Valuing Energy Options in a One Factor Model Fitted to Forward Prices

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Abstract

In this paper we develop a single-factor modeling framework which is consistent with market observable forward prices and volatilities. The model is a special case of the multi-factor model developed in Clewlow and Strickland [1999b] and leads to analytical pricing formula for standard options, caps, floors, collars and swaptions. We also show how American style and exotic energy derivatives can be priced using trinomial trees, which are constructed to be consistent with the forward curve and volatility structure. We demonstrate the application of the trinomial tree to the pricing of a European and American Asian option. The analysis in this paper extends the results in Schwartz [1997] and Amin, et al. [1995].

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1 Introduction

In this paper we develop a pricing framework that enables the valuation of general energy contingent claims. There are currently two streams to the pricing literature. The first starts from a stochastic representation of the energy spot asset and other key variables, such as the convenience yield on the asset and interest rates (see for example Gibson and Schwartz [1990], Schwartz [1997], and Hilliard and Reis [1998]), and derives the prices of energy contingent claims consistent with the spot process. However, one of the problems of implementing these models is that often the state variables are unobservable - even the spot price is hard to obtain, with the problems exasperated if the convenience yield has to be jointly estimated. The second stream of the literature models the evolution of the forward or futures curve[1]. Forward or futures contracts are widely traded on many exchanges with prices easily observed - often the nearest maturity futures price is used as a proxy for the spot price with longer dated contracts used to imply the convenience yield. The framework of this paper resides in this second stream, simultaneously modeling the evolution of the entire forward curve conditional on the initially observed forward curve. As such it allows a unified approach to the pricing and risk management of a portfolio of energy derivative positions. Our

1 When interest rates are deterministic, as we assume in this paper, futures prices are equal to forward prices and so all our results for forward prices also apply to futures prices. The model can be extended to the case of stochastic interest rates using the results of Amin and Jarrow [1992].

framework is therefore closer to that of Cortazar and Schwartz [1994], and Amin, et al [1995], although, as we show in this paper, the two approaches are related.

We introduce our model, which is a special case of the multi-factor model in Clewlow and Strickland [1999b], in section 2. The model can be seen as an extension of the first model in Schwartz [1997], in the same way that the Heath, Jarrow, and Morton [1992] framework can be viewed as an extension of, say, the Vasicek [1977] model. The volatility structure of forward prices is the same, and reflects the mean reverting nature of energy prices, but the initial forward curve can be whatever the market dictates – unlike the Schwartz model, where the curve is endogenously determined. In section 3 we derive analytical pricing formulae for European options on the spot asset, options on forward contracts, caps, floors, collars, and swaptions. Section 4 presents our methodology for building recombining trinomial trees for the spot price process consistent with the forward curve. In section 5 we show how European and American style path dependent energy options can be priced using the tree with Asian options used as an example and with market data for crude oil and gas. The analysis of this paper significantly extends the analysis of both the Schwartz paper, which only looks at pricing futures contracts, and the paper of Amin, et al. (1995) which briefly outline how to price American options only when the term structure of futures prices has a flat volatility structure.

2 The Model

The starting point for our analysis is the stochastic evolution of the energy forward curve, $F(t,T)$. In a risk-neutral world investors price all claims as the expected future value discounted at the riskless rate. Since forward contracts do not require any initial investment, in a risk neutral world, the expected change in the forward price must be zero. Also, in order to obtain a Markovian spot price process the volatilities of forward prices must have a negative exponential form[2]. These observations lead to the following stochastic differential equation (SDE) for the forward price curve;

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2 See Carverhill [1992] for the proof of this in the context of the HJM framework.

$$\frac{dF(t,T)}{F(t,T)} = s e^{-a(T-t)} dz(t) \tag{2.1}$$

This is a more general version of the 1 factor version looked at by Schwartz [1997]. In that paper he proposes a process for the spot energy price and derives the forward price curve and the volatility curve to have particular forms. The model in equation (2.1) has two volatility parameters; σ determines the level of spot and forward price return volatility, whilst α determines the rate at which the volatility of increasing maturity forward prices decline and is also the speed of mean reversion of the spot price. These parameters can be estimated directly from the prices of options on the spot price of energy or forward contracts using the results in section 3 of this paper or, alternatively, by best fitting to historical volatilities of forward prices (an approach we use in section 5).

Any specification of the whole forward price dynamics implies a process for the spot price. For the specification in equation (2.1) the implied spot price process is shown in Appendix A to be;

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} + a\left(\ln F(0,t) - \ln S(t)\right) + \frac{s^2}{4}\left(1 - e^{-2at}\right)\right]dt + s\,dz(t) \tag{2.2}$$

The single factor model for the spot asset in Schwartz [1997] has the following defining SDE;

$$\frac{dS(t)}{S(t)} = a\left[m - \ln S\right]dt + s\,dz(t) \tag{2.3}$$

Therefore, equation (2.2) attains consistency with the initial forward curve $F(0,T)$ by making the long term risk adjusted drift, μ, the following function of time;

$$m(t) = \frac{\partial \ln F(0,t)}{\partial t} + \ln F(0,t) + \frac{s^2}{4}\left(1 - e^{-2at}\right) \tag{2.4}$$

We show in Appendix B that the forward curve at date t is given by;

$$F(S(t),t,T) = F(0,T)\left(\frac{S(t)}{F(0,t)}\right)^{\exp[-a(T-t)]} \exp\left[-\frac{s^2}{4a}e^{-aT}\left(e^{2at}-1\right)\left(e^{-aT}-e^{-at}\right)\right] \quad (2.5)$$

Thus, the forward curve at any future time is simply a function of the spot price at that time, the initial forward curve and the volatility function parameters. This result is computationally extremely useful, as it means that when pricing derivatives using trees the payoff of the derivatives can be evaluated analytically. It also allows us to obtain an analytical formula for the price of European swaptions in section 3.4.

3 Pricing European Options

In this section we discuss the pricing of European options on both the spot energy price and on forward contracts. Related results for standard European options have previously appeared in Amin and Jarrow [1991,1992] and Amin, et al. [1995].

