Role of the Minimal State Variable Criterion

in Rational Expectations Models

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I. Introduction

It is well known that Bob Flood's second paper with Peter Garber (Flood and Garber, 1980b) was an influential pioneering work in empirical testing for the existence of bubble phenomena in rational expectations macroeconomics. It is not so well known, by contrast, that his paper with Burmeister and Garber (Burmeister, Flood, and Garber, 1983) provided one of the earliest steps toward a useful and general classification of rational expectations solutions, theirs focussing on the distinction between bubble and bubble-free (or fundamentals) solutions.[1] The present paper amounts to an extension of this type of classificational analysis, together with an attempt to establish the scientific merits of one particular scheme.

For many years now it has been commonplace knowledge that many dynamic models with rational expectations (RE) feature a multiplicity of paths that satisfy all of the conditions for intertemporal equilibrium. Indeed, most dynamic RE models that are not based on explicit optimization analysis of individuals' behavior fall into that category and so do some that involve full-fledged general equilibrium analysis with optimizing agents.[2] But in many applications the analyst is not specifically concerned with this multiplicity—often interpreted as the possible existence of "bubbles"—and wishes to focus attention on one particular path that is presumed to be

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[1] What, it might be asked, is the definition of a bubble in a rational expectations model? The basic idea of Burmeister, Flood, and Garber (1983) is that a bubble is an extra component that arises in addition to the component that reflects "market fundamentals," an important implication of which is that bubble components are not necessarily explosive. Unfortunately, the identification of market fundamentals has to be made on a model-specific basis, although there is rarely any disagreement. Below it will be argued that the MSV solution procedure is constructed so as to yield the market fundamentals solution, thereby providing a method for defining bubbles in particular cases.

[2] Leading examples of the latter type include real asset price bubbles in overlapping-generations models, as demonstrated by Calvo (1978) and Woodford (1984), and price level bubbles in infinite-horizon monetary models, as in Brock (1975), Flood and Garber (1980b), Gray (1986), and Obstfeld and Rogoff (1983).

of economic relevance, e.g., if bubbles were absent.[3] Consequently, several alternative criteria have been proposed for selection of the path on which to focus. Among these are Taylor's (1977) "minimum-variance" criterion, the "expectational-stability" criterion of Evans (1985) (1986), the "minimal-state variable" criterion made explicit in McCallum (1983), and the popular "saddle path" or "stability" criterion. The latter is favored by Sargent (1987), Whiteman (1983), Blanchard and Kahn (1980), Blanchard and Fischer (1989), and many others, and is often used in computation algorithms such as King and Watson (1995) or Klein (1997).

In practice, analysts are often unclear as to which of the criteria is being utilized, when attention is focused on a single solution, because in many cases the last three of the four above-listed criteria all point to the same solution. Some analysts are explicit, however, and a sampling of the literature suggests that the most frequently adopted of the criteria, in these cases of explicit justification, is that of stability or non-explosiveness. The stability criterion has been recommended, moreover, in the influential textbooks of Sargent (1987, pp. 197-9, 306-7) and Blanchard and Fischer (1989, pp. 225, 260).

One purpose of the present paper is to consider the strengths and weaknesses for scientific research of these alternative criteria. In particular, it will be argued that the stability and minimum-variance criteria are inherently unsatisfactory. By contrast, the minimal-state-variable (MSV) criterion is scientifically attractive, according to our argument, for it provides a classificational scheme that is designed to be useful in terms of positive analysis. The criterion of expectational stability, finally, will be characterized as reflecting a substantive behavioral hypothesis rather than a classification scheme, so its attractiveness is an empirical issue rather than a question of constructive scientific practice.

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[3] Although empirical testing is attractive in principle, this practice is in fact extremely common. This seems to be recognized by Blanchard and Fischer (1989, p. 260).

A second purpose of the paper is to emphasize that the minimal-state-variable (MSV) criterion generally identifies a single solution that can reasonably be interpreted as the unique solution that is free of bubble components, i.e., the fundamentals solution. It can accordingly be used as the basis for tests of a substantive hypothesis to the effect that bubble solutions are not of empirical relevance. This hypothesis would remain of interest, moreover, even if the association of the MSV criterion with the bubble-free property were not accepted.

In conducting this argument, it will be expositionally useful to provide illustrations in the context of a particular example. Consequently, one will be developed in Section II. The unsatisfactory nature of the minimum variance and stability criteria will then be argued in Section III. Section IV will make the case for the MSV criterion, with attention being devoted to a critical argument of Froot and Obstfeld (1991), and Section V will consider the "expectational stability" criterion of Evans (1985, 1988). Next, Section VI will demonstrate how unique MSV solutions can be defined and calculated in a very wide class of linear rational expectations models, after which Section VII will describe the relevance of the foregoing analysis for some prominent recent research. Finally, Section VIII will provide a brief summary.

II. An Illustrative Model

As a vehicle for illustrating several of the points to be made below, consider the familiar Cagan money demand function

$$m_t - p_t = \gamma + \alpha E_t \Delta p_{t+1} + \xi_t, \qquad \alpha<0 \tag{1}$$

where m_t and p_t are logs of an economy's nominal money stock and its price level. Also, $E_t(\cdot)$ is defined as $E(\cdot|\Omega_t)$, where Ω_t includes $m_t, m_{t-1}, \ldots, p_t, p_{t-1}, \ldots$, and $\xi_t, \xi_{t-1}, \ldots$. The disturbance ξ_t, which reflects random behavioral demand shifts, will be assumed to be a random walk variate so that $\Delta\xi_t = u_t$ is white noise. For our purposes it is of no consequence whether or not one conceives of (1) as resulting from an explicit maximization problem, since there are such models that give rise to multiple solutions and our points are designed to be relevant for any model with multiple solutions—with correct account being taken of all non-negativity requirements, transversality conditions, and anything else that might eliminate some paths from contention as solutions.

To represent policy behavior that generates the money supply, we will adopt a rule of the following form:

(2) $\Delta m_t = \mu_0 + \mu_1 \Delta p_{t-1}$.

Thus the money stock growth rate in each period is related to inflation in the previous period. One would expect sensible policy behavior to involve a negative value of μ_1, so that money creation is slowed when recent inflation has been rapid, and a value that is not too large (so as to avoid instrument instability). But for the present we shall adopt only the restriction $\mu_1 \leq (\alpha-1)^2 /(-4\alpha)$, which is necessary (as we shall see) for the Δp_t solution values to involve real (i.e., non-complex) numbers. It would of course be possible to include a random disturbance term in (2) as well as (1), but nothing would be gained and clutter would be added. To complete the model, it needs to be specified that it pertains to all periods t = 1,2,... with m_0 and Δp_0 given. The specified type of policy behavior can therefore only be adopted after an economy is already in existence so that Δp_0 and m_0 will be well defined. Inserting (2) into the first difference of (1) yields

(3) $\mu_0 + \mu_1 \Delta p_{t-1} = \Delta p_t + \alpha E_t \Delta p_{t+1} - \alpha E_{t-1} \Delta p_t + u_t$,

and for present purposes it will suffice to consider solutions of the form[4]

(4) $\Delta p_t = \pi_0 + \pi_1 \Delta p_{t-1} + \pi_2 u_t$.

