DYNAMICS OF BELIEFS AND LEARNING UNDER $\mathbf{a}_L$-PROCESSES – THE HOMOGENEOUS CASE

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ABSTRACT. This paper studies a class of models in which agents' expectations influence the actual dynamics while the expectations themselves are the outcome of some learning process. Under the assumptions that agents have homogeneous expectations (or beliefs) and that they update their expectations by least-squares L- and general $\mathbf{a}_L$-processes, the dynamics of the resulting expectations and learning schemes are analyzed. It is shown how the dynamics of the system, including stability, instability and bifurcation, are affected by the learning processes. The cobweb model with a simple homogeneous expectation scheme is employed as an example to illustrate the stability results, the various types of bifurcations and the routes to complicated price dynamics.

Keywords: Homogeneous beliefs, least-squares L-process, general $\mathbf{a}_L$-process, stability, instability, bifurcation, cobweb model.

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1. INTRODUCTION

Many dynamic economic models form an expectations feedback system. Expectations affect actual outcomes, actual outcomes affect expectations through learning, and so on. The standard economic models, such as the capital asset pricing model and the Black-Scholes option valuation model, rely on the assumption of rational homogeneous expectations. However, the past decade has witnessed a rapidly increasing interest in work on boundedly rational expectations. Boundedly rational agents use simple learning schemes to form their expectations. In the boundedly rational world, stability of expectations and learning schemes becomes important in many models of finance and economics.

Among various learning schemes that boundedly rational agents may use, the properties of least-squares learning processes[1] under homogeneous expectations have been studied extensively (see, for example, Balasko and Royer (1996), Bray (1983), Evans and Honkapohja (1999), Evans and Ramey (1992), Lucas (1978) and Marcet and Sargent (1989)). In his survey paper, Grandmont (1998) considers stability and convergence to self-fulfilling expectations in large socioeconomic systems and suggests a kind of general *'Uncertainty Principle' – Learning is bound to generate local instability of self-fulfilling expectations, if the influence of expectations on the dynamics is significant.* When learning processes are involved, as pointed out by Balasko and Royer (1996), *'the properties of the (Walrasian) equilibrium with respect to the convergence of least-squares learning processes and, more generally, of recursive processes have hardly been studied'.* This paper intends to add to the literature on this problem. In particular, the dynamics of the (Walrasian) equilibrium is analyzed when the learning processes follow the least-squares L- and the general $\mathbf{a}_L$-processes (see the following section for the definition).

[1]The term learning is being used in a very particular, and perhaps restricted sense here. It refers to a situation in which agents adopt a rule to come of with an expectation of next period's price. A broad use of the term learning would envisage a situation in which agents are able to switch strategies in light of prediction errors, for example as in Brock and Hommes (1997).

This paper considers a deterministic (nonlinear) framework and focuses on an extremely simple case, in which the state of the system is completely described at every date by a single real number x_t. Depending upon the context, the state variable x may stand for a price, a rate of inflation, a real rate of interest etc. Traders plan one period ahead. To abstract from all forms of uncertainty, the traders' expectations or forecasts follow finite least-squares L- or general $\mathbf{a}_L$-processes.

Balasko and Royer (1996) consider local stability under a homogeneous least-squares L-learning process, which is formed from the past L values of the state variable. They determine a relationship between stability and the parameter L. Intuition would suggest that the larger is the number of observations L, the more stable is the equilibrium, and they show that this intuition is essentially correct for the least squares L-process. As in Balasko and Royer (1996), this paper tries to determine the stability properties associated with different values of L and different $\mathbf{a}_L$-processes and it is seen to what extent Balasko and Royer's result still holds. The learning processes considered are of finite memory (i.e. $L \geq 1$ is finite). Related studies on homogeneous learning can be found in Barucci et.al. (1999), Grandmont (1985) and Grandmont and Laroque (1986) for finite memory and Balasko and Royer (1996), Bischi and Gardini (2000), Chiarella (1988) and Hommes (1991), (1994) for infinite memory.

Under homogeneous expectations, this paper concentrates on how the stability of the system is affected differently by the least-squares L and the general $\mathbf{a}_L$ learning processes with different lag length L and weight vector $\mathbf{a}$. In particular, it is shown that, with the least-squares (learning) L-process, the (local) stability region of the fixed equilibrium can be completely characterized by the lag length L and the parameters of the system, while the instability of the fixed equilibrium generates a $1 : L+1$ resonance.[2] However, for the general $\mathbf{a}_L$-process, much more rich dynamics arise. In the simplest cases of $L = 2$ and 3, it is found that, depending on the weight vector $\mathbf{a}$, the stability region can be different and the instability of the fixed equilibrium can generate $p : q$-resonances (for almost any p, q) and quasi-periodic orbits.

[2]The sense in which the term resonance is being used in this paper will be explained below.

The paper is organized as follows. Section 2 introduces first a temporary equilibrium relation with homogeneous beliefs and, following Balasko and Royer (1996), describes homogeneous least-squares L-processes and general $\mathbf{a}_L$-processes. As an example, a nonlinear cobweb model under a homogeneous least-squares L-process and a simple $\mathbf{a}_L$-process is then presented. Under the assumption of homogeneous expectations, Section 3 investigates the dynamics of the system, including stability, instability and bifurcations, when the agents follow least-squares L-processes. An analysis of the dynamics of the general $\mathbf{a}_L$-process then follows in Section 4. In Section 5, we apply the results obtained in the previous sections to the nonlinear cobweb model introduced in Section 2. Various bifurcation phenomena and routes to chaos are analyzed in detail. Section 6 concludes. The proofs of the various propositions are gathered in the appendix.

2. Homogeneous Beliefs and Learning

In this section, least squares L- and general homogeneous $\mathbf{a}_L$-processes as learning processes for a general temporary equilibrium model are introduced first, then a nonlinear cobweb model under a homogeneous least-squares L-process is presented. Stability and bifurcation of the (Walrasian) equilibrium of the general model with these processes are studied in the following sections.

For the convenience of the discussion, the state variable x_t is treated as the price at period t. In this section, all traders' expectations at date t about the future state are assumed to be identical and denoted as x^e_{t+1}. Assume that the current equilibrium state x_t depends on the common forecast through the temporary equilibrium relation[3]

$$T(x_t, x^e_{t+1}) = 0. \tag{2.1}$$

This paper focuses on the case $x^e_{t+1} = {}_{t-1}x^e_{t+1}$, indicating that the information set used to form the expectations includes information only up to and including time $t-1$. The conditioning on information up to and including $(t-1)$ arises in situations where the

[3]To be more precise, $x^e_{t+1} = {}_{t-1}x^e_{t+1}$ if x_t is not included in the information set and $x^e_{t+1} = {}_{t}x^e_{t+1}$ if x_t is in the information set.

equilibrium price is a result of the Walrasian auctioneer process. Agents are assumed to form expectations before the auctioneer announces the equilibrium price.

Let the expectation function be defined by

$$_{t-1}x^e_{t+1} = \Psi(x_{t-1}, \cdots, x_{t-L}). \tag{2.2}$$

Assume there exists x^* such that $T(x^*, y^*) = 0$ with $y^* = \Psi(x^*, x^*, \cdots, x^*)$, that is x^* is a fixed equilibrium of $T(x, \Psi(x, \cdots, x)) = 0$. It is also assumed that the map T is well defined and continuously differentiable locally at (x^*, y^*), the expectation function Ψ is well defined and continuously differentiable locally near $(x_1, x_2, \cdots, x_L) = (x^*, x^*, \cdots, x^*)$. Denote

$$B = \frac{\partial T(x,y)}{\partial x}\bigg|_{(x^*,y^*)}, \quad C = \frac{\partial T(x,y)}{\partial y}\bigg|_{(x^*,y^*)}, \quad d_j = \frac{\partial \Psi(x_1, \cdots, x_L)}{\partial x_j}\bigg|_{(x^*,\cdots,x^*)}$$

with $j = 1, \cdots, L$ and assume B, $C \neq 0$. Then the linearization of (2.1) and (2.2) at the fixed point x^* is given by

$$B(x_t - x^*) + C\sum_{j=1}^{L} d_j(x_{t-j} - x^*) = 0. \tag{2.3}$$

Consider L real numbers $a_j \geq 0$[4] satisfying $\sum_{j=1}^{L} a_j = 1$ and denote by $\mathbf{a}_L = (a_1, a_2, \cdots, a_L)$ the (L-dimensional) weight vector. The following definition is introduced in Balasko and Royer (1996).

Definition 2.1. The **general homogeneous**[5] **recursive (finite) $\mathbf{a}_L$-process** is defined by (2.2) and

$$\Psi(x_1, \cdots, x_L) = g(a_1x_1 + \cdots a_Lx_L), \quad 0 \leq a_j \leq 1, \quad \sum_{j=1}^{L} a_j = 1 \tag{2.4}$$

[4]Here a_j are treated as the weights (or probabilities) of the past states and therefore are assumed to be nonnegative.

