On Correlation Effects and Default Clustering in Credit Models

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Abstract

We establish Markovian models in the Heath, Jarrow and Morton paradigm where the credit spreads curves of multiple firms and the term structure of interest rates can be represented analytically at any point in time in terms of a finite number of state variables. The models make no restrictions on the correlation structure between interest rates and credit spreads. In addition to diffusive and jump-induced default correlations, default events can impact credit spreads of surviving firms. This feature allows a greater clustering of defaults. Numerical implementations highlight the importance of taking interest rate-credit spread correlations, credit-spread impact factors and the full credit spread curve information into account when building a unified model framework that prices any credit derivative.

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This paper investigates Markovian models in the Heath, Jarrow and Morton (1992) (hereafter HJM) paradigm that can be used to price credit derivatives on both single and multiple names. The models we develop have the following properties. First, they fully incorporate the current riskless term structure information as well as the full credit spread curve information for each firm. Second, the models, being Markovian, permit the riskless and risky credit spread curves to be analytically computed at any point in time, based on a finite collection of state variables. Third, the models allow for arbitrary correlation between riskless interest rates and credit spreads as well as arbitrary correlations between credit spreads of different firms. Fourth, interest rate and credit spread volatilities could be time homogeneous, level dependent, and can be initialized to term structures of volatilities. Fifth, we allow shocks to the economy that cause interest rates to jump as well as credit spreads of firms to change. Finally, we permit the default of some firms to cause jumps in the term structure of credit spreads of other surviving firms. These features collectively allow defaults to cluster over time.

Duffie and Singleton (1999a), Schönbucher (2000), and others, have shown how the HJM paradigm can be extended to include risky debt. Specifically, necessary restrictions on the dynamics of drift terms of forward rates and risky forward credit spreads have been identified that permit risky bonds to be priced in an arbitrage-free environment. Unfortunately, the resulting dynamics of all riskless forward rates and risky forward credit spreads are not in general Markov in a finite number of state variables. As a result, implementing these models, even via Monte Carlo simulation, is delicate and computationally intensive. The problem is compounded further if the derivative security that needs to be priced depends on the credit spreads of multiple names. In this paper, we generate an m-factor model for the riskless term structure and a correlated n-factor model for forward credit spreads, in such a way that riskless and risky bond prices can be recovered analytically in terms of their initial values and a finite collection of underlying state variables.

When the credit-spread dynamics and jumps are shut down, our model reduces to the multivariate extensions of Ritchken and Sankarasubramanian (1995), developed by Inui and Kijima (1998).[1] With credit spreads, our HJM models become more interesting, especially if credit spreads are correlated with interest rates, and if jumps are permitted. For such cases we identify volatility restrictions that ensure that finite-state-variable models can be identified which make implementation issues associated with the HJM paradigm easy to address.[2] Using time series data on term structures of both riskless rates and credit spreads, we provide evidence that suggests the restrictions are not severe.

[1]For further Markovian models of the riskless term structure, see Bhar and Chiarella (1995) and Cheyette (1995).

[2]Specifying a HJM model actually requires specifying structures for the volatilities of forward rates, and a family of forward rate curves, such as the Nelson-Siegel family, under which the forward rate curve is initialized. The specific model and family of curves for the calibration are said to be consistent if all forward rates produced by the model are contained in the family of forward rate curves used in calibration. A series of interesting papers have addressed this consistency issue, including Bjork and Christensen (1999), Bjork and Svensson (2002), La-Chioma and Picoli (2007) and the references therein.

The analysis is trivial when interest rates and credit spreads are uncorrelated. As a result, our analysis would be of limited interest if the correlation effects between interest rates and credit spreads had little effect on prices of credit derivatives. Therefore, the credit derivatives pricing examples that we consider are geared towards illustrating how some credit derivatives products, such as options on defaultable bonds and contingent credit default swaps, are extremely sensitive to correlations between interest rates and credit spreads.

In addition to establishing credit-derivatives models for single names, this paper focuses on models of credit contracts that depend on multiple names. Default correlation can typically be directly specified through the joint dynamics of the default intensities. Firms default rates are independent conditional on the realization of the state variables. Examples of this Conditional Independent Defaults (CID) approach include Bakshi, Madan and Zhang (2005), who assume intensities are driven by interest rates and a firm-specific factor, such as leverage; Janosi, Jarrow and Yildirim (2003) who model the intensity as a function of interest rates and a market index; Duffee (1999), who assumes that the intensities depend on factors relating to interest rates alone; and Driessen (2005) who adds additional common factors for all firms as well as firm-specific factors.

The main drawbacks attributed to the CID approach is the low level of default correlation they generate compared with empirical levels.[3] Duffie and Singleton (1999a) provide an alternative approach that uses separate point processes, some of which trigger joint defaults, while others reflect firm-specific defaults. Contagion models extend the conditional-independence approach to account for the fact that default clustering takes place. Jarrow and Yu (2001) and Yu (2007) extend these models so that a default can trigger jumps in the intensities of other firms' default processes. Such contagion models arise because of commercial or financial relationships between firms, or because levels of overall default risk in the economy may have increased in certain periods, as in Davis and Lo (2001).

Yu (2007) argues that the apparent low correlation is not a problem of the CID approach, but rather a problem with the choice of state variables. Specifically, a limited set of state variables or factors may not be sufficient to model the changes in intensities, and perhaps additional state variables are necessary. More recently, Duffie, Eckner, Horel and Saita (2008) estimate frailty models in which firms could be jointly exposed to unobservable risk factors.

The empirical evidence suggests that contagion of some form is important. Collin-Dufresne, Goldstein and Helwege (2003) and Jorion and Zhang (2007) find that a major credit event at one firm is associated with significant increases in spreads of other firms. Das, Duffie, Kapadia and Saita (2007) test whether default events can be modeled as conditionally independent and reject this hypothesis. More recently, Lando and Nielsen (2008), using the same sample, cannot reject this hypothesis, but, using alternative tests they do find support for contagion effects that take place through firm covariates as opposed to domino effects.

[3]See, for example, Hull and White (2001) and Schönbucher and Schubert (2002).

In our models, we introduce default dependence in three ways. First, we correlate the intensity processes of different firms. Second, we permit jumps in the riskless yield curve as well as corresponding jumps in credit spreads of individual firms. Third, we permit jumps to occur in the intensities of some firms when particular events, such as bankruptcies of certain firms, occur. In this regard, our models are an extension of Jarrow and Yu (2001), and especially Yu (2007). These *infection* models assume that the intensity process jumps at the time of a default of another firm, and this leads to a repricing of the bond. In our approach, we assume that if a default event occurs, the entire credit spread curve of any particular surviving firm could be affected, with the size of the jump depending on the surviving firm, the firm that defaulted and on the maturity of the forward credit spread.

With these additions, clustering of defaults is permissible. Moreover, since our models are Markovian, and have the property that they are automatically calibrated to existing yield and credit spread curves, pricing of derivatives contracts based on a portfolio of credits can be efficiently accomplished. Moreover, unlike most models of portfolios of credits, our models fully incorporate all information on credit curves for all individual names in the portfolio.

We provide several applications of our models that illustrate the role of correlations, jumps, and clustering, which are of interest in their own right. As an example, we investigate the pricing of a recent market innovation, namely a contingent credit default swap (CCDS). This contract can be viewed as an insurance policy that protects an investor against loses on derivatives that arise because the counterparty defaults. We value the counterparty credit risk associated with possible non-performance of an interest-rate swap, and also compute the change in value of the CCDS when the protection seller's credit is correlated with the credit of the counterparty in the underlying derivative. Counterparty credit risk has become an important asset class on Wall Street, and our numerical results highlight the necessity of including contagion or clustering effects in our models in order to generate the observed range of compensation for bearing such risk.

We do not need exotic contracts to highlight properties of our model. We illustrate the important role of correlation between interest rates and credit spreads when the underlying instrument is an option on a risky bond, and we investigate the impact of altering default clustering on the behavior of the prices of various tranches of a CDS index. Further, unlike the majority of competing models, our models of CDS index tranches incorporate the full term structure of credits for the individual names in the portfolio. We demonstrate that the distribution of credit spread shapes within the underlying CDS index structure can make an enormous difference in their valuation. Finally, since our models permit events to occur which can trigger large interest or credit risk shocks that may permeate through all bonds in a sector, they may be of interest in studies of value at risk where the impact of small probability events that cause large correlated losses within industries is currently of much interest, especially in light of the credit crunch of 2007-8.

The paper proceeds as follows. In Section 1 we describe our general HJM models of riskless and risky term structures. In Section 2 we present our main results that allow us to price derivative contracts on single names using Markovian HJM models. We also provide some empirical evidence that shows how well models can fit the term structures of volatilities, and that indicates that the Markovian restrictions are not severe. In Section 3 we extend the analysis to include portfolios of risky bonds where defaults of any one bond could impact the credit spreads of other bonds. In Section 4 we implement our model to price an array of products that are dependent on both interest rates and credit spreads, or on credit correlations, and highlight the fact that the models can accommodate a high degree of default clustering. Section 5 concludes the paper.

1 HJM Models for Defaultable Bonds

Consider a collection of I firms. The default state of these firms is summarized by the process $Y(t) = (Y_1(t), Y_2(t), ..., Y_I(t))$, where $Y_i(t) = 1$ if firm i has defaulted by time t, and $Y_i(t) = 0$ otherwise. Set $\tau_i = inf\{t|Y_i(t) = 1\}$, and let $X(t)$ be a vector of state variables that influence riskless yields and credit spreads of corporate debt. The default intensity for firm i at date t is $\eta_i(X(t), Y(t))$. Given the state variables, $(X(t), Y(t))$, the defaults of surviving firms over the next time increment are independent, time-inhomogeneous Poisson events. In addition to the default intensity for each firm, we also require assumptions on recovery given default. Let $\ell_i(t)$ denote the loss rate upon default. This loss rate could also depend on state variables, $(X(t), Y(t))$. The instantaneous credit spread, $\lambda_i(t)$, relates to the default intensity $\eta_i(t)$ by $\lambda_i(t) = \eta_i(t)\ell_i(t)$. In order to obtain tractable models that permit interesting dependence between defaults some additional structure on the interacting nature of defaults is required, as well as the identification and dynamics of the state variables for $X(t)$.

We begin our analysis, by first focusing on an important element of $X(t)$, namely the riskless term structure that certainly affects the yields on corporate debt. Rather than pushing back uncertainty to fundamental macroeconomic variables, we directly model the risk-neutral dynamics of riskless forward rates and risky credit spreads by a jump diffusion process in a Heath-Jarrow-Morton framework, where the term structures are initialized to their observable values. Specifically, we allow credit spread curves to respond to jumps in riskless yields and we also permit spreads of some firms to jump in response to defaults of primary firms. The modeling of these credit spread curves can be delicate because it is here where interesting dependence between defaults arises, and care has to be taken to ensure that feedback effects between $Y(t)$ and the state variables $X(t)$ are properly taken into account.

Let $P(t, T)$ be the price at date t of a pure riskless discount bond that pays \$1 at date T. Then:

$$P(t, T) = e^{-\int_t^T f(t,u)du}, \tag{1}$$

where $f(t, u)$ represents the date-t forward rate for the future time increment $[u, u+dt]$. We assume that forward rates follow a jump-diffusion of the form:

$$df(t, T) = \mu_f(t, T)dt + \sigma_f(t, T)dz_f(t) + c_f(t, T)dN_f(t) \text{ given } f(0, T), \quad \forall T \leq T^*. \tag{2}$$

T^* is a distant time horizon, and $z_f(t) = (z_1(t), z_2(t), ..., z_m(t))'$ is an m-dimensional standard Wiener process with

$$E(dz_f(t)dz_f(t)') = I_{m\times m}dt.$$

$N_f(t)$ is an independent Poisson process that models jump events, with

$$dN_f(t) = \begin{cases} 1 & \text{with probability} \quad \eta_f dt \\ 0 & \text{with probability} \quad 1-\eta_f dt. \end{cases} \tag{3}$$

We assume that $\mu_f(t,T)$, $\sigma_f(t,T)$, and $c_f(t,T)$ are regular enough to allow differentiation under the integral sign, interchange of the order of integration, partial derivatives with respect to the T variable, and have the property that the resulting bond prices are bounded. The volatility structure, $\sigma_f(t,T)$, is a predictable $1\times m$ vector process that at date t depends on observable state variables, while $c_f(t,T)$ is a simple deterministic function of time to maturity, $T-t$. The instantaneous spot rate at date t is $r(t) = f(t,t)$.

The dynamics of the bond price follows by applying Ito's lemma for jump-diffusion processes:

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= \left(r(t) + \frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) - \int_t^T \mu_f(t,u)du\right)dt - \sigma_p(t,T)dz_f(t) \\ &\quad +(e^{-K_p(t,T)}-1)dN_f(t), \end{aligned} \tag{4}$$

where

$$\sigma_p(t,T) = \int_t^T \sigma_f(t,u)du, \tag{5}$$

$$K_p(t,T) = \int_t^T c_f(t,u)du. \tag{6}$$

Now consider a risky zero-coupon corporate bond. Its yield can be broken down into a riskless yield and a credit spread. We first consider firms whose credit spreads do not depend on whether other firms have defaulted. For such a firm, firm A say, that has not defaulted prior to date t, we have:

$$dY_A(t) = \begin{cases} 1 & \text{with probability} \quad \eta_A(X_t)dt \\ 0 & \text{with probability} \quad 1-\eta_A(X_t)dt, \end{cases} \tag{7}$$

Such firms are called *primary* firms. Clearly, the default intensity for primary firms should depend on more factors than those that determine the riskless forward rate curve. Our model for a primary bond allows the default intensity, $\eta_A(t)$, to depend on riskless factors and on an additional n factors as well.

Specifically, let $\Pi_A(t,T)$ represent the date-t price of the bond issued by primary firm A that promises to pay \$1 at date T. The time to default is a stopping time, τ_A, say. Define $Y_A(t) = 1_{\tau_A\le t}$. We assume that $Y_A(t)$ has intensity $\eta_A(t)$. If a default occurs at time t, the loss rate is $\ell_A(t)$. With $\lambda_A(t) = \eta_A(t)\ell_A(t)$, we have:

$$\Pi_A(t,T) = V_A(t,T)1_{\tau_A>t}, \tag{8}$$

where

$$V_A(t,T) = e^{-\int_t^T (f(t,u)+\lambda_A(t,u))du} = P(t,T)S_A(t,T). \quad (9)$$

Here $f(t,T)$ is the date-t riskless forward interest rate for date T, as before, and $\lambda_A(t,T)$ is the forward credit spread for firm A. The risk-neutral dynamics of the riskless forward rates are given by (2), and the risk-neutral dynamics of the credit spreads are given by

$$d\lambda_A(t,T) = \mu_A(t,T)dt + \sigma_A(t,T)dz_A(t) + c_{fA}(t,T)dN_f(t), \ \forall t \leq \tau_A, \quad (10)$$

with the date-0 riskless forward curve and the date-0 firm A credit spread curve initialized to their observable values. The forward credit spreads are driven by a continuous diffusive term, $dz_A(t)$, where $z_A(t) = (z_{A_1}(t), \ldots, z_{A_n}(t))'$ is an n-dimensional standard Wiener process with $E(dz_A(t)dz'_A(t)) = I_{n\times n}dt$. It is correlated with the diffusive riskless term according to $E(dz_f(t)dz'_A(t)) = \Sigma^A_{m\times n}dt$, where:

$$(\Sigma^A)_{ij} = \rho^A_{ij}. \quad (11)$$

Further, when there is a jump in the riskless curve, then there is a corresponding jump in the credit spread curve. That is, jump risk in riskless rates could transmit to shocks in the credit spreads as well. Similar to the riskless forward rates, the volatility factor, $\sigma_A(t,T)$, is predictable, while $c_{fA}(t,T)$ is a deterministic function of time to maturity, $T-t$. Finally, the default of the risky bond occurs at some random stopping time, τ_A.