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[4] This point will be explained below, in Section IV.

The latter implies $E_t\Delta p_{t+1} = \pi_0 + \pi_1(\pi_0 + \pi_1\Delta p_{t-1} + \pi_2 u_t)$ so substitution into (3) yields

(5) $\mu_0 + \mu_1\Delta p_{t-1} = \pi_0 + \pi_1\Delta p_{t-1} + \pi_2 u_t + \alpha\pi_0 + \alpha\pi_1(\pi_0 + \pi_1\Delta p_{t-1} + \pi_2 u_t) - \alpha(\pi_0 + \pi_1\Delta p_{t-1}) + u_t.$

Thus for (4) to be a solution it must be true that

(6a) $\mu_0 = \pi_0 + \alpha\pi_1\pi_0$

(6b) $\mu_1 = \pi_1 + \alpha\pi_1^2 - \alpha\pi_1$

(6c) $0 = \pi_2 + \alpha\pi_1\pi_2 + 1.$

The second of these clearly implies that[5]

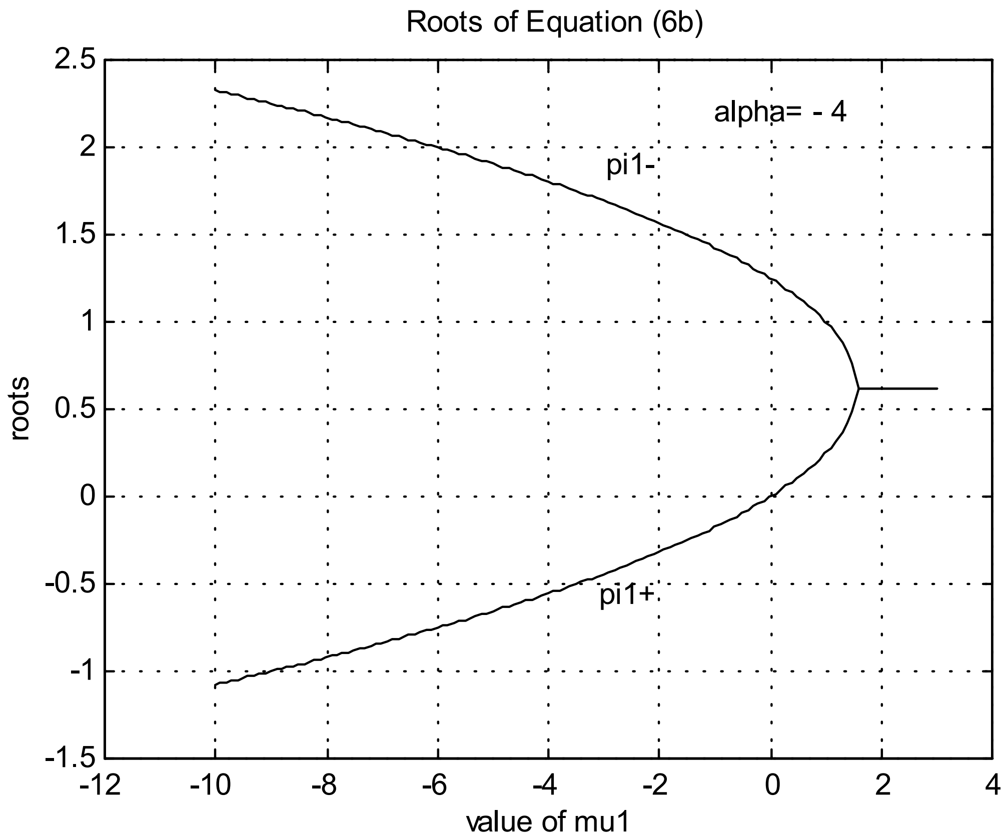
(7) $$\pi_1 = \frac{(\alpha - 1) \pm [(\alpha - 1)^2 + 4\alpha\mu_1]^{1/2}}{2\alpha}$$

Once it is decided whether to add or subtract the positive term $d \equiv [(\alpha-1)^2 + 4\alpha\mu_1]^{1/2}$, the values of π_0 and π_2 will be defined uniquely. But that decision is crucial for determining the model's implied behavior of Δp_t. That fact is illustrated in Figure 1, where $\pi_1^+ = (\alpha - 1 + d)/2\alpha$ and $\pi_1^- = (\alpha - 1 - d)/2\alpha$ are plotted for $\alpha = -4$ (representative for all $\alpha < -1$) against μ_1. Clearly, values of π_1^+ (the lower branch) and π_1^- (the upper branch) lie both within and outside of the range $-1 < \pi_1 < 1$ that is necessary for dynamic stability. (In the somewhat unrealistic case with $-1 < \alpha < 0$, not illustrated in Figure 1, π_1^- exceeds 1.0 for all μ_1 that give real roots.)

A particularly simple and transparent special case of this example occurs when $\mu_1 = 0$ in (2), so that the money stock growth rate is constant. In that case one might expect Δp_{t-1} to be absent from (4), since it does not appear in the model and can affect the value of Δp_t only if it is (arbitrarily) expected by the economy's participants to affect Δp_t. Thus we are led to look for solutions of the form $\Delta p_t = \pi_0 + \pi_2 u_t$ in this case, and we find that $\Delta p_t = \mu_0 - u_t$. This result is of course consistent

[5] From (7) we see that $\mu_1 > (\alpha-1)/(-4\alpha)$ would give complex roots.

Figure 1



Roots of Equation (6b)
alpha= - 4
pi1-
pi1+
roots
value of mu1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-12
-10
-8
-6
-4
-2
0
2
4

with our more general example. Indeed, the solutions in (7) for π_1 are $\pi_1^+ = 0$ and $\pi_1^- = (\alpha\text{-}1)/\alpha$ when $\mu_1 = 0$, the first of which implies the absence of Δp_{t-1} from (4) and duplicates the solution just found.[6] The second value, $\pi_1^- = (\alpha\text{-}1)/\alpha$, is with $\alpha < 0$ unambiguously greater than 1.0, so it implies an explosive, dynamically unstable path. Furthermore, this value π_1^- will support an infinity of unstable paths. This may be seen by supposing that $\pi_3 u_{t-1}$ is added to the conjectured solution in (4) and then verifying that this expression is consistent with all of the model's equations for any value of π_3 (upon which π_2 depends).[7] If $\pi_1^+ = 0$ is taken as the relevant value for π_1, however, it is implied that $\pi_3 = 0$ and $\pi_2 = -1$.

Note that in the special case in which $\mu_1 = 0$, the solution involving π_1^+ (i.e., $\Delta p_t = \mu_0 - u_t$) is clearly the one that would be regarded as the bubble-free or fundamentals solution by Burmeister, Flood, and Garber (1983).[8] Indeed, analogous solutions are so regarded quite generally in the literature in examples similar to our special case. By contrast, the solutions involving π_1^- would generally be regarded, in this special case, as bubble solutions—i.e., solutions that add bubble components to the fundamentals solution. McCallum (1983, pp. 147, 161) proposed a general extension of the bubble vs. bubble-free terminology to cases analogous to those in which $\mu_1 \neq 0$ in the example at hand; that extension will be utilized below.