[5]Note the double use of the word homogeneous in this paper. Earlier to indicate that all agents have identical expectations. Here to indicate a particular expectation formation rule.

with some (locally near x^*) continuously differentiable function g. The **homogeneous least-squares L-process** is simply the $\mathbf{a}_L$-process defined by the weight vector $\mathbf{a}_L = (1/L, 1/L, \cdots, 1/L)$.

When $g(x) = x$ and $\mathbf{a}_L = (1, 0, \cdots, 0)$, the expectation corresponds to the naive expectation—the expected price equals the most recently observed price x_{t-1}. In general, any forecast or expectation of future prices will be some function of past prices and in this context the general $\mathbf{a}_L$-process is the simplest expectations scheme. This expectation is also known as *Linear Backward-Looking Expectations* or *Distributed Lag Expectations* (e.g. Hommes (1998)). There are many possibilities for the distribution of weights a_j. When $a_j = 1/L$, weights are distributed evenly and the expectations scheme corresponds to the least-squares L-process as already stated. In adaptive learning schemes, more weight is given to more recent observations, that is $a_j \geq a_{j+1}, 1 \leq j \leq L-1$, e.g. the arithmetic and geometric weights considered in Section 4. For more related discussion, the reader is referred to Balasko and Royer (1996) and Hommes (1998).

Following from (2.3) and (2.4), the linearization of (2.1) and (2.2) (near x^*) with the general homogeneous recursive (finite) $\mathbf{a}_L$-process becomes

$$B(x_t - x^*) + Cg_o \sum_{j=1}^{L} a_j (x_{t-j} - x^*) = 0, \tag{2.5}$$

where $g_o = g'(x^*) = [dg(x)/dx]|_{x=x^*}$[6]. Replacing $x_t - x^*$ by x_t in (2.5), the local stability of the fixed equilibrium x^* of (2.5) is then equivalent to the stability of the zero solution of the equation

$$x_t + \alpha \sum_{j=1}^{L} a_j x_{t-j} = 0 \qquad \text{with} \qquad \alpha = Cg_o/B. \tag{2.6}$$

The parameter α measures the combined influence of both the expectation (through C/B) and extrapolation (through g_o) on the price x_t locally (near x^*). Therefore the

[6]In the context of Grandmont (1998), the parameter g_o may be referred to as the extrapolation rate and B, C as the parameters of the system. In particular, C/B measures the local (near the fixed equilibrium x^*) dependence of the price (x_t) on the expectation (x^e_{t+1}).

local stability of the general homogeneous recursive (finite) $\mathbf{a}_L$-process is generically governed by the eigenvalues of the characteristic polynomial of (2.6)

$$\Gamma_L(\lambda) \equiv \lambda^L + \alpha \sum_{j=1}^{L} a_j \lambda^{L-j} = 0. \tag{2.7}$$

Equation (2.7) is an L-th order polynomial and hence the zero solution of (2.6) is Locally Asymptotically Stable (**LAS** hereafter) if and only if all roots $\lambda_j (j = 1, \cdots, L)$ of (2.7) satisfy $|\lambda_j| < 1$ $(j = 1, \cdots, L)$. It is in general difficult to obtain explicit necessary and sufficient conditions in terms of the coefficients of $\Gamma(\lambda)$. However, for the homogeneous least-squares L-process, an explicit necessary and sufficient condition is derived in terms of the coefficients of $\Gamma(\lambda)$ in the following section. A complete analysis of the stability and bifurcation of $\mathbf{a}_2$ and $\mathbf{a}_3$ is carried out in Section 4. For the more general $\mathbf{a}_L$ processes, it only seems possible to obtain some sufficient conditions about stability.

As both an illustration of the results obtained in the following sections and an important application in its own right, consider a general cobweb class of models where the market equilibrium price is determined by (see Hommes (1998)) for details)

$$D(p_t) = S({}_{t-1}p^e_{t+1}), \tag{2.8}$$

and D and S are demand and supply functions. Let price expectations be given by

$${}_{t-1}p^e_{t+1} = H(\overrightarrow{P}_{t-1}), \tag{2.9}$$

where $\overrightarrow{P}_{t-1} = (p_{t-1}, p_{t-2}, \cdots, p_{t-L})$ is a vector of past prices (with lag L) and H is called the expectation function or the perceived law of motion. Combining (2.8) and (2.9) yields

$$D(p_t) = S(H(\overrightarrow{P}_{t-1})). \tag{2.10}$$

To keep the model as simple as possible, throughout the paper a linear demand function is assumed, namely

$$D(p_t) = a - bp_t. \tag{2.11}$$

Following Chiarella (1988) and Hommes (1998), a nonlinear, but monotonic, S-shaped supply curve

$$S(x) = Tanh(\beta x) = \frac{e^{\beta x} - e^{-\beta x}}{e^{\beta x} + e^{-\beta x}} \tag{2.12}$$

is selected, where the parameter β tunes the steepness of the S-shape.[7] Also, assume that

$$H(\overrightarrow{P}_{t-1}) = g_o \sum_{j=1}^{L} a_j p_{t-j} \tag{2.13}$$

with $a_j \geq 0$, $\sum_{j=1}^{L} a_j = 1$, $g_o \in \mathbb{R}$ for $j = 1, \cdots, L$. The parameter g_o is allowed to be both positive (trend chasing expectations) and negative (contrarian expectations). Then, the resulting nonlinear difference equation is given by

$$p_t = \frac{1}{b}[a - S(g_o \sum_{j=1}^{L} a_j p_{t-j})]. \tag{2.14}$$

Following the notation in (2.1),

$$T(x, y) = bx + S(y) - a. \tag{2.15}$$

The fixed equilibrium p^* satisfies

$$a - bp^* = S(g(p^*)) = Tanh(\beta g_o p^*). \tag{2.16}$$

For $g_o \geq 0$, the fixed equilibrium is unique. However, for $g_o < 0$, there exists $g_o^* < 0$ such that there is unique positive fixed equilibrium p^* for $g_o \in (g_o^*, 0)$ and three equilibria p_j^* $(j = 1, 2, 3)$ satisfying $p_3^* < p_2^* < 0 < p_1^*$. When $g_o = g_o^*$, there are two equilibria $p_2^* < 0 < p_1^*$. Accordingly, the characteristic equation of the linearized equation of (2.14) at the fixed equilibrium p^* has the form of (2.7) with

$$B = b,\ C = \frac{4\beta}{(e^{\beta g_o p^*} + e^{-\beta g_o p^*})^2},\ \alpha = \frac{C g_o}{B} = \frac{4\beta g_o / b}{e^{\beta g_o p^*} + e^{-\beta g_o p^*\,2}} \quad \text{and} \quad g'(p^*) = g_o.$$

[7]We are indebted to an anonymous referee for pointing out that application of the results developed in the following two sections to the cobweb model is independent from the functional forms for both the demand and supply.

Here B and C are the slopes of the demand and supply function at the fixed equilibrium p^*, respectively.

3. Dynamics of Homogeneous Least-Squares L-Processes

This section investigates the dynamics of the system with the least-squares L-process, focusing in particular on the local stability and bifurcation analysis in the parameter space of the system and the lag length.

3.1. **Local Stability.** The following Lemma 3.1, obtained in Chiarella and He (2000*c*), is used to characterize the local stability.

Lemma 3.1. *Let*

$$Q_L(\lambda) \equiv \lambda^L + \gamma\lambda^{L-1} + \gamma\lambda^{L-2} + \cdots + \gamma\lambda + \gamma. \tag{3.1}$$

Then the zeros of $Q_L(\lambda)$ lie inside the unit circle if and only if

$$-\frac{1}{L} < \gamma < 1. \tag{3.2}$$

The simple condition (3.2) characterizes completely the stability of the corresponding higher order difference equation and, more importantly, the condition connects the parameter γ to the order L of the system. For the homogeneous least-squares L-process, $a_j = 1/L$ for $j = 1, \cdots, L$, the corresponding characteristic polynomial $\Gamma_L(\lambda)$ has the form of (3.1) with $\gamma = \alpha/L = \frac{Cg_o}{BL}$. Applying Lemma 3.1 leads to the following result on the LAS of the homogeneous least-squares L-process.

Proposition 3.2. *The fixed (Walrasian) equilibrium x^* of the homogeneous least squares L-process is locally asymptotically stable (LAS) if and only if*

$$-1 < \alpha = \frac{Cg_o}{B} < L. \tag{3.3}$$

Denote by $D_L(\alpha) = \{\alpha : -1 < \alpha < L\}$ the stability region for the parameter α corresponding to the homogeneous least-squares L-process. Then, for the homogeneous least-squares L-process, the LAS of the fixed equilibrium x^* is completely characterized by $D_L(\alpha)$. Obviously, $D_L(\alpha) \subset D_{L'}(\alpha)$ for $L < L'$, that is, L-stability implies L'-stability for $L < L'$. In other words, the larger is the number of observations L (used in the least-squares learning process) the more stable is the fixed equilibrium. This is in particular the case when $\alpha > 0$. However, when $\alpha < 0$, the condition (3.3) is independent of L and hence increasing the lag length L does not necessarily improve the local stability of the fixed equilibrium.