If there are several primary firms, then the correlation between the credit spread innovations will presumably be determined by the nature of the operations and the capital structure of the firms. However, the credit spread of any specific firm at any point in time will not be influenced by defaults of any other primary firm. From a computational point of view then, the price of a zero-coupon bond of such a firm is not dependent on the path of credit spreads of other firms up to that date, but is a function of the riskless yield curve up to that date, as well as of the dynamics of the firm-specific credit spreads.

Application of Ito's lemma for the jump-diffusion processes specified above yields:

$$\frac{dS_A(t,T)}{S_A(t,T)} = \left(\lambda_A(t) + \frac{1}{2}\sigma_{S_A}(t,T)\sigma'_{S_A}(t,T) - \int_t^T \mu_A(t,u)du\right)dt - \sigma_{S_A}(t,T)dz_A(t) + (e^{-K_{fA}(t,T)} - 1)dN_f(t), \quad (12)$$

where the j^{th} element of $\sigma_{S_A}(t,T)$ is

$$\sigma_{S_{A_j}}(t,T) = \int_t^T \sigma_{A_j}(t,u)du, \quad (13)$$

and where

$$K_{fA}(t,T) = \int_t^T c_{fA}(t,u)du. \quad (14)$$

Proposition 1

Assume the dynamics of forward rates and risky forward credit spreads under the risk-neutral measure are given by equations (2) and (10). Assume that at the time of default of the risky bond, the recovery value is proportional to the market value of the bond just prior to default. Then, to avoid arbitrage opportunities, the following drift restrictions must hold:

$$\mu_f(t,T) = \sigma_p(t,T)\sigma'_f(t,T) - c_f(t,T)e^{-K_p(t,T)}\eta_f \quad (15)$$

$$\mu_A(t,T) = \sigma_{S_A}(t,T)\sigma'_A(t,T) + \sigma_f(t,T)\Sigma\sigma'_{S_A}(t,T) + \sigma_p(t,T)\Sigma\sigma'_A(t,T) + g_A(t,T), \quad (16)$$

where

$$g_A(t,T) = \eta_f\left(c_f(t,T)e^{-K_p(t,T)} - (c_f(t,T) + c_{fA}(t,T))e^{-(K_p(t,T)+K_{fA}(t,T))}\right).$$

Proof: See Appendix A.

Equations (15) and (16) curtail the drift expressions in terms of the volatility structures. The restriction on the drift terms for riskless forward rates under the risk-neutral measure were first identified by Heath, Jarrow and Morton (1992). The restrictions for risky forward rates were identified by several authors, including Schönbucher (2000). In general, these restrictions imply that the dynamics of riskless and risky bonds are not Markovian in a finite collection of state variables. This creates computational difficulties since the entire riskless and risky term structures have to be stored along all the paths that are generated.

2 Markovian Models For Defaultable Bonds

To obtain tractable models we need to curtail the volatility structures, $\sigma_f(t,T) = (\sigma_{f_1}(t,T), \ldots, \sigma_{f_m}(t,T))$ and the impact factor, $c_f(t,T)$, to plausible forms for riskless debt, as well as the structures $\sigma_A(t,T) = (\sigma_{A_1}(t,T), \ldots, \sigma_{A_n}(t,T))$ and $c_{fA}(t,T)$ for risky debt. We assume:

$$\sigma_{f_i}(t,T) = h_{f_i}(t)e^{-\kappa_{f_i}(T-t)}, \quad (17)$$

$$\sigma_{A_j}(t,T) = h_{A_j}(t)e^{-\kappa_{A_j}(T-t)}, \quad (18)$$

and jump-impact factors of the form

$$c_f(t,T) = c_f e^{-\gamma_f(T-t)}, \quad (19)$$

$$c_{fA}(t,T) = c_{fA}e^{-\gamma_{fA}(T-t)}, \quad (20)$$

where $h_{f_j}(t)$ and $h_{A_j}(t)$ are predictable functions that depend on state variables at date t.

Substituting these expressions into equations (5), (6), (13) and (14), the volatility expressions, $\sigma_p(t,T) = (\sigma_{p_1}(t,T), \ldots, \sigma_{p_m}(t,T))$ and $\sigma_A(t,T) = (\sigma_{A_1}(t,T), \ldots, \sigma_{A_n}(t,T))$ are given by

$$\sigma_{p_i}(t,T) = h_{f_i}(t)K(t,T;\kappa_{f_i}),$$

$$\sigma_{S_{A_j}}(t,T) = h_{A_j}(t)K(t,T;\kappa_{A_j})$$

and the impact factors $K_p(t,T)$ and $K_{fA}(t,T)$ can be computed as:

$$\begin{aligned} K_p(t,T) &= c_f K(t,T;\gamma_f) \\ K_{fA}(t,T) &= c_{fA} K(t,T;\gamma_{fA}) \end{aligned}$$

where:

$$K(t,T;x) = \frac{1}{x}(1 - e^{-x(T-t)})$$

With these volatility and impact restrictions, Markovian models can be obtained for riskless bonds and risky debt of primary firms.

Proposition 2

(i) Under the risk-neutral dynamics (2), with the volatility and impact restrictions given in (17) and (19), the riskless bond price at date t is linked to the forward price of the bond at date 0 by:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\sum_{i=1}^{2}\sum_{j=1}^{m} H_{ij}(t,T)\psi_{ij}(t) - H_3(t,T)\psi_3(t) + H_J(t,T)}, \tag{21}$$

where

$$\begin{aligned} H_{1j}(t,T) &= \frac{1}{\kappa_{f_j}} K(t,T;\kappa_{f_j}), \quad \textit{for } j = 1,\ldots,m \\ H_{2j}(t,T) &= -\frac{1}{\kappa_{f_j}} K(t,T;2\kappa_{f_j}), \quad \textit{for } j = 1,\ldots,m \\ H_3(t,T) &= c_f K(t,T;\gamma_f), \\ H_J(t,T) &= c_f \eta_f \int_t^T L_f(t,u)du. \end{aligned}$$

The dynamics of the state variables, initialized to 0 at date 0, are given by:

$$\begin{aligned} d\psi_{1j}(t) &= (h_{f_j}^2(t) - \kappa_{f_j}\psi_{1j}(t))dt + \kappa_{f_j} h_{f_j}(t) dz_{f_j}(t) \\ d\psi_{2j}(t) &= (h_{f_j}^2(t) - 2\kappa_{f_j}\psi_{2j}(t))dt \\ d\psi_3(t) &= -\gamma_f \psi_3(t)dt + dN_f(t), \end{aligned}$$

and

$$L_f(t,T) = \frac{e^{-\frac{c_f}{\gamma_f}}}{c_f}\left(e^{\frac{c_f}{\gamma_f}e^{-\gamma_f(T-t)}} - e^{\frac{c_f}{\gamma_f}e^{-\gamma_f T}}\right).$$

(ii) Given the risk-neutral dynamics (2) and (10), the volatility and impact structures specified in (17) through (20), and assuming that at the time of default the recovery value is proportional to the market value of the bond just prior to default, the price of a defaultable zero-coupon bond is given by $\Pi_A(t,T) = V_A(t,T)1_{\tau_A > t}$, where $V_A(t,T) = P(t,T)S_A(t,T)$ and

$$\begin{aligned} S_A(t,T) &= \frac{S_A(0,T)}{S_A(0,t)} e^{-A_0(t,T) - \sum_{j=1}^{n}(K_{0,j}(t,T)\xi_{0,j} - K_{1,j}(t,T)\xi_{1,j})} \\ &\quad \times e^{\sum_{i=1}^{m}\sum_{j=1}^{n}(K_{2,ij}(t,T)\xi_{2,ij} - K_{3,ij}(t,T)\xi_{3,ij} - K_{5,ij}(t,T)\xi_{5,ij})} \times e^{-K_4(t,T)\xi_4(t)}. \end{aligned}$$

Here, $A_0(t,T) = \int_t^T G_A(t,u)du$, *where* $G_A(t,u) = \int_0^t g_A(v,u)dv$, *and*

$$
\begin{aligned}
K_{0,j}(t,T) &= \frac{1}{\kappa_{A_j}} K(t,T;\kappa_{A_j}), \quad \textit{for } j = 1,\ldots,n \\
K_{1,j}(t,T) &= \frac{1}{\kappa_{A_j}} K(t,T;2\kappa_{A_j}), \quad \textit{for } j = 1,\ldots,n \\
K_{2,ij}(t,T) &= \frac{\rho^A_{ij}(\kappa_{f_i}+\kappa_{A_J})}{\kappa_{A_j}\kappa_{f_i}} K(t,T;\kappa_{f_i}+\kappa_{A_j}), \quad \textit{for } i = 1,\ldots,m; j = 1,\ldots,n \\
K_{3,ij}(t,T) &= \frac{\rho^A_{ij}}{\kappa_{A_j}} K(t,T;\kappa_{f_i}), \quad \textit{for } i = 1,\ldots,m; j = 1,\ldots,n \\
K_4(t,T) &= c_{fA} K(t,T;\gamma_{fA}), \\
K_{5,ij}(t,T) &= \frac{\rho^A_{ij}}{\kappa_{f_i}} K(t,T;\kappa_{A_j}), \quad \textit{for } i = 1,\ldots,m; j = 1,\ldots,n.
\end{aligned}
$$

The dynamics of the state variables, all initialized at date 0 *to be* 0*, are:*

$$
\begin{aligned}
d\xi_{0,j}(t) &= (h^2_{A_j}(t) - \kappa_{A_j}\xi_{0j}(t))dt + \kappa_{A_j} h_{A_j}(t) dz_{A_j}(t) \\
d\xi_{1,j}(t) &= (h^2_{A_j}(t) - 2\kappa_{A_j}\xi_{1j}(t))dt \\
d\xi_{2,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - (\kappa_{A_j}+\kappa_{f_i})\xi_{2,ij}(t))dt \\
d\xi_{3,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - \kappa_{f_i}\xi_{3,ij}(t))dt \\
d\xi_4(t) &= -\gamma_{fA}\xi_4(t) + dN_f(t) \\
d\xi_{5,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - \kappa_{A_j}\xi_{5,ij}(t))dt.
\end{aligned}
$$

Proof: See Appendix A.

The first part of the Proposition shows that, given an initial riskless term structure, riskless bond prices at any future date are fully characterized by $2m+1$ state variables, namely, $X(t) = (\psi_1(t), \psi_2(t), \psi_3(t))$, where $\psi_1(t)$ and $\psi_2(t)$ are of size m. The second part of Proposition 2 states that forward credit spreads of all maturities at date t are linked to the credit spread curve at date 0 through a total of $2n+3mn+1$ state variables. Risky bond prices are determined by these $2n+3mn+1$ state variables in addition to the $2m+1$ state variables for the riskless factors. Collectively then, the state variable vector, $X(t)$, consists of at most $3mn+2(m+n+1)$ state variables.

The existence of correlations between riskless rates and credit spreads creates significantly more state variables. Indeed, if uncorrelated, the total number of state variables declines by $3mn$. Further reductions in the number of state variables can occur if certain parameters are curtailed or if some volatility structures are deterministic. For the special interest where $m=1$ and $n=1$, as shown below, the number of state variables for the credit spread reduces to five, and the number of state variables for computing riskless and risky bonds reduces to eight.

Corollary

For the case $m = 1$ and $n = 1$, equivalent representations for riskless bond prices $P(t,T)$ and risky bond prices, $\Pi_A(t,T) = V_A(t,T)1_{\tau_A > t}$, where $V_A(t,T) = P(t,T)S_A(t,T)$ are:

$$
\begin{aligned}
P(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-K(t,T;\kappa_f)(r(t)-f(0,t)) - \frac{1}{2}K^2(t,T;\kappa_f)\psi_2(t) + c(K(t,T;\kappa_f) + c_f K(t,T;\gamma_f)\psi_3(t)} \\
&\quad \times e^{-c\eta_f (K(t,T;\kappa_f)L_f(t,t) - \int_t^T L_f(t,u)du)} \qquad (22) \\
S_A(t,T) &= \frac{S_A(0,T)}{S_A(0,t)} e^{-\left(A_0(t,T) + K_0(t,T)\lambda_A(t) + \sum_{j=1}^4 K_j(t,T)\xi_j(t)\right)}. \qquad (23)
\end{aligned}
$$

The terms in the exponent of $S_A(t,T)$ are given as

$$
\begin{aligned}
A_0(t,T) &= \int_t^T G_A(t,u)du - K(t,T;\kappa_A)(\lambda_A(0,t) + G_A(t,t)) \\
K_0(t,T) &= K(t,T;\kappa_A) \\
K_1(t,T) &= \frac{1}{2}K(t,T;\kappa_A)^2 \\
K_2(t,T) &= \rho^A \left(\frac{1}{\kappa_A} + \frac{1}{\kappa_f}\right)(K(t,T;\kappa_A) - K(t,T;\kappa_A + \kappa_f)) \\
K_3(t,T) &= \frac{\rho^A}{\kappa_A}(K(t,T;\kappa_f) - K(t,T;\kappa_A)) \\
K_4(t,T) &= c_{fA}(K(t,T;\gamma_{fA}) - K(t,T;\kappa_A)),
\end{aligned}
$$

Proof: See Appendix A

Note that the model delivers exponential affine riskless and risky bond prices even though the short rate is not necessarily affine, since $h_f(\cdot)$ and $h_A(\cdot)$ can be arbitrary. As can be seen, the state variables for the riskless term structure are now $\Phi(t)$, say, where

$$\Phi(t) = (r(t), \psi_2(t), \psi_3(t)),$$

and the state variables for the price of a bond issued by a primary firm is $X_A(t)$, say, where

$$X_A(t) = \{(\Phi(t), \Upsilon_A(t))\},$$

where $\Upsilon_A(t) = \{\lambda_A(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)\}$ are the additional state variables for the credit spreads.

When the stochastic drivers of credit spreads are shut down ($n = 0$) and interest rates are driven by one stochastic driver ($m = 1$) with no jumps in the interest rates, the model reduces to Ritchken and Sankarasubramanian (1995). For the slightly more general case of $m > 1$, $n = 0$ and no jumps, the model corresponds to that of Inui and Kijima (1998). With interest rate jumps we get a modest extension. When n is released from 0, we get new models for credit spreads and risky bond prices. For the above model with $m = n = 1$ and jumps, the total number of state variables is eight. When jumps are not allowed, the number of state variables drops to six. When interest rates and credit spreads are uncorrelated, the number of state variables drops to four. And if the predictable functions h_{f_j} and h_{A_i} are constants, then all path statistics fall away. In that case, the framework reduces to a generalized Vasicek (1977) model for both riskless and risky bond prices, with just two state variables.

Clearly, as m and n increase, the number of state variables increases rather rapidly. For example, if the riskless term structure is driven by $m = 3$ stochastic drivers and credit spreads are driven by $n = 2$ stochastic drivers, in addition to the one riskless jump process, the total number of state variables is 30. Since there are only five diffusive stochastic drivers, and single jump processes for interest rates and default, most of these state variables merely serve as path statistics that can be updated extremely rapidly. For pricing derivatives on a single name, this number of state variables is not large enough to provide an excessive computational burden.

To highlight the computational effort involved with these models, consider a rather simple problem where potential cash flows occur monthly over a 10-year time horizon, and where the terminal cash flow of the derivative depends on the riskless and risky discount function going out another 20 years. Consider a simple HJM model with two factors, one for interest rates ($m = 1$) , one for credit spreads ($n = 1$) and no jumps in riskless interest rates, but where the volatility structures are not of our form. The forward rates of $30 \times 12 = 360$ monthly forward rates as well as the risky credit spreads need to be tracked. As such, the model is Markovian in 720 state variables. If the time partitions are refined to weeks, then weekly forward rates and spreads must be computed and the number of state variables increases by a factor of four to $2,880$. In contrast, with the Markovian models, a maximum of eight state variables need to be maintained, and this number does not change as the partition is refined.