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[6] Note that $\pi_1^+ = 0$ because $(\alpha\text{-}1) + [(\alpha\text{-}1)^2]^{1/2} = (\alpha\text{-}1) - (\alpha\text{-}1)$ since $[(\alpha\text{-}1)^2]^{1/2}$ is by convention a positive number and α-1 is in the present case negative.

[7] The undetermined-coefficient conditions are (6a), (6b), $0 = \pi_2 + \alpha\pi_1\pi_2 + \alpha\pi_3 + 1$, and $0 = \pi_3 + \alpha\pi_1\pi_3 - \alpha\pi_3$. With $\pi_1 = (\alpha\text{-}1)/\alpha$, the last of these is satisfied for any π_3 and the next to last relates π_2 to π_3.

[8] Burmeister, Flood, and Garber (1983) work in the context of a Cagan-style model similar to (1), except with a white-noise rather than a random-walk disturbance, and define the bubble-free or fundamentals solution as the one that depends only upon "current and expected future values of money and the disturbance" (1983, p. 312).

III. The Stability and Minimum Variance Criteria

As it happens, extensive utilization of the foregoing example will be briefly delayed, for our argument concerning the stability and minimum-variance criteria can be developed without reference to any particular model. Let us begin with Taylor's (1977) minimum-variance criterion. According to the latter, the choice among multiple solutions should be dictated by the unconditional variance of a variable analogous to Δp_t in the foregoing example. But there are two serious flaws with this proposal, the first of which is its ambiguity. Specifically, in many models there will be more than one endogenous variable of interest. (In fact, even in the example of Section II—despite the appearance of equation (3)—there are two endogenous variables, Δp_t and Δm_t.) But in such cases the minimum variance criterion will not be well specified, because the various endogenous variables may indicate different solutions. Indeed, in some cases there may even exist some ambiguity as to whether the (possibly detrended) level or first difference of a given variable is relevant. Second, the minimum-variance criterion is presumably intended to pertain to the solution path that would be empirically relevant. But that would of course suggest that the modeled economy's agents are motivated to choose the minimum-variance solution over others, and it is not the case that agents will typically be so motivated. Indeed, the minimum-variance criterion evidently pertains to some social desideratum, not anything that could be affected by any single agent's choice. Consequently, the model's agents will have no incentive to select this solution path, so there is no particular reason to believe that it would in fact be empirically relevant.

Turning now to the case of the stability criterion, our argument is quite different. Here the problem is that the criterion is, to a significant extent, self-defeating. For the criterion is precisely that the selected solution path must be non-explosive—dynamically stable—under the natural presumption that exogenous driving variables (such as shocks and policy instruments) are non-explosive. Yet one important objective of dynamic economic analysis is to determine whether particular hypothetical policy rules—or institutional arrangements—would lead to desirable economic performance, which will usually require stability. Or, to express the point somewhat differently, the purpose of a theoretical analysis will often be to determine the conditions under which a system will be dynamically stable and unstable. But, obviously, the adoption of the stability criterion for selection among solutions would be logically incompatible with use of the models' solution to determine if (or under what conditions) instability would be forthcoming. To the extent, then, that this objective of analysis is important, the stability criterion is inherently unsuitable. One cannot use a model to determine whether property "A" would be forthcoming, if the model includes a requirement that "A" must not obtain.

In addition, there are a substantial number of cases in which there exists an infinity of solution paths all of which are stable. In such cases, then, the stability criterion fails to select a single path on which to focus as the bubble-free or fundamentals path.[9] That failure would be defensible if it were true that no single path has special characteristics that justify labeling it as bubble-free, but it is not. Even in these cases the MSV criterion provides a clear demarcation between one path and the others. To develop that argument is the purpose of the next section.

[9] Because of their use of the stability criterion, Blanchard and Fischer (p. 260) suggest that if bubble paths are explosive then "unless the focus is specifically on bubbles, assume that the economy chooses the [stable] path, which is the fundamental [bubble-free] solution"—and do so even if there is no aspect of the model that explicitly disqualifies the explosive paths. But then in cases in which the bubble paths are not explosive, they are unable to recommend among various courses of action. Instead, they retreat to a hope—a "working assumption"—that "the conditions needed to generate stable multiplicities of equilibria are not met in practice" (p. 261). But we know that in various cases this hope is not justified.

IV. The MSV Criterion

The MSV criterion is designed to yield a single bubble-free solution *by construction*. Its definition begins by limiting solutions to those that are linear[10] functions—analogous to (4) in the example of Section II—of a minimal set of "state variables," i.e., predetermined or exogenous determinants of current endogenous variables. For a set of state variables to be minimal, it must be "one from which it is impossible to delete... any single variable, or group of variables, while continuing to obtain a solution valid for all admissible parameter values" (McCallum, 1983, p. 145). Here the language is somewhat convoluted because there is not in general a unique minimal set of state variables, even though there is a unique MSV solution. Two or more different sets of variables may span the same space, of course, with neither being a proper subset of the other.

But relying upon a minimal set of state variables is not the only requirement (in addition to linearity) for a MSV solution. In cases in which the minimal set includes a lagged value of an endogenous variable there will typically be more than one solution to the undetermined-coefficient identities analogous to equations (6) above. So one part of the definition of the MSV solution is a rule for selection of the appropriate solution. That rule is that the solution continues to be based on a minimal set of state variables for all special cases of the parameter values. Typically, some admissible sets of parameter values will include zero coefficients in all structural equations for a lagged endogenous variable. But in any such case, this lagged value will not be part of a minimal set, so its solution-equation coefficient analogous to π_1 will be zero for the MSV solution in that special case. Thus the MSV solution must be, to pertain for all admissible parameter values, the one that is the MSV solution in that special case.

[10] In linear models, that is.

To illustrate this determination, consider the choice between π_1^+ and π_1^- in the example of Section II. In the special case in which $\mu_1 = 0$ in (2), the variable Δp_{t-1} does not appear in model (and in fact appears to be an irrelevant bygone). Thus Δp_{t-1} can in this case affect the value of Δp_t only if it is—arbitrarily—expected by the economy's participants to affect Δp_t. Thus it does not appear in the minimal set of state variables in this special case with $\mu_1 = 0$, so $\pi_1 = 0$ is implied. But from the perspective of the general case, it is π_1^+ that yields the value 0 in this special case, π_1^- instead being equal to $(\alpha\text{-}1)/\alpha$. Consequently, it is the solution to equations (6) with $\pi_1 = \pi_1^+$ that makes (4) the MSV solution expression for Δp_t in this model.