Proposition 3.2 improves the related result obtained in Balasko and Royer (1996). Through a delicate analysis, they obtain an inclusion relation on the geometry of the stability domains D_L of the 1-dimensional *complex valued* linear least-squares L-process and show that L-stability implies L'-stability for $L < L'$. However, under their assumption, the geometry of the stability domains D_L is in fact characterized by 1-dimensional *real valued*, instead of complex valued, processes and therefore Proposition 3.2 can be applied and the stability regions $D_L(\alpha)$ can be characterized explicitly. Consequently, the inclusion relationship that L-stability implies L'-stability for $L < L'$ is easily established.

3.2. **Bifurcation Analysis.** Under the homogeneous least squares L-process, the characteristic polynomial has the form

$$\Gamma_L(\lambda) \equiv \lambda^L + \frac{\alpha}{L}[\lambda^{L-1} + \cdots + \lambda + 1] = 0. \tag{3.4}$$

It follows from Proposition 3.2 that the fixed point x^* becomes unstable for $\alpha \notin (-1, L)$. Therefore $\alpha = -1$ and L are the critical bifurcation values.

For $\alpha = -1$, $\Gamma_L(\lambda) = \lambda^L - [\lambda^{L-1} + \cdots + \lambda + 1]/L$. Obviously, $\lambda = 1$ is an eigenvalue. It can be shown that the remaining $L - 1$ eigenvalues satisfy $|\lambda| < 1$. Thus, for $\alpha = -1$, the fixed point x^* becomes unstable and $\lambda = 1$ corresponds to a fold bifurcation in general.

For $\alpha = L$, $\Gamma_L(\lambda) = \lambda^L + \lambda^{L-1} + \cdots + \lambda + 1$. Obviously, $\lambda = 1$ is not an eigenvalue. In fact, $\Gamma_L(\lambda) = [\lambda^{L+1} - 1]/[\lambda - 1]$, which implies that the L eigenvalues of $\Gamma_L(\lambda)$ satisfy $\lambda^{L+1} = 1$ with $\lambda \neq 1$, that is $\lambda_k = e^{2kw\pi i}$ with $w = p/q$, $p = 1, q = L+1$ and $k = 1, 2, \cdots, L$. Geometrically, the L eigenvalues correspond to the $L+1$ unit roots distributed evenly on the unit circle, excluding $\lambda = 1$. When $L = 1$, a flip or period-doubling bifurcation occurs. When $L = 2$, $p = 1$ and $q = 3$ the system is a map in $\mathbb{R}^2$. In such a case, according to Kuznetsov (1995) (pages 104 and 350), $\alpha = L = 2$ leads to a $1:3$ *strong resonance*.[8] The dynamics of a strong resonance can be exceedingly complicated (see Kuznetsov (1995)). In general, for $L \geq 2$, according to Sonis (2000), $\alpha = L$ lead the L-dimensional map to have $1 : L+1$ periodic resonances. Theoretical analysis for such types of bifurcation of higher dimensional discrete systems can be exceedingly complicated and are not yet completely understood. In Section 5, a numerical approach is employed for the cobweb model to demonstrate a range of complicated dynamics caused by such resonance bifurcations.

4. Dynamics of General Homogeneous $\mathbf{a}_L$-Processes

Now turn to the stability of the general homogeneous $\mathbf{a}_L$ process. As mentioned earlier, it seems in general impossible to derive explicit necessary and sufficient conditions. However, Rouche's theorem (which is stated in Appendix A.1) can be employed to obtain some sufficient conditions for LAS.

Proposition 4.1. *For the general homogeneous $\mathbf{a}_L$-process (2.1), (2.2) and (2.4),*

- *the fixed equilibrium x^* is LAS if*

$$|\alpha| < 1. \tag{4.1}$$

[8]For a map in $\mathbb{R}^2$, when all the eigenvalues are on the unit circle, there is no *"(strong) resonance"* if there is an eigenvalue, say $\bar{\lambda}$, satisfying $\bar{\lambda}^q \neq 1$ for $q = 1, 2, 3, 4$. Otherwise, we say the map has a $1 : q$ *(strong) resonance* ($q = 1, 2, 3, 4$). When the nonresonance condition is satisfied, for a $\mathbb{R}^2$ map depending on one parameter, as the eigenvalues of the fixed equilibrium move off the unit circle, there appears a closed invariant curve — all the iterates of any point on the curve remain on the curve — encircling the fixed point. Such a bifurcation corresponds to the Poincare-Andronov-Hopf bifurcation, see also Hale and Kocak (1991) for more discussion. When the nonresonance condition is not satisfied, a $1 : q$ (strong) resonance bifurcation in $\mathbb{R}^2$ can generate a two orbits of period q — one orbit is a sink and the other is a saddle. For the cobweb model, based on the analysis in Section 5, as α increases through 2, the fixed equilibrium is bifurcated to a $1 : 3$ resonance bifurcation, which means that two orbits of period 3 bifurcate from the fixed point: a sink and a saddle, as shown in Fig.5.2.

- *the fixed equilibrium x^* is unstable if there exists at least one $a_j : 1 \leq j \leq L$ such that*

$$2|\alpha|a_j > 1 + |\alpha|. \tag{4.2}$$

The proof of Proposition 4.1 is given in Appendix A.2. Following a different argument, a more general sufficient condition on the local stability is derived in Grandmont (1998). Our sufficient condition on the stability can also be obtained from the Proposition 2.2 in Grandmont (1998). However, our instability result is quite different to that of Proposition 2.1 in Grandmont (1998), which relates the eigenvalues of the actual system to the perfect foresight eigenvalues.

Comparing the conditions (3.3) and (4.1), one can see that the LAS of the least-squares 1-process implies the LAS of the general $\mathbf{a}_L$-processes for any L. However the stability regions defined by the least-squares L-process and the general $\mathbf{a}_L$-process can be independent of each other in general [9], as shown in the following analysis of the homogeneous $\mathbf{a}_2, \mathbf{a}_3$-processes. If $\alpha < 0$, the LAS of the least-squares L-process implies the LAS of the general $\mathbf{a}_L$-process, but if $\alpha > 0$ this is not true in general.

Denote by $D_L(\alpha, \mathbf{a})$ the region, in terms of the parameter α and weight vector $\mathbf{a}$, for the LAS of the fixed equilibrium x^* of the general $\mathbf{a}_L$-process. As the parameters move across the boundaries of the (local) stability region $D_L(\alpha, \mathbf{a})$, various bifurcations can be generated. When $\alpha = -1$, it follows from (2.7) that $\lambda = 1$ is an eigenvalue for any lag L. Therefore, when $\alpha = -1$ belongs to part of the stability boundaries (which is the case for $L = 2, 3$ based on the following discussion), a fold bifurcation occurs. The dynamics of the general $\mathbf{a}_L$-process can be very complicated, as indicated by the numerical results for the cobweb model in Section 5.

To understand the dynamics, in the following, consider the simple cases $L = 2$ and 3 so that the stability region and bifurcation can be characterized explicitly. In these cases, related discussions on the stability region have been considered in Hommes

[9] Note that the condition (4.2) can be written as $[2a_j - 1]|\alpha| > 1$, so, if either the parameter α or one of the weights a_j is large enough (such that this condition holds) then the equilibrium x^* is unstable. This indicates that certain selections of the weights a_j of the general $\mathbf{a}_L$-process can lead the equilibrium to be unstable, while at the same time it can be stable under the least-squares L-process.

(1998) (for $L = 2$) and Grandmont and Laroque (1990)) (for $L = 3$). However, the following analysis, in particular on the bifurcation, gives further insights into the dynamics and how they lead to periodic resonances, quasi-periodic orbits and chaotic behavior.

4.1. **Stability and Bifurcation of the $\mathbf{a}_2$-Process.**

For $L = 2$, the characteristic polynomial has the form

$$\Gamma_2(\lambda) = \lambda^2 + \alpha[a_1\lambda + a_2], \qquad 0 \le a_1, a_2 \le 1, \qquad a_1 + a_2 = 1. \tag{4.3}$$

4.1.1. *Local Stability.*

Proposition 4.2. *For $L = 2$ and $\mathbf{a}_2 = (a_1, a_2) = (1 - a_2, a_2)$,*

$$D_2(\alpha, \mathbf{a}) = \{(\alpha, a_2) : \alpha > -1, \quad a_2\alpha < 1, \qquad \alpha(1 - 2a_2) < 1\}.$$

It is easy to check that in the case of the homogeneous least-squares 2-process (so $a_2 = 1/2$), Proposition 4.2 leads to $D_2(\alpha) = \{\alpha : -1 < \alpha < 2\}$. Consider more closely the region $D_2(\alpha, \mathbf{a})$.

$$\begin{aligned} D_2(\alpha, \mathbf{a}) = \Big\{(\alpha, a_2) : -1 <& \alpha < \frac{1}{1 - 2a_2} \quad \text{for} \quad 0 \le a_2 \le \frac{1}{3} \qquad \text{and} \\ -1 <& \alpha < \frac{1}{a_2} \qquad\quad \text{for} \quad \frac{1}{3} \le a_2 \le 1\Big\}. \end{aligned}$$

The stability region $D_2(\alpha, \mathbf{a})$ is plotted on the (a_2, α)-plane in Fig.4.1.