3.1 Options on the Spot

From the standard risk-neutral pricing results (Cox and Ross [1976], Harrison and Pliska [1981]) the price of any contingent claim on the spot price, $C(t,S(t);\Theta)$, is given by the expectation of the discounted payoff under the risk neutral measure[3]

$$C(t,S(t);\Theta) = E_t\left[P(t,T)C(T,S(T);\Theta)\right] \quad (3.1)$$

where $P(t,T) = \exp\left(-\int_t^T r(u)du\right)$ and Θ is a vector of constant parameters. Therefore for a standard European call option $c(t,S(t);K,T)$ with strike price K and maturity date T on the asset $S(.)$ we have

$$c(t,S(t);K,T) = \mathrm{E}_t\left[P(t,T)\max(0,S(T)-K)\right] \quad (3.2)$$

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3 We make the standard assumptions regarding the filtration (see for example Amin and Jarrow [1992]).

Equation (2.1) can be integrated to give

$$F(t,T) = F(0,T)\exp\left[-\frac{1}{2}\int_0^t s^2 e^{-2a(T-u)}du + \int_0^t s e^{-a(T-u)}dz(u)\right] \tag{3.3}$$

The process for the spot can be obtained by setting $T = t$;

$$S(t) = F(t,t) = F(0,t)\exp\left[-\frac{1}{2}\int_0^t s^2 e^{-2a(t-u)}du + \int_0^t s e^{-a(t-u)}dz(u)\right] \tag{3.4}$$

From this we can see that the natural logarithm of the spot price is normally distributed;

$$\begin{aligned} \ln S(T) \sim\ & N\left[\ln F(0,T) - \frac{1}{2}\int_0^T s^2 e^{-2a(T-u)}du, \int_0^T s^2 e^{-2a(T-u)}du\right] \\ =\ & N\left[\ln F(0,T) - \frac{s^2}{4a}\left[1-e^{-2aT}\right], \frac{s^2}{2a}\left[1-e^{-2aT}\right]\right] \end{aligned} \tag{3.5}$$

Since interest rates are deterministic and $\ln S(T)$ is normally distributed we can use the results of Black and Scholes [1973] to obtain the following analytical formula for a standard European call option

$$c(t,S(t);K,T) = P(t,T)[F(t,T)N(h) - KN(h-\sqrt{w})] \tag{3.6}$$

where

$$h = \frac{\ln\left(\frac{F(t,T)}{K}\right) + \frac{1}{2}w}{\sqrt{w}} \quad , \quad w = \frac{s^2}{2a}\left[1-e^{-2a(T-t)}\right],$$

A special case of equation (2.1) is where $s(t,T) = s$. This is the restriction of Amin, et al. In this case $w = s^2(T-t)$.

The formula for standard European put options on the spot can be easily obtained by put-call parity.

3.2 Options on Forwards and Futures

Many options in the energy markets are on forward or futures contracts. In this section we derive the price at time *t* of a European call option with strike price *K* that matures at time *T* on a forward contract that matures at time *s*. Options are again priced using the standard methods. At date *t* the European call has the price

$$c(t, F(t,s); K, T, s) = \mathrm{E}_t\left[P(t,T)\max(0, F(T,s) - K)\right] \qquad (3.7)$$

Using the methodology of section 3.1 it is straightforward to show that the solution is

$$c(t, F(t,s); K, T, s) = P(t,T)[F(t,s)N(h) - KN(h - \sqrt{w})] \qquad (3.8)$$

where

$$h = \frac{\ln\left(\frac{F(t,s)}{K}\right) + \frac{1}{2}w}{\sqrt{w}}$$

w^2 is now given by the integral of the forward price return variance over the life of the option;

$$\begin{aligned} w^2(t,T,s) &= \int_t^T s^{\,2} e^{-2a(s-u)} du \\ &= \frac{s^{\,2}}{2a}\left(e^{-2a(s-T)} - e^{-2a(s-t)}\right) \end{aligned} \qquad (3.9)$$

This extends the results in Schwartz [1997] to pricing European options. Note that the results of section 3.1 are actually a special case of the results in this section with $s = T$.

3.3 Caps, Floors and Collars

Energy price caps, floors and collars are popular instruments for energy risk management. An energy price cap limits the floating price of energy the holder will pay on a predetermined set of dates $T+i\Delta T$; $i=1,\ldots,N$ to a fixed cap level *K*. A cap is therefore a portfolio of standard European call options with its price given by

$$Cap(t;K,T,N,\Delta T)=\sum_{i=1}^{N} c(t,F(t,T+i\Delta T);K,T+i\Delta T,T+i\Delta T) \qquad (3.10)$$

Conversely, an energy price floor limits the minimum price the holder will pay and is therefore a portfolio of standard European put options. A collar is simply a portfolio of a long position in a cap and a short position in a floor.

3.4 Options on Swaps

We define the time *t* value of an energy swaption, with maturity date *T*, to swap a series of floating spot price payments on dates $T+i\Delta T$ for a fixed strike price *K* to be

$$Swpn(t;K,T,s,N,\Delta T)=P(t,T)\mathrm{E}_t\left[\max\left(0,\left\{\frac{1}{N}\sum_{i=1}^{N} F(T,T+i\Delta T)\right\}-K\right)\right] \qquad (3.11)$$

We show in Appendix C that the value of the swaption defined in equation (3.11) is given by

$$Swpn(t;K,T,s,N,\Delta T)=\frac{1}{N}\sum_{i=1}^{N} c(t,F(t,T+i\Delta T);K_i,T,T+i\Delta T) \qquad (3.12)$$

where $K_i = F(S^*,T,T+i\Delta T),\ i=1,\ldots,N$ and $F(S^*,T,s)$ is the forward price at time *T* for maturity *s* when the spot price at time *T* is S^* and is given by the solution to;

$$\frac{1}{N}\sum_{i=1}^{N} F(S^*,T,T+i\Delta T)=K \qquad (3.13)$$

4 Building Trinomial Trees for the Spot Process

In this section we propose a general, robust and efficient procedure involving the use of trinomial trees for modelling the spot process (2.2) so that it is consistent with initial market data. The procedure is similar to constructing trinomial trees for the short rate, as outlined by Hull and White [1994a, 1994b], and described in detail in Clewlow and Strickland [1998].

These trees can then be used for pricing American style and path dependent options. American option valuation requires evaluation of the following expression

$$C(t) = \underset{q \in \Psi[t,T]}{Max} \; \tilde{E}_t\left[\exp(-\int_t^q r(u)du)g(q)\right] \qquad (4.1)$$

where $g(q)$ is the payoff of the option when it is exercised at date θ and $\Psi[t,T]$ is the class of all early exercise strategies (stopping times) in [t,T]. The early exercise strategy, and hence the option price, can be easily determined from the tree for the spot energy price.

