FISCAL HEDGING WITH NOMINAL ASSETS

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ABSTRACT

We analyze optimal fiscal and monetary policy in an economy with distortionary labor income taxes, nominal rigidities and nominal debt of various maturities. Optimal policy prescribes the exclusive use of long term debt. Such debt mitigates the distortions associated with hedging fiscal shocks by allowing the government to allocate them efficiently across states and periods.

I. Introduction

Governments have traditionally financed deficits by selling *nominal* bonds of varied maturities. A long standing policy question concerns the optimal management of such liabilities. Various contributors have posited a role for short term nominal debt. Campbell (1995) argues that a cost-minimizing government should respond to a steeply sloped nominal yield curve by shortening the maturity structure since high yield spreads tend to predict high expected bond returns in the future. Barro (1997) emphasizes tax smoothing considerations. He asserts that governments can reduce their risk exposure and better smooth taxes by shortening the maturity structure when the inflation process becomes more volatile and persistent. Barro characterizes the reduction in the average maturity of US Federal bonds between 1946 and 1976 as an optimal response to changes in the inflation process. Both lines of argument treat the processes for inflation and nominal interest rates exogenously.

In this paper, we explore optimal maturity management in a fully specified general equilibrium model. We identify a motive for issuing long term nominal debt and give calibrated examples in

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which there is exclusive use of the longest term nominal debt available. In these examples, the management of nominal interest rates departs from the Friedman rule. A switch from a favorable to an unfavorable fiscal environment, triggered by an adverse shock[1] is followed by increases in current and future short term nominal interest rates, with increases in the latter concentrated in future adverse shock states. When a spell of adverse fiscal shocks begins, the yield curve takes a corresponding humped shape, with the hump occurring at the longest traded debt maturity. It reverts to a lower level and a flatter shape when this spell ends or when the debt outstanding at the beginning of the spell has matured. Optimal policy implies that long term nominal debt is riskier than short term debt. However, the volatility of long term debt returns is deliberate and managed so as to hedge the fiscal risk the government faces. The risk premium on this debt resembles an insurance premium paid by the government; it does not provide a motive for shortening the maturity structure.

Since our focus is the management of the nominal maturity structure, we consider an economy in which households are borrowing-constrained and can only buy non-contingent nominal debt. This assumption implies that the government must hedge fiscal shocks indirectly through contemporaneous inflations or variations to the nominal term structure. Following Siu (2004), we introduce two nominal rigidities that enrich the government's policy problem.[2] First, we assume that some firms set their prices before the realization of the current state. This rigidity implies that contemporaneous innovations to inflation are associated with costly misallocations of production across firms. The government must trade such distortions off against the hedging benefits that inflation innovations provide. Second, we assume that households face a cash-in-advance constraint applied to some goods (cash goods), but not others (credit goods). Variations in the nominal term structure imply positive short term nominal interest rates after some histories and, hence, misallocations of consumption across cash and credit goods. The government must trade the hedging benefits of these variations off against the consumption distortions they induce.

Absent borrowing constraints on households, an allocation in the neighborhood of the optimal complete markets one can be implemented by taking arbitrarily large positions in debt markets. With these constraints in place, such positions are no longer possible and the hedging of risk requires more substantial movements in inflation and nominal interest rates. We use a simple example with a single shock to isolate an advantage of long term debt in this case: it allows the government to

[1] Here, adverse fiscal shocks comprise positive shocks to government spending and negative shocks to productivity.

[2] Absent these rigidities all hedging could be achieved through contemporaneous adjustments in inflation and nominal debt would effectively function as a real contingent claim. See, for example, Chari et al (1991).

postpone the costly positive interest rates used to hedge shocks. As noted, calibrated numerical examples indicate that the government relies almost exclusively on the longest term debt available. Such debt permits the postponement of positive nominal interest rates and their concentration in states where they can contribute to the hedging of multiple past shocks. Such postponement and concentration effects underpin a gradual upward response of short term nominal interest rates during spells of adverse fiscal shocks.

The literature on optimal fiscal and monetary policy has made various assumptions about the asset structure confronting the government. Our paper is closest to Siu (2004). We follow him in restricting the government to the use of nominal debt and incorporating frictions that render state-contingent inflations distortionary. In contrast to Siu, we allow the government to trade nominal debt of more than one period maturity. Thus, we are able to consider the optimal maturity structure. Additionally, in our model the government can influence the price of outstanding nominal bonds via current and future nominal interest rate policy. This opens up a second channel for hedging fiscal shocks that is absent in Siu's earlier contribution.