It is important to recognize that this definition for the MSV solution involves a procedure that makes it unique by construction. It is logically possible to dispute whether this solution warrants being termed the bubble-free or fundamentals solution, although the answer seems to the present writer to be a clear "yes."[11] But it makes no logical sense to argue that the MSV solution is not unique.[12]

In that regard, Froot and Obstfeld (1991) have suggested that the MSV solution is not unique by demonstrating an example in which there is a non-linear function of the single state variable that constitutes a minimal set. That demonstration does not provide a valid counterexample to the claim of the last paragraph above, however, because linearity of the solution expressions such as (4) is required for the MSV solution. It is not surprising, it should be said, that Froot and Obstfeld

[11] The reason, of course, is that all other solutions involve—at least in special case—"extraneous" state variables, ones not in a minimal set. Thus the solution values involve variables that do not appear in the model's structural equations and therefore affect the endogenous variables only because they are (arbitrarily) expected to do so. I would also claim that the MSV solution corresponds to the bubble-free or fundamental values in all the standard, non-contentious examples in the literature. This claim cannot be proved correct, of course, but I am happy to put it forth as a refutable conjecture.

[12] Recall that our argument is presuming a linear model. It is possible to distinguish MSV solutions in some nonlinear models, but no general analysis has yet been developed.

would have misinterpreted the definition given in McCallum (1983), because the latter mistakenly took it for granted that only linear expressions would provide solutions in the class of linear models considered. But the outlined procedure, which defines the MSV solution, was expressly designed to yield a unique solution. So the restriction of linearity would have been explicitly included if the author had realized that it was needed.

The example presented in Section II was chosen, as one would expect, to illustrate points concerning the contrast between MSV and other solution criteria. In particular, for values of $\mu_1 < 2\alpha-1$, the MSV solution features dynamic instability since $\pi_1^+ < -1$. Thus this case demonstrates that the set of solutions selected by the MSV criterion, but ruled out by the stability criterion, is not empty. It is, moreover, intuitively plausible that instability would obtain in this case, as it reflects a very strong application of policy feedback response—which when excessive induces "instrument instability." Indeed, this is an example of the type of determination that a dynamic model should be able to provide—i.e., the conditions under which feedback is destabilizing. Alternatively, the example of Section II also illustrates the possibility of non-exploding bubble solutions, which occur when $1 < \mu_1 < (\alpha-1)^2/(-4\alpha)$.

At this point in the discussion it should be clear that the MSV criterion may be regarded as a classification scheme, i.e., a technique for delineating the solution that is of a bubble-free or fundamental nature from those that include bubble components. This scheme is intended to be scientifically useful, by providing a single solution that the researcher may focus upon if he/she is engaged in an investigation such that the possibility of bubbles is deliberately excluded at the outset. In addition, the classification scheme serves a second scientific purpose by providing the basis for a substantive hypothesis to the effect that market outcomes in actual economies are generally of the bubble-free variety. Even though RE general equilibrium analysis provides no general theoretical basis for ruling out bubble solutions, it is a coherent plausible substantive hypothesis that such solutions do not occur in practice.

The plausibility of that hypothesis is emphasized by the undetermined-coefficient method of deriving the MSV solution. The relevant point is that, in the space of π_1 values, the bubble-free value π_1^+ is of measure 1/2. And this continues to be true in the special case with $\mu_1 = 0$ in which there is an infinity of non-MSV bubble paths. In that case, the MSV solution features $\pi_1^+ = 0$, yielding $\Delta p_t = \mu_0 - u_t$. Use of the value $\pi_1^- = (\alpha\text{-}1)/\alpha$, however, gives rise in this case to an infinity of solutions of the form

(8) $\Delta p_t = \pi_0 + \pi_1 \Delta p_{t-1} + \pi_2 u_t + \pi_3 u_{t-1}$,

where the multiplicity arises because any value of π_3 will satisfy the model when π_1 equals $(\alpha\text{-}1)/\alpha$ (given that $\mu_1 = 0$). For some researchers, it is a common practice in such cases to presume that the outcome—the particular path realized in the market—is determined by an "initial condition" Δp_0 that serves to pin down π_2. From that perspective there is only a single value of Δp_0 that will imply $\pi_2 = -1$, and also that $\pi_1 = \pi_3 = 0$, thereby yielding the bubble-free solution.[13] In the space of initial conditions, then, the bubble-free outcome is of measure zero. But is entirely unclear which of these spaces is relevant to the market's solution outcome. It is thus a plausible hypothesis that bubble-free solutions will obtain generally.[14] The generation of that hypothesis is the second scientific contribution of the MSV solution criterion.

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[13] Recall that we are discussing the case with $\pi_1 = 0$. The solution value for π_3 when $\pi_1 = (\alpha\text{-}1)/\alpha$ is undetermined.

[14] Application to the striking argument of Woodford (1994, pp. 105-111), Bernanke and Woodford (1997, pp. 669-675), and Clarida, Gali, and Gertler (1997, pp. 20-23), is considered below in Section VII.

V. The Expectational Stability Criterion

The last alternative criterion to be explicitly discussed is that of "expectational stability" as developed by Evans (1985, 1986).[15] The basic idea is to determine whether there is convergence of an iterative procedure toward a RE solution; if there is such convergence the RE solution approached is the one selected by this criterion. It is not entirely clear whether the steps in the iterative process are supposed to reflect sequential positions in calendar time or in some type of conceptual meta-time, but to this reader the latter seems more appropriate. In any event, the sequence of calculations begins with a function, analogous to the expression just below (4), that determines expectations—but with coefficients that differ somewhat from those implied by RE in the model at hand. Then the model and this expectation function imply a "law of motion" for the model's endogenous variables. This law of motion, which may not be fully consistent with the expectation function used in its derivation, is then adopted as the basis for a revised expectation function to be used (in the same way) in the next round of the iterative process. Expectational stability obtains when this process converges to the RE solution under consideration.[16] In fact there are two variants: weak expectational stability obtains if the original expectation function is specified so as to include the same determining "state variables" as the RE solution under consideration, whereas strong expectational stability obtains when additional variables are permitted in the expectations function.

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[15] For more recent developments see Evans (1989) and Evans and Honkapohja (1992, 1997).

[16] Actually, it is shown by Evans (1989) and Evans and Honkapohja (1997) that expectational stability obtains when the differential equation analog of this difference equation converges. This will be the case under a somewhat broader set of conditions, so convergence of the iterative procedure is sufficient but not necessary of expectational stability. This result draws on Marcet and Sargent (1989).