To make closer comparisons between the general and restricted HJM models, we can easily map the path statistics in the restricted models to unique points on the term structure, As an illustration, consider the one factor model with no jumps in interest rates. From equation (22) we can obtain the forward rate expression for date T=t+m:[4]

$$f(t, t+m) = f(0, t+m) + e^{-\kappa_f m}(r(t) - f(0,t)) + \frac{e^{-\kappa_f m}}{\kappa_f}(1 - e^{-\kappa_f m})\psi_2(t)$$

We then can replace the state variable $\psi_2(t)$ with the forward rate, $f(t, t+m)$. Other forward rates can then be expressed as affine combinations of the new state variables $r(t)$ and $f(t, t+m)$. In particular, for maturity $t + n$, we have:

$$f(t,t+n) = f(0,t+n) + [e^{-\kappa_f n} - W(m,n;\kappa_f)](r(t) - f(0,t)) + W(m,n;\kappa_f)(f(t,t+m) - f(0,t+m)) \quad (24)$$

where $W(m, n; \kappa_f) = \frac{(1-e^{-\kappa_f n})}{(1-e^{-\kappa_f m})}$. Hence changes in all forward rates can be expressed a functions of changes in *two* forward rate points on the curve. This is in contrast with the general HJM model

[4]This equation is derived in the Appendix as equation (A.3) with $T = t + m$

which requires all points on the forward rate curve be carried.[5]

Similarly, for the credit spread curve, we could compare a one factor model, under our restrictions, with a one factor model under no restrictions. For the case of no jumps, with interest rates and intensities uncorrelated, the credit spread curve would take on the exact same form as equation (24). For the more interesting case, when correlations are not zero, we have, from equation (23) for $T = t + m_i$:[6]

$$\lambda_A(t, t + m_i) = \lambda_A(0, t + m_i) + e^{-\kappa_A m_i}(\lambda_A(t) - \lambda_A(0, t)) + \sum_{j=1}^{3} D_j(m_i, \kappa_A, \kappa_f, \rho)\xi_j(t)$$

where $D_j(m_i, \kappa_A, \kappa_f, \rho)$ are the coefficients of $\xi_j(t)$ which are provided in the appendix as the coefficients in equation (A.4). It can be immediately seen, that the three state variables $\xi_j(t); j = 1, 2, 3$ can be mapped onto three forward rates, with maturities m_1, m_2, m_3. Hence, all points on the credit spread curve can be represented as maturity dependent combinations of just four points on the credit spread curve. This, again, is in contrast to the more general HJM model that requires all points on the credit curve.

With multi-factor models, the computational burden on a non-Markovian model rapidly becomes immense and as time increments are refined, the number of state variables explodes. In contrast, the number of state variables to keep track of for the Markovian models remains relatively small and does not increase as the time partition is refined.

2.1 Are the Volatility Restrictions Severe?

Unlike, many of the non-Gaussian models for credit spreads, our Markovian models make no assumptions on correlations between the stochastic drivers for credit spreads and interest rates.[7] However, the volatility structures of riskless forward rates and risky credit spreads need to be curtailed. One reasonable volatility structure in the Markovian class is given by:

$$\begin{aligned} \sigma_f(t, T) &= \sigma[r(t)]^\gamma e^{-\kappa_f(T-t)} \\ \sigma_A(t, T) &= \sigma_A[\lambda_A(t)]^\gamma e^{-\kappa_A(T-t)} \end{aligned}$$

When $\gamma = 0$ the forward rates collapses to the Generalized Vasicek model. When $\gamma = 0.5$ short interest rates are given by a square root process similar to Cox, Ingersoll and Ross. Finally, a proportional model obtains when $\gamma = 1$. Note that the volatility structure of forward rates remains time invariant, and decays exponentially with maturity. Ritchken and Sankarasubramanian (1995b) show that within the class of Markovian models, all calibrated to fit the same term structure,

[5] For more details of transforming the state variables to points on the forward curve, see Bliss and Ritchken (1996), and especially Chiarella and Kwon (2001).

[6] This expression is also derived as equation (A.4) in the appendix.

[7] For example, a multifactor CIR model for credit spreads requires positive correlation between riskless rates and spread.

prices of caps and floors could still vary considerably according to the specific volatility structures within their family. As a result the selection of a specific volatility structure within the restricted Markovian family is still important.

Empirical evidence suggests that there could be a hump in the volatility structure of forward rates.[8] This can easily be obtained in a two stochastic driver model. For example, a simple two factor model for interest rates with volatility structures $\sigma_{f_j}(t,T) = \sigma_{f_j} e^{-\kappa_{f_j}(T-t)}$ for $j = 1,2$ would provide humped forward rate volatility curves.[9] Empirical evidence for such models, often referred to as double mean reverting models, has been provided by Jagadeesh and Pennacchi (1996), Ritchken and Chuang (2002), and Bakshi, Madan and Zhang (2002).[10]

It should be noted that in the models that we have developed, the volatility factors are exponentially dampened functions across the maturity spectrum. While this restriction may not be severe in a multifactor model, we want to emphasize that this restriction can easily be removed, by generalizing Proposition 2, and allowing for arbitrary shapes in the term structure of volatilities.

For example, in a single factor model for riskless forward rates, we could have:

$$\sigma_f(t,T) = h_f(t)[\sum_{j=1}^{k} a_j e^{-\kappa_j(T-t)}]$$

With sums of exponential functions, we can easily permit humped structures for volatilities. In this case, under the risk neutral measure, we can still obtain a Markovian representation of forward rates, although now there will be more state variables.

To illustrate this, consider the case where $k = 2$. By mixing two exponential functions we can certainly obtain a hump shaped curve, so it might not be necessary to have $k > 2$. Substituting the above equation into the HJM drift restriction, equation (15), using the resulting expression in equation (2) and then integrating equation (2) eventually leads to:

$$\begin{aligned} f(t,T) &= f(0,T) + \sum_{j=1}^{2} d_1^{(j)} e^{-\kappa_j(T-t)} \psi_1^{(j)}(t) - \sum_{j=1}^{2} d_2^{(j)} e^{-2\kappa_j(T-t)} \psi_2^{(j)}(t) - d_{12} e^{-(\kappa_1+\kappa_2)(T-t)} \psi_{12}(t) \\ &\quad + c_f e^{-\gamma_f(T-t)} \psi_3(t) - c_f \eta_f L_f(t,T) \end{aligned}$$

[8] Several researchers report a hump in the volatility structure that peaks at around the two year maturity. Litterman and Scheinkman (1991) use a principal component analysis of interest movements, to reveal a humped volatility form. Heath, Jarrow, Morton and Spindel (1992) provide cursory evidence of such a hump. Amin and Morton (1994) use Eurodollar futures and options and obtain negative estimates of κ over the short end of the curve. Since negative estimates over the entire maturity spectrum are not plausible, they argue that there is a hump in the structure. Goncalves and Issler (1996) estimate the term structure of volatility using a simple Generalized Vasicek model. Their historical analysis of forward rates also reveals a hump.

[9] Alternatively, a humped structure in volatilities can be obtained in a one factor model by replacing the exponential requirement, $e^{-\kappa(T-t)}$ with a form, $\frac{a(T)}{a(t)}$, and specifying $a(t) = \alpha_0 + \alpha_1 e^{-\kappa t}$.

[10] For an excellent discussion of these models and further empirical studies see Brigo and Mercurio (2000).

where

$$\psi_1^{(j)}(t) = d_1^{(j)} \int_0^t h^2(u) e^{-\kappa_j(t-u)} du + \kappa_j d_i^{(j)} \int_0^t h(u) e^{-\kappa_j(t-u)} dz_f(u) \text{ for } j = 1,2.$$
$$\psi_2^{(j)}(t) = \int_0^t h^2(u) e^{-2\kappa_j(t-u)} du \text{ for } j = 1,2.$$
$$\psi_{12}(t) = \int_0^t d_{12} e^{-(\kappa_1+\kappa_2)(t-u)} du$$

and $d_1^{(j)} = a_j(\frac{a_1}{\kappa_1} + \frac{a_2}{\kappa_2}$, $d_2^{(j)} = \frac{a_j^2}{\kappa_j}$ for $j = 1,2$, and $d_{12} = a_1 a_2(\frac{1}{\kappa_1} + \frac{1}{\kappa_2})$. Compared to our earlier single factor model of forward riskless rates, the entire term structure requires three additional state variables so as to permit a Markovian representation. The volatility parameters a_j, κ_j, for $j = 1,2$ can easily be chosen so as to closely match the volatility term structure of forward rates.[11] In a similar way, the volatility term structures for credit spreads can be generalized to permit volatility humps as well.

2.2 Empirical Evidence on Volatility Structures

Using cap and swaption data, Fan, Gupta and Ritchken (2003) conduct empirical tests on one and two factor Markovian HJM interest rate models and conclude that these models can explain the volatility skew in derivative markets very well. Fan, Gupta and Ritchken (2007) also show that a one factor Markovian HJM model, with two state variables priced caps and swaptions as well as four factor models where the volatility structure was identified from a principal component analysis as in Longstaff, Santa-Clara and Schwartz (2001).

The above studies illustrate that estimates of the volatility parameters of riskless interest rates can be obtained from cross sectional information on interest rate caps and swaptions, and that the volatility restrictions for riskless securities may not be severe.

The volatility structure of credit spreads has not come under the same scrutiny as the volatility of interest rates. Theoretical option models, starting with Merton (1974), Longstaff and Schwartz (1995) and Jarrow, Lando and Turnbull (1997), among others, permit credit-spread curves to be increasing, decreasing, or hump-shaped, and allow volatility structures for forward credit spreads to take on shapes that typically decay with maturity. Given the liquidity of the CDS markets for individual names, not only is the credit spread curve identifiable, but, like the riskless term structure, the volatility term structure can be analyzed. If derivatives such as options on credit spreads or options on credit default swaps trade, then it is might be possible to imply out estimates of volatility parameters from the prices of these traded instruments.

[11] For $k > 2$, similar results can be obtained, with the number of state variables expanding. For further discussion on how the parameters of such models could be evaluated see Fan, Gupta and Ritchken (2007) and the references therein.

To illustrate whether the restrictions are reasonable for riskless yields and for risky credit spreads, we briefly turn to the data. Under the true data generating process we consider a process with no jumps:

$$d \ln P(t, t+m) = \mu(\cdot)dt - \sigma_p(t, t+m)dz_f(t),$$

Under our one factor model structure we could have $\sigma_p(t, t+m) = \frac{\sigma}{\kappa} h_f(t)(1 - e^{-\kappa_f m})$, where $h_f(t)$ for example could be a function of any rate or set of rates drawn from the yield curve at date t. The following discretized process results:

$$\Delta \ln P(t, t+m) \sim N(\mu(\cdot)\Delta t, \quad \frac{\sigma_f^2}{\kappa_f^2} h_f(t)^2 (1 - e^{-\kappa_f m})^2 \Delta t)$$

Transforming this process, leads to:

$$Y_f(t, m) = \frac{\Delta \ln P(t, t+m)}{h_f(t)} \sim N(\mu^*(\cdot)\Delta t, \quad F_f^2(m)\Delta t)$$

where $F_f(m) = \frac{\sigma_f}{\kappa_f}(1 - e^{-\kappa_f m})$. For high frequency data, the impact of the drift term is typically negligible, and can often be ignored.

With data collected on k_1 different maturities on each date, and assuming we have changes over k_2 consecutive periods, we then have $n = k_1 \times k_2$ data points. The logarithm of the likelihood function is proportional to

$$L_n(\kappa_f, \sigma_f) = -\sum_{j=1}^{n} \ln(F(m_j)\sqrt{\Delta t}) - \frac{1}{2}\sum_{j=1}^{n}\left(\frac{Y_f(m_j)}{F(m_j)\sqrt{\Delta t}}\right)^2$$

where m_j is the maturity of the j^{th} data point, $j = 1, ..., n$

The exact same logic can be used for the credit spreads on a single name. Specifically we have:

$$Y_A(t, m) = \frac{\Delta \ln S_A(t, t+m)}{h_A(t)} \sim N(\mu^*(\cdot)\Delta t, F_A^2(m)\Delta t)$$

where $F_A(t, m) = \frac{\sigma_A}{\kappa_A}(1 - e^{-\kappa_A m})$.

We briefly illustrate the feasibility of estimating volatility parameters using this approach by turning to the data. We obtain zero-coupon Treasury yields of maturities 1 through 20 years from Gurkaynak, Sack and Wright (2006).[12] We also obtain the time series of credit default swaps for an illustrative firm, Time Warner, from Datastream. This data is weekly, starting in July 2004 and ending in August 2008. While the most liquid contracts are for 5-years, Datastream provides credit spreads data for 1 through 10 year maturities. Figure 1 shows the time series of riskless yields and credit spreads used for this illustration.

Figure 1 Here

[12] Their daily Treasury yield curves over this range of dates are available from July 1981 to the present, in our case July 2008, and can be downloaded from http://www.federalreserve.gov/pubs/feds/2006.

From the daily riskless data, we compute the time series, for $Y_f(t,m)$. To do this we need to specify a particular form for $h_f(\cdot)$. We choose $h_f(t) = \sqrt{y(t,t+1)}$, where $y(t,t+1)$ is the one year riskless yield.

The top panel in Figure 2 shows the box plots of $Y_f(t,m)$ values for maturities ranging from 1 year to 20 years. As can be seen the means are close to zero and the spread of the distributions increase with maturity. The right panel shows the actual volatilities by maturity, computed over the entire period, and compares the volatility structure to the fitted theoretical volatility structure, where the parameters are estimated by maximum likelihood.

Figure 2 Here

As can be seen, the model fits the historical term structure of volatilities very precisely. The maximum likelihood estimates are $\kappa_f = 0.0106$ and $\sigma_f = 0.0625$.

The bottom panel compares the historically computed time series of the volatilities of changes in the logarithmic prices of bonds of different maturities, using a rolling window of one year, with the forward looking predicted values of the volatility structure given by $\sigma_f(\sqrt{y(t,t+1)})F_f(t,m)$ The figure shows that while the term structure of the historical, rolling volatilities takes on more shapes then the term structures of the one factor model, the overall patterns are quite similar. Clearly, additional empirical work would be necessary to establish whether a second factor would be beneficial.

Figure 3 repeats the same analysis using the credit spread data on Time Warner. The box and whisker plots illustrate that $Y_A(t,m)$ values have dispersions that increase with maturity, and their averages are close to zero. Similar to the riskless rates the assumed structure for $h_A(t)$ is a square root model of the form $h_A(t) = \sqrt{s(t,t+1)}$ where $s(t,t+1)$ is the one year spread at date t. The volatility structure for $Y_A(t,m)$l is $F_A(t,m)$. The estimates obtained were $\kappa_A = 0.0013$ and $\sigma_A = 0.1103$. The right graph shows how the values of the volatility structure obtained using maximum likelihood compare with the actual volatilities computed using the full data set. The fit appears reasonable with some bias at the short and long maturities.

Figure 3 Here

The bottom panel compares the historically computed time series of the volatilities of changes in the logarithmic prices, $S_A(t,t+m)$ for different maturities, m, using a rolling window of one year, with the forward looking predicted values of the volatility structure given by $\sigma_A(\sqrt{s(t,t+1)})F_A(t,m)$. The figure shows that the theoretical model closely matches the shape of the actual volatility and the rise in credit spread volatility in 2007 was reasonably well accounted for.