The plan for the paper is as follows. Sections II and III describe the environment and characterize competitive allocations. Section IV gives the Ramsey problem for our economy and contrasts it with those obtained under alternative asset market structures. Section V identifies a motive for using long term debt in a simple example, while Section VI provides a general recursive formulation. Section VII uses this formulation to obtain optimal policy in calibrated economies.

II. A model with sticky prices

The economy is inhabited by infinitely-lived households, firms and a government. Let $s_t \in S = \{\widehat{s}_i\}_{i=1}^N$ denote a period t shock and $s^t \in S^{t+1}$ a t-period history of shocks. We assume that s_0 is distributed according to π^0 and that subsequently shocks evolve according to a Markov process with transition π. The implied probability distribution over shock histories s^t is denoted π^t.

A. Households

Preferences Households have preferences over stochastic sequences of cash goods $\{c_{1t}\}_{t=0}^{\infty}$, credit goods $\{c_{2t}\}_{t=0}^{\infty}$ and labor $\{l_t\}_{t=0}^{\infty}$ of the form:

$$E[\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}, l_t)], \tag{1}$$

where $U : \mathbb{R}^2_+ \times [0, T] \to \mathbb{R}$ is twice continuously differentiable on the interior of its domain, strictly concave, strictly increasing in its first two arguments and decreasing in its third argument. We assume that U satisfies the Inada conditions for $j = 1, 2$ and each (c_i, l), $i \neq j$, $\lim_{c_j \to 0} \frac{\partial U}{\partial c_j}(c_1, c_2, l) = \infty$ and for each (c_1, c_2), $\lim_{l \to T} \frac{\partial U}{\partial l}(c_1, c_2, l) = -\infty$. Finally, we assume that U is homothetic in (c_1, c_2) and weakly separable in l. Let U_{jt}, $j = 1, 2, l$ denote the derivatives of U with respect to each of its arguments at date t and let U_{jkt}, $j, k = 1, 2, l$ denote its second derivatives at t.

Trading Each household enters period t with a portfolio of money $M_t \geq 0$ and nominal (zero coupon) bonds $\{B_t^k\}_{k=1}^K \in \mathbb{R}^K_+$, where the superscript k denotes the maturity of the bond and K is the maximal maturity traded. The shock s_t is then realized. Asset market trading occurs in two rounds during the course of the day. The first *liquidity trading round* occurs in the morning immediately after the shock realization. In this, households are able to liquidate their bond holdings in light of their post-shock cash needs. On the other side of the market, the government (one may think of it as the Fed) responds to these needs by trading bonds for money. The household's budget constraint in this trading round is

$$A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_t^{k+1}(s^{t-1}) \geq \widetilde{M}_t(s^t) + \sum_{k=1}^{K} Q_t^k(s^t) \widetilde{B}_t^k(s^t), \tag{2}$$

where Q_t^k is the nominal price of the k-th maturity bond, $A_t = B_t^1 + M_t$ and $\widetilde{M}_t$ and $\{\widetilde{B}_t^k\}_{k=1}^K$ denote the portfolio of money and bonds purchased by households. Households then shop for cash and credit goods, exert effort in production and receive after-tax wage and dividend income. Since money is required for cash goods consumption, households face the cash-in-advance constraint:

$$P_t(s^t) c_{1t}(s^t) \leq \widetilde{M}_t(s^t). \tag{3}$$

In the afternoon, asset markets reopen allowing households to settle credit balances accrued whilst shopping and invest income. This second *hedging trading round* also allows the government (one may now think of it as the Treasury) to finance its budget deficit and purchase a portfolio that hedges itself against future shocks. Define: $\widetilde{A}_t(s^t) \equiv \widetilde{B}_t^1(s^t) + \{\widetilde{M}_t(s^t) - P_t(s^t) c_{1t}(s^t)\} - P_t(s^t) c_{2t}(s^t) + (1 - \tau_t(s^t)) I_t(s^t)$. Here P_t is the period t price level, τ_t is the income tax rate and I_t is the household's nominal income. The latter is given by $I_t = W_t l_t + \int_0^1 \Pi_{i,t} di$, where W_t is the nominal wage and $\Pi_{i,t}$ is the nominal profit of intermediate goods firm i at date t.[3] The household's budget constraint in the hedging trading round is:

$$\widetilde{A}_t(s^t) + \sum_{k=2}^{K} \widetilde{Q}_t^k(s^t)\widetilde{B}_t^k(s^t) \geq A_{t+1}(s^t) + \sum_{k=2}^{K} \widetilde{Q}_t^k(s^t)B_{t+1}^k(s^{t-1}), \tag{4}$$

where $\widetilde{Q}_t^k$ denotes the price of a k-maturity bond in this round.