The process can be illustrated with the model of Section II. With a RE solution of form (4), whether or not it is the MSV solution, expectations will conform to $E_t\Delta p_{t+1} = \pi_0 + \pi_1\Delta p_t$ so the iterative procedure assumes that expectations at t of Δp_{t+1} satisfy

(9) $\Delta p_{t+1}^{e,n} = \phi_0^n + \phi_1^n \ \Delta p_t,$

where n indexes the iterations. Now with (9) prevailing, Δp_t will be determined by the analog of (3), namely, $\mu_0 + \mu_1\Delta p_{t-1} = \Delta p_t + \alpha(\phi_0^n + \phi_1^n\Delta p_t) - \alpha\Delta p_t^e + u_t$, where Δp_t^e is given from the past.[17] The last equation can be written as

(10) $\Delta p_t = (1 + \alpha\phi_1^n)^{-1} \ [\mu_0 - \alpha\phi_0^n + \mu_1\Delta p_{t-1} + \alpha\Delta p_t^e - u_t]$

and it suggests that expectations for Δp_{t+1} should satisfy

(11) $\Delta p_{t-1}^{e,n+1} = (1-\alpha + \alpha\phi_1^n)^{-1} \ [\mu_0 - \alpha\phi_0^n + \mu_1\Delta p_t]$

since u_t is white noise. Then writing the right-hand side of the latter in form (9) gives

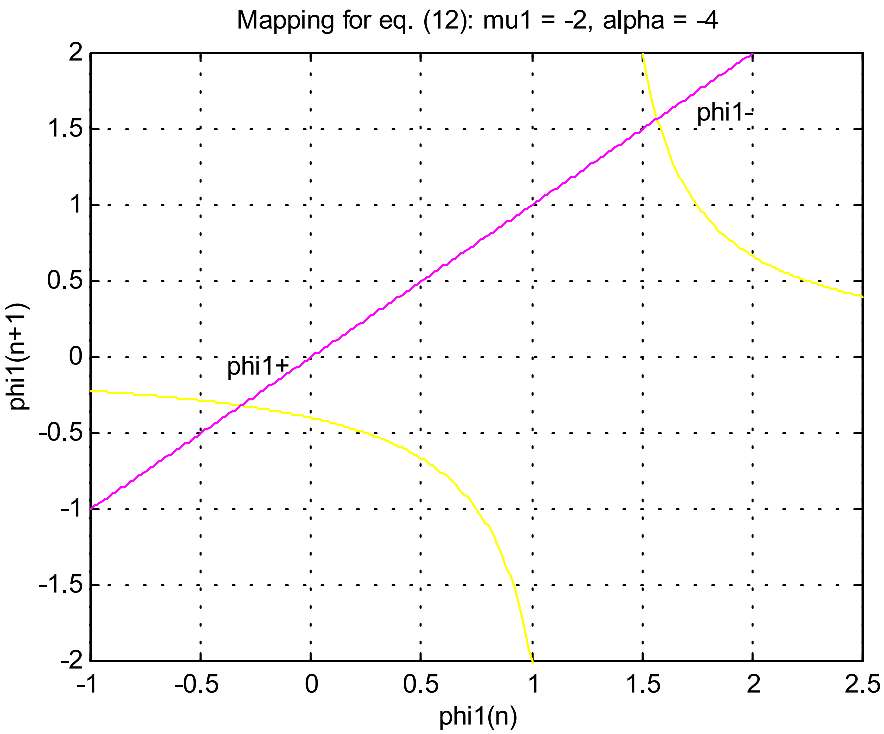
(12) $\phi_0^{n+1} = (1-\alpha + \alpha\phi_1^n)^{-1} (\mu_0 - \alpha\phi_0^n)$ $\qquad$ $\phi_1^{n+1} = (1-\alpha + \alpha\phi_1^n)^{-1} \ \mu_1,$

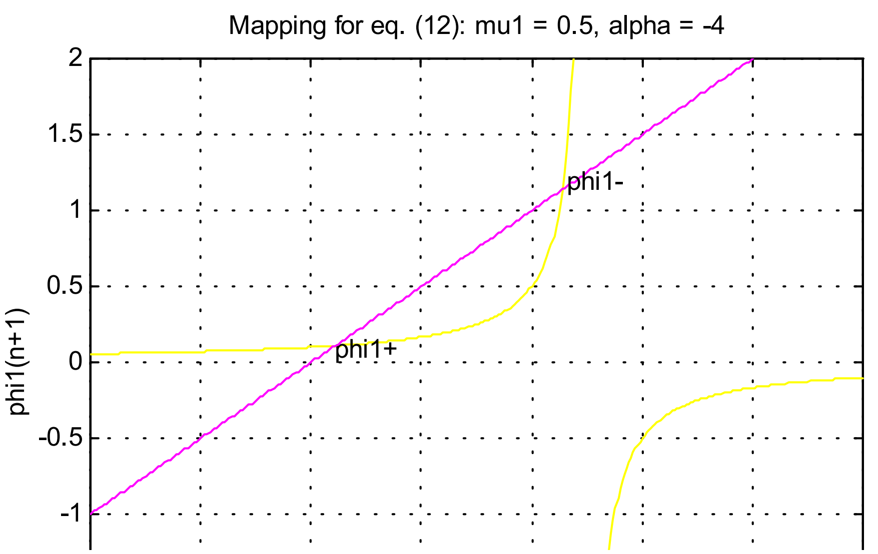
which define an iterative process for the values.

From the second of expressions (12), we see that the stationary values for ϕ_1 are the same as the two roots in (7). The expectational stability analysis selects the one—if there is one—for which the difference equation in ϕ_1^n is dynamically stable, i.e., the one that would be approached by the iterative process. From plots of ϕ_1^{n+1} vs. ϕ_1^n such as those in Figure 2, we can see that the root ϕ_1^+ is (locally) stable, since the slope is less than 1.0 in absolute value for all $\mu_1 <(\alpha-1)^2 /(-4\alpha)$. At the

[17] It is not entirely clear whether Evans and Honkapohja (1992, 1997) would agree with this derivation, as their examples do not include expectations formed at different times. But in the present model, Δp_t^e is clearly meant to represent the expectation of Δp_t formed in period t-1. So it is not what the iterative procedure at t is concerned with! Thus it would seem incorrect to write $\phi_0^{n-1} + \phi_1^{n-1} \Delta p_{t-1}$ in place of Δp_t^e in (10).

Figure 2


Mapping for eq. (12): mu1 = -2, alpha = -4
phi1(n+1)
phi1(n)
phi1-
phi1+
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1
1.5
2
2.5


Mapping for eq. (12): mu1 = 0.5, alpha = -4
phi1(n+1)
phi1-
phi1+
2
1.5
1
0.5
0
-0.5
-1

root ϕ_1^-, by contrast, the slope will exceed 1.0 in absolute value so the iterative process will not be convergent. With $\phi_1 = \phi_1^+$, moreover, the behavior of ϕ_0^n is stable for all parameter values yielding real roots in (6b). In this example, then, the expectational stability criterion points to the same solution as does the MSV criterion as long as $\mu_1<1$.

It is not entirely clear, however, just how much emphasis should be placed on that agreement. One reason is that Evans and Honkapohja (1992) argue that there are some cases in which expectational stability does not point to the MSV solution. I am not entirely persuaded that these cases include any well-motivated economic models, but in any event that is not the main point. The point, instead, is that if the analysis calls for focus on the bubble-free solution, then that would still be accomplished by means of the MSV criterion. If expectational stability provides an accurate guide to the behavior in actual economies, then a non-MSV bubble solution would prevail in such cases. But that would provide no reason for changing the classification of bubble vs. non-bubble solutions. And it is far from certain that expectational stability does provide a guide for actual economic behavior, for that hypothesis requires that this particular iterative process, among all those that could be conjectured, is empirically relevant. Nevertheless, while the amount of warranted emphasis is unclear, it is the case that in most—if not all—sensible models the expectational stability criterion does point to the MSV solution.[18]

[18] The example in Evans and Honkapohja (1992) is an exception but is not, I would suggest, as well motivated as the model in the present paper, which differs in its assumptions regarding the times (actually, information sets) relevant for forming expectations of Δp_t and Δp_{t+1}.