Amin et al [1995] show how to derive a binomial tree to be consistent with the implied spot process when the volatilities of the forward prices are constant. This section extends their analysis to the mean reverting model of section 1 and to trinomial trees.

4.1 The Tree Building Procedure

The spot price process (2.2) can be written in terms of its natural log, $x(t) = \ln(S(t))$, after an application of Ito's lemma as follows;

$$dx(t) = \left[\frac{\partial \ln F(0,t)}{\partial t} + a\left(\ln F(0,t) - x(t)\right) + \frac{s^2}{4}\left(1 - e^{-2at}\right) - \frac{1}{2}s^2\right]dt + s\,dz(t) \qquad (4.2)$$

which we write as

$$dx(t) = [q(t) - a x(t)]dt + s\,dz(t) \qquad (4.3)$$

where $q(t) = \left(\frac{\partial \ln F(0,t)}{\partial t} + a \ln F(0,t) + \frac{s^2}{4}\left(1 - e^{-2at}\right) - \frac{1}{2}s^2\right)$

The tree building procedure consists of two stages. First, a preliminary tree is built for x assuming that $q(t)$=0 $\forall$ t and the initial value of x is zero. The resulting 'simplified' process for this new variable, $\bar{x}$,is given by

$$d\bar{x}(t) = -a\bar{x}(t)dt + s\,dz(t) \qquad (4.4)$$

The time values represented in the tree are equally spaced and have the form $t_i = i\Delta t$ where i is a non-negative integer and Δt is the time step. The levels of $\bar{x}$ (and consequently x) are equally spaced and have the form $\bar{x}_{i,j} = j\Delta x$ where Δx is the space step[4]. Any node in the tree can therefore be referenced by a pair of integers, that is the node at the ith time step and jth level we refer to as node (i,j). The trinomial tree technique is basically an explicit finite difference scheme and from stability and convergence considerations, a reasonable choice for the relationship between the space step and the time step is given by[5]:

$$\Delta x = s\sqrt{3\Delta t} \tag{4.5}$$

The trinomial branching process and the associated probabilities are chosen to be consistent with the drift and volatility of the process (4.3). The three nodes which can be reached by the branches emanating from node (i,j) are $(i+1, k-1)$, $(i+1, k)$, and $(i+1, k+1)$ where k is chosen so that the value of $\bar{x}$ reached by the middle branch is as close as possible to the expected value of $\bar{x}$ at time t_{i+1}. The expected value of $\bar{x}_{i,j}$ is $\bar{x}_{i,j} - a\bar{x}_{i,j}\Delta t$.

Let $p_{u,i,j}$, $p_{m,i,j}$, and $p_{d,i,j}$ define the probabilities associated with the lower, middle, and upper branches emanating from node (i,j) respectively. We show in Appendix C that the probabilities are given by:

$$\begin{aligned}
p_{u,i,j} &= \frac{1}{2}\left[\frac{s^2\Delta t + a^2 x_{i,j}^2 \Delta t^2}{\Delta x^2} + (k-j)^2 - \frac{a x_{i,j}\Delta t}{\Delta x}(1-2(k-j)) - (k-j)\right] \\
p_{d,i,j} &= \frac{1}{2}\left[\frac{s^2\Delta t + a^2 x_{i,j}^2 \Delta t^2}{\Delta x^2} + (k-j)^2 + \frac{a x_{i,j}\Delta t}{\Delta x}(1+2(k-j)) + (k-j)\right] \\
p_{m,i,j} &= 1 - p_{u,i,j} - p_{d,i,j}
\end{aligned} \tag{4.6}$$

The procedure described so far applies to the process $\bar{x}$ with $\theta(t) = 0$ and $\bar{x} = 0$.

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[4] The methodology generalises in a straightforward way to non-constant time and space steps (see Clewlow and Strickland [1998], Chapter 5.

[5] See Hull and White [1993].

The second stage in the tree building procedure consists of displacing the nodes in the simplified tree in order to add the proper drift and to be consistent with the observed forward prices[6]. We can introduce the correct time varying drift, by displacing the nodes at time $i\Delta t$ by an amount a_i. The a_i's are chosen to ensure that the tree correctly returns the observed forward price curve. The value of x at node (i, j) in the new tree equals the value of $\bar{x}$ at the corresponding node in the original tree plus a_i; the probabilities remain unchanged. The key to this stage is the use of forward induction and state prices to ensure that the tree returns the current market forward prices.

Define the state price $Q_{i,j}$ as the value, at time 0, of a security that pays 1 unit of cash if node (i,j) is reached, and zero otherwise. State prices are the building blocks of all securities; in particular, the price today $C(0)$ of any European claim with payoff function $C(S)$ at time step i in the tree is given by;

$$C(0) = \sum_j Q_{i,j} C(S_{i,j}) \tag{4.7}$$

where the summation takes place across all of the nodes j at time i. The state prices are obtained by forward induction[7]:

$$Q_{i+1,j} = \sum_{j'} Q_{i,j'} p_{j',j} P(i\Delta t, (i+1)\Delta t) \tag{4.8}$$

where $p_{j',j}$ is the probability of moving from node (i, j') to node $(i+1, j)$ and $P(i\Delta t, (i+1)\Delta t)$ denotes the price at time $i\Delta t$ of the pure discount bond maturing at time $(i+1)\Delta t$. The summation takes place over all nodes j, at time step i which branch to node $(i+1, j)$. In order to use the state prices to match the forward curve we use the following special case of equation (4.7);

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[6]From equation (4.3) we have an analytical solution for $q(t)$. However, we prefer not to use this, as it is the continuous time adjustment and would fail to return the observed forward prices in the tree exactly due to discretisation involved in the tree construction.