We now turn to correlations. The top panel of Table 1 shows the correlation among changes in selected weekly riskless interest rates and credit spreads of Time Warner. The correlations between interest rates and credit spreads are negative, significantly different from zero, but their magnitude is not large. The second panel shows a similar analysis for a second firm, AT&T, in which the correlations between interest rates and credit spreads is further from zero. The bottom panel shows positive correlations between changes in credit spreads of the two firms.

Table 1 Here

The above example illustrates the potential size of volatilities and correlations. In Section 4, we will consider specific examples of pricing that highlight implementation issues associated with pricing derivatives sensitive to interest rates, credit spreads and counterparty risk.

3 Pricing Risky Debt of Secondary Firms

In addition to primary firms there may be secondary firms in the marketplace. These firms are more susceptible to defaults of primary firms. For example, a secondary firm may carry significant debt from a primary firm; or a significant portion of its sales may flow through a primary firm. So, when a primary firm defaults, the credit spreads of the secondary firm may jump up. Alternatively, the default of a primary firm could be good news for a secondary firm, in that it now may play a bigger role in the competitive market. In this case, an unanticipated default of its large competitor could result in an unanticipated downward shock to all credit spreads along the maturity spectrum of the secondary firm.

Assume that there are m_B primary firms whose default could affect the credit spreads of our secondary firm, firm B. Assume that the primary firms are labeled $A_1, A_2, ...A_{m_B}$. For primary bond A_i, the state variables are $X_{A_i}(t) = \{(\Phi(t), \Upsilon_{A_i}(t))\}$. For our secondary firm B, the number of factors influencing its credit spread curve is significantly enhanced. In addition to $\Phi(t)$, we may need to know the credit spreads of all surviving primary firms. Let

$$X_B(t) = \{\Phi(t), \Upsilon_{A_1}(t), ..., \Upsilon_{A_{m_B}}(t), \Upsilon_B(t), Y(t)\},$$

where $\Upsilon_{A_i}(t)$, is empty if firm i has defaulted prior to date t. Unlike primary firms, to model a secondary firm's process we need to know the correlation structure among all the firm-specific diffusive factors, as well as how these diffusive terms correlate with the secondary firm's diffusive component. In addition we need to track the primary firms status. Once we have $\lambda_A(t)$, for a primary firm, and knowing the loss function $\ell_A(t)$, we can compute the instantaneous risk-neutral default probability, $\eta_A(t) = \lambda_A(t)/\ell_A(t)$. It is this probability that determines the likelihood of default in the next time increment.

The risk-neutral dynamics of the riskless forward rate and the credit spread of firm B are given by (2) and

$$\begin{aligned} d\lambda_{A_i}(t,T) &= \mu_{A_i}(t,T)dt + \sigma_A(t,T)dz_{A_i}(t) \\ &\quad + c_{fA_i}(t,T)dN_f(t), \ \forall t \le \tau_{A_i}, i = 1, \ldots, m_B \qquad (25) \\ d\lambda_B(t,T) &= \mu_B(t,T)dt + \sigma_B(t,T)dz_B(t) + c_{fB}(t,T)dN_f(t) \\ &\quad + \sum_{i=1}^{m_B} c_{A_iB}(t,T)(1 - Y_{A_i}(t))dY_{A_i}(t), \ \forall t \le \tau_B. \qquad (26) \end{aligned}$$

The date-0 riskless forward curve, the date-0 credit spread curves of firms A_i and the date-0 firm B credit spread curve are initialized to their observable values. The forward credit spread of firm B is driven by a continues diffusive term, $dz_B(t)$, where $z_B(t) = (z_{B_1}(t), \ldots, z_{B_n}(t))'$ is an n-dimensional standard Wiener process with

$$E(dz_B(t)dz_B'(t)) = I_{n\times n}dt.$$

It is correlated with the diffusive riskless term according to $E(dz_f(t)dz_B'(t)) = \Sigma^B_{m\times n}dt$, where

$$(\Sigma^B)_{ij} = \rho^B_{ij}. \qquad (27)$$

In addition, it is correlated with firm A's diffusive term according to $E(dz_A(t)dz_B'(t)) = \Sigma^{AB}_{n\times n}dt$, where $(\Sigma^{AB})_{ij} = \rho^{AB}_{ij}$. When there is a jump in the riskless curve, then there is a corresponding jump in the credit spread curve of firm B. Further, default of any of the primary firms A_i could transmit to shocks in the credit spreads of the secondary firm. As before, the volatility factor, $\sigma_B(t,T)$, is predictable, while $c_{fB}(t,T)$ and $c_{A_iB}(t,T)$ are deterministic functions of time to maturity, $T-t$. More specifically, we impose volatility structures $\sigma_B(t,T) = (\sigma_{B_1}(t,T), \ldots, \sigma_{B_n}(t,T))$ of the form

$$\sigma_{B_j}(t,T) = h_{B_j}(t)e^{-\kappa_{B_j}(T-t)}, \qquad (28)$$

and jump impact factors given by

$$c_{fB}(t,T) = c_{fB}e^{-\gamma_{fB}(T-t)}, \qquad (29)$$

where $h_{B_j}(t)$ is a predictable function that depends on a set of state variables.

Repeating the same steps as before, we obtain the dynamics for $V_B(t,T)$:

$$\begin{aligned} \frac{dV_B(t,T)}{V_B(t,T)} &= \mu_{V_B}(t,T)dt - \sigma_p(t,T)dz_f(t) - \sigma_{S_B}(t,T)dz_B(t) \\ &\quad + (e^{-K_p(t,T)-K_{fB}(t,T)} - 1)dN_f(t) \\ &\quad + \sum_{i=1}^{m_B}(e^{-K_{A_iB}(t,T)} - 1)(1 - Y_{A_i}(t))dY_{A_i}(t), \qquad (30) \end{aligned}$$

where $\sigma_{S_B}(t,T)$ and $K_{fB}(t,T)$ are defined analogous to (13) and (14), and where

$$K_{A_iB}(t,T) = \int_t^T c_{A_iB}(t,u)du. \qquad (31)$$

We have

$$\mu_{V_B}(t,T) = \lambda_B(t) - \int_t^T \mu_B(t,u)du + \frac{1}{2}\sigma_{S_B}(t,T)\sigma'_{S_B}(t,T) + r(t) - (e^{-K_p(t,T)} - 1)\eta_f + \sigma_p(t,T)\Sigma^B\sigma'_{S_B}(t,T).$$

To avoid riskless arbitrage opportunities, the instantaneous expected return under the risk neutral measure is $(r(t)+\lambda_B(t))dt$. Substituting this constraint into the above equation, and following the same steps as for primary bonds yields:

$$\mu_B(t,T) = \sigma_{S_B}(t,T)\sigma'_B(t,T) + \sigma_f(t,T)\Sigma^B\sigma'_{S_B}(t,T) + \sigma_p(t,T)\Sigma^B\sigma_B(t,T) + g_B(t,T) - \sum_{i=1}^{m_B}\eta_{A_i}(t)c_{A_iB}(t,T)e^{-K_{A_iB}(t,T)}(1-Y_{A_i}(t)), \quad (32)$$

where

$$g_B(t,T) = \eta_f\left(c_f(t,T)e^{-K_p(t,T)} - (c_f(t,T)+c_{fB}(t,T))e^{-K_p(t,T)-K_{fB}(t,T)}\right).$$

Following the same logic as used to develop the price of a primary bond, the date-t credit spread of a secondary bond is linked to its date-0 value by:

$$\lambda_B(t,T) = \lambda_B(0,T) + \int_0^t \mu_B(u,T)du + \int_0^t \sigma_B(u,T)dz_B(u) + \int_0^t c_{fB}(u,T)dN_f(u) + \sum_{i=1}^{m_B}\int_0^t c_{A_iB}(u,T)(1-Y_{A_i}(u))dY_{A_i}(u). \quad (33)$$

Assume that the impact of a primary firm's default on firm B is a constant, $c_{A_iB}(t,T) = c_{A_iB}$. Then $\sum_{i=1}^{m_B}\eta_{A_i}(t)c_{A_iB}(t,T)e^{-K_{A_iB}(t,T)}(1-Y_{A_i}(t))$ in equation (32) and $\sum_{i=1}^{m_B}\int_0^t c_{A_iB}(u,T)(1-Y_{A_i}(u))dY_{A_i}(u)$ in (33) simplify to $\sum_{i=1}^{m_B}\eta_{A_i}(t)c_{A_iB}e^{-c_{A_iB}(T-t)}(1-Y_{A_i}(t))$ and $\sum_{i=1}^{m_B}c_{A_iB}Y_{A_i}(t)$, respectively.

Proposition 3 below, now provides a tidy representation linking date-t credit spreads for a secondary firm to its date-0 credit spread curve, through a finite collection of state variables. Specifically, we have:

Proposition 3

Given the risk-neutral dynamics (2), (25) and (26) together with the volatility and impact structures specified in (17) through (20), (28) and (29), the price of a defaultable zero-coupon bond issued by a secondary firm at date t, $\Pi_B(t,T)$, is given by $\Pi_B(t,T) = V_B(t,T)1_{\tau_B>t}$, where $V_B(t,T) = P(t,T)S_B(t,T)$ and

$$S_B(t,T) = \frac{S_B(0,T)}{S_B(0,t)}e^{-B_0(t,T)-\sum_{j=1}^{n}(K^B_{0,j}(t,T)\xi^B_{0,j}-K^B_{1,j}(t,T)\xi^B_{1,j})} \times e^{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(K^B_{2,ij}(t,T)\xi^B_{2,ij}-K^B_{3,ij}(t,T)\xi^B_{3,ij}-K^B_{5,ij}(t,T)\xi^B_{5,ij}\right)} \times e^{-K^B_4(t,T)\xi^B_4(t)} \times e^{\sum_{j=1}^{m_B}\left(\left(1-e^{-c_{A_jB}(T-t)}\right)U_{A_jB}(t)-c_{A_jB}(T-t)Y_{A_j}(t)\right)}. \quad (34)$$

Here, $B_0(t,T) = \int_t^T G_B(t,u)du$, where $G_B(t,u) = \int_0^t g_B(v,u)dv$. The K^B variables and the ξ^B state variables are defined as in Proposition 2, but now with respect to the secondary bond, B, rather than the primary bond, A, and where

$$U_{A_jB}(t) = \int_0^{t\wedge\tau_{A_j}} \eta_{A_j}(u)e^{-c_{A_jB}(t-u)}du, \quad \textit{for } j = 1,\ldots,m_B. \tag{35}$$

Proof See Appendix A.

Unlike primary bonds, the prices of secondary bonds depend on the status of primary bonds as well as the timing of primary bond defaults. Specifically, the credit-spread curves of secondary firms depend on the collections of state variables that also include all the primary firms' state variables. Computing $U_{A_jB}(t)$ requires knowledge of the risk-neutral probability $\eta_{A_j}(t)$ which in turn requires knowledge of the state variables for the primary firm j.

The actual linkage of credit spreads at date t, for a surviving secondary firm, is connected to date 0 information by:

$$\begin{aligned}
\lambda_B(t,T) \;=\;& \lambda_B(0,T) + G_B(t,T) + \sum_{j=1}^{n}\frac{1}{\kappa_{B_j}}e^{-\kappa_{B_j}(T-t)}\xi^B_{0,j}(t) - \sum_{j=1}^{n}\frac{1}{\kappa_{B_j}}e^{-2\kappa_{B_j}(T-t)}\xi^B_{1,j}(t)\\
&- \sum_{i=1}^{m}\sum_{j=1}^{n}\rho^B_{ij}\left(\frac{1}{\kappa_{f_i}}+\frac{1}{\kappa_{B_j}}\right)e^{-(\kappa_{B_j}+\kappa_{f_i})(T-t)}\xi^B_{2,ij}(t)\\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\rho^B_{ij}}{\kappa_{B_j}}e^{-\kappa_{f_i}(T-t)}\xi^B_{3,ij}(t) + c_{fB}e^{-\gamma_{fB}(T-t)}\xi^B_4(t) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\rho^B_{ij}}{\kappa_{f_i}}e^{-\kappa_{B_j}(T-t)}\xi^B_{5,ij}(t)\\
&+ \sum_{i=1}^{m_B}c_{A_iB}Y_{A_i}(t) - \sum_{i=1}^{m_B}c_{A_iB}e^{-c_{A_iB}(T-t)}U_{A_iB}(t),
\end{aligned} \tag{36}$$

Setting $T = t$ in (36) yields the spot credit spread at date t:

$$\begin{aligned}
\lambda_B(t) \;=\;& \lambda_B(0,t) + G_B(t,t) + \sum_{j=1}^{n}\frac{1}{\kappa_{B_j}}(\xi^B_{0,j}(t) - \xi^B_{1,j}(t))\\
&- \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\rho^B_{ij}\left(\frac{1}{\kappa_{f_i}}+\frac{1}{\kappa_{B_j}}\right)\xi^B_{2,ij}(t) - \frac{\rho^B_{i,j}}{\kappa_{B_j}}\xi^B_{3,ij}(t) - \frac{\rho^B_{i,j}}{\kappa_{f_i}}\xi^B_{5,ij}(t)\right) + c_{fB}\xi^B_4(t)\\
&+ \sum_{i=1}^{m_B}c_{A_iB}(Y_{A_i}(t) - U_{A_iB}(t)).
\end{aligned} \tag{37}$$

Once $\lambda_B(t)$ is known, and given loss given default, the risk neutral probability of default, $\eta_B(t)$ can be computed.

Of special interest is the case where $m = n = 1$. For this scenario, we drop the subscripts $(i,j) = (1,1)$. Using the above expression, we can represent $\xi^B_{0,1}(t)$ in terms of the other variables.

Substituting the expression into equation (36) and rearranging, leads to:

$$\begin{aligned}
\lambda_B(t,T) &= \lambda_B(0,T) + G_B(t,T) + e^{-\kappa_B(T-t)}\left(\lambda_B(t) - \lambda_B(0,t) - G_B(t,t)\right) \\
&+ \frac{1}{\kappa_B}e^{-\kappa_f(T-t)}(1-e^{-\kappa_B(T-t)})\xi_1^B(t) + \rho^B(\frac{1}{\kappa_B}+\frac{1}{\kappa_f})e^{-\kappa_B(T-t)}(1-e^{-\kappa_f(T-t)})\xi_2^B(t) \\
&+ \frac{\rho^B}{\kappa_B}(e^{-\kappa_f(T-t)} - e^{-\kappa_B(T-t)})\xi_3^B(t) + c_{fB}(e^{-\gamma_{fB}(T-t)} - e^{-\kappa_B(T-t)})\xi_4^B(t) \\
&+ \sum_{i=1}^{m_B} c_{A_iB}(1-e^{-\kappa_B(T-t)})Y_{A_i}(t) + \sum_{i=1}^{m_B} c_{A_iB}(e^{-\kappa_B(T-t)} - e^{-cA_iB(T-t)})U_{A_iB}(t). \qquad (38)
\end{aligned}$$

The valuation formula for risky debt issued by a secondary firm then follows directly by substituting the credit spread and forward rate expressions into equation (38), and simplifying. Specifically, $\Pi_B(t,T) = V_B(t,T)1_{\tau_B>t}$, where $V_B(t,T) = P(t,T)S_B(t,T)$ and

$$\begin{aligned}
S_B(t,T) &= \frac{S_B(0,T)}{S_B(0,t)}e^{-\left(A_0^B(t,T)+K_0^B(t,T)\lambda_B(t)+\sum_{j=1}^4 K_j^B(t,T)\xi_j^B(t)\right)} \\
&\times e^{-(T-t)\sum_{i=1}^{m_B} c_{A_iB}Y_{A_i}(t)+\frac{1}{\kappa_B}(1-e^{-\kappa_B(T-t)})\sum_{i=1}^{m_B} c_{A_iB}Y_{A_i}(t)} \\
&\times e^{-\sum_{i=1}^{m_B}\left(\frac{1}{\kappa_B}(1-e^{-\kappa_B(T-t)})-\frac{1}{c_{A_iB}}(1-e^{-c_{A_iB}(T-t)})\right)c_{A_iB}U_{A_iB}(t)}. \qquad (39)
\end{aligned}$$

Here, bond prices are Markovian, although not in eight state variables as is the case for primary bonds but in $8 + 2m_B$ state variables. Note, too, that in order for the model to hold, the default of any primary bond cannot lead to a default of the secondary bond. If such a situation is possible, then the no-jump condition that is necessary for discounting to proceed at the rate $r(t) + \lambda_B(t)$ is violated.[13]

4 Model Implementation

In this section, we implement our model to price an array of single-name and multi-name credit-sensitive products. Our primary goal is to illustrate the importance of incorporating the interaction between credit spreads and interest rates into our model structure; allowing for a greater clustering of defaults by including credit-spread impact factors; and taking into account the full credit spread curve information, and the distribution of these curves across firms. We discuss these in order.