VI. General Derivation in Linear Models

The argument above has relied on the proposition that there is a unique MSV solution in a wide class of linear models; the main purpose of this section is to demonstrate the validity of that claim. In addition, a second purpose is to present a compact and easily understood exposition of a convenient and practical computational procedure for solving linear rational expectations models. This procedure, which is applicable to a class of models that is broad enough to include most cases of practical interest, can be implemented by means of a MATLAB routine provided by Paul Klein (1997).[19] The present exposition departs from Klein's, however, by relying upon the elementary undetermined-coefficients (UC) approach used throughout the present paper. In a sense, the current exposition could be viewed as merely an extension to the appendix of McCallum (1983). It is an extension that is nontrivial, however, and essential for practical (i.e., computational) purposes. Here it is accomplished by use of the generalized Schur decomposition theorem discussed by Klein. The UC reasoning utilized here is, however, much more elementary mathematically than Klein's.[20]

Let y_t be a M×1 vector of non-predetermined endogenous variables, k_t be a K×1 vector of predetermined endogenous variables, and u_t be a N×1 vector of exogenous variables. The model

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[19] Klein's (1997) approach builds upon earlier contributions of King and Watson (1995) and Sims (1996). Other significant recent contributions are Uhlig (1997) and Binder and Pesaran (1995), which use UC analysis. The Uhlig paper also features a useful procedure for linearizing models that include nonlinear relationships.

[20] An earlier draft of this paper included a demonstration that closed-form representations of MSV solutions can be obtained by means of formulae developed by Whiteman (1983). This demonstration was illustrated in the context of the simple example of Section II, in line with the much more extensive analysis in McCallum (1985). That analysis was more tedious and less useful that that of the present section, however, since the latter is based on a convenient computational algorithm. The present discussion is taken in large part from McCallum (1998).

can then be written as

(14) $A_{11} E_t y_{t+1} = B_{11} y_t + B_{12} k_t + C_1 u_t$

(15) $u_t = R u_{t-1} + \varepsilon_t$

where A_{11} and B_{11} are square matrices while ε_t is a N×1 white noise vector.[21] Thus u_t is formally a first-order autoregressive process, which can of course be defined so as represent AR processes of higher orders for the basic exogenous variables. Also, for the predetermined variables we assume

(16) $k_{t+1} = B_{21} y_t + B_{22} k_t + C_2 u_t$.

If only once-lagged values of y_t were included in k_t, then we would have $B_{21} = I$, $B_{22} = 0$, and $C_2 = 0$, but the present setup is much more general. Crucially, the matrices A_{11}, B_{21}, and B_{22} may be singular; that is what makes the setup convenient in practice.

In this setting a UC solution will be of the form

(17) $y_t = \Omega k_t + \Gamma u_t$

(18) $k_{t+1} = \Pi_1 k_t + \Pi_2 u_t$,

[21] Here, as above, $E_t y_{t+1}$ is the expectation of y_{t+1} conditional upon an information set that includes all of the model's variables dated t and earlier.

where the Ω, Γ, Π_1, and Π_2 matrices are real. Therefore, $E_t y_{t+1} = \Omega E_t k_{t+1} + \Gamma E_t u_{t+1} = \Omega(\Pi_1 k_t + \Pi_2 u_t) + \Gamma R u_t$. Substitution into (14) and (16) then yields

(19) $\quad A_{11}[\Omega(\Pi_1 k_t + \Pi_2 u_t) + \Gamma R u_t] = B_{11}[\Omega k_t + \Gamma u_t] + B_{12} k_t + C_1 u_t$

and

(20) $\quad (\Pi_1 k_t + \Pi_2 u_t) = B_{21}(\Omega k_t + \Gamma u_t) + B_{22} k_t + C_2 u_t.$

Collecting terms in k_t, it is implied by UC reasoning that

(21) $$\begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega\Pi_1 \\ \Pi_1 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}$$

whereas the terms in u_t imply

(22) $\quad A_{11}\Omega \Pi_2 + A_{11}\Gamma R = B_{11}\Gamma + C_1$

(23) $\quad \Pi_2 = B_{21}\Gamma + C_2.$

Let A and B denote the two square matrices in (21), and assume that $|B-\lambda A|$ is nonzero for some complex number λ. This last condition will not hold if the model is poorly formulated (i.e., fails to place any restriction on some endogenous variable); otherwise it will be satisfied even with singular A_{11}, B_{21}, B_{22}.[22] Then the generalized Schur decomposition theorem guarantees the

[22] See King and Watson (1995) or Klein (1997).

existence of unitary (therefore invertible) matrices Q and Z such that QAZ = S and QBZ = T, where S and T are triangular.[23] The ratios t_{ii}/s_{ii} are generalized eigenvalues of the matrix pencil $B - \lambda A$;[24] they can be rearranged without contradicting the foregoing theorem. Such rearrangements correspond to selection of different UC solutions as discussed in McCallum (1983, pp. 145-147 and 165-166). We shall return to this topic below; for the moment let us assume that the eigenvalues t_{ii}/s_{ii} (and associated columns of Q and Z) are arranged in order of their moduli with the largest values first.

Now premultiply (21) by Q and define $H \equiv Z^{-1}$. Then since QA = SH and QB = TH, the resulting equation is

$$(24)\qquad \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega\Pi_1 \\ \Pi_1 \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}$$

and its first row can be written as

$$(25)\qquad S_{11}(H_{11}\Omega+H_{12})\Pi_1 = T_{11}(H_{11}\Omega+H_{12}).$$

The latter will be satisfied for Ω such that

$$(26)\qquad \Omega = -H_{11}^{-1}H_{12} = -H_{11}^{-1}\left(-H_{11}Z_{12}Z_{22}^{-1}\right) = Z_{12}Z_{22}^{-1},$$

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[23] See Golub and Van Loan (1996, p. 377).

[24] Or, in the terminology used by Uhlig (1997), are eigenvalues of B with respect to A.

where the second equality results because HZ = I. Thus we have a solution for Ω, provided that Z_{22}^{-1} exists.[25]

Next, writing out the second row of (24) we get

(27) $S_{21}(H_{11}\Omega+H_{12})\Pi_1 + S_{22}(H_{21}\Omega+H_{22})\,\Pi_1 = T_{21}(H_{11}\Omega+H_{12}) + T_{22}(H_{21}\Omega+H_{22})$.

Then using (26) and HZ = I we can simplify this to

(28) $S_{22}\ Z_{22}^{-1}\ \Pi_1 = T_{22}\ Z_{22}^{-1}$

so since S_{22}^{-1} exists by construction [26] we have

(29) $\Pi_1 = Z_{22}\ S_{22}^{-1}\ T_{22}\ Z_{22}^{-1}$.