Fig.4.1 indicates that, when the homogeneous expectation follows the $\mathbf{a}_2$-process (i.e. ${}_{t-1}x^e_{t+1} = g(a_1 x_{t-1} + a_2 x_{t-2})$ with $0 \le a_1, a_2, \le 1$ and $a_1 + a_2 = 1$), the stability region, in terms of α, of the fixed equilibrium depends on the weight vector $\mathbf{a} = (a_1, a_2)$. When $a_1 = 0$, the stability region is given by $\alpha \in (-1, 1)$. As a_1 increases, the stability region of α becomes larger ($\alpha \in (-1, 1/(1 - a_1))$). When $a_1 = 2/3$, the largest stability region $\alpha \in (-1, 3)$ is attained. However, as a_1 increases further, the stability region of α becomes smaller and ends up as $\alpha \in (-1, 1)$ when $a_1 = 1$. This

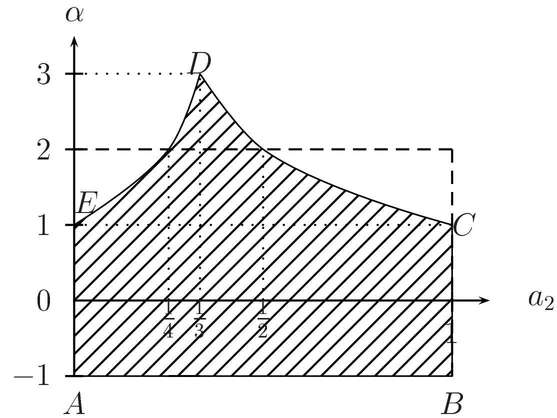

FIGURE 4.1. Local stability region of the fixed equilibrium of the homogeneous $\mathbf{a}_2$-process

indicates that, for $a_1 \in (0, 2/3)$ an increasing a_1 enlarges the stability region of α, but this is no longer the case for $a_1 \in (2/3, 1)$. Compared with the homogeneous least-squares 2-process, the general $\mathbf{a}_2$-process generates smaller (larger) stability region of α for $a_2 \in (0, 1/4) \cup (1/2, 1)$ ($a_2 \in [1/4, 1/2]$).

4.1.2. *Bifurcation Analysis.* The stability region of the homogeneous $\mathbf{a}_2$-process in Fig.4.1 shows that the fixed equilibrium is LAS in the region $ABCDE$. However, when the parameters (a_2, α) moves out of the region (along AB or CDE), the fixed equilibrium becomes unstable and various types of bifurcation occur. Along AB, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -a_2$ and this implies that, when the parameters (a_2, α) move across AB, a flop bifurcation is generated in general. The curve AB is called a *divergence (or flop) curve*. Along DE, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -a_1/(1 - 2a_2)$. So, when the parameters (a_2, α) move across DE, a flip (or period doubling) bifurcation occurs. This curve is called a *flip curve*.

Along CD, the eigenvalues $\lambda_{1,2} \in \mathbb{C}$ satisfy $|\lambda_j| = 1$. That is, $\lambda_j = \cos(2w\pi) \pm i\sin(2w\pi)$ for $j = 1, 2$ and $0 \le w \le 1$. If w is a rational fraction: $w = p/q$ then so-called $p : q$-periodic resonances[10] occur. Indeed, along CD the system has $p : q$-periodic resonances for (p, q) =(1,2), (1,3), (1,4), (2,5), (2, 7), (3,7), (4,9), (3,10), (3, 11), (4,11)[11]. In particular, $a_2 = 1/3$ corresponds to a period doubling bifurcation

[10]See Sonis (2000) for the details.

[11]The existence of $p : q$-periodic resonance is determined by the value of $\rho \equiv \lambda_1 + \lambda_2 = 2\cos(2w\pi)$ and hence the value of a_2. In fact, $\rho \equiv \lambda_1 + \lambda_2 = 2\cos(2w\pi) = -\alpha a_1$. Along CD, $\alpha = 1/a_2$ and

(with $(p, q) = (1, 2)$), $a_2 = 1/2$ leads to a strong 1:3-periodic resonance and $a_2 = 1$ leads to a strong 1:4-periodic resonance. Other rational fraction (of $w = p/q$) represents fixed points of weak resonances. For example, $a_2 = 0.38197$ corresponds to a 2:5-periodic weak resonance and $a_2 = 0.69202, 0.35689$ correspond to a 2:7, 3:7-periodic weak resonance, respectively. If w is irrational, one obtains quasi-periodic orbits. The curve CD is thus called a *flutter-saddle curve*. Therefore, along the flutter-saddle curve, the system generates various types of resonances and quasi-periodic orbits, while only $1 : 3$ strong resonance occur for the least-squares 2-process.

4.2. **Stability and Bifurcation of the $\mathbf{a}_3$-Process.** For $L = 3$, the characteristic equation has the form

$$\Gamma(\lambda) \equiv \lambda^3 + \alpha[a_1\lambda^2 + a_2\lambda + a_3] = 0. \tag{4.4}$$

Following a recent bifurcation analysis for three-dimensional discrete dynamical systems by Sonis (2000), the local stability region can be characterized completely (see Appendix A.3 for the proof).

Proposition 4.3. *For* $L = 3$ *and* $\mathbf{a}_3 = (a_1, a_2, a_3) = (1 - a_2 - a_3, a_2, a_3)$,

$$\begin{aligned} D_3(\alpha, \mathbf{a}) = \{(\alpha, a_2, a_3) : \quad & 0 \le a_2, a_3 \le 1, \quad a_2 + a_3 \le 1, \\ & \alpha > -1, \quad (1 - 2a_2)\alpha < 1, \quad 1 - a_2\alpha + a_3(1 - a_2 - 2a_3)\alpha^2 > 0\}. \end{aligned}$$

Furthermore, $\mathbf{a} = (3/7, 3/7, 1/7)$ *leads to the largest stability interval for* $\alpha \in (-1, 7)$.

The bounded stability region $D_3(\alpha, \mathbf{a})$ in the (a_2, a_3, α)-space is shown in Fig.4.2. In particular, for $a_3 = 0$, it reduces to the stability region $D_2(\alpha, \mathbf{a})$ in Fig.4.1. As a_3 increases, the stability region in terms of (α, a_2) becomes more peaked. When $a_3 = 1/7$, the surface reaches its peak. In particular, with $\mathbf{a} = (3/7, 3/7, 1/7)$, α has the largest stability region $\alpha \in (-1, 7)$, which enlarges the stability region ($\alpha \in (-1, 3)$) of the least-squares 3-process significantly. However, as a_3 increases further,

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hence $\rho = -(1 - a_2)/a_2$ for $a_2 \in [1/3, 1]$. Therefore $a_2 = 1/(1 - \rho) \in [1/3, 1]$ iff $\rho \in [-2, 0]$. Then, checking with the Table 1 in Sonis (2000), one can find various types of periodicity.

the stability region in terms of (α, a_2) shrinks and ends up with $\alpha \in (-1, 1)$ when $a_3 = 1, a_2 = 0$.

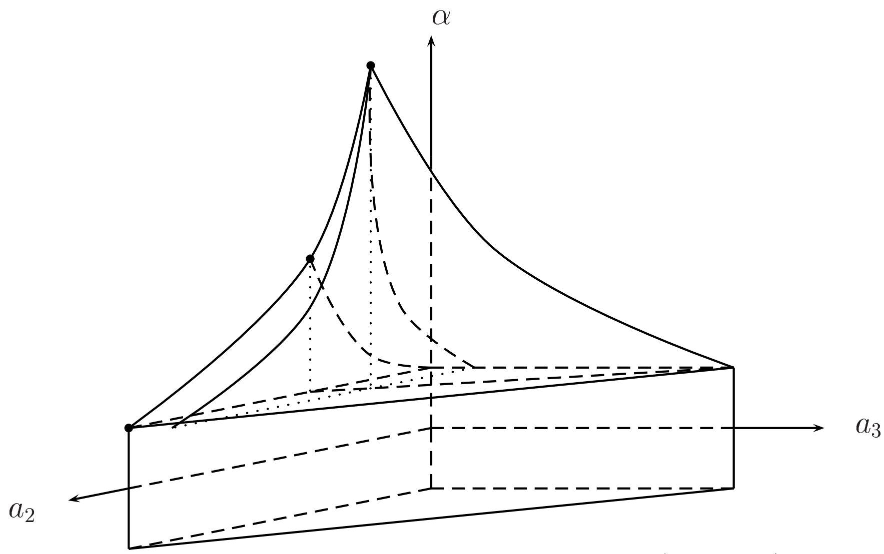


FIGURE 4.2. Local stability regions, in terms of (a_2, a_3, α), of the homogeneous $\mathbf{a}_3$-process