4.1 The Importance of Interest Rate-Credit Spread Correlations

Our first example focuses on pricing a traditional derivatives contract, namely an option on a defaultable bond, using the two-facto, eight state variable model introduced in the Corollary in

[13]In such cases, a change of measure is needed in order to price defaultable bonds, as discussed in Collin-Dufresne, Goldstein and Hugonnier (2004).

Section 2. To underline the validity of our findings, we also examine the price sensitivity of innovative products that are more recent to the credit-derivatives landscape. In particular, our second example illustrates how the price of insurance against counterparty credit risk changes as the diffusive correlation between credit spreads and interest rates changes.

4.1.1 Bond Options

We simulate the price of an European call option on a five-year zero-coupon bond issued by some primary firm A, with an exercise date in three years. Figure 4 shows the percentage change in the value of the at-the-money option as the correlation between the riskless and the risky diffusive terms, ρ^A, moves away from 0. As ρ^A becomes negative, as is the usual case in U.S. corporate bond markets (see, for example, Duffee (1999)), the price of the at-the-money option decreases. As ρ^A decreases from 0 to -0.9, option prices decline dramatically by almost 40 percent. The owner of the call option profits from low interest rates and low credit spreads at expiration, which implies that the value of the call option increases as ρ^A increases. Note that the sensitivity to ρ^A diminishes rather rapidly as the option moves into the money.

Figure 4 Here

In Table 2, we compare the sensitivity of bond option prices to ρ^A to the sensitivity of the option prices to the jump intensity, η_f and to the impact factor, c_{fA}. We find that changes in η_f have almost no impact on bond option prices, as long as the jumps impact interest rates and credit spreads with opposite signs. And although we detect a small upward trend in option prices as c_{fA} moves away from zero (due to the increase in volatility in future spreads), the sensitivity is substantially smaller than that for ρ^A. The fact that among the parameters that capture the interaction between interest rates and credit spreads only ρ^A allows us to generate a wide range of option prices highlights the importance of allowing for diffusive interest rate-credit spread correlations when calibrating term-structure models to data.

Table 2 Here

4.1.2 Contingent Credit Default Swaps

Counterparty credit risk is one of the fastest growing asset classes on Wall Street (see Terán (2007)), and therefore the focus of our second application. Specifically, we value contingent credit default swaps (CCDS). A CCDS is a credit default swap (CDS) whose notional is linked to the present value of an over-the-counter (OTC) derivative. It provides protection on the OTC derivative by ensuring that the instrument will be fully replaced upon the default of the counterparty. In other words, it is an OTC derivative instrument with a knock-in feature, which is ignited upon the default of the counterparty of the underlying derivative.

The fact that a growing proportion of banks' risk exposures are neither investment grade nor collateralized, together with the fact that their OTC business in derivatives has expanded, has increased counterparty credit risk. While some of this risk is diversifiable, there is a notion that since many banks are conducting similar strategies, the correlations among counterparties may have tightened, exacerbating volatilities and reducing diversification benefits. At the same time, due to adoption of fair-value accounting for credit derivative contracts, where counterparty exposures are being marked to market, banks that do not hedge counterparty risk are faced with fluctuating values. As a result insuring counterparty credit risk has become an important activity, and CCDS contracts have become more prevalent.

Derivatives typically referenced by CCDS are plain vanilla interest-rate swaps. For our numerical implementation, we assume that firm X enters a five-year floating-for-fixed interest-rate swap with counterparty Y_A, a primary firm in the sense of Section 2. If X is concerned that firm Y_A will default before maturity of the swap contract, X can buy counterparty-risk insurance from firm Z using the CCDS market. A five-year CCDS contract stipulates that if firm Y_A defaults within the next five years, the protection seller Z pays to the protection buyer X the market value of the floating-for-fixed swap, as long as it is positive at the time of default. In return, X pays Z a quarterly insurance premium until the end of the five-year term or until default of firm Y_A, whichever occurs first. In what follows, we ignore any potential default risk associated with X or Z. Appendix B describes how to compute at-market CCDS rates.

Figure 4 shows the percentage changes in CCDS rates as the correlation between the riskless and the risky diffusive terms, ρ^A, moves away from 0. A negative correlation between the riskless term structure and the credit curve implies that at times when default risk is high (and hence the CCDS is likely to trigger payment), interest rates are low, and the value of the swap to X is high. Conversely, if the correlation is positive, then at times when default risk is high, interest rates are high, and the value of the swap to A is low, possibly negative. As a result, the CCDS rate increases as correlation decreases. Figure 4 shows that the relative effect of ρ^A on CCDS rates is of similar magnitude as the effect on the value of the at-the-money bond option described in Section 4.1.1.

Table 3 reports simulation results when η_f or c_{fA} move away from zero. Although we detect a small upward trend in CCDS rates, in both scenarios the sensitivity is substantially smaller than for ρ^A, making the latter the only correlation parameter that allows us to generate a wide range of CCDS rates.

Table 3 Here

In summary, our two examples stress the fact that a number of important credit-sensitive products are highly sensitive to the diffusive correlation between interest rates and credit spreads. A unified framework that prices all kinds of credit derivatives therefore needs to be flexible enough to allow for interest rate-credit spread correlations.

4.2 The Importance of Default Clustering

In the section, we shift our focus to multi-name products and highlight the importance of incorporating default correlations into our model structure. In our first example, we revisit the pricing of a CCDS contract, but now explicitly take into account the default risk of the protection seller Z. We show that if default of the protection seller negatively impacts the credit risk of the counterparty in the underlying swap contract, insurance premia rise dramatically. In our second application, we price tranches of CDS indices, effectively extending the two-firm setting of the first example to a much larger portfolio of 125 credits.

4.2.1 Counterparty Risk in Insurance Contracts

Revisiting the CCDS example of the previous section, we now explicitly take into account the default risk of the protection seller Z and simulate the price of the counterparty risk associated with the CCDS contract itself. This source of counterparty risk borne by the protection buyer, X, should not be confused with the counterparty risk in the underlying swap contract. Instead, it is the risk associated with default of the protection seller, Z, in the insurance contract prior to default of the counterparty in the underlying interest-rate swap contract.

To better mimic reality, we now assume that the protection seller is a primary firm of good credit quality, denoted by Z_A, and that the counterparty to the underlying swap contract, Y_B, is a riskier secondary firm. The value V of insurance against default of the protection seller in the CCDS contract is computed as the value of the underlying floating-for-fixed interest-rate swap (if positive) at the time of default of firm Y_B, given that firm Z_A defaulted prior to Y_B. V is computed as of time zero. Figure 5 shows the percentage changes in V in response to changes in ρ^A and ρ^B, ρ^{AB}, c_{AB} and η_f, whereas Table 4 reports the associated estimates and standard deviations for V. We obtain striking results with regard to the credit-spread impact factor c_{AB}. If the impact of default of the protection seller Z_A on the credit spread of firm Y_B is increased from zero to a jump in instantaneous credit spreads of 0.1, V increases by almost 90 percent. An instantaneous credit spread of 0.1 or higher translates into a 80 basis points or higher likelihood of default within the next month. As c_{AB} increases to 1 or higher, that number increases to 8 percent or higher. The appropriate size of c_{AB} ultimately depends on the closeness of the relationship between the primary firm (Z_A) and the secondary firm (Y_B). For example, c_{AB} might be higher if both firms belong to the same sector, and it might be lower if they are close competitors. As before, we also find a significant negative relationship between the prices of insurance against default of the protection seller in the CCDS contract and diffusive correlations between interest rates and credit spreads, as measured by ρ^A and ρ^B. The impact of ρ^{AB} and η_f is comparatively small.

Figure 5 Here

Table 4 Here

4.2.2 CDS Index Tranches

To further emphasize the effectiveness of the credit-spread impact factor c_{AB} in generating a wide range of prices for multi-name credit derivatives, we extend the two-firm setting of the previous section to a much larger portfolio of credits. Specifically, we now investigative the sensitivity of CDS index tranche prices to c_{AB}. We mimic the setup of the five-year investment-grade CDX index (ticker CDX.NA.IG; for details see www.markit.com) by considering a portfolio of 125 five-year investment-grade CDS contracts, and pricing tranches on its loss distribution. CDX.NA.IG indices are sliced into five tranches: the equity tranche that incurs the first 0-3% of losses, two mezzanine tranches that aggregated are responsible for the subsequent 3-10% of losses, and two senior tranches that together account for the next 10-30% of losses.

Figure 6 shows simulated tranche prices as a function of c_{AB}. As the credit spread impact factor increases from 0 to 0.1, senior tranche prices increase from 0 to 349 basis points due to an increase in the likelihood that more than 10% of the firms in the portfolio default. At the same time, the value of the junior tranche decreases dramatically from 32% to 15%. The decrease in value is due to the fact that the payments by the junior tranche investor, by a given quarter, are negatively related to the value of an European put option with strike price equal to the upper attachment point of the junior tranche. Since an increase in c_{AB} yields a higher volatility in the performance of the underlying collateral pool, and hence a higher option value, it ultimately lowers the value of the junior tranche. As in Section 4.2.1, we also analyze the tranche price sensitivity to ρ^A and ρ^B, ρ^{AB}, and η_f. The results (not reported) indicate that these alternative correlation parameters have a significantly smaller impact, if any, on CDS index tranche prices.

Figure 6 Here

In summary, our results highlight the importance of including credit-spread impact factors in our models that permit jumps to occur in the intensities of secondary bonds when a primary bond defaults. The senior tranche investor in a CDS index has to make payments only if a relatively large number of defaults occur within a rather short time frame. Similarly, the protection seller in the CCDS contract fails to make a promised payment only in the scenario where it defaults before the counterparty in the underlying swap contract, that is, in cases where the default of Z_A is followed within a rather short time period by a default of Y_B. To further emphasize the effectiveness of the credit-spread impact factor c_{AB} in generating such default clustering, Figure 7 shows sample paths for the timing of default events over a five-year period. Each sample path is associated with a particular value for c_{AB}, while everything else is kept the same. The figure indicates that for higher credit-spread impact factors, defaults are more likely to occur in clusters, and that they are spread out more evenly across time if no feedback effects from defaults of primary firms are allowed.

Figure 7 Here

4.3 The Importance of the Initial Credit Spread Curve Distribution

We conclude this section by investigating how sensitive multi-name products are to the distribution, across firms, of initial credit spread curves. Our goal is to demonstrate the importance of taking into account the full credit spread curve information for each firm, and their distribution across firms. The latter should be of particular concern when pricing multi-name credit derivatives.

Continuing with our CDX example from Section 4.2.2, Table 5 shows the simulated tranche prices of a five-year CDX.NA.IG index for different distributions of initial credit spread curves. In particular, we consider two simplified scenarios: we first investigate the case where the initial credit spread curve is flat for all firms, but possibly at different levels. The credit-spread level for each firm is simulated from a uniform distribution that is centered around 100 basis points, say. Second, we simulate tranche prices under the assumption that the initial credit spread curve is a linear function of time, with a fixed point of 100 basis points at 2.5 years (half the contract term).

Our results indicate that the junior and the mezzanine tranche prices increase significantly as the probability of higher initial forward credit spreads at the short end of the term structure increases. (For the chosen model specification, movements in senior tranche spreads are insignificant.) In the first example, the junior (mezzanine) tranche price increases by 5% (7%) when moving from a flat initial credit spread curve at 100 basis points for all firms to a flat initial credit spread curve at a level that is uniformly distributed between 50 and 150 basis points.

The effect of changes in the slope of the initial credit spread curves is even more dramatic. For our second set of distributions, the results displayed in the bottom panel of Table 5 show that the junior (mezzanine) tranche price increases by 45% (15%) as the initial spread curve, for each firm, is tilted from being flat at 100 basis points to being downward sloping from 175 basis points (at 0-year maturity) to 25 basis points (at 5-year maturity). Similarly, junior and mezzanine tranche price increases dramatically as initial spread curves are tilted from being flat to being upward sloping.

5 Conclusion

The market for credit derivatives on individual names and on portfolios of names has increased dramatically over the last several years. Pricing credit derivatives relative to given term structures of interest rates and credit spreads is very important. To accomplish this, it is often the case that researchers adopt models where interest rates are uncorrelated with credit spreads. While this typically does lead to simplifications, it can be the source of large errors. A common valuation approach is to use the HJM paradigm that permits full information on the current term structures to be incorporated into the model. However, without curtailing the structure of volatilities and correlations, this paradigm leads to massive path dependence in pricing and hedging. Our contribution here has been to curtail volatility structures in such a way that the path dependence can be readily captured by a finite set of state variables.

In particular, we extend the HJM Markovian models of riskless bonds to Markovian models of risky bonds. First, we establish a multi-factor Markovian model for the case where interest rates are driven by m stochastic drivers and forward credit spreads are driven by n correlated stochastic drivers with jumps affecting interest rates and credit spreads. Analytical expressions for both risky and riskless term structures were derived in terms of $2(m + n + 1) + 3mn$ state variables. The resulting models have desirable properties. In particular, the volatility restrictions are not that severe. They allow the initial term structures to take on shapes consistent with the data, and they allow for levels to fluctuate with the levels of state variables that include a set of forward rates. The importance of correlation among spreads and riskless rates was highlighted by pricing options on risky bonds and contingent credit default swaps, where volatilities were level dependent and correlation could easily be adjusted between -1 to $+1$.

This paper also extended the analysis to consider multiple bonds, where default of any primary bond could impact the entire credit spread curve of others. To illustrate the model we considered the pricing of counterparty credit risk in contracts which in turn insure against the possible non-performance of derivatives. The feasibility of our models was also illustrated by considering the valuation of tranches of CDS indices that are comprised of 125 names. Such an analysis would not be possible for general HJM models; but with a relatively small collection of state variables, our models can easily be implemented to price the appropriate tranches. Interestingly, since all 125 term structures of credit spreads can be matched, our models incorporate more information than most in pricing CDS index tranches and allows us to explore more precisely the impact of heterogeneity in the composition of the index.

Finally, our model also permits clustering of defaults to occur through a variety of channels including correlations, jumps and impact factors. This is an important aspect that allows stress tests to be conducted on portfolios containing risky debt or credit default swaps, and allows counterparty credit risk to be assessed. It remains for future empirical work to identify simple parsimonious volatility structures for forward riskless and risky rates within our large family, that jointly capture the primary dynamics in interest and credit markets.