[7] Equation (4.8) is a discrete version of the Kolmogorov forward equation.

$$P(0,i\Delta t)F(0,i\Delta t)=\sum_j Q_{i,j}S_{i,j} \tag{4.9}$$

In Appendix D we show that the adjustment term needed to ensure that the tree correctly returns the observed forward curve is given explicitly as

$$a_i=\ln\left(\frac{P(0,i\Delta t)F(0,i\Delta t)}{\sum_j Q_{i,j}e^{\bar{x}_{i,j}}}\right) \tag{4.10}$$

4.2 Examples of Trinomial Trees Fitted to Market Forward Curves

We have fitted the spot rate tree to a number of different market forward curves. Figure 4.1 shows 3 market curves that are representative of; a downward sloping forward price curve (NYMEX Light, Sweet Crude Oil Futures Contracts, 1 October 1997), an upward sloping curve (NYMEX Light, Sweet Crude Oil Futures Contracts, 17 December 1997), and an approximately flat forward curve which exhibits seasonality (NYMEX Henry Hub Natural Gas Futures Contracts, 17 December, 1997). Two years worth of monthly maturity contracts are used to construct the curves.

Figure 4.1 Market Forward Curves

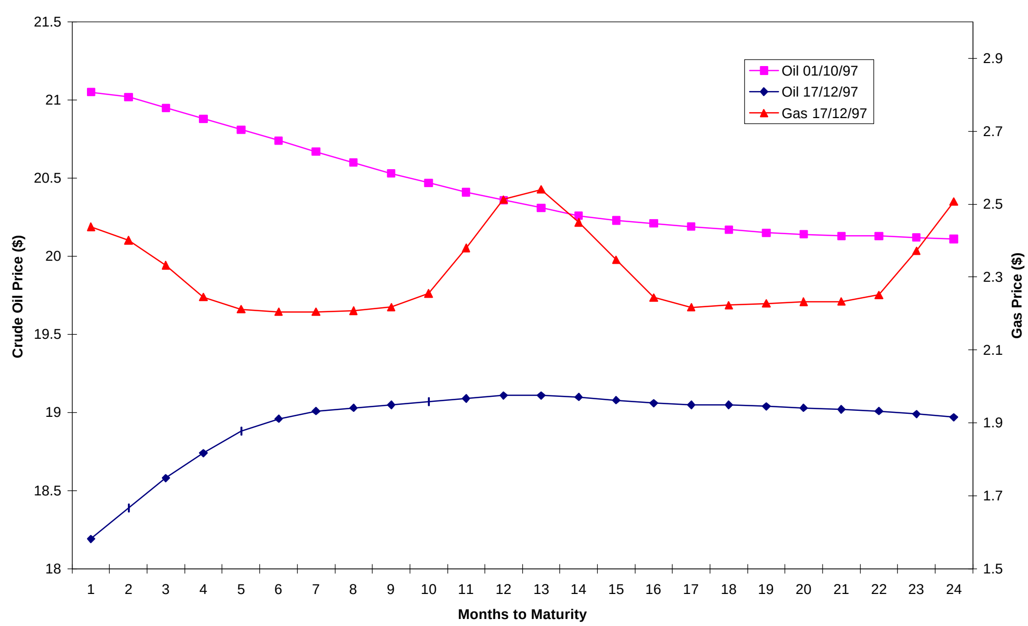
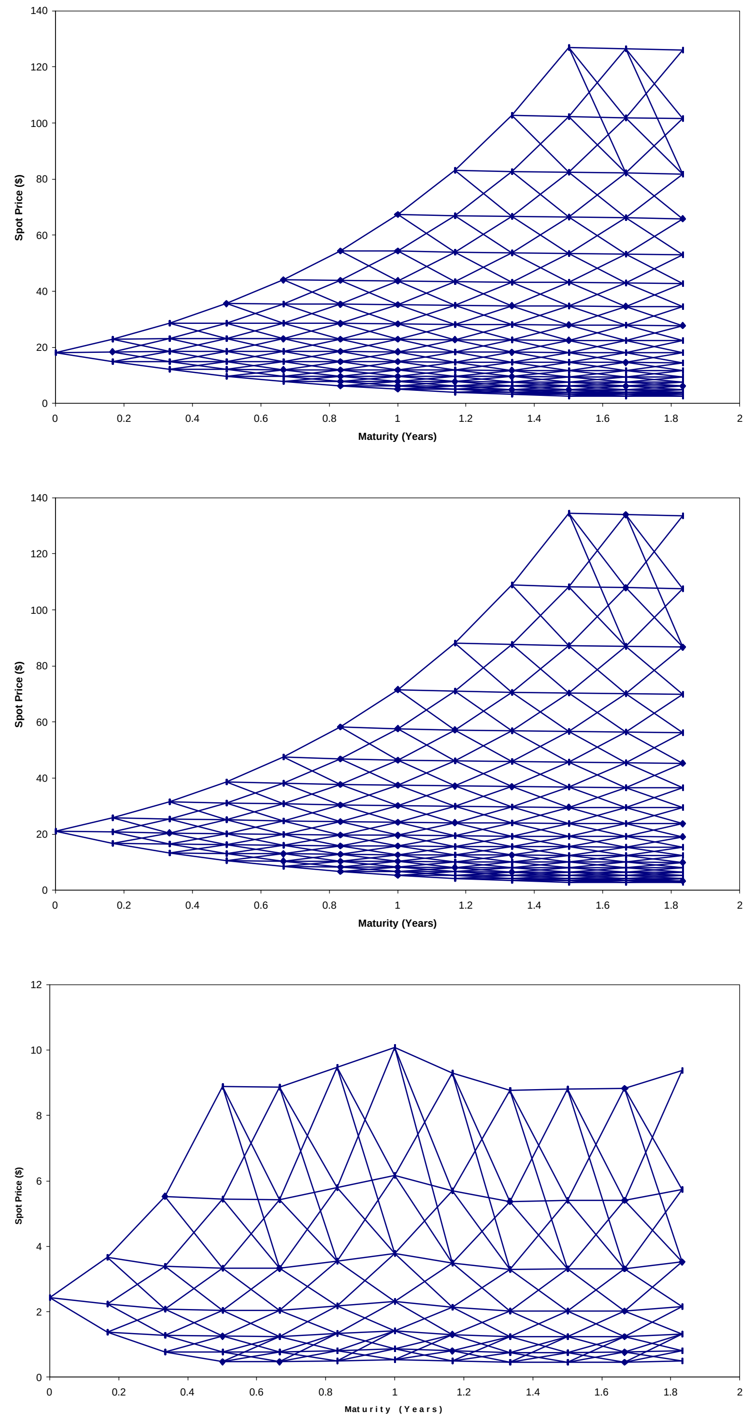


Figure 4.2 shows the resulting trees with time steps every two months.