The boundaries of $D_3(\alpha, \mathbf{a})$ constitute the bifurcation surfaces of the fixed equilibrium x^* given by $\Pi_j = 0$ for $j = 1, 2, 3$, where

$$\Pi_1 \equiv 1 + \alpha, \qquad \Pi_2 \equiv 1 - (1 - 2a_2)\alpha, \quad \Pi_3 \equiv 1 - a_2\alpha + a_3(1 - a_2 - 2a_3)\alpha^2$$

and $0 \leq a_2, a_3 \leq 1, a_2 + a_3 \leq 1$. In fact, on the plane $\Pi_1 = 0$, at least one of the eigenvalues is equal to 1, i.e., the dynamics become divergent – this is a *divergence plane*. On the surface $\Pi_2 = 0$ at least one of the eigenvalues is equal to -1, i.e., the dynamics become oscillatory – this is a *flip surface*. Each point on the flip surface corresponds to a two-period cycle, and the movement of the fixed point through it generates the Feigenbaum type period doubling sequence in three dimensions, leading to chaos (Feigenbaum (1978)). On the surface $\Pi_3 = 0$, the eigenvalues satisfy

$$\lambda_{1,2} = \cos(2w\pi) \pm i\sin(2w\pi), \qquad \lambda_3 = r_o, \qquad r_o \in [-1, 1] \qquad w \in [0, 1]. \quad (4.5)$$

Let $\rho = 2\cos(2w\pi)$, then $\rho \in [-2, 2]$. Similar to the discussion for the $\mathbf{a}_2$-process, if w is a rational fraction: $w = p/q$, $p : q$-periodic resonances occur, which corresponds to a continuous curve on the surface $\Pi_3 = 0$. For a given (p, q), ρ is determined by

$\rho = 2\cos(2\pi p/q)$, and the corresponding weights that generate a $p : q$ resonance bifurcation can be identified in the following way.

From the characteristic polynomial (4.4) and (4.5),

$$\begin{cases} a_1\alpha & = -(\lambda_1+\lambda_2+\lambda_3) = -2cos(2w\pi) - r_o = -\rho - r_o \\ a_2\alpha & = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 = 1 + 2r_o cos(2w\pi) = 1 + r_o\rho \\ a_3\alpha & = -\lambda_1\lambda_2\lambda_3 = -r_o, \end{cases}$$

i.e.

$$(1-a_2-a_3)\alpha = -\rho - r_o, \qquad a_2\alpha = 1 + r_o\rho, \qquad a_3\alpha = -r_o.$$

Eliminating r_o from the equations, one has

$$a_2 = \frac{1+(\rho^2-2\rho-2)a_3}{1-\rho} \qquad \text{for} \qquad \rho \neq 1. \tag{4.6}$$

Hence, for given ρ, equation (4.6) gives the corresponding condition on the weight vector $\mathbf{a} = (1-a_2-a_3, a_2, a_3)$. In particular, for $(p,q) = (1,3)$, $\rho = -1$ and hence the weights satisfying $a_2 = [1+a_3]/2$ with $a_3 \in [0, 1/3]$ correspond to the strong 1:3-periodic resonance. For $(p,q) = (1,4)$, $\rho = 0$ and hence the weights satisfying $a_2 = 1 - 2a_3$ with $a_3 \in [0, 1/2]$ correspond to the strong 1:4-periodic resonance. For the period doubling bifurcation (i.e., $(p,q) = (1,2)$, $\rho = -2$), the weights satisfy $a_3 = (1+6a_3)/3$ with $a_3 \in [0, 2/9]$. Fig.4.3 indicates those weights for $(p,q) = (1,2), (1,3), (1,4)$. Other rational fractions represent weak resonances, for example, $w = 1/q$ with $q = 5, 6, \cdots$. Similarly, one can derive the weight equations for any given p and q (and hence ρ). If w is irrational, quasi-periodic orbits are generated. Hence the surface $\Pi_3 = 0$ is called a *flutter-saddle surface.*

4.3. **Remark.** The previous analysis leaves open many interesting and important questions. Among which, two questions are of particular interest:

(a). Is there always an $\mathbf{a}_L$ process which can generate a larger stability region than the least-squares L-process does?

(b). Is it possible to characterize the stability and bifurcation as $L \to \infty$?

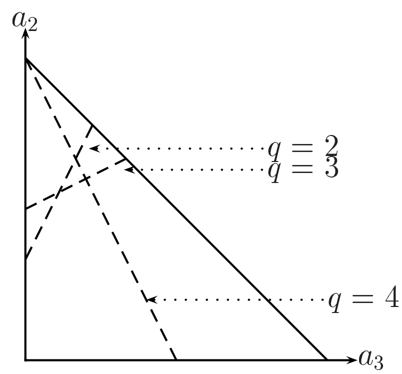


FIGURE 4.3. The weights which give the periodic doubling bifurcation ($(p, q) = (1, 2)$) and strong 1:3 or 1:4 periodic resonances ($p = 1, q = 3, 4$) for the homogeneous $\mathbf{a}_3$-process

To have complete answers to these two questions is not a easy task. We attempt to gain some insights by selecting two special weighting processes — arithmetic weights and geometric weights.

Consider first the **arithmetic weights** (e.g. Hommes (1998))

$$a_j = \frac{2(L+1-j)}{L(L+1)}, \qquad j = 1, 2, \cdots, L \tag{4.7}$$

so that the weights form a declining arithmetic sequence summing to 1. With such selections, one can verify that[12]

$$\begin{aligned} D_2(\alpha, \mathbf{a}) &= D_3(\alpha, \mathbf{a}) = \{\alpha : -1 < \alpha < 3\}; \\ D_4(\alpha, \mathbf{a}) &= D_5(\alpha, \mathbf{a}) = \{\alpha : -1 < \alpha < 5\}; \\ D_6(\alpha, \mathbf{a}) &= D_7(\alpha, \mathbf{a}) = \{\alpha : -1 < \alpha < 7\}; \\ D_8(\alpha, \mathbf{a}) &= D_9(\alpha, \mathbf{a}) = \{\alpha : -1 < \alpha < 9\}. \end{aligned} \tag{4.8}$$

That is, the arithmetic weights produce larger stability regions for the general $\mathbf{a}_L$-process than for the least-squares L-process when $L = 2, 4, 6, 8$, while for $L = 3, 5, 7$, their stability regions are the same. This observation would suggest the possibility of a general result on the stability region for the general $\mathbf{a}_L$ process with arithmetic

[12]The results here can be verified by using Jury's test (e.g. Elaydi (1996) (pages 180–181)) and Maple^{TM}.

weights:

$$D_{2L}(\alpha, \mathbf{a}) = D_{2L+1}(\alpha, \mathbf{a}) = \{\alpha : -1 < \alpha < 2L + 1\} \tag{4.9}$$

for $L = 1, 2, \cdots$. However, whether this conjecture holds or not is still an open question.

One can see from the stability regions of $\mathbf{a}_2$ and $\mathbf{a}_3$-processes that, unlike the least-squares process, the inclusion relation $D_L(\alpha, \mathbf{a}) \subset D_{L'}(\alpha, \mathbf{a})$ for $L < L'$ is not true, in general.

Consider now the **geometric weights**

$$a_1 = a, \quad a_j = a\omega^{j-1} \quad \text{for} \quad j = 2, \cdots, L,$$

where $\omega \in (0, 1)$. That is, the weight associated with the past observations decay geometrically. From $\sum_{j=1}^{L} a_j = 1$, we have $a = (1-\omega)/(1-\omega^L)$. Applying Propositions 4.2 and 4.3 for $L = 2, 3$, respectively, we obtain the local stability regions of the fixed equilibrium

$$\begin{aligned} D_2 \equiv & \ \{(\alpha, \omega) : -1 < \alpha < \min\{1 + 1/\omega, (1 + \omega)/(1 - \omega)\}\} \\ D_3 \equiv & \ \{(\alpha, \omega) : -1 < \alpha, \quad \alpha < \tfrac{1+\omega+\omega^2}{1-\omega+\omega^2}, \\ & \qquad 1 - \tfrac{\omega}{1+\omega+\omega^2}\alpha + \tfrac{\omega^2(1-\omega^2)}{(1+\omega+\omega^2)^2}\alpha^2 > 0\}. \end{aligned}$$

for $L = 2$ and 3, respectively. In both cases, the local stability region of the fixed equilibrium is bounded, see Fig.4.4(a), (b).