To find Γ and Π_2 we return to (22) and (23). Combining them we have

(30) $G\Gamma + A_{11}\Gamma R = F$

[25] This is the same condition as that required by Klein (1997, p. 13) and King and Watson (1995, pp. 9-11). It appears to provide no difficulties in practice. The King and Watson example of a system in which the condition does not hold is one in which $B_{12} = 0$ in my notation so the MSV solution has $\Omega = 0$ and the other solution matrices follow easily.

[26] By the arrangement of generalized eigenvalues, S_{22} has no zero elements on the diagonal (and is triangular).

where $G \equiv A_{11}\Omega B_{21} - B_{11}$ and $F \equiv C_1 - A_{11}\Omega C_2$. If G^{-1} exists, which it typically will with nonsingular B_{11}, the latter becomes

$$(31) \qquad \Gamma + G^{-1}A_{11}\Gamma R = G^{-1}F.$$

This can be solved for Γ by the steps given in McCallum (1983, p. 163) or can be obtained as

$$(32) \qquad \text{vec}(\Gamma) = [I + R'\otimes G^{-1}A_{11}]^{-1} \text{ vec}(G^{-1}F),$$

as in Klein (1997, p. 28).[27] Finally, Π_2 is obtained from (23). In sum, the UC solution for a given ordering of the eigenvalues is given sequentially by equations (26), (24), (32), and (23).

Different values of Ω, and thus different solutions, will be obtained for different orderings of the generalized eigenvalues t_{ii}/s_{ii}. What ordering should be used to obtain the economically relevant solution? Many writers, following Blanchard and Kahn (1980), arrange them in order of decreasing modulus and conclude that a unique solution obtains if and only if the number with modulus less than 1.0 ("stable roots") equals K, the number of predetermined variables. The minimal-state-variable (MSV) procedure, by contrast, is to choose the arrangement that would yield $\Omega = 0$ if it were the case that $B_{12} = 0$—this step relying upon the continuity of eigenvalues with

[27] This uses the identity that if A, B, C are real conformable matrices, $\text{vec}(ABC) = (C'\otimes A)\text{ vec}(B)$. See Golub and Van Loan (1996, p. 180).

respect to parameters.[28] Uhlig (1997, p. 17) correctly notes that this procedure is difficult to implement and also that in many cases it will lead to the same solution as the Blanchard-Kahn stability criterion. Adoption of the decreasing-value arrangement will therefore often be attractive, even for MSV adherents. In such cases it seems unnecessary, however, to limit one's attention to problems in which there are exactly K stable roots. If there are fewer than K stable roots, the MSV criterion will produce a single explosive solution whereas if there are more than K stable roots, it will yield the single stable solution that is bubble-free—both of these being solutions that may be of particular scientific interest. In those exceptional cases in which an MSV analyst suspects that the Blanchard-Kahn and MSV criteria would call for different solutions, he/she could replace B_{12} with αB_{12}, plot eigenvalues for various values of α between 1 and 0, and then adjust the ordering if necessary.

VII. Relevance for Recent Issues

The example of Section II is simple and clearly related to much of the existing bubble literature, but may seem remote from most monetary policy discussions of the late 1990s. To show that such is not the case—that the example is in fact highly relevant—is the purpose of the present section.

Let us begin by considering the following model, in which y_t denotes the log of output relative to capacity, R_t is a nominal interest rate, and v_t is a white-noise disturbance:

(33) $y_t = b_0 + b_1 (R_t - E_t\Delta p_{t+1}) + v_t$ $\qquad b_1<0$

(34) $\Delta p_t = (1-\theta) E_t\Delta p_{t+1} + \theta\Delta p_{t-1} + \alpha y_t$ $\qquad \alpha>0$

(35) $R_t = \mu_0 + \mu_1 (E_t\Delta p_{t+1} - \Delta p^*) + \mu_2 y_t$ $\qquad \mu_1,\mu_2>0$

[28] With $B_{12} = 0$, k_t does not appear in the system (14) (19), in this case so k_t represents extraneous variables of a bootstrap, bubble, or sunspot nature.

Here (33) is a textbook-style IS function,[29] (34) is a price-adjustment relation that with $0 \leq \theta < 1$ can represent either the specification of Calvo (1983) and Rotemberg (1982) or the Fuhrer-Moore (1995) setup, and (35) is an interest-rate policy rule that can reflect pure inflation targeting (with $\mu_2 = 0$) or a rule of the more general Taylor (1993) variety.

Substitution of (35) into (33) and elimination of y_t then yields a linear equation that includes the variables Δp_t, $E_t\Delta p_{t+1}$, Δp_{t-1}, and v_t. That list differs from the one pertaining to equation (5) by not including $E_{t-1}\Delta p_{t+1}$, but that difference is of no consequence for the issues at hand because the distinction between $E_t\Delta p_{t+1}$ and $E_{t-1}\Delta p_{t+1}$ is irrelevant for the condition analogous to (6b) that determines the value of the crucial coefficient on Δp_{t-1} in the RE solution expression. Indeed, it can be verified that for some admissible parameter values the system has two stable solutions.[30] Interestingly, large values of μ_1 do not generate explosive MSV solutions with the policy rule (35), but if Δp_{t-1} is entered in place of $E_t\Delta p_{t+1}$ then large μ_1 values will induce instability, just as in the example of Section II.

An issue that has attracted considerable attention recently is the so-called "Woodford warning" of possible solution "indeterminacy" when policy feedback rules relate to market expectations of inflation or some other target variable, a problem emphasized by Woodford (1994), Kerr and King (1996), Bernanke and Woodford (1997), Clarida, Gali, and Gertler (1997), and Svensson (1998). An example can be presented in the following system, which is adapted from Clarida, Gali, and Gertler (1997, p. 16):

(36) $\quad y_t = E_t y_{t+1} + b_1 (R_t - E_t\Delta p_{t+1}) + v_t \qquad\qquad b_1<0$

[29] It would be more desirable theoretically to use an expectational IS relation, as argued in McCallum and Nelson (1997) and elsewhere, but that would lead to a cubic equation for the coefficient on Δp_{t-1} in the MSV solution without altering the basic message.

[30] Two stable solutions exist if the parameters are $\alpha = 0.2$, $b_1 = 0.5$, $\theta = 0.2$, and $\mu_1 = 0.5$.