Figure 4.2 Spot Price Trees Fitted to Market Forward Curves

(Downward sloping, Upward Sloping, and Seasonal)



The volatility parameters used in the tree construction were chosen by best fitting, in a least squares sense, the negative exponential forward price volatility function to sample standard deviations of one years worth of historical daily futures returns. The resulting parameters for the speed of mean reversion and spot price volatility were 0.34 and 0.31 respectively for crude oil, and 1.42 and 0.69 for the gas data.

Table 4.1 shows the results of pricing a one year at-the-money (forward) option on crude oil. The tree was constructed to fit the downward sloping forward curve of crude oil on the 1st October 1997 from Figure 4.1. Prices for European and American exercise options on both the spot and options on a 1.5 year forward contract are determined from the tree for different numbers of time steps. The volatility parameters used in the tree construction were chosen by a best fit to sample standard deviations for one year of historical data prior to 1st October 1997. Interest rates are assumed to be 6%.

Table 4.1 Value of European and American Options Calculated From the Tree

	Options on Spot				**Options on Future**			
Steps/ Year	**Euro Call**	**Euro Put**	**Amer Call**	**Amer Put**	**Euro Call**	**Euro Put**	**Amer Call**	**Amer Put**
20	1.925	1.925	2.401	2.097	1.550	1.694	1.577	1.728
40	1.918	1.918	2.395	2.093	1.543	1.688	1.573	1.722
60	1.914	1.914	2.385	2.089	1.539	1.684	1.569	1.719
80	1.911	1.911	2.389	2.087	1.537	1.681	1.567	1.717
100	1.909	1.909	2.385	2.086	1.535	1.679	1.565	1.715
120	1.907	1.907	2.387	2.085	1.533	1.678	1.564	1.714
140	1.906	1.906	2.383	2.084	1.532	1.677	1.563	1.713
160	1.905	1.905	2.386	2.083	1.531	1.676	1.562	1.712
180	1.904	1.904	2.384	2.082	1.530	1.675	1.561	1.711
200	1.904	1.904	2.383	2.082	1.530	1.675	1.561	1.711
Analytical	1.904	1.904			1.530	1.675		

We also compare the prices of European options calculated from the tree with the analytical values calculated via equations (3.6) and (3.7). Table 4.1 illustrates that prices calculated from the tree converge rapidly to the analytical price. It can also be seen from Table 4.1 that there is an early exercise premium associated with both options on the spot price and on the forward price due to the fact that the downward sloping forward curve implies a significant convenience yield on the spot asset.

The nature of the construction of the tree implies that hedge parameters can be quickly and easily calculated. If we calculate hedge parameters with respect to some 'shift' in the forward curve, then this shift only affects the displacement coefficients - it doesn't effect the position of the branches relative to the central branch or the probabilities associated with the branches.

5 Pricing General Path Dependent Options in Spot Price Trees

Having constructed trinomial trees for the spot energy process we show in this section how to price general path dependent options using the techniques developed in Hull and White [1993] (HW) for a Black and Scholes [1973] world and extended by Clewlow and Strickland [1999a] (CS) to multi-factor interest rate models.

5.1 Pricing General Path Dependent Contingent Claims

Assume we wish to price a general path dependent option whose payoff depends on some function $G(F(t,s); 0 \le t \le T, t \le s)$ of the path of the forward price curve. The procedure developed in HW and CS follows a number of steps. Firstly, the user determines the range (i.e. the minimum and maximum) of the possible values of $G(.)$ which can occur for every node in the tree. This is achieved by stepping forward through the tree from the origin to the maturity date computing, at each node, the minimum and maximum value of $G(.)$ given the value at the nodes at the previous time step which have branches to the current node and the forward curve at the current node.

Secondly, we choose an appropriate set of values of $G(.)$ between the minimum and maximum possible for each node. In choosing this set of values we note that the nodes which lie on the upper and lower edges of the tree have only one path which reaches them and therefore there can be only one value of $G(.)$. The largest range of values will typically occur in the central section of the tree. The number of values we consider should in general increase only linearly with the number of time steps and also decrease linearly from the central nodes of the tree down to one at the edges of the tree in order to control the computational requirements. Let $n_{i,j}$ be the number of values we store at node (i,j) and $G_{i,j,k}$, $k = 1,\ldots, n_{i,j}$ be the values of $G(.)$ where $G_{i,j,1}$ is the minimum and $G_{i,j,n_{i,j}}$ is the maximum. Clewlow and Strickland [1998] suggest choosing $n_{i,j}$ to be

$$n_{i,j} = 1 + b\left(i - abs(j)\right) \tag{5.1}$$

so that $n_{i,j}$ will always be one at the edges of the tree and $1+bi$ in the centre of the tree. In this way we can increase β to increase the accuracy of the approximation by considering more values of $G(.)$. In choosing the actual set of $n_{i,j}$ values for each node we should consider the distributional properties of the function $G(.)$. This will vary depending on the nature of $G(.)$ and therefore must be considered on a case by case basis.

The third step in the procedure is to set the value of the option at maturity at every node and for every value of $G(.)$

$$C_{N,j,k} = \mathrm{C}(t_N, F_{N,j,k}); \forall j,k \tag{5.2}$$

Finally, we step back through the tree computing discounted expectations and applying the early exercise condition at every node and for every value of $G(.)$

$$C_{i,j,k} = e^{-f(i,i+1)\Delta t}\left(p_{u,i,j}C_{i+1,j+1,u} + p_{m,i,j}C_{i+1,j,m} + p_{d,i,j}C_{i+1,j-1,d}\right) \tag{5.3}$$

where $f(i,i+1)$ denotes the one period forward rate from time step i to time step i+1 and where $C_{i+1,j+1,u}$, $C_{i+1,j,m}$, $C_{i+1,j-1,d}$ are the values of the option at time step i+1, given the current $G_{i,j,k}$, for upward, middle and downward branches of the spot price. These are obtained by computing the value of $G(.)$, given the current value, after upward, middle and downward branches $G_{i+1,j+1,u}$, $G_{i+1,j,m}$, $G_{i+1,j-1,d}$.