Following Chari and Kehoe (1993), we assume that household participation in bond markets is anonymous, so that bonds issued by households are unenforceable and no one is willing to buy them.[4] Formally, we assume, for all t, s^t and k,

$$B_t^k(s^{t-1}) \geq 0, \quad \widetilde{B}_t^k(s^t) \geq 0. \tag{5}$$

This constraint precludes lending by the government to households in the bond market and we will refer to it as a *no lending constraint.* Both the repayment of government loans and the payment of taxes are transfers to the government. Ramsey models typically assume that the first is lump sum, while the second is not. This distinction is arbitrary. In practice, costs associated with enforcing repayments and monitoring household effort and productivity are likely to render loan repayments contingent on observed income or consumption. Hence, they will distort household decisions just as taxes do. We do not explicitly model such costs, rather we simply rule government loans out.[5]

Households maximize (1) subject to the constraints for all i, t, $c_{it} \geq 0$, $l_t \in [0, T]$ and (2)- (5).

B. Final goods firms

Final goods firms produce output Y_t from intermediate goods Y_{it} using the technology: $Y_t = [\int_0^1 Y_{it}^{\frac{1}{\mu}} di]^\mu$, $\mu > 1$. Intermediate goods are produced by sticky price firms who set their price P_{st} before s_t is realized, and flexible price ones who set their price P_{ft} after s_t is learned. Letting ρ denote the fraction of sticky price firms and assuming symmetry across each type of intermediate good firm, the total output of final goods firms is given by: $Y_t = [(1-\rho)Y_{ft}^{1/\mu} + \rho Y_{st}^{1/\mu}]^\mu$, where Y_{ft} and Y_{st} are, respectively, the amount of flexible and sticky price intermediate good used. Final

[3] We assume that the latter is paid as a dividend to the household. To economize on space and without loss of generality, we omit a detailed description of the stock market and assume instead that households own a diversified and non-tradeable portfolio of shares.

[4] On the other hand, we do allow households to borrow from local stores to finance credit good consumption.

[5] Weaker restrictions on government lending of the form $B_t^k(s^{t-1}) \geq -B$ would lead to qualitatively similar results.

goods firms are competitive and choose quantities of intermediate goods to maximize their profits:

$$\sup_{Y_{ft}(s^t),Y_{st}(s^t)} P_t(s^t)\left[(1-\rho)Y_{ft}(s^t)^{\frac{1}{\mu}}+\rho Y_{st}(s^t)^{\frac{1}{\mu}}\right]^{\mu}-(1-\rho)P_{ft}(s^t)Y_{ft}(s^t)-\rho P_{st}(s^{t-1})Y_{st}(s^t). \quad (6)$$

C. Intermediate goods

Intermediate goods are produced with labor according to the technology: $Y_{it}=\theta_t L_{it}^{\alpha}$, where $\theta_t(s^t)=\theta(s_t)$, $\theta : S \rightarrow \mathbb{R}_+$, is a productivity shock. Substituting this and the demand curves stemming from (6) into its objective, a flexible price intermediate goods firm chooses its price $P_{ft}(s^t)$ to solve:

$$\sup_{P_{ft}(s^t)} P_{ft}(s^t)\left(P_{ft}(s^t)/P_t(s^t)\right)^{\frac{-\mu}{\mu-1}}Y_t(s^t)-W_t(s^t)\left\{\left(P_{ft}(s^t)/P_t(s^t)\right)^{\frac{-\mu}{\mu-1}}Y_t(s^t)/\theta_t(s^t)\right\}^{\frac{1}{\alpha}}.$$

In contrast, a sticky price firm chooses its price $P_{st}(s^{t-1})$ before s_t is determined, so as to solve:

$$\sup_{P_{st}(s^{t-1})} E_{s^{t-1}}\left[(1-\tau_t)U_{2t}/P_t\left(P_{st}\left(P_{st}/P_t\right)^{\frac{-\mu}{\mu-1}}Y_t-W_t\left\{\left(P_{st}/P_t\right)^{\frac{-\mu}{\mu-1}}Y_t/\theta_t\right\}^{\frac{1}{\alpha}}\right)\right].$$