Appendices

A Proofs

Proof of Proposition 1

Under the risk-neutral measure, the expected instantaneous return of the riskless bond should equal the riskless rate, $r(t)$. Absence of arbitrage therefore requires the expected instantaneous return in equation (4) to be equal to $r(t)dt$. Hence:

$$r(t) + \frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) - \int_t^T \mu_f(t,u)du + (e^{-K_p(t,T)} - 1)\eta_f = r(t),$$

from which:

$$\int_t^T \mu_f(t,u)du = \frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) + (e^{-K_p(t,T)} - 1)\eta_f.$$

Differentiating with respect to T, we obtain:

$$\mu_f(t,T) = \sigma_p(t,T)\sigma_f'(t,T) - c_f(t,T)e^{-K_p(t,T)}\eta_f.$$

This restriction on the drift term is the classic Heath-Jarrow-Morton restriction for forward rates under the risk-neutral measure when the dynamics follow a jump diffusion of the form in equation (2).

Now turn to the risky bond. Under our assumptions, the dynamics for $V_A(t,T)$ are:

$$\begin{aligned}\frac{dV_A(t,T)}{V_A(t,T)} &= \mu_{V_A}(t,T)dt - \sigma_p(t,T)dz_f(t) - \sigma_{S_A}(t,T)dz_A(t) \\ &\quad + (e^{-K_p(t,T)-K_{fA}(t,T)} - 1)dN_f(t),\end{aligned}$$

where

$$\begin{aligned}\mu_{V_A}(t,T) &= \lambda_A(t) - \int_t^T \mu_A(t,u)du + \frac{1}{2}\sigma_{S_A}(t,T)\sigma_{S_A}'(t,T) + r(t) \\ &\quad - (e^{-K_p(t,T)} - 1)\eta_f + \sigma_p(t,T)\Sigma\sigma_{S_A}'(t,T).\end{aligned}$$

If we assume that at the time of default the recovery value is proportional to the market value of the bond just prior to default, absence of arbitrage implies that we can price the risky bond as an expectation, under a risk-neutral measure Q, of discounted cash flows. The discount rate is modified from r to $r + \lambda_A$ to reflect default risk (see Duffie and Singleton (1999b)). Given that

$$\begin{aligned}E^Q\left(\frac{dV_A(t,T)}{V_A(t,T)}\right) &= (r(t) + \lambda_A(t))dt \\ &= \left(\mu_{V_A}(t,T) + (e^{-K_p(t,T)-K_{fA}(t,T)} - 1)\eta_f\right)dt,\end{aligned}$$

under the risk-neutral measure we have:

$$\int_t^T \mu_A(t,u)du = \frac{1}{2}\sigma_{S_A}(t,T)\sigma'_{S_A}(t,T) + \sigma_p(t,T)\Sigma\sigma'_{S_A}(t,T) + (e^{-K_p(t,T)-K_{fA}(t,T)} - e^{-K_p(t,T)})\eta_f.$$

Differentiating this expression with respect to T yields:

$$\mu_A(t,T) = \sigma_{S_A}(t,T)\sigma'_A(t,T) + \sigma_f(t,T)\Sigma\sigma'_{S_A}(t,T) + \sigma_p(t,T)\Sigma\sigma'_A(t,T) + g_A(t,T),$$

where

$$g_A(t,T) = \eta_f\left(c_f(t,T)e^{-K_p(t,T)} - (c_f(t,T) + c_{fA}(t,T))e^{-(K_p(t,T)+K_{fA}(t,T))}\right).$$

Proof of Proposition 2

(i) First consider the riskless bond. Substituting the volatility and impact expressions into equation (15) and using (2) leads, upon simplification, to:

$$f(t,T) = f(0,T) + \sum_{j=1}^{m}\frac{1}{\kappa_{f_j}}e^{-\kappa_{f_j}(T-t)}\left(\psi_{1j}(t) - e^{-\kappa_{f_j}(T-t)}\psi_{2j}(t)\right) + c_f e^{-\gamma_f(T-t)}\psi_3(t) - c_f\eta_f L_f(t,T), \quad \text{(A.1)}$$

where

$$\psi_{1j}(t) = \int_0^t h^2_{f_j}(u)e^{-\kappa_{f_j}(t-u)}du + \kappa_{f_j}\int_0^t h_{f_j}(u)e^{-\kappa_{f_j}(t-u)}dz_{f_j}(u) \text{ for } j = 1,\ldots,m$$

$$\psi_{2j}(t) = \int_0^t h^2_{f_j}(u)e^{-2\kappa_{f_j}(t-u)}du \text{ for } j = 1,\ldots,m$$

$$\psi_3(t) = \int_0^t e^{-\gamma_f(t-u)}dN_f(u),$$

and

$$L_f(t,T) = \int_0^t e^{-\gamma_f(T-u)-\frac{c_f}{\gamma_f}(1-e^{-\gamma_f(T-u)})}du = \frac{e^{-\frac{c_f}{\gamma_f}}}{c_f}\left(e^{\frac{c_f}{\gamma_f}e^{-\gamma_f(T-t)}} - e^{\frac{c_f}{\gamma_f}e^{-\gamma_f T}}\right).$$

The result follows by substituting equation (A.1) into equation (1), and simplifying.

(ii) The forward credit spread, $\lambda_A(t,T)$, can be expressed as

$$\lambda_A(t,T) = \lambda_A(0,T) + \int_0^t \mu_A(u,T)du + \int_0^t \sigma_A(u,T)dz_A(u) + \int_0^t c_{fA}(u,T)dN_f(u)$$

Substituting the volatility restrictions given in equations (17), (19), (18) and (20), into the drift restriction, and using the above equation, we obtain, upon simplification:

$$\begin{aligned}
\lambda_A(t,T) &= \lambda_A(0,T) + G_A(t,T) + \sum_{j=1}^{n} \frac{1}{\kappa_{A_j}} e^{-\kappa_{A_j}(T-t)} \xi_{0,j}(t) - \sum_{j=1}^{n} \frac{1}{\kappa_{A_j}} e^{-2\kappa_{A_j}(T-t)} \xi_{1,j}(t) \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n} \rho_{ij}^{A} \left(\frac{1}{\kappa_{f_i}} + \frac{1}{\kappa_{A_j}} \right) e^{-(\kappa_{A_j}+\kappa_{f_i})(T-t)} \xi_{2,ij}(t) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\rho_{ij}^{A}}{\kappa_{A_j}} e^{-\kappa_{f_i}(T-t)} \xi_{3,ij}(t) \\
&\quad + c_{fA} e^{-\gamma_{fA}(T-t)} \xi_4(t) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\rho_{ij}^{A}}{\kappa_{f_i}} e^{-\kappa_{A_j}(T-t)} \xi_{5,ij}(t). \qquad \text{(A.2)}
\end{aligned}$$

Here,

$$\begin{aligned}
\xi_{0,j}(t) &= \int_0^t e^{-\kappa_{A_j}(t-u)} h_{A_j}^2(u)du + \kappa_{A_j} \int_0^t e^{-\kappa_{A_j}(t-u)} h_{A_j}(u) dz_{A_j}(u) \;\text{ for } j = 1,\ldots,n \\
\xi_{1,j}(t) &= \int_0^t e^{-2\kappa_{A_j}(t-u)} h_{A_j}^2(u)du \;\text{ for } j = 1,\ldots,n \\
\xi_{2,ij}(t) &= \int_0^t e^{-(\kappa_{A_j}+\kappa_{f_i})(t-u)} h_{A_j}(u) h_{f_i}(u) du \;\text{ for } j = 1,\ldots,n; i = 1,\ldots,m \\
\xi_{3,ij}(t) &= \int_0^t e^{-\kappa_{f_i}(t-u)} h_{A_j}(u) h_{f_i}(u) du \;\text{ for } j = 1,\ldots,n; i = 1,\ldots,m \\
\xi_4(t) &= \int_0^t e^{-\gamma_{fA}(t-u)} dN_f(u) \\
\xi_{5,ij}(t) &= \int_0^t e^{-\kappa_{A_j}(t-u)} h_{A_j}(u) h_{f_i}(u) du \;\text{ for } j = 1,\ldots,n; i = 1,\ldots,m
\end{aligned}$$

and $G_A(t,T) = \int_0^t g_A(u,T)du$. Note that if $\gamma_{fA} = \gamma_f$, analytic solutions are available for $G_A(t,T)$.

The result then follows by substituting the above equation into equation (9), and simplifying.

Proof of Corollary to Proposition 2

From equation (A.1), consider the case $m = 1$ and put T=t, to obtain

$$r(t) = f(0,t) + \frac{1}{\kappa_f}(\psi_1(t) - \psi_2(t)) + c_f \psi_3(t) - c_f \eta_f L_f(t,t),$$

where second subscripts on the ψ variables have been dropped. From this, an expression for $\psi_1(t)$ can be obtained. Substituting, it back into equation (A.1), yields:

$$\begin{aligned}
f(t,T) &= f(0,T) + e^{-\kappa_f(T-t)}(r(t) - f(0,t)) + \frac{1}{\kappa_f}(e^{-\kappa_f(T-t)} - e^{-2\kappa_f(T-t)})\psi_2(t) \\
&\quad + c_f(e^{-\gamma(T-t)} - e^{-\kappa_f(T-t)})\psi_3(t) + c_f \eta_f (e^{-\kappa_f(T-t)} L_f(t,t) - L_f(t,T)). \qquad \text{(A.3)}
\end{aligned}$$

from which the riskless bond price follows.

From equation (A.2) and for the case $n = 1$ put T=t. This allows us to express $\xi_{01}(t)$ in terms of the other variables. Substitute the resulting expression back into equation (A.2), dropping unnecessary subscripts and rearranging terms leads to

$$
\begin{aligned}
\lambda_A(t,T) &= \lambda_A(0,T) + G_A(t,T) + e^{-\kappa_A(T-t)}\left(\lambda_A(t) - \lambda_A(0,t) - G_A(t,t)\right) \\
&+ \frac{1}{\kappa_A} e^{-\kappa_A(T-t)}(1 - e^{-\kappa_A(T-t)})\xi_1(t) \\
&+ \rho^A \left(\frac{1}{\kappa_A} + \frac{1}{\kappa_f}\right) e^{-\kappa_A(T-t)}(1 - e^{-\kappa_f(T-t)})\xi_2(t) \\
&+ \frac{\rho^A}{\kappa_A}(e^{-\kappa_f(T-t)} - e^{-\kappa_A(T-t)})\xi_3(t) \\
&+ c_{fA}(e^{-\gamma_{fA}(T-t)} - e^{-\kappa_{A_1}(T-t)})\xi_4(t). \qquad \text{(A.4)}
\end{aligned}
$$

The risky bond price, $\Pi_A(t,T) = P(t,T)S_A(t,T)1_{\tau_A > t}$ can then be computed, since $S_A(t,T)$ follows by integrating the above equation.

Proof of Proposition 3

The proof follows by substituting the volatility and jump impact factor restrictions in (28) and (29) into the drift expression, and using the simplifications obtained from assuming constant impact factors to credit spread curves of secondary firms when primary firms default. This leads to the credit spread at date t, linked to the date 0 forward credit spread by the expression:

$$
\begin{aligned}
\lambda_B(t,T) &= \lambda_B(0,T) + G_B(t,T) + \sum_{j=1}^{n} \frac{1}{\kappa_{B_j}} e^{-\kappa_{B_j}(T-t)} \xi^B_{0,j}(t) - \sum_{j=1}^{n} \frac{1}{\kappa_{B_j}} e^{-2\kappa_{B_j}(T-t)} \xi^B_{1,j}(t) \\
&- \sum_{i=1}^{m}\sum_{j=1}^{n} \rho^B_{ij} \left(\frac{1}{\kappa_{f_i}} + \frac{1}{\kappa_{B_j}}\right) e^{-(\kappa_{B_j}+\kappa_{f_i})(T-t)} \xi^B_{2,ij}(t) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\rho^B_{ij}}{\kappa_{B_j}} e^{-\kappa_{f_i}(T-t)} \xi^B_{3,ij}(t) + c_{fB} e^{-\gamma_{fB}(T-t)} \xi^B_4(t) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\rho^B_{ij}}{\kappa_{f_i}} e^{-\kappa_{B_j}(T-t)} \xi^B_{5,ij}(t) \\
&+ \sum_{i=1}^{m_B} c_{A_iB} Y_{A_i}(t) - \sum_{i=1}^{m_B} c_{A_iB} e^{-c_{A_iB}(T-t)} U_{A_iB}(t), \qquad \text{(A.5)}
\end{aligned}
$$

where $G_B(t,T) = \int_0^t g_B(u,T)du$.

The result then follows by substituting the above equation into equation (1), and simplifying.

B Valuing Contingent Credit Default Swaps

Let us assume that firm X enters a T-year floating-for-fixed interest-rate swap with counterparty Y, and that X buys insurance against firm Y's default from firm Z in the CCDS market. The T-year CCDS contract stipulates that if firm Y defaults at or prior to T, the protection seller Z pays to the protection buyer X the market value of the floating-for-fixed swap, as long as it is positive at the time of default. In return, X pays Z a quarterly insurance premium until T or until default of firm Y, whichever occurs first. In what follows, we ignore any potential default risk associated with X or Z.

We assume that the interest-rate swap pays every six months, in arrears. Let $T_1 = 6$ months, $T_2 = 1$ year, etc. until $T_K = T$ years. The ex-coupon market value of the swap at some coupon date T_i is given by:

$$S \sum_{j=i+1}^{K} P(T_i, T_j) - (1 - P(T_i, T)),$$

where S denotes the at-market 6-month swap rate determined at time 0. At T_i, firms X and Y also pay interest in the amount of $1/P(T_{i-1}, T_i) - 1$ and S, respectively.

If firm Y defaults at time τ, $T_{i-1} < \tau < T_i$, then the time-τ market value of the interest-rate swap, $W(\tau)$, can be computed as:

$$W(\tau) \quad = \quad S \sum_{j=i}^{K} P(\tau, T_j) - (P(\tau, T_i) - P(\tau, T)) - P(\tau, T_i) \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right). \tag{B.1}$$

For sample path z, $z = 1, \ldots, Z$, the time-0 value of the protection leg of the CCDS is

$$V_{seller}^{(z)} = D^{(z)}(\tau) \max\{0, W^{(z)}(\tau)\}. \tag{B.2}$$

where $D^{(z)}(t) = e^{-\int_0^t r^{(z)}(u)\, du}$.

If the protection buyer pays in quarterly installments, the fair market value of the insurance payments can be computed as CV_{buyer}, where C is the annualized CCDS rate and

$$V_{buyer} \quad = \quad \frac{1}{4} \sum_{j=3,6,\ldots,4T} P(0, j/4) S_Y(0, j/4) + \text{Accrued Interest}. \tag{B.3}$$

The accrued interest component is also a result of the simulation step. It is estimated as $\frac{1}{Z} \sum_{z=1}^{Z} AI^{(z)}$, where

$$AI^{(z)} \quad \approx \quad D_{\tau^{(z)}}^{(z)} \left(\tau^{(z)} - \frac{\lfloor 4\tau^{(z)} \rfloor}{4} \right) 1_{\{\tau^{(z)} \leq T\}}. \tag{B.4}$$

and $\lfloor x \rfloor$ denotes the largest integer less than x.

Tables

Table 1: **Correlations Among Interest Rates and Credit Spreads** The top panel shows the correlation among riskless interest rates, among credit spreads of Time Warner (ticker TWX), and the correlations between credit spreads and interest rates. The second panel repeats the analysis for AT&T (ticker SBC). The third panel shows the correlations between the spreads of Time Warner and AT&T. All correlations significantly different from 0, at the 5% level of significance are starred. The data comes from Datastream over the period from July 2004 through the end of July 2008, and the correlations are based on weekly changes.