The values $G_{i+1,j+1,u}$, $G_{i+1,j,m}$, $G_{i+1,j-1,d}$ .and therefore also the option values $C_{i+1,j+1,u}$, $C_{i+1,j,m}$, $C_{i+1,j-1,d}$, will not in general be stored at the upward, middle and downward nodes and therefore must be obtained by interpolation. For example using linear interpolation we have

$$C_{i+1,j+1,u} = C_{i+1,j+1,k_l} + \left(\frac{C_{i+1,j+1,k_u} - C_{i+1,j+1,k_l}}{G_{i+1,j+1,k_u} - G_{i+1,j+1,k_l}}\right)\left(G_{i+1,j+1,u} - G_{i+1,j+1,k_l}\right) \tag{5.4}$$

where k_l and k_u are such that $G_{i+1,j+1,k_l} \le G_{i+1,j+1,u} \le G_{i+1,j+1,k_u}$ and $k_u = k_l + 1$. That is the two values of $G(.)$ which lie closest to either side of $G_{i+1,j+1,u}$ are found and a linear interpolation between these is done to obtain an estimate for $C_{i+1,j+1,u}$. The value of the path dependent contingent claim is read from the tree as the value of $C_{0,0,0}$.

5.2 Pricing Asian Options in a Trinomial Tree

As a specific example of the generalised methodology outlined in section 5.1 we price European and American versions of an average price call option, where the average is taken over the spot energy price on the fixing dates t_l, $l = 1, \ldots, L$.

Let there be a total of *N* time steps from the start of the life of the option until its maturity. In order to find the range of values of the average at each node we step forward through the tree from *i*=0 to *i*=*N*. If we have found the range for all nodes up to time step *i-1* then for any node (*i*,*j*) the minimum average is determined by the minimum average of the lowest node at time step *i-1* with a branch to the current node and the spot price at the current node. The minimum average is given by

$$G_{i,j,1} = \begin{cases} \dfrac{G_{i-1,j_l,1} m_{i-1} + S_j}{m_i} & \text{if } t_i = t_{m_i} \text{ i.e. a fixing date} \\ G_{i-1,j_l,1} & \text{otherwise} \end{cases} \tag{5.5}$$

where m_i is the number of fixing dates which have occurred up to time step *i* and node $(i-1, j_l)$ is the lowest node with a branch to node (*i*,*j*). Similarly the maximum average is determined by the maximum average of the highest node at time step *i-1* with a branch to the current node and the energy spot price at the current node

$$G_{i,j,n} = \begin{cases} \dfrac{G_{i-1,j_u,n} m_{i-1} + S_j}{m_i} & \text{if } t_i = t_{m_i} \text{ i.e. a fixing date} \\ G_{i-1,j_u,n} & \text{otherwise} \end{cases} \tag{5.6}$$

where node $(i-1, j_u)$ is the highest node with a branch to node (*i*,*j*). Now since the arithmetic average of the spot price is essentially a sum of lognormally distributed prices it will also be approximately lognormally distributed. We therefore choose a log-linear set for the $n_{i,j}$ values of the average at each node (*i*,*j*) which gives

$$G_{i,j,k} = G_{i,j,1} e^{(k-1)h} \tag{5.7}$$

where $h = \dfrac{\ln(G_{i,j,n}) - \ln(G_{i,j,1})}{n_{i,j} - 1}$.

In order to determine the option values of equation (5.4) we first compute what the average would be, given the current average, after upward, middle and downward branches $G_{i+1,j+1,u}$, $G_{i+1,j,m}$, $G_{i+1,j-1,d}$

$$G_{i+1,j+1,u} = \begin{cases} \dfrac{G_{i,j,k} m_i + S_{j+1}}{m_{i+1}} & \text{if } \; t_{i+1} = t_{m_{i+1}} \text{ i.e. a fixing date} \\ G_{i,j,k} & \text{otherwise} \end{cases} \tag{5.8}$$

$$G_{i+1,j,m} = \begin{cases} \dfrac{G_{i,j,k} m_i + S_{j}}{m_{i+1}} & \text{if } \; t_{i+1} = t_{m_{i+1}} \text{ i.e. a fixing date} \\ G_{i,j,k} & \text{otherwise} \end{cases} \tag{5.9}$$

$$G_{i+1,j-1,d} = \begin{cases} \dfrac{G_{i,j,k} m_i + S_{j-1}}{m_{i+1}} & \text{if } \; t_{i+1} = t_{m_{i+1}} \text{ i.e. a fixing date} \\ G_{i,j,k} & \text{otherwise} \end{cases} \tag{5.10}$$

5.3 A Numerical Example

In this section we price European and American versions of a fixed strike average price call option on crude oil with 1 year to maturity and where the terminal payoff is determined by the daily average of the crude oil price during the last month of the life of the option. The valuation date is the 1st October 1997, the tree is constructed to be consistent with the downward forward curve in Figure 4.1, using the same parameters as used for Table 4.1. Table 5.1 contains the results.

Table 5.1 Convergence of European and American Fixed Strike Average Rate Call Options

European

	Max. Number of Values for Average		
Steps/Year	4	12	20
12	1.869	1.869	1.869
60	1.877	1.854	1.852
108	1.911	1.858	1.853
168	1.958	1.865	1.856
216	1.998	1.872	1.859

American

12	1.922	1.922	1.922
60	1.969	1.949	1.947
108	2.037	1.986	1.981
168	2.113	2.008	2.000
216	2.173	2.022	2.009

Table 5.1 shows the convergence of both the European and American option values as we increase both the numbers of time steps per year and also the maximum number of averages at each node (see equation (5.1)). A further increase in either of these dimensions does not achieve greater accuracy of the option value.

5 Summary and Conclusions

In this paper we have developed a single-factor modeling framework which is consistent with market observable forward prices and volatilities. We derived analytical formulae for the forward price curve at a future date, standard European options on spot and forward prices, caps, floors, collars and swaptions. We have also shown how American style and exotic energy derivatives can be priced using trinomial trees, which are constructed to be consistent with the forward curve and volatility structure. As an example of the application of the trinomial tree technique we described the pricing of European and American Asian options and gave an illustrative example of the convergence properties of the procedure. The analysis in this paper extends the results in Schwartz [1997] and Amin, et al. [1995].