D. Government

The government faces a stochastic process for government spending $\{G_t\}_{t=0}^{\infty}$ of the form $G_t(s^t)=G(s_t)$, where $G : S \rightarrow \mathbb{R}_+$. The government finances its spending by levying taxes on labor and trading non-contingent nominal bonds. Its budget constraint from the first liquidity trading round is $A_{gt}(s^{t-1})+\sum_{k=1}^{K-1}Q_t^k(s^t)B_{gt}^{k+1}(s^{t-1})\leq \widetilde{M}_t(s^t)+\sum_{k=1}^{K}Q_t^k(s^t)\widetilde{B}_{gt}^k(s^t)$, where we use a g subscript to distinguish elements of the government's portfolio and $A_{gt}=B_{gt}^1+M_t$. Its budget constraint in the second hedging trading round is: $\widetilde{A}_{gt}(s^t)+\sum_{k=2}^{K}\widetilde{Q}_t^k(s^t)\ \widetilde{B}_{gt}^k(s^t)\leq A_{gt+1}(s^t)+\sum_{k=2}^{K}\widetilde{Q}_t^k(s^t)B_{gt+1}^k(s^t)$, where $\widetilde{A}_{gt}(s^t)=\widetilde{M}_t(s^t)+\widetilde{B}_{gt}^1(s^t)-\tau_t(s^t)I_t(s^t)+P_t(s^t)G(s_t)$.

E. Competitive equilibria and allocations

Define, respectively, an *allocation* and an $s^{t-1}-$*continuation allocation* to be sequences $e^{\infty}=\{c_{1t},c_{2t},L_{f,t},L_{s,t}\}_{t=0}^{\infty}$ and $e^{\infty}(s^{t-1})=\{c_{1t+r}(s^{t-1},\cdot),c_{2t+r}(s^{t-1},\cdot),L_{ft}(s^{t-1},\cdot),L_{st}(s^{t-1},\cdot)\}_{r=0}^{\infty}$.

Definition 1. *$\{c_{1t},\ c_{2t},\ l_t,\ L_{ft},\ L_{st},\ \tau_t,\ W_t,\ P_{st+1},\ P_{ft},\ P_t,\ \{Q_t^k\}_{k=1}^K,\ \{\widetilde{Q}_t^k\}_{k=1}^K,\ \{B_t^k\}_{k=1}^K,\ \{B_{gt}^k\}_{k=1}^K,\ M_t,\ \{\widetilde{B}_t^k\}_{k=1}^K,\ \{\widetilde{B}_{gt}^k\}_{k=1}^K,\ \widetilde{M}_t\}_{t=0}^{\infty}$ is a **competitive equilibrium** at $\{P_{s0},\ M_0,\ \{B_0^k\}_{k=1}^K\}$ if each $c_{it}\geq 0$, $l_t\in[0,T]$ and*

1. *$\{c_{1t},c_{2t},l_t,\{B_t^k\}_{k=1}^K,M_t,\{\widetilde{B}_t^k\}_{k=1}^K,\widetilde{M}_t\}_{t=0}^{\infty}$ solves the household's problem given $\{P_{s0},\ M_0,\ \{B_0^k\}_{k=1}^K\}$;*

2. *the sequence of input amounts* $\{L^{\alpha}_{ft}\}^{\infty}_{t=0}$ *and* $\{L^{\alpha}_{st}\}^{\infty}_{t=0}$ *solve the final goods firm's problem; the price sequences* $\{P_{ft}\}^{\infty}_{t=0}$ *and* $\{P_{s,t+1}\}^{\infty}_{t=0}$ *solve the intermediate firms' problems;*

3. *the government's budget constraints hold at each date;*

4. *the labor, bonds and goods markets clears:* $\forall t,\ s^t,\ l_t = (1-\rho)L_{ft} + \rho L_{st},\ B^k_t = B^k_{gt},\ \widetilde{B}^k_t = \widetilde{B}^k_{gt},$ $c_{1t} + c_{2t} + G_t = \theta_t[(1-\rho)L^{\alpha/\mu}_{ft} + \rho L^{\alpha/\mu}_{st}]^{\mu};$

5. *the no lending constraints hold:* $\forall t,\ s^t,\ k,\ B^k_{g,t}(s^{t-1}) \geq 0,\ \widetilde{B}^k_{g,t}(s^t) \geq 0.$

e^{∞} *is a competitive allocation if it is part of a competitive equilibrium.*

III. Characterizing competitive allocations

Proposition 2 provides a set of conditions that characterize competitive allocations. Before stating the proposition, we discuss those conditions that are new and refer the reader to Siu (2004) for further details of the other more standard conditions.

A. Implementability and Measurability constraints

Implementability and measurability constraints are central elements of any dynamic Ramsey taxation model. We describe and interpret these conditions under our asset market structure and then contrast them with the corresponding constraints from earlier work.