TWX							
		Riskless Rate			Credit Spread		
		1 year	5 year	10 year	1 year	5 year	10 year
	1 year	1	0.86*	0.69*	-0.13*	-0.14*	-0.22*
Riskless Rate	5 year		1	0.94*	-0.25*	-0.26*	-0.23*
	10 year			1	-0.08	-0.17*	-0.16*
	1 year				1	0.53*	0.59*
Credit Spread	5 year					1	0.88*
	10 year						1
SBC							
		Riskless Rate			Credit Spread		
		1 year	5 year	10 year	1 year	5 year	10 year
	1 year	1	0.86*	0.69*	-0.33*	-0.27*	-0.21*
Riskless Rate	5 year		1	0.94*	-0.30*	-0.29*	-0.28*
	10 year			1	-0.15*	-0.18*	-0.19*
	1 year				1	0.70*	0.53*
Credit Spread	5 year					1	0.80*
	10 year						1
Correlation between Spreads of TWX and SBC							
		TWX					
		1 year	5 year	10 year			
	1 year	0.36*	0.51*	0.40*			
SBC	5 year	0.30*	0.65*	0.56*			
	10 year	0.19*	0.56*	0.52*			

Table 2: **The Importance of Interest Rate-Credit Spread Correlations: Bond Options.** Simulation results for the prices C of an at-the-money European call option on a zero-coupon defaultable bond with a notional amount of \$100 and an exercise date in three years, under different parameter specifications. Standard errors are reported in parentheses. The benchmark set of parameter values is given by $\gamma_f = \gamma_{fA} = 0.5$, $c_f = -0.01$, $\kappa_f = \kappa_A = 0.2$, $h_f(t) = 0.03\sqrt{r(t)}$, $h_A(t) = 0.09\sqrt{\lambda_A(t)}$, and $\ell_A = 1$. We set $f(0,t) = 0.10$, and initialize the credit spread curve at $\lambda_A(0,0) = 0.05$, with monthly increases of 0.001. The first set of results assumes $\eta_f = 0$, the second set of results assumes $\rho^A = 0$ and $c_{fA} = 0.01$, and the third set of results assumes $\rho^A = 0$ and $\eta_f = 0.05$. We study contracts with a maturity of five years. Our Monte Carlo simulations use 100,000 sample paths with antithetic sampling.

ρ^A	C	η_f	C	c_{fA}	C
-0.9	0.568	0	0.914	0	0.902
	(0.003)		(0.005)		(0.005)
-0.7	0.661	0.01	0.913	0.01	0.913
	(0.003)		(0.005)		(0.005)
-0.5	0.741	0.02	0.912	0.02	0.927
	(0.004)		(0.005)		(0.005)
-0.3	0.813	0.03	0.913	0.03	0.945
	(0.004)		(0.005)		(0.005)
-0.1	0.882	0.04	0.913	0.04	0.963
	(0.004)		(0.005)		(0.005)
0	0.914	0.05	0.913	0.05	0.983
	(0.005)		(0.005)		(0.005)
0.1	0.947	0.06	0.913	0.06	1.005
	(0.005)		(0.005)		(0.005)
0.3	1.003	0.07	0.912	0.07	1.028
	(0.005)		(0.005)		(0.005)
0.5	1.060	0.08	0.913	0.08	1.051
	(0.005)		(0.005)		(0.005)
0.7	1.116	0.09	0.912	0.09	1.075
	(0.006)		(0.005)		(0.005)
0.9	1.167	0.1	0.913	0.1	1.104
	(0.006)		(0.005)		(0.005)

Table 3: **The Importance of Interest Rate-Credit Spread Correlations: CCDS.** Simulation results for at-market rates C in basis points of CCDSs on a floating-for-fixed interest-rate swap, under different parameter specifications. Standard errors are reported in parentheses. The benchmark set of parameter values is given by $\gamma_f = \gamma_{fA} = 0.5$, $c_f = -0.01$, $\kappa_f = \kappa_A = 0.2$, $h_f(t) = 0.03\sqrt{r(t)}$, $h_A(t) = 0.09\sqrt{\lambda_A(t)}$, and $\ell_A = 1$. We set $f(0,t) = 0.10$, and initialize the credit spread curve at $\lambda_A(0,0) = 0.05$, with monthly increases of 0.001. The first set of results assumes $\eta_f = 0$, the second set of results assumes $\rho^A = 0$ and $c_{fA} = 0.01$, and the third set of results assumes $\rho^A = 0$ and $\eta_f = 0.05$. We study contracts with a maturity of five years. Our Monte Carlo simulations use 100,000 sample paths with antithetic sampling.

ρ^A	C	η_f	C	c_{fA}	C
-0.9	7.77	0	5.81	0	5.84
	(0.07)		(0.06)		(0.06)
-0.7	7.32	0.01	5.83	0.01	5.88
	(0.07)		(0.06)		(0.06)
-0.5	6.89	0.02	5.85	0.02	5.91
	(0.06)		(0.06)		(0.06)
-0.3	6.44	0.03	5.86	0.03	5.95
	(0.06)		(0.06)		(0.06)
-0.1	6.03	0.04	5.87	0.04	5.98
	(0.06)		(0.06)		(0.06)
0	5.81	0.05	5.88	0.05	6.02
	(0.06)		(0.06)		(0.06)
0.1	5.60	0.06	5.89	0.06	6.05
	(0.06)		(0.06)		(0.06)
0.3	5.21	0.07	5.91	0.07	6.08
	(0.05)		(0.06)		(0.06)
0.5	4.79	0.08	5.92	0.08	6.10
	(0.05)		(0.06)		(0.06)
0.7	4.41	0.09	5.93	0.09	6.13
	(0.05)		(0.06)		(0.06)
0.9	4.05	0.1	5.94	0.1	6.15
	(0.05)		(0.06)		(0.06)

Table 4: **The Importance of Default Correlation: Counterparty Risk in Insurance Contracts.** Simulation results for values V of insurance against default of counterparty in CCDSs on a floating-for-fixed interest-rate swap, under different parameter specifications. The notional on the interest-rate swap is \$1mm. Standard errors are reported in parentheses. The benchmark set of parameters is given by $\rho^A = \rho^B = \rho^{AB} = 0$, $\eta_f = 0$, $\gamma_f = \gamma_{fA} = \gamma_{fA} = 0.5$, $c_f = -0.01$, $c_{fA} = c_{fB} = 0.01$, $c_{AB} = 0$, $\kappa_f = \kappa_A = \kappa_B = 0.2$, $h_f(t) = 0.03\sqrt{r(t)}$, $h_A(t) = 0.04\sqrt{\lambda_A(t)}$, $h_B(t) = 0.09\sqrt{\lambda_A(t)}$, and $\ell_A = \ell_B = 1$. We set $f(0,t) = 0.10$, $\lambda_A = 0.01$, and initialize the secondary credit spread curve at $\lambda_B(0,0) = 0.05$, with monthly increases of 0.001. As before, the maturity of the CCDS contract is set equal to five years and the simulations use 100,000 sample paths with antithetic sampling.

$\rho^A = \rho^B$	V	ρ^{AB}	V	c_{AB}	V	η_f	C
-0.9	72.61	-0.9	36.15	0	40.26	0	40.26
	(3.86)		(2.49)		(2.63)		(2.63)
-0.7	63.73	-0.7	39.36	0.01	43.03	0.01	40.78
	(3.58)		(2.62)		(2.68)		(2.67)
-0.5	55.48	-0.5	39.15	0.02	46.40	0.02	40.86
	(3.26)		(2.59)		(2.79)		(2.68)
-0.3	48.86	-0.3	38.99	0.03	50.12	0.03	40.78
	(2.98)		(2.58)		(2.90)		(2.68)
-0.1	42.03	-0.1	39.96	0.04	54.16	0.04	42.55
	(2.69)		(2.62)		(3.05)		(2.79)
0	40.26	0	40.26	0.05	58.53	0.05	42.95
	(2.63)		(2.63)		(3.22)		(2.81)
0.1	38.09	0.1	41.13	0.06	61.62	0.06	43.19
	(2.55)		(2.65)		(3.29)		(2.83)
0.3	33.71	0.3	41.10	0.07	65.27	0.07	43.61
	(2.35)		(2.65)		(3.42)		(2.84)
0.5	28.96	0.5	42.21	0.08	69.17	0.08	44.30
	(2.12)		(2.67)		(3.54)		(2.91)
0.7	25.64	0.7	42.15	0.09	73.06	0.09	44.24
	(1.98)		(2.66)		(3.67)		(2.94)
0.9	21.43	0.9	41.57	0.1	76.26	0.1	43.96
	(1.78)		(2.64)		(3.74)		(2.94)

Table 5: **The Importance of the Initial Credit Spread Curve Distribution** Simulated tranche prices of the five-year CDX.NA.IG index, for different distributions of initial credit spread curves across the firms in the index. The benchmark set of parameters is given by $\rho^A = \rho^B = \rho^{AB} = 0$, $\eta_f = 0$, $\kappa_f = \kappa_A = \kappa_B = 0.2$, $h_f(t) = 0.025\sqrt{r(t)}$, $h_A(t) = 0.05\sqrt{\lambda_A(t)}$, $h_B(t) = 0.05\sqrt{\lambda_B(t)}$, and $\ell_A = \ell_B = 1$. We set $f(0,t) = 0.05$ and $\lambda_A(0,t) = \lambda_B(0,t) = 0.01$. There are 25 primary firms and 100 secondary firms. At the time of default of a particular firm, the responsible tranche investors pay 60% of notional. Our Monte Carlo simulations use 10,000 sample paths with antithetic sampling.

Distribution of initial credit spread curves	tranche spreads (basis points)		
	junior	mezzanine	senior
$\lambda(0,t) = 0.01$	3,177.98	356.05	0.37
$\lambda(0,0) \sim$ Uniform$(0.005, 0.015)$ and $\lambda(0,t) = \lambda(0,0)$	3,265.57	369.42	0.43
$\lambda(0,0) \sim$ Uniform$(0, 0.02)$ and $\lambda(0,t) = \lambda(0,0)$	3,326.61	380.12	0.44
$\lambda(0,t)$ increases linearly from $\lambda(0,t) = 0.0025$ to $\lambda(0,t) = 0.0175$	2,204.16	328.02	0.31
$\lambda(0,t)$ increases linearly from $\lambda(0,t) = 0.005$ to $\lambda(0,t) = 0.015$	2,484.05	336.37	0.26
$\lambda(0,t)$ increases linearly from $\lambda(0,t) = 0.0075$ to $\lambda(0,t) = 0.0125$	2,811.99	345.43	0.30
$\lambda(0,t) = 0.01$	3,177.98	356.05	0.37
$\lambda(0,t)$ decreases linearly from $\lambda(0,t) = 0.0125$ to $\lambda(0,t) = 0.0075$	3,585.48	366.08	0.41
$\lambda(0,t)$ decreases linearly from $\lambda(0,t) = 0.015$ to $\lambda(0,t) = 0.005$	4,069.97	381.98	0.39
$\lambda(0,t)$ decreases linearly from $\lambda(0,t) = 0.0175$ to $\lambda(0,t) = 0.0025$	4,610.59	408.18	0.41

Figures

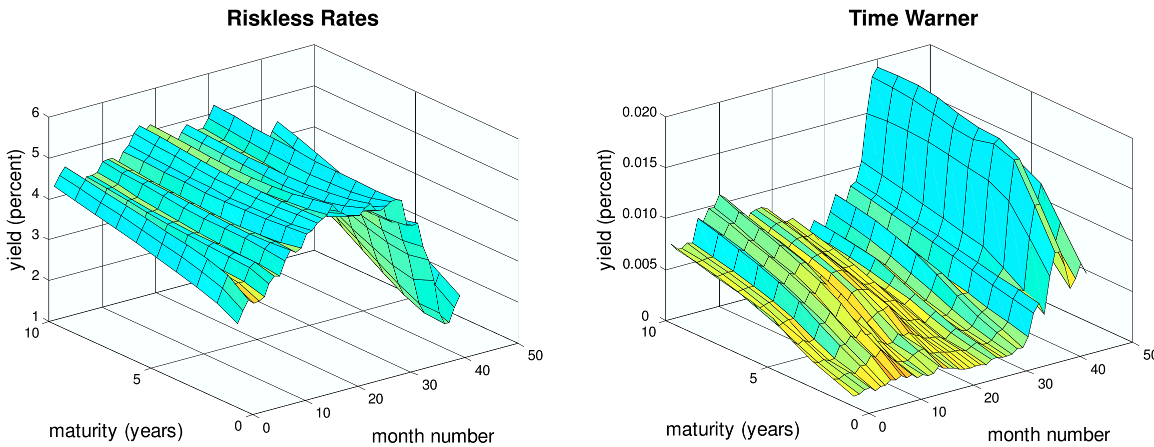

Figure 1: **Riskless Yield Curves and Risky Credit Spreads** The left panel shows the time series of riskless yields to maturity at each month from July 2004 to July 2008. The data for the analysis comes from the daily term structures provided by Gurkaynak, Sack and Wright (2006). The right panel shows the monthly time series of credit spreads for Time Warner. This data is provided by Datastream and also spans the time period from July 2004 to July 2008.

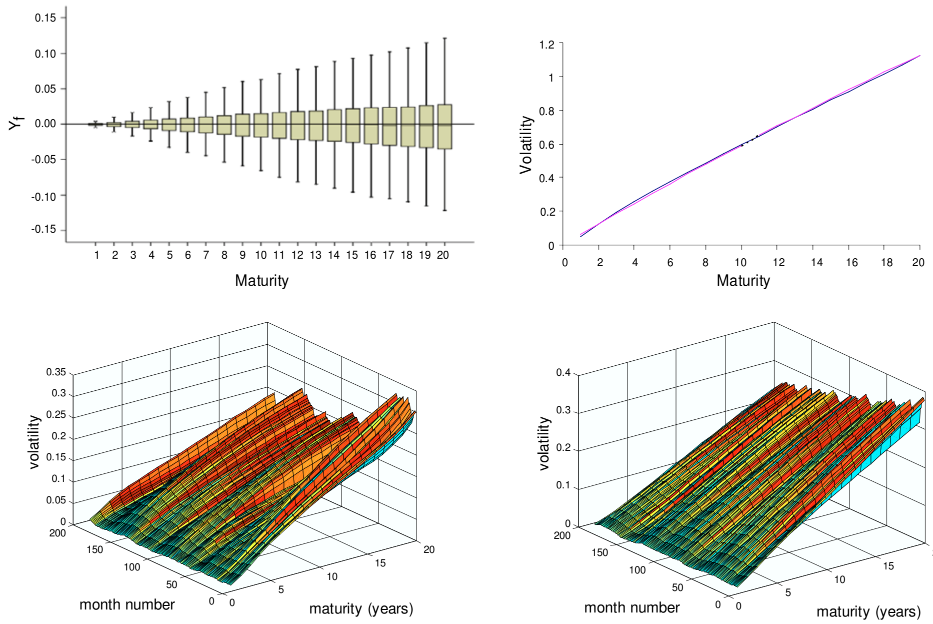

Figure 2: **The Term Structure of Riskless Volatilities** The top left panel shows the box whisker plots of the normalized daily changes in the logarithmic prices, $Y_f(t, m)$, for maturities m, ranging from 1 year through 20 years. The plots clearly reveal that their volatilities increase with maturity and that the means of the distributions are close to zero. The right graph shows the actual term structure of the volatilities of Y_f for each maturity, m, (solid line) and compares it to the almost identical fitted values (dashed line) obtained by maximum likelihood estimation. The bottom panel compares the historical term structures of volatilities of changes in the logarithm of bond prices of different maturities, where the volatilities are computed on a rolling basis over a historical time period of one year, to the predicted term structures of volatilities based on the one factor model. The data for the analysis comes from the daily term structures provided by Gurkaynak, Sack and Wright (2006) and spans the period from July 1981 to July 2008.