Appendix A : Proof of the Spot Price SDE

From equation (2.1) we have that forward prices satisfy the following SDE;

$$\frac{dF(t,T)}{F(t,T)} = s\,(t,T)dz(t) \tag{A.1}$$

This lognormal specification allows the following solution for the forward price;

$$F(t,T) = F(0,T)\exp\left[-\frac{1}{2}\int_0^t s\,(u,T)^2\,du + \int_0^t s\,(u,T)dz(u)\right] \tag{A.2}$$

The process for the spot can be obtained by setting $T = t$;

$$S(t) = F(t,t) = F(0,t)\exp\left[-\frac{1}{2}\int_0^t s\,(u,t)^2\,du + \int_0^t s\,(u,t)dz(u)\right] \tag{A.3}$$

Differentiating we obtain;

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} - \int_0^t s\,(u,t)\frac{\partial s\,(u,t)}{\partial t}du + \int_0^t \frac{\partial s\,(u,t)}{\partial t}dz(u)\right]dt + s\,(t,t)dz(t) \tag{A.4}$$

For the specific single factor model of this paper we have;

$$s\,(t,T) = s\,e^{-a\,(T-t)} \tag{A.5}$$

$$\frac{\partial s\,(t,T)}{\partial T} = -a\,s\,e^{-a\,(T-t)} \tag{A.6}$$

Let

$$\frac{dS(t)}{S(t)} = y(t)dt + s\,(t,t)dz(t) \tag{A.7}$$

where

$$y(t) = \left[\frac{\partial \ln F(0,t)}{\partial t} - \int_0^t s\,(u,t) \frac{\partial s\,(u,t)}{\partial t} du + \int_0^t \frac{\partial s\,(u,t)}{\partial t} dz(u) \right] \qquad \text{(A.8)}$$

Therefore we have;

$$y(t) = \frac{\partial \ln F(0,t)}{\partial t} + a s^{2} \int_0^t e^{-2a(t-u)} du - a \int_0^t \left(s e^{-a(t-u)} \right) dz(u) \qquad \text{(A.9)}$$

From (A.3), we have;

$$\ln S(t) = \ln F(0,t) - \frac{1}{2} \int_0^t s^{2} e^{-2a(t-u)} du + \int_0^t s e^{-a(t-u)} dz(u) \qquad \text{(A.10)}$$

implying

$$a \int_0^t s e^{-a(t-u)} dz(u) = a \left[\ln S(t) - \ln F(0,t) + \frac{1}{2} \int_0^t s^{2} e^{-2a(t-u)} du \right] \qquad \text{(A.11)}$$

Therefore

$$y(t) = \frac{\partial \ln F(0,t)}{\partial t} + a s^{2} \int_0^t e^{-2a(t-u)} du - a \left[\ln S(t) - \ln F(0,t) + \frac{1}{2} \int_0^t s^{2} e^{-2a(t-u)} du \right] \qquad \text{(A.12)}$$

Now $\int_0^t e^{-2a(t-u)} du = \frac{1}{2a} \left[1 - e^{-2at} \right]$

and so, after rearranging, we obtain;

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} + a \left(\ln F(0,t) - \ln S(t) \right) + \frac{s^{2}}{4} \left(1 - e^{-2at} \right) \right] dt + s\,dz(t) \qquad \text{(A.13)}$$

$$F(t, \quad F(0\ T)\exp\Big\lfloor -\frac{}{2}\int^{t} s \quad {}^{-} \quad {}^{T\ u}\ du + \int^{t}\ e^{-\quad T-u} \quad (u \ \Big\rfloor \tag{B.1}$$

$$\int_0^t s^{\,2} e^{-2a(T-u)}du = \frac{s^{\,2}}{2a} e^{-2aT}\left[e^{-2at} - 1\right] \tag{B.2}$$

From equation (3.4) we have;

$$S(t) = F(0,t)\exp\left[-\frac{1}{2}\int_0^t s^{\,2} e^{-2a(t-u)}du + \int_0^t s e^{-a(t-u)}dz(u)\right] \tag{B.3}$$

Now $\int_0^t s^{\,2} e^{-2a(t-u)}du = \frac{s^{\,2}}{2a}\left[1 - e^{-2at}\right]$ which implies that

$$\int_0^t s e^{-a(t-u)}dz(u) = \ln\left(\frac{S(t)}{F(0,t)}\right) + \frac{s^{\,2}}{4a}\left[1 - e^{-2at}\right] \tag{B.4}$$

Also $\int_0^t s e^{-a(T-u)}dz(u) = \int_0^t s e^{-aT}e^{au}dz(u) = \frac{e^{-aT}}{e^{-at}}\int_0^t s e^{-a(t-u)}dz(u)$, substituting from equation (B.4) we obtain;

$$\int_0^t s e^{-a(T-u)}dz(u) = e^{-a(T-t)}\left[\ln\left(\frac{S(t)}{F(0,t)}\right) + \frac{s^{\,2}}{4a}\left[1 - e^{-2at}\right]\right] \tag{B.5}$$

Substituting into equation (B.1), using equations (B.2) and (B.5), and simplifying we obtain;

$$F(t,T) = F(0,T)\left(\frac{S(t)}{F(0,t)}\right)^{\exp[-a(T-t)]}\exp\left[-\frac{s^{\,2}}{4a}e^{-aT}\left(e^{2at} - 1\right)\left(e^{-aT} - e^{-at}\right)\right] \tag{B.6}$$

Appendix C : Proof of the Analytical Formula for a Swaption

From equation (3.11) we have;

$$Swpn(t;K,T,s,N,\Delta T)=P(t,T)\mathrm{E}_t\left[\max\left(0,\left\{\frac{1}{N}\sum_{i=1}^{N}F(T,T+i\Delta T)\right\}-K\right)\right] \tag{C.1}$$

Let S^* be given by the solution to the following;

$$\frac{1}{N}\sum_{i=1}^{N}F(S^*,T,T+i\Delta T)=K \tag{C.2}$$

Now let K_i be given by;

$$K_i=F(S^*,T,T+i\Delta T),\ \ i=1,\ldots,N \tag{C.3}$$

Since the forward price $F(S(T),T,s)$ is monotonically increasing in $S(T)$ (see equation (2.5)) then we have;

$$Swpn(t;K,T,s,N,\Delta T)=\frac{1}{N}\sum_{i=1}^{N}c(t,F(t,T+i\Delta T);K_i,T,T+i\Delta T) \tag{C.4}$$