(37) $\Delta p_t = \beta E_t \Delta p_{t+1} + \alpha E_t y_t$ $\alpha>0,\ 0<\beta<1$

(38) $R_t = \mu_1 E_t \Delta p_{t+1}$ $\mu_1>0$

Here we have an expectational IS function, a Calvo-Rotemberg price adjustment specification, and a pure inflation-forecast targeting rule.[31] For simplicity, constants are eliminated by normalization and v_t is again taken to be white noise. In this system there are no predetermined variables so the MSV solution is of the form $y_t = \phi_1 v_t$, $\Delta p_t = \phi_2 v_t$. Trivial calculations show that $\phi_1 = 1$, $\phi_2 = \alpha$ so the solution is $y_t = v_t$, $\Delta p_t = \alpha v_t$. The policy coefficient μ_1 does not appear in the solution equations because policy is responding to the expected future inflation rate, which is a constant (normalized to zero). A caveat must be applied to the foregoing, however: the MSV solution is defined only for $\mu_1>1.0$. Values of $\mu_1<1.0$ are inadmissible for "process consistency" reasons, introduced by Flood and Garber (1980a) and discussed in McCallum (1983, pp. 159-160).

But suppose that the researcher looks for solutions of the form

(39) $y_t = \phi_{11}\Delta p_{t-1} + \phi_{12} v_t$

(40) $\Delta p_t = \phi_{21}\Delta p_{t-1} + \phi_{22} v_t$.

Then $E_t y_{t+1} = \phi_{11}(\phi_{21}\Delta p_{t-1}\ \phi_{22} v_t)$, $E_t\Delta p_{t+1} = \phi_{21}(\phi_{21}\Delta p_{t-1} + \phi_{22} v_t)$, and the undetermined-coefficient conditions analogous to (6) are

(41a) $\phi_{11} = \phi_{11}\phi_{21} + b_1(\mu_1 - 1)\phi_{21}^2$

(41b) $\phi_{12} = \phi_{11}\phi_{22} + b_1(\mu_1 - 1)\phi_{21}\ \phi_{22} + 1$

(41c) $\phi_{21} = \beta\phi_{21}^2 + \alpha\phi_{11}$

(41d) $\phi_{22} = \beta\phi_{21}\phi_{22} + \alpha\phi_{12}$.

From the first and third of these we obtain the crucial requirement

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[31] Clarida, Gali, and Gertler (1997) also include terms involving y_t and R_{t-1} on the right-hand side of (38). They are omitted here only to keep the example as simple and transparent as possible.

(42) $\phi_{21} = \beta\phi_{21}^2 + \alpha b_1(\mu_1-1)\phi_{21}^2/(1-\phi_{21})$.

Clearly, one root of the foregoing is $\phi_{21} = 0$, which implies $\phi_{11} = 0$ and consequently gives the MSV solution. But (42) is also satisfied by values of ϕ_{21} such that

(43) $$\phi_{21} = \frac{\delta \pm [\delta^2 - 4\beta]^{1/2}}{2\beta}, \qquad \delta = 1 + \beta + \alpha b_1(\mu_1 - 1).$$

Here $\delta^2 - 4\beta$ is positive for $\mu_1 < 1$ and $\mu_1 > 1 + [2\beta^{1/2} - (1+\beta)]/(-b_1\alpha)$.[32] So for those values, there are non-zero real roots for ϕ_{21} and thus solutions in addition to the MSV solution. That this possibility obtains for large values of μ_1 represents a problem for monetary policy, according to the non-MSV analysis of the authors mentioned above. But under the hypothesis that the MSV solution prevails, large values of μ_1 pose no problem: the solution remains $y_t = v_t$, $\Delta p_t = \alpha v_t$. Since $\mu_1 \to \infty$ is conceptually akin to setting R_t such that $E_t\Delta p_{t+1} = 0$, where 0 is the implicit target rate of inflation, the MSV hypothesis seems more consistent with the inflation forecast targeting prescription of Svensson (1998) than does the non-MSV analysis of Bernanke and Woodford (1997) or Clarida, Gali, and Gertler (1997). This conclusion pertains, I conjecture, to this entire body of analysis, not just the single (and extreme) case considered above. In any event, it should be emphasized that if a multiplicity of solutions is found by considering non-MSV procedures, it has nothing to do with the phenomenon of "nominal indeterminacy"—i.e., cases in which a model determines values of real variables but not nominal variables. For a recent discussion of this distinction, see McCallum (1997).

Finally, we might also mention the "fiscal theory of price level determination," due principally to Woodford (1995) and Sims (1994), which has been attracting a good bit of attention.

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[32] Note that with the values $\beta = .99$, $\alpha = .3$, $b_1 = -1$ used by Clarida, Gali, and Gertler (1997), this last expression equals $1 + [1.96 + 1.99]/0.3 = 14.2$, precisely as reported in their Table 4 for this special case.

In this regard, the argument presented in Section 7 of McCallum (1997) indicates that adoption of the fiscal theory of price level determination, in contrast to the more traditional "monetarist" approach, amounts to acceptance of the hypothesis that a non-MSV or bubble solution is empirically relevant. The MSV solution is also available,[33] however, and implies fully traditional price level-money stock relationships and behavior.

VIII. Conclusions

Let us conclude with a brief summary. This paper has been concerned with the minimal-state-variable (MSV) criterion for selection among solutions in linear rational expectations models that feature a multiplicity of paths that satisfy all conditions for equilibrium. The paper compares the MSV criterion with others that have been proposed, including Taylor's (1977) minimum-variance criterion, the expectational stability criterion of Evans (1985, 1986), and the saddle-path or non-explosiveness (i.e., dynamic stability) criterion favored by Blanchard and Kahn (1980), Blanchard and Fischer (1989), Sargent (1987), and Whiteman (1983) and utilized in practice by a large number of researchers. It is emphasized that the MSV criterion can be viewed as a classification scheme, one that delineates the unique solution that is of a bubble-free nature—i.e., reflecting only market fundamentals—from those that include bubble or bootstrap components.

It is argued that the MSV classification scheme is of scientific value in two ways. First, it provides a unique solution upon which a researcher may focus attention if the project at hand suggests or permits the a priori exclusion of bubble solutions. Second, it provides the basis for a substantive hypothesis to the effect that market outcomes in actual economies are generally of a bubble-free nature. In describing the latter role, the paper argues that the possibility that

[33] The example cited is one in which the model is not linear, so the MSV concept has to be extended and the generality of Section VI cannot be claimed.

bubble-free solutions dominate empirically is much more plausible than is suggested by solution approaches that parameterize different solutions by (possibly irrelevant) initial conditions rather than by undetermined-coefficient parameter values. It also explains the basis of McCallum's (1983) "subsidiary principle" that is used to make the MSV solution unique by construction.

In the process of demonstrating the uniqueness of the MSV solution, the paper presents a convenient and practical computational procedure for solving linear rational expectations models of a very broad class. This exposition, which utilizes the generalized Schur decomposition theorem, is developed by means of the mathematically simple undetermined-coefficients approach. In addition, examples are provided that illustrate the applicability and importance of the MSV criterion to issues of current concern in the analysis of monetary policy rules.

Finally, it should be recognized that some readers may be unwilling to accept the paper’s interpretation of the MSV solution as the bubble-free or fundamentals solution. In that case, it remains true that the MSV approach provides a unique solution upon which a researcher may focus attention, if desired, and provides the basis for a substantive hypothesis to the effect that actual outcomes generally conform to the MSV solution. If this hypothesis is in fact true, then several classes of problems discussed in the literature are empirically irrelevant.

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