\right)^{T}$$

and the matrices A_j and C_i are given by

$$A_j = \begin{pmatrix} a_{21}^j & a_{31}^j & & & & \\ a_{12}^j & a_{22}^j & a_{32}^j & & & \\ & \ddots & \ddots & \ddots & & \\ & & & a_{1,I-2}^j & a_{2,I-2}^j & a_{3,I-2}^j \\ & & & & a_{1,I-1}^j & a_{2,I-1}^j \end{pmatrix},$$

$$C_i = \begin{pmatrix} c_{21}^i & c_{31}^i & & & & \\ c_{12}^i & c_{22}^i & c_{32}^i & & & \\ & \ddots & \ddots & \ddots & & \\ & & & c_{1,J-2}^i & c_{2,J-2}^i & c_{3,J-2}^i \\ & & & & c_{1,J-1}^i & c_{2,J-1}^i \end{pmatrix}.$$

Therefore, the solution to equations (3) can be obtained by solving successively the following series of equations

$$\mathrm{A}_j \mathrm{x}_\mathrm{j} = \mathrm{B}_\mathrm{j}, \qquad j = 1, \cdots, J-1,$$

$$\mathrm{C}_i \mathrm{y}_\mathrm{i} = \mathrm{D}_\mathrm{i}, \qquad i = 1, \cdots, I-1.$$