To consider the limit case of $L \to \infty$, we introduce a new variable y_t for the geometric moving average:

$$y_{t-1} = a[x_{t-1} + \omega x_{t-2} + \omega^2 x_{t-3} \cdots + \omega^{L-1} x_{t-L}].$$

Then

$$y_t = \omega y_{t-1} + a[x_t - \omega^L x_{t-L}]. \tag{4.10}$$

Since $\omega \in (0, 1)$, as $L \to \infty$, the limiting equation of (4.10) is given by

$$y_t = \omega y_{t-1} + (1 - \omega) x_t. \tag{4.11}$$

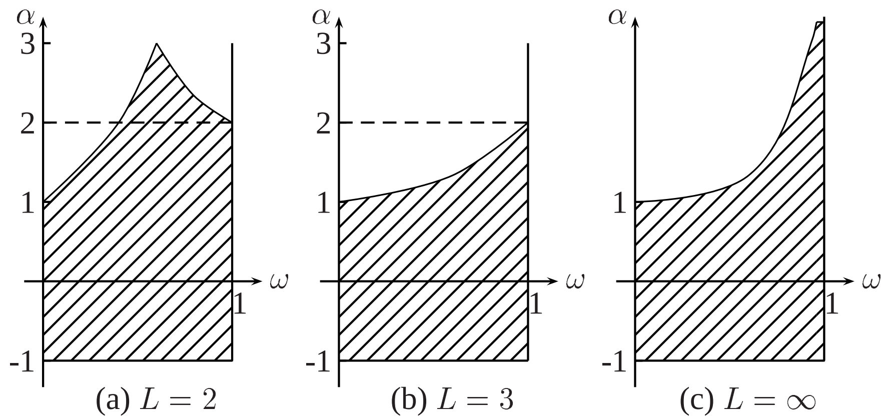


FIGURE 4.4. Local stability of fixed equilibrium with geometric weights and (a) $L = 2$, (b) $L = 3$ and (c) the limiting case $L = \infty$.

This, together with (2.6), leads to a two-dimensional system

$$\begin{cases} x_t = & -\alpha y_{t-1} \\ y_t = & [\omega - \alpha(1-\omega)]y_{t-1}. \end{cases} \tag{4.12}$$

Thus the zero solution of the system (4.12) is LAS if and only if

$$-1 < \alpha < \frac{1+\omega}{1-\omega}.$$

In this case, the local stability region of the fixed equilibrium is unbounded, see Fig.4.4(c). One can see that, in terms of the parameter α, the stability region becomes larger as the weight, ω, associated with the most recent observation increases. Furthermore, $\alpha = -1$ corresponds to fold bifurcations and $\alpha = \frac{1+\omega}{1-\omega}$ corresponds to period doubling bifurcations.

5. COBWEB DYNAMICS WITH HOMOGENEOUS LEARNING PROCESS

As application of the results obtained in the previous sections and illustration of the complex global dynamics generated by various types of bifurcations, the cobweb model introduced in Section 2 is considered in this section.

5.1. **The Homogeneous Least-Squares Process.** Applying Proposition 3.2 to the cobweb model leads to the following stability result.

Corollary 5.1. *The fixed equilibrium p^* of (2.14) with the homogeneous least squares L-process is LAS if and only if*

$$-1 < \alpha = \frac{4\beta g_o}{b}[e^{\beta g_o p^*} + e^{-\beta g_o p^*}]^{-2} < L. \tag{5.1}$$

To undertake a numerical bifurcation analysis, let $a = 0.65, \beta = 1, b = 1$. The focus of the analysis will be on the sensitivity of the dynamics to the parameter g_o. Table 1 shows the critical value g^*_α of g_α[13] for $\alpha = -1$ and $\alpha = 1, 2, 3, 4$:

α	-1	1	2	3	4
g^*_α	-5.87499...	1.12915...	2.52289...	4.06055...	5.67050....

TABLE 1. Critical values of g_α for $\alpha = -1, 1, 2, 3, 4$

The number of the fixed equilibria depends on g_o. It can be shown that for $g_0 > g^*_{-1} = -5.87499...$ (corresponding to $\alpha = -1$), there is a unique positive equilibrium p^*, which is LAS for $-1 < \alpha < L$. When $g_o < g^*_{-1}$, there exist three equilibria p^*_j $(j = 1, 2, 3)$ satisfying $p^*_1 < p^*_2 < 0 < p^*_3$ with both p^*_1 and p^*_3 locally stable and p^*_2 unstable. When $g_o = g^*_{-1}$, there are two equilibria p^*_1 and p^*_2 satisfying $p^*_1 < 0 < p^*_2$ (with p^*_2 locally stable). For $L = 1, 2$ and 3, numerical simulations on the nonlinear model suggest the stability and bifurcation behavior near g^*_{-1} indicated as in Fig. 5.1. Therefore, as g_o decreases further from g^*_{-1}, a saddle-node bifurcation [14] appears — the stability of the positive fixed equilibrium continues to hold, however two new (negative) fixed equilibria appear with one stable and the other unstable. Therefore, $\alpha = -1$ (so that one of the eigenvalues is 1) does not generate a flop bifurcation in this case.

[13]The g^*_α is calculated in the following way. Let p^* be the fixed equilibrium and let $y^* = \beta g_o p^*, A = b/\beta g_o$. Then y^* and A satisfy $Ay^* + \tanh y^* = a$ and $\alpha A[e^{y^*} + e^{-y^*}]^2 = 4$. For given α, solving for y^* from the equation $\alpha[a - \tanh y^*][e^{y^*} + e^{-y^*}]^2 = 4y^*$ leads to $g^*_\alpha = b/(\beta A) = by^*/[\beta(a - \tanh y^*)]$.

[14]The saddle-node bifurcation results from a symmetry breaking of the map. In fact, for $a = 0$, the map T satisfies $T(-x, -y) = -T(x, y)$ and the type of bifurcation of fixed equilibrium for such symmetry maps is a pitchfork. However, this symmetry no longer holds when $a \neq 0$, leading to a saddle-node bifurcation.

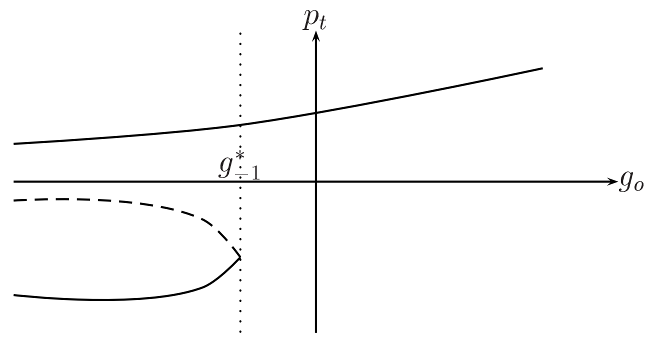


FIGURE 5.1. Saddle-node bifurcation diagram: solid lines indicate local stable fixed equilibria and the dotted line indicates an unstable fixed equilibrium.

Consider $L = 1$. Numerical simulations show that, for g_o near g_1^* corresponding to $\alpha = 1$, all the solutions p_t (locally) converge to the fixed point p^* for $g_o < g_1^*$. However, as g_o increases above g_1^*, a locally stable period doubling is generated.

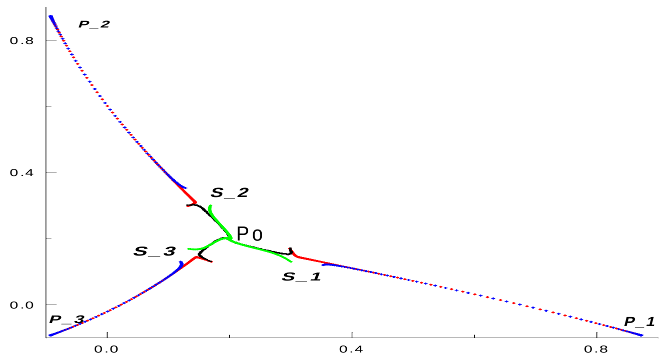


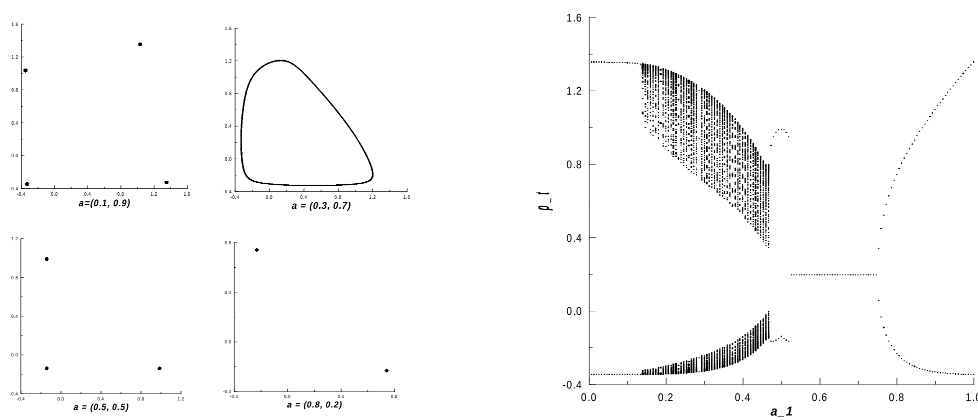
FIGURE 5.2. Phase plot of (p_t, p_{t-1}) for the least-Squares 2-process: an LAS fixed equilibrium P_o for $g_o < g_2^*$ and 2 sets of three-cycles with $\{S_1, S_2, S_3\}$ unstable and $\{P_1, P_2, P_3\}$ (locally) stable.

Consider $L = 2$. For g_o near $g_o = g_2^*$ corresponding to $\alpha = 2$, all the solutions converge to the fixed point for $g < g_2^*$. For $g_o = g_2^*$, a strong 1:3-periodic resonance appears. As g_o increases above g_2^*, there is bifurcation to two sets of period three cycles, one set of the three period cycle $\{P_1, P_2, P_3\}$ is (locally) stable and the other set $\{S_1, S_2, S_3\}$ is unstable (see Fig. 5.2). The dynamics are very similar to those found in the analysis of the cobweb learning model investigated in Chiarella and He (2000*c*) where a normal form analysis is used to investigate the stability of the strong 3-periodic resonance.