Appendix D : Proof of the Transition Probabilities

Under the simplified process for $\bar{x}$ of section 4.1 we have

$$E[\Delta x] = -a x_{i,j} \Delta t \tag{D.1}$$

$$E[\Delta x^2] = s^2 \Delta t + E[\Delta x]^2 \tag{D.2}$$

Recall from section 4.1 that *k* determines the destination level of $\bar{x}$ of the middle branch from node (*i*,*j*), therefore equating the first and second moments of Δx in the tree with the values given by equations (D.1) and (D.2) we obtain;

$$p_{u,i,j}((k+1)-j)\Delta x + p_{m,i,j}(k-j)\Delta x + p_{d,i,j}((k-1)-j)\Delta x = -a x_{i,j}\Delta t \tag{D.3}$$

$$p_{u,i,j}((k+1)-j)^2\Delta x^2 + p_{m,i,j}(k-j)^2\Delta x^2 + p_{d,i,j}((k-1)-j)^2\Delta x^2 = s^2\Delta t + (-a x_{i,j}\Delta t)^2 \tag{D.4}$$

Also, we require that the sum of the probabilities should be equal to one;

$$p_{u,i,j} + p_{m,i,j} + p_{d,i,j} = 1 \tag{D.5}$$

Solving the system of equations (D.3), (D.4) and (D.5) we obtain;

$$\begin{aligned}
p_{u,i,j} &= \frac{1}{2}\left[\frac{s^2\Delta t + a^2 x_{i,j}^2 \Delta t^2}{\Delta x^2} + (k-j)^2 - \frac{a x_{i,j}\Delta t}{\Delta x}(1-2(k-j)) - (k-j)\right] \\
p_{d,i,j} &= \frac{1}{2}\left[\frac{s^2\Delta t + a^2 x_{i,j}^2 \Delta t^2}{\Delta x^2} + (k-j)^2 + \frac{a x_{i,j}\Delta t}{\Delta x}(1+2(k-j)) + (k-j)\right] \\
p_{m,i,j} &= 1 - p_{u,i,j} - p_{d,i,j}
\end{aligned} \tag{D.6}$$

Appendix E : Proof of the Adjustment Term for a[i]

From equation (4.9) we have

$$P(0,i\Delta t)F(0,i\Delta t)=\sum_j Q_{i,j}S_{i,j} \tag{E.1}$$

Expressing the spot price $S_{i,j}$ in terms of $x_{i,j}$ we obtain;

$$\begin{aligned} P(0,i\Delta t)F(0,i\Delta t) &= \sum_j Q_{i,j}e^{x_{i,j}} \\ &= \sum_j Q_{i,j}e^{(\bar{x}_{i,j}+a_i)} \end{aligned} \tag{E.2}$$

$$P(0,i\Delta t)F(0,i\Delta t)=e^{a_i}\sum_j Q_{i,j}e^{\bar{x}_{i,j}} \tag{E.3}$$

Rearranging equation (E.3) yields;

$$a_i=\ln\left(\frac{P(0,i\Delta t)F(0,i\Delta t)}{\sum_j Q_{i,j}e^{\bar{x}_{i,j}}}\right) \tag{E.4}$$

References

Amin K and R A Jarrow, 1991, "Pricing Foreign Currency Options Under Stochastic Interest Rates", *Journal of International Money and Finance*, Vol. 10, pp. 310-329.

Amin K, and R A Jarrow, 1992, "Pricing Options on Risky Assets in a Stochastic Interest Rate Economy", *Mathematical Finance*, Vol. 2, No. 4, pp. 217-237.

Amin K, V Ng and S C Pirrong, 1995, "Valuing Energy Derivatives" in *Managing Energy Price Risk*, Risk Publications, London.

Black, F, and M Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol. 81, pp. 637-659.

Clewlow L, and C Strickland, 1998, *Implementing Derivatives Models*, John Wiley and Sons, London.

Clewlow L and C Strickland, 1999a, "Pricing Interest Rate Exotics in Multi-Factor Markovian Short Rate Trees", Working Paper, School of Finance and Economics, University of Technology Sydney.

Clewlow L and C Strickland, 1999b, "A Multi-Factor Model for Energy Derivatives Risk Management", Working Paper, School of Finance and Economics, University of Technology Sydney.

Cortazar G and E S Schwartz, 1994, "The Valuation of Commodity Contingent Claims", *The Journal of Derivatives*, Vol. 1, No. 4, 27-39.

Cox J C, J E Ingersoll and S A Ross, 1981, "The Relation between Forward Prices and Futures Prices", *Journal of Financial Economics*, Vol. 9, pp. 321-346.

Cox, J C and S A Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes", *Journal of Financial Economics*, Vol. 3, pp. 145 - 166.

Gibson R and E S Schwartz, 1990, “Stochastic Convenience Yield and the Pricing of Oil Contingent Claims”, *Journal of Finance*, Vol. 45, pp. 959-976.

Heath, D, R Jarrow and A Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation", *Econometrica*, Vol. 60, No. 1, pp. 77-105.

Harrison, J M and S Pliska, 1981, “Martingales and Stochastic Integrals in the Theory of Continuous Trading”, Stochastic Processes and Their Applications, Vol. 11, pp. 215-260.

Hilliard J E and J Reis, 1998, “Valuation of Commodity Futures and Options under Stochastic Convenience Yields, Interest Rates, and Jump Diffusions in the Spot”, *Journal of Financial and Quantitative Analysis*, Vol. 33, No. 1, pp. 61-86.

Hull J, and A White, 1993, “Efficient Procedures For Valuing European and American Path-Dependent Options, *The Journal of Derivatives*, Fall, pp. 21-31.

Hull J, and A White, 1994a, “Numerical Procedures For Implementing Term Structure Models II: Two-Factor Models, *The Journal of Derivatives*, Winter, pp. 37-48.

Hull J, and A White, 1994b, “Numerical Procedures For Implementing Term Structure Models I: Single-Factor Models, *The Journal of Derivatives*, Fall, pp. 7-16.

Schwartz E S, 1997, “The Stochastic Behaviour of Commodity Prices: Implications for Pricing and Hedging”, *The Journal of Finance*, Vol. 52, No. 3, pp. 923 –973.

Vasicek O, 1977, "An Equilibrium Characterisation of the Term Structure", *Journal of Financial Economics*, Vol. 5, pp. 177-188.