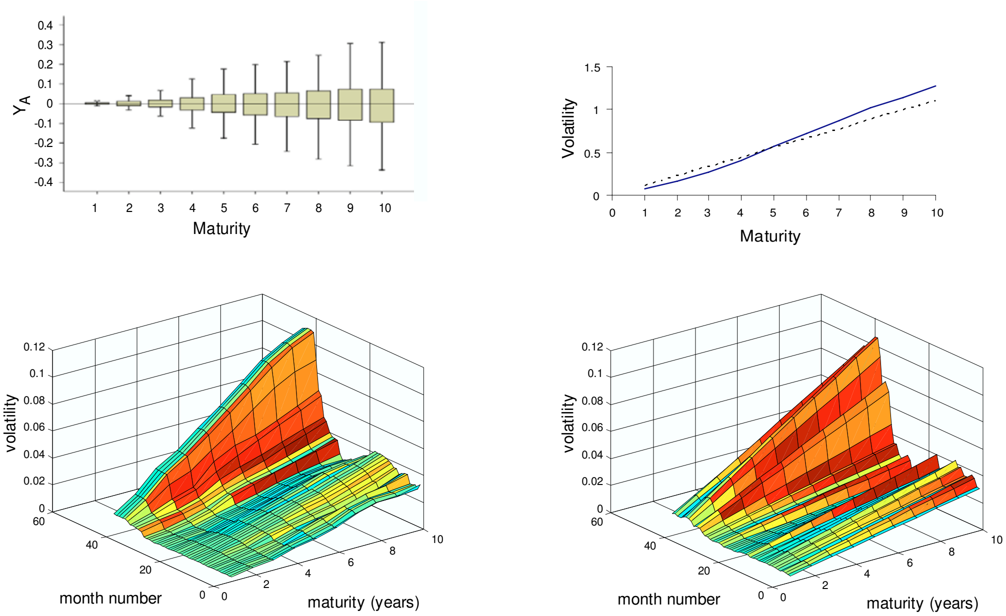

Figure 3: **The Term Structure of Credit Spread Volatilities** The top left panel shows the box whisker plots of the normalized daily changes in the logarithmic prices, $Y_A(t, m)$, for maturities m, ranging from 1 year through 20 years. The plots clearly reveal that their volatilities increase with maturity and that the means of the distributions are close to zero. The right panel shows the actual term structure of the volatilities of Y_A for each maturity, m, (solid line) and compares it to the fitted values (dashed lines) obtained by maximum likelihood estimation. The bottom panel compares the historical term structures of volatilities of changes in the logarithm of bond prices of different maturities, where the volatilities are computed on a rolling basis over a one year historical time period, to the predicted volatilities based on the one factor model. The data for the analysis comes from the weekly term structures of credit default swaps on Time Warner provided by Datastream over the period from July 2004 to July 2008.

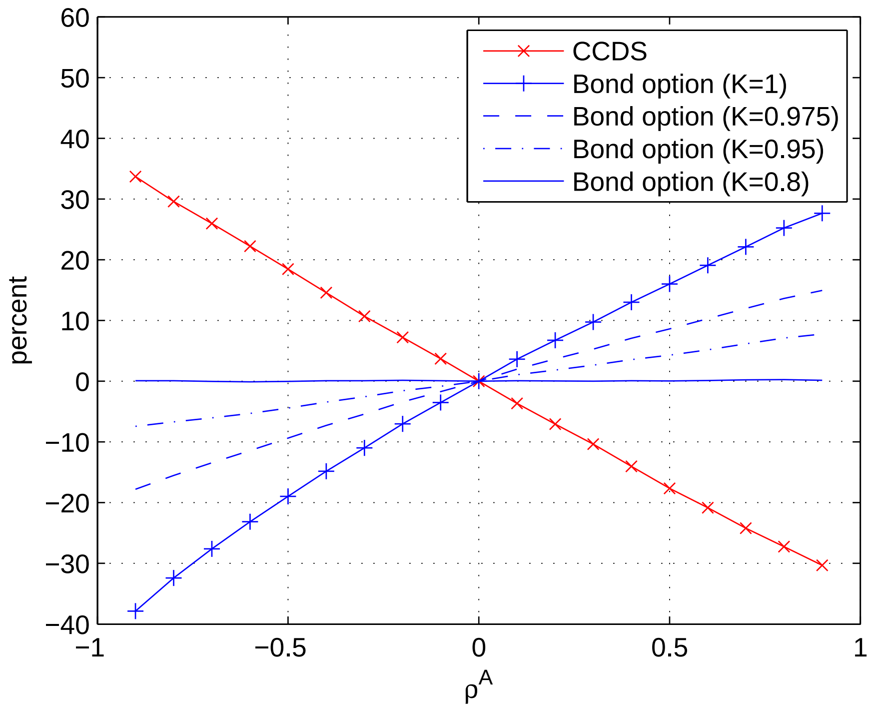

Figure 4: **The Importance of Interest Rate-Credit Spread Correlations** Percentage changes in the prices of European call options on zero-coupon bonds and in at-market CCDS rates for varying ρ^A, relative to their respective values at $\rho^A = 0$. Strike prices of the bond options are equal to K times the forward price, for different K. The benchmark set of parameter values is given by $\gamma_f = \gamma_{fA} = 0.5$, $c_f = -0.01$, $\kappa_f = \kappa_A = 0.2$, $h_f(t) = 0.03\sqrt{r(t)}$, $h_A(t) = 0.09\sqrt{\lambda_A(t)}$, and $\ell_A = 1$. We set $f(0,t) = 0.10$, and initialize the credit spread curve at $\lambda_A(0,0) = 0.05$, with monthly increases of 0.001. The first set of results assumes $\eta_f = 0$, the second set of results assumes $\rho^A = 0$ and $c_{fA} = 0.01$, and the third set of results assumes $\rho^A = 0$ and $\eta_f = 0.05$. We study contracts with a maturity of five years. The time until expiration of the bond options is three years. Our Monte Carlo simulations use 100,000 sample paths with antithetic sampling.

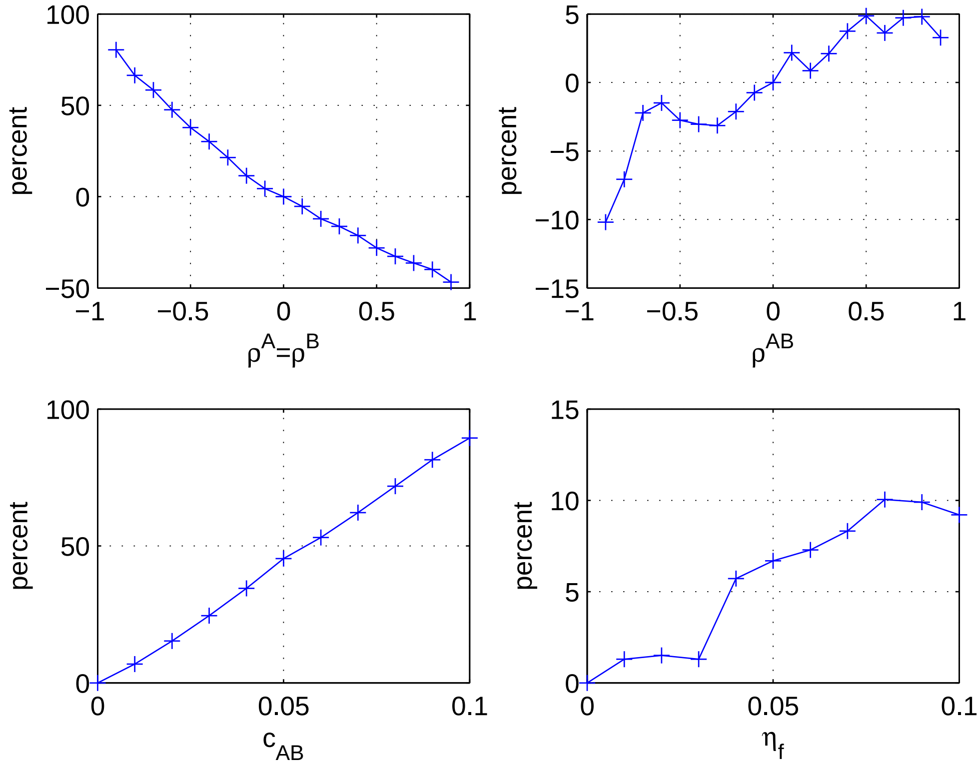

Figure 5: **The Importance of Default Correlation: Counterparty Risk in Insurance Contracts.** Percentage changes in the value of insurance against default of the protection seller in a CCDS contract for varying $\rho^A = \rho^B$, ρ^{AB}, c_{AB} and η_f. The benchmark set of parameters is given by $\rho_{fA} = \rho_{fB} = \rho_{AB} = 0$, $\eta_f = 0$, $\gamma_f = \gamma_{fA} = \gamma_{fA} = 0.5$, $c_f = -0.01$, $c_{fA} = c_{fB} = 0.01$, $c_{AB} = 0$, $\kappa_f = \kappa_A = \kappa_B = 0.2$, $h_f(t) = 0.03\sqrt{r(t)}$, $h_A(t) = 0.04\sqrt{\lambda_A(t)}$, $h_B(t) = 0.09\sqrt{\lambda_A(t)}$, and $\ell_A = \ell_B = 1$. We set $f(0,t) = 0.10$, $\lambda_A = 0.01$, and initialize the secondary credit spread curve at $\lambda_B(0,0) = 0.05$, with monthly increases of 0.001. As before, the maturity of the CCDS contract is set equal to five years and the simulations use 100,000 sample paths with antithetic sampling.

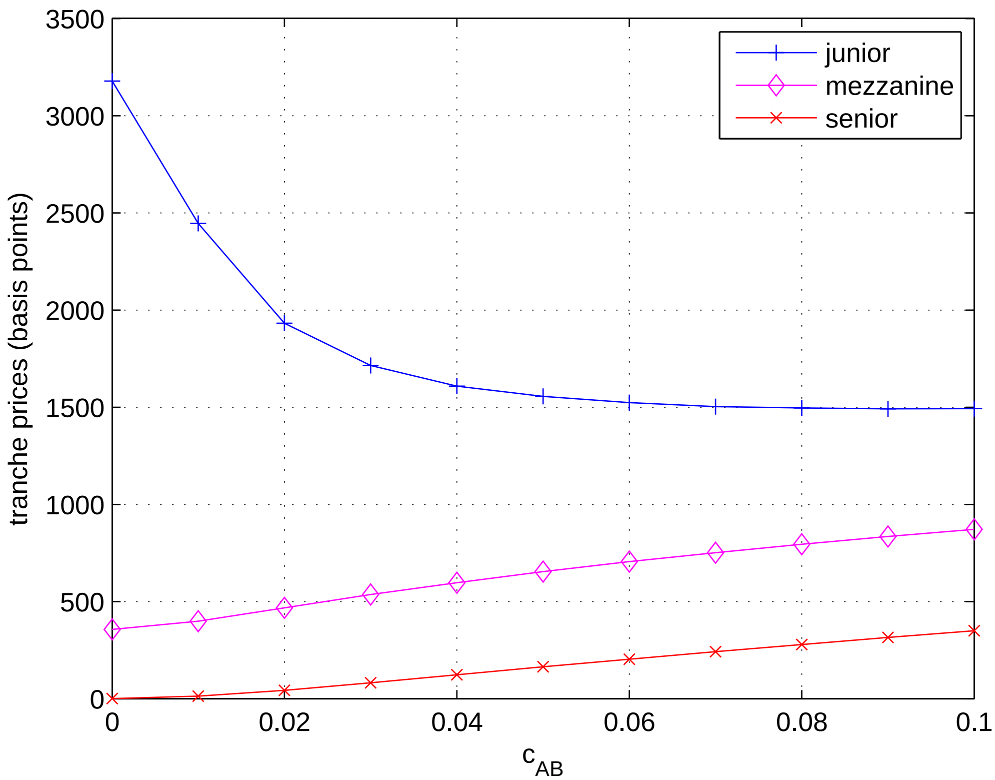

Figure 6: **The Importance of the Credit-Spread Impact Factor: CDS Index Tranches.** Simulated tranche prices of the five-year CDX.NA.IG index, as a function of c_{AB}. The benchmark set of parameters is given by $\rho^A = \rho^B = \rho^{AB} = 0$, $\eta_f = 0$, $\kappa_f = \kappa_A = \kappa_B = 0.2$, $h_f(t) = 0.025\sqrt{r(t)}$, $h_A(t) = 0.05\sqrt{\lambda_A(t)}$, $h_B(t) = 0.05\sqrt{\lambda_B(t)}$, and $\ell_A = \ell_B = 1$. We set $f(0,t) = 0.05$ and $\lambda_A(0,t) = \lambda_B(0,t) = 0.01$. There are 25 primary firms and 100 secondary firms. At the time of default of a particular firm, the responsible tranche investors pay 60% of notional. Our Monte Carlo simulations use 10,000 sample paths with antithetic sampling.

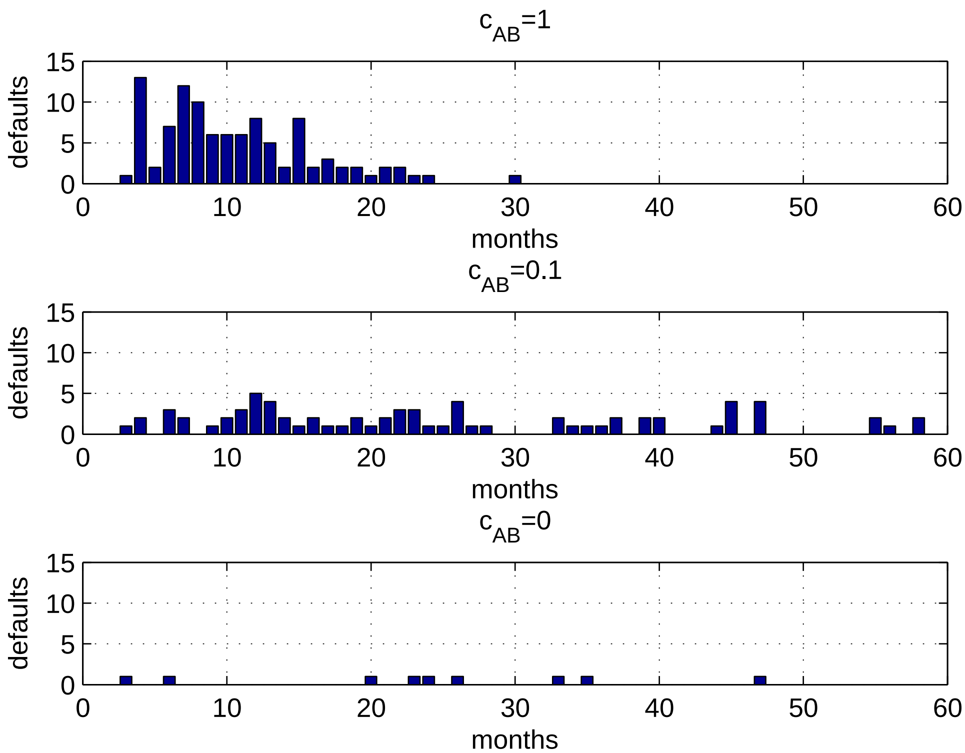

Figure 7: **Generating Default Clustering** Sample path of the number of defaults across time. Parameter values are specified as $\rho^A = \rho^B = \rho^{AB} = 0$, $\eta_f = 0$, $\kappa_f = \kappa_A = \kappa_B = 0.2$, $h_f(t) = 0.025\sqrt{r(t)}$, $h_A(t) = 0.05\sqrt{\lambda_A(t)}$, $h_B(t) = 0.05\sqrt{\lambda_B(t)}$, and $\ell_A = \ell_B = 1$. We set $f(0,t) = 0.05$ and $\lambda_A(0,t) = \lambda_B(0,t) = 0.01$. At time 0, there are 25 primary firms and 100 secondary firms.

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