For $L = 3, 4$, $g_o = g_3^*, g_4^*$ lead to strong 1:4 and 1:5 periodic resonances, respectively. Similar dynamics near the critical values g_3^*, g_4^* are found except that the bifurcated cycles are period four and five cycles, respectively.

5.2. **The Homogeneous $\mathbf{a}_2$-Process.** The cobweb model under the homogeneous $\mathbf{a}_2$-process is studied in Hommes (1998) where various strange attractors have been obtained through different combinations of the parameters β and $\mathbf{a} = (a_1, a_2)$. The following bifurcation and numerical analysis give further insights into the dynamics of the cobweb model as the extrapolation rate g_o and learning $\mathbf{a} = (a_1, a_2)$-process change. In particular, the routes to various resonances and quasi-periodic orbits, even chaotic behavior are investigated.

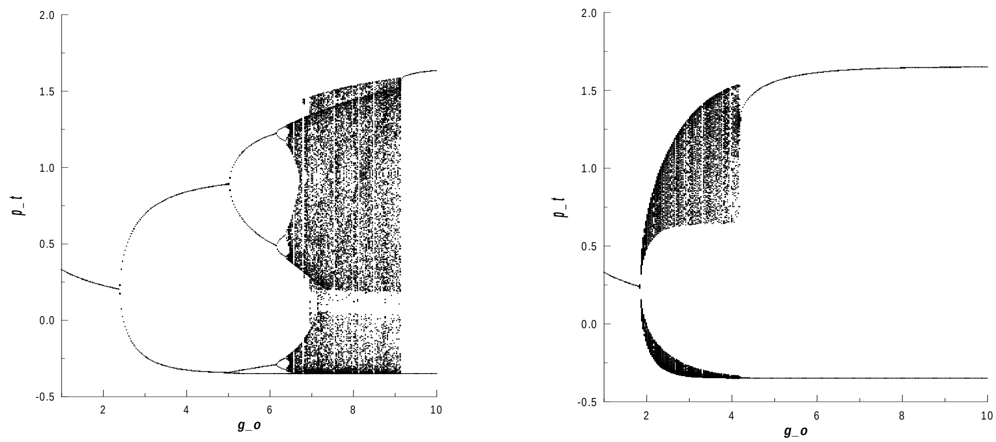
As before, let $a = 0.65, b = 1$ and $\beta = 1$. With α defined by (5.1), the stability region is given by Fig.4.1. When the parameters (a_2, α) move cross AB, the flop (or divergence) curve, from the stability region, two new equilibria appear and the system has similar dynamics as the case with the least-squares process (see Fig. 5.1).



(a) (b)

FIGURE 5.3. (a) Phase plot of (p_t, p_{t-1}) for $\mathbf{a} = (0.1, 0.9), (0.3, 0.7), (0.5, 0.5), (1.8, 0.2)$ respectively; (b) Bifurcation diagram for a_1 and $\alpha = 2$.

Now g_o is selected so that $\alpha = 2$. Then, across the line $\alpha = 2$ (hence different a_2, $\mathbf{a} = (a_1, a_2)$-processes), lead to various-types of periodic resonances and quasi-periodic orbits. Fig. 5.3(a) shows the phase plot for $\mathbf{a} = (0.1, 0.9), (0.3, 0.7), (0.5, 0.5)$



(a) (b)

FIGURE 5.4. Bifurcation diagram for (a): $\mathbf{a} = (0.76, 0.24)$ and (b): $\mathbf{a} = (0.35, 0.65)$.

and $(0.8, 0.2)$, respectively. They correspond to strong 1:4 periodic resonance, quasi-periodic (almost closed) orbit, strong 1:3 periodic resonance and period doubling, respectively. With the fixed α (hence g_o), the bifurcation diagram for a_1 is plotted in Fig. 5.3(b). Fig. 5.3 shows that there are two different types of bifurcation when a_2 moves out of the stable region (which is (1/4, 1/3) for $\alpha = 2$). When a_2 crosses DE – the flip curve, a period doubling bifurcation is generated. However, when a_2 crosses CD – the flutter saddle curve, various resonances and quasi-periodic orbits occur. For example $a_2 = 0.5$ corresponds to a strong 1:3 periodic resonance (the solutions converge (locally) to a 3-cycle). When a_2 moves further, say $a_2 = 0.7$, the solutions converge to either higher periodic or quasi-periodic cycles, indicated by the (almost) closed curve on the phase plot of (p_t, p_{t-1}) for $(a_1, a_2) = (0.3, 0.7)$. When a_2 is close to 1 (that is, almost all weight is on p_{t-2}), the solutions converge to 4-cycles.

Based on the analysis of the $\mathbf{a}_2$-process in the last section, along the flutter-saddle curve CD, there exist $p : q$ resonances with $(p, q) = (1, 2), (1, 3), (1, 3), (2, 5), (2, 7), (3, 7), (10, 3)$ and so on. For example, for $(p, q) = (2, 7), (3, 7)$ the corresponding $(a_2, \alpha) = (0.69202..., 1.44504...), (0.35689..., 2.80193...)$ and hence g_o =1.724513..., 3.748733..., respectively[15]. Numerical simulations show (not reported here) that, in

[15] For a given (p, q), on the curve CD, $a_2 = 1/(1 - \rho)$ and $\alpha = 1/a_2$ with $\rho = 2\cos(2\pi p/q)$. From which, the corresponding g_o can be calculated (as before)

both cases, periodic 7 orbits appear for different initial values, they either tend to the fixed equilibrium or form piece-wise or (almost) closed orbits on the phase plane. Also, for fixed a_2, further increasing g_o leads the price (x_t, x_{t-1}) to different attractors for $p = 2$ and 3. For $p = 2$, the attractors appear chaotic, while for $p = 3$, the attractors are either piece-wise, or (almost) closed orbits, or regular circles.

The dynamics caused by the extrapolation rate g_o can be seen in the bifurcation diagrams plotted in Fig. 5.4 (a) for $\mathbf{a} = (0.76, 0.24)$ and (b) for $\mathbf{a} = (0.35, 0.65)$, respectively. They show different bifurcation routes from stability to complicated (and then to a simple) dynamics. In fact, the complicated price dynamics of the cobweb model caused by g_o has been investigated numerically by Hommes (1998). For many parameter values the cobweb model with $\mathbf{a}_2$-process has a strange attractor. Numerically, he shows that there are two different bifurcation routes to chaos when g_o increases — one is the well known period doubling route and the other is the breaking of an invariant circle. Our analysis gives some theoretical insight into these different routes to chaos. In particular, the second route to chaos is through various resonances and quasi-periodic orbits.

5.3. **The Homogeneous $\mathbf{a}_3$-Process.** As analyzed in the previous section, various bifurcations occur when the parameters (a_2, a_3, α) cross either the divergence, or flip, or flutter saddle surface. In particular, bifurcation from the flip surface can lead to chaos by the period doubling route. While the bifurcation from the flutter saddle surface leads to various $p : q$ resonances and quasi-periodic orbits, which in turn lead to complicated dynamics and chaos. To illustrate the dynamics of the homogeneous $\mathbf{a}_3$-process, a bifurcation diagram in terms of a_2 is shown in Fig.5.5, where a, b and β are selected as before and $g_o = 9.5$, $a_3 = 0.21$ and $a_2 \in [0, 0.79]$. In general, the $\mathbf{a}_3$-process can have all types of bifurcations generated from the $\mathbf{a}_2$-process. Furthermore, the switching between simple and complicated price dynamics can be more interesting, as shown in Fig.5.5.

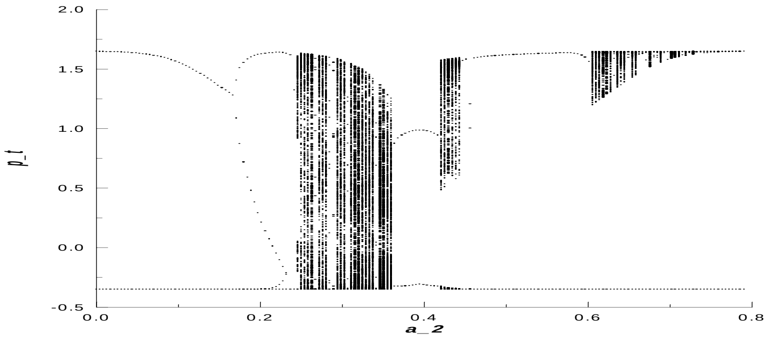


FIGURE 5.5. Bifurcation diagram for the homogeneous $\mathbf{a}_3$-process with $g_o = 9.5, a_3 = 0.21$ and $a_2 \in [0, 0.79]$

6. CONCLUSION

This paper investigates the dynamics of agents' learning via least-squares L-processes and general $\mathbf{a}_L$-processes. In the case of homogeneous beliefs, it provides an explicit study of how the local stability of the equilibrium is affected by the least-squares L-process and $\mathbf{a}_L$-process and shows various types of bifurcation routes, in the case of the cobweb model, to complicated dynamics. Our results might be summarized as follows:

- When the agents follow the homogeneous least-squares L-process, a complete picture can be drawn in terms of (local) stability and bifurcation: the stability region of the fixed equilibrium is completely characterized by the lag length L of the least-squares learning process and the parameters of the model, in particular the parameter α, which is related to the extrapolation rate of the traders in the cobweb model. When the fixed equilibrium becomes unstable, a $1 : L + 1$ resonance is generated.
- When the agents follow the general homogeneous $\mathbf{a}_L$-process, the stability of the fixed equilibrium and bifurcations depends essentially upon the weight vector $\mathbf{a}_L$. In the simplest cases of $L = 2$ and 3, it is found that, depending on the weight vector $\mathbf{a}$, the stability region can be different and the instability of the fixed equilibrium can generate various resonances and quasi-periodic orbits, which in turn lead to complicated dynamics and chaos by different routes.

The expectations functions considered in this paper are some of the simplest learning processes in which all the weights on the past states are constants. However, they yield very rich dynamics in terms of the stability, bifurcation and routes to complicated dynamics. In reality economies are populated by heterogeneous agents who form their expectations and learning processes differently. Furthermore these agents revise their expectations by adapting the weights in accordance to their observations of how the other agents are performing. How heterogeneous expectations and learning affects the dynamics in this situation is studied to some extent in Chiarella and He (2000*a*), (2000*b*). However many interesting questions remain for future research.

Appendix A

A.1. **Rouche's Theorem.** If the complex functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve γ, and if $|g(z)| < |f(z)|$ on γ, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside γ.

A.2. **Proof of Proposition 4.1.** To prove the first part, let $f_1(z) = z^L$ and $g_1(z) = \alpha[a_1 z^{L-1} + a_2 z^{L-2} + \cdots + a_{L-1} z + a_L]$. Then, following $a_j \geq 0$ and $\sum_{j=1}^L a_j = 1$, on $|z| = 1$, $|f_1(z)| = 1$ and $|g_1(z)| \leq |\alpha| \sum_{j=1}^L a_j = |\alpha|$. Thus, under the condition $|\alpha| < 1$, $|g_1(z)| < |f_1(z)|$ on $|z| = 1$. Note that $f(z)$ has h zeros inside $|z| = 1$. It follows from Rouche's Theorem that both $f_1(z)$ and $\Gamma(z) = f_1(z) + g_1(z)$ have the same number of zeros inside $|z| = 1$, which implies that all the eigenvalues of $\Gamma(z)$ lie inside $|z| = 1$. Therefore, x^* is LAS.

To prove the second part, let $f_2(z) = a_j z^{L-i}$ and $g_2(z) = \Gamma(z) - f_2(z)$. Note that $a_j \geq 0$, $\sum_{j=1}^L a_j = 1$ and $f_2(z)$ has $L-i$ zeros inside $|z| = 1$. Also, under the condition (4.2), $|g_2(z)| \leq 1 + |\alpha| \sum_{j=1, j\neq i}^L a_j < a_j|\alpha| = |f_2(z)|$ on $|z| = 1$. Then, following Rouche's Theorem, $\Gamma(z) = f_2(z) + g_2(z)$ has only $L-i$ zeros inside $|z| = 1$ (hence there exists at least one zero satisfying $|z| \geq 1$). Therefore, the equilibrium x^* is not LAS. *Q.E.D*

A.3. **Proof of Proposition 4.3.** It follows from Sonis (2000) that all the eigenvalues λ of the characteristic polynomial $\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$ satisfy $|\lambda| < 1$ iff

$b_o \equiv 1 + c_1 + c_2 + c_3 > 0, b_3 \equiv 1 - c_1 + c_2 - c_3 > 0$ and $\Delta_2 \equiv 1 - c_2 + c_1 c_3 - c_3^2 > 0$. Applying this result to the $\mathbf{a}_3$-process gives us the stability region $D_3(\alpha, \mathbf{a})$ defined in Proposition 4.3.

The largest stability interval for α is given by $\alpha \in (-1, \alpha^*)$, where $\alpha^* = \min\{\alpha_1, \alpha_2\}$ and α_j solve the equations $\Pi_2 = \Pi_3 = 0$. Eliminate a_2 from the equations $\Pi_2 = \Pi_3 = 0$,

$$\alpha_j = \frac{(1 - a_3) \pm |1 - 7a_3|}{2a_3(1 - 4a_3)}.$$

Obviously, when $a_3 = 1/7$, α^* has its largest value of 7. Correspondingly, $a_1 = a_2 = 3/7$. Therefore, $\mathbf{a} = (3/7, 3/7, 1/7)$ leads to the largest stability interval for $\alpha \in (-1, 7)$. *Q.E.D*

References

Balasko, Y. and Royer, D. (1996), 'Stability of competitive equilibrium with respect to recursive and learning processes', *Journal of Economic Theory* **68**, 319–348.

Barucci, E., G.I.Bischi and L.Gardini (1999), 'Endogenous fluctuations in a bounded rationality economy: Learning non perfect foresight equilibria', *Journal of Economic Theory* **87**, 243–253.

Bischi, G. and Gardini, L. (2000), Equilibrium selection and transient dynamics under adaptive and statistical learning, Working Paper 9, Universita' Degli Studi di Parma.

Bray, M. (1983), *Convergence to rational expectations equilibria*, Cambridge Univ. Press, Cambridge, UK. in *Individual Forecasting and Aggregate Outcomes*, Eds. R. Frydman and E.S. Phelps.

Brock, W. and Hommes, C. (1997), 'A rational route to randomness', *Econometrica* **65**, 1059–1095.

Chiarella, C. (1988), 'The cobweb model, its instability and the onset of chaos', *Economic Modelling* **5**, 377–384.

Chiarella, C. and He, X. (2000*a*), 'Asset pricing and wealth dynamics under heterogeneous expectations', *School of Finance and Economics, University of Techonology Sydney* . Working Paper.

Chiarella, C. and He, X. (2000*b*), 'Dynamics of beliefs and learning under $\mathbf{a}_l$-process – the heterogeneous case', *School of Finance and Economics, University of Techonology Sydney* . Working Paper.

Chiarella, C. and He, X. (2000*c*), *The Dynamics of the Cobweb when Producers are Risk Averse Learners*, Physica-Verlag, pp. 86–100. in *Optimization, Dynamics, and Economic Analysis*, E.J. Dockner, R.F. Hartl, M. Luptacik and G. Sorger (Eds).

Elaydi, S. (1996), *An Introduction to Difference Equations*, Springer, New York.

Evans, G. and Honkapohja, S. (1999), *Learning Dynamics*, chapter 7, pp. 449–542. in *Handbook of Macroeconomics*, Eds. J.B. Taylor and M. Woodford.

Evans, G. and Ramey, G. (1992), ‘Expectation calculation and macroeconomics dynamics’, *American Economic Review* **82**, 207–224.

Feigenbaum, M. (1978), ‘Qualitative universality for a class of nonlinear transformations’, *Journal of Statistical Physics* **19**, 25–52.

Grandmont, J. and Laroque, G. (1990), *Stability, Expectations and Predetermined Variables*, Vol. 1, MIT Press, Cambridge, MA, chapter 3, pp. 71–92. in *Essays in Honor of E. Malinvaud*, Eds. P. Champsaur, et,al.

Grandmont, J.-M. (1985), ‘On endogenous competitive business cycles’, *Econometrica* **53**, 995–1045.

Grandmont, J.-M. (1998), ‘Expectations formation and stability of large socioeconomic systems’, *Econometrica* **66**(4), 741–781.

Grandmont, J.-M. and Laroque, G. (1986), ‘Stability of cycles and expectations’, *Journal of Economic Theory* **40**, 138–151.

Hale, J. and Kocak, H. (1991), *Dynamics and bifurcations*, Vol. 3 of *Texts in Applied Mathematics*, Springer-Verlag, New York.

Hommes, C. (1991), ‘Adaptive learning and roads to chaos: The case of the cobweb’, *Economics Letters* **1991**, 127–132.

Hommes, C. (1994), ‘Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand’, *Journal of Economic Behavior and Organization* **24**, 315–335.

Hommes, C. (1998), ‘On the consistency of backward-looking expectations: The case of the cobweb’, *Journal of Economic Behavior and Organization* **33**, 333–362.

Kuznetsov, Y. (1995), *Elements of applied bifurcation theory*, Vol. 112 of *Applied mathematical sciences*, SV, New York.

Lucas, R. (1978), ‘Asset prices in an exchange economy’, *Econometrica* **46**, 1229–1145.

Marcet, A. and Sargent, T. (1989), ‘Convergence of least squares learning mechanisms in self referential linear stochastic models’, *Journal of Economic Theory* **48**, 337–368.

Sonis, M. (2000), ‘Critical bifurcation surfaces of 3d discrete dynamics’, *Discrete Dynamics in Nature and Society* **4**, 333–343.