Primary Surplus Value First, define the *primary surplus value*: $\xi_t(s^t) \equiv E_{s^t}[\sum^{\infty}_{j=0} \beta^{t+j}\Lambda_{t+j}(s^{t+j})]$, where $\Lambda_{t+j} = U_{1t+j}c_{1t+j} + U_{2t+j}c_{2t+j} + U_{lt+j}\Upsilon_{t+j}$ and $\Upsilon_{t+j} \equiv \frac{\mu}{\alpha}[(1-\rho)L_{ft+j} + \rho L^{1-\frac{\alpha}{\mu}}_{ft+j} L^{\frac{a}{\mu}}_{st+j}]$. $\xi_t(s^t)$ gives the present discounted value of future primary surpluses accruing to the government after s^t. To see this, note that the definition of Λ_t, the household's first order conditions, the expression for profits from an intermediate goods firm and the resource constraint imply: $\Lambda_t = U_{2t}\ \{i^1_t \frac{M_t}{P_t} + [\tau_t \frac{I_t}{P_t} - G_t]\}$, where $i^1_t = \frac{1}{Q^1_t} - 1$ is the one period nominal interest rate.

Inflation Shocks Define: $N_t(s^t) \equiv ([(1-\rho)L^{\frac{\alpha}{\mu}}_{ft}(s^t) + \rho L^{\frac{\alpha}{\mu}}_{st}(s^t)]^{\mu}/L^{\alpha}_{st}(s^t))^{\frac{\mu-1}{\mu}}$. In a competitive equilibrium, $N_t(s^t) = \frac{P_{st}(s^{t-1})}{P_t(s^t)}$ and the sequence $\{N_t\}$ thus describes the shocks to inflation implied by a (competitive) allocation.[6]

[6] We formally prove this and other statements made in the current section in the appendix.

Bond Pricing Define the sequence $\{D^k_{t+1}\}_{k=1}^K$ by $D^1_{t+1} = 1$ and for $k > 1$,

$$D^k_{t+1}(s^t) \equiv \sum_{s^{t+k-1}} \left[\prod_{j=1}^{k-1} \frac{U_{2t+j}(s^{t+j})}{U_{1t+j}(s^{t+j})} \prod_{j=1}^{k-1} \left\{ \frac{N_{t+j}(s^{t+j}) U_{1t+j}(s^{t+j})}{E_{s^{t+j-1}}[N_{t+j}U_{1t+j}]} \right\} \pi^{k-1}(s^{t+k-1}|s^t) \right].$$

In competitive equilibria, the liquidity trading round bond price $Q^k_t(s^t)$ equals $\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D^k_{t+1}(s^t)$[7] and thus equals the expected product of cash-credit good wedges under the "distorted" probability measure:

$$\tilde{\pi}^{k-1}(s^{t+k-1}|s^t) = \prod_{j=1}^{k-1} \left\{ \frac{N_{t+j}(s^{t+j}) U_{1t+j}(s^{t+j})}{E_{s^{t+j-1}}[N_{t+j}U_{1t+j}]} \right\} \pi^{k-1}(s^{t+k-1}|s^t). \tag{7}$$

This measure weights states with higher than average values of $\prod_{j=1}^{k-1} N_{t+j}(s^{t+j}) U_{1t+j}(s^{t+j})$- i.e. states in which inflation shocks are modest and cash goods scare - more heavily.

Portfolio Weights Finally, define the *portfolio weights* $a_t(s^{t-1}) \equiv A_t(s^{t-1})/P_{s,t}(s^{t-1})$ and for each k, $b^k_t(s^{t-1}) \equiv B^k_t(s^{t-1})/P_{s,t}(s^{t-1})$.

Using this notation, the implementability/measurability constraints are, for all t, s^t:

$$\underbrace{\frac{\xi_t(s^t)}{U_{1t}(s^t)}}_{primary\ \ surplus} = \underbrace{N_t(s^t) \left\{ a_t(s^{t-1}) + \frac{U_{2t}(s^t)}{U_{1t}(s^t)} \sum_{k=1}^{K-1} b^{k+1}_t(s^{t-1}) D^k_{t+1}(s^t) \right\}}_{liabilities}. \tag{8}$$

At date 0, the portfolio weights are predetermined, whereas at dates $t > 0$ they are chosen as part of a competitive equilibrium. In the latter case, they are measurable with respect to information at date $t-1$ and (8) places cross state restrictions on the process for ξ_t. Following Aiyagari et al (2002), we refer to the date 0 version of (8) as the *implementability constraint* and to the date $t > 0$ versions as *measurability constraints.* The left and right hand sides of (8) can be interpreted, respectively, as government primary surplus and liability values[8]; (8) asserts that these values must be equal after all histories.

[7] The inability of households to borrow on bond markets introduces potential indeterminacy in bond prices. We resolve this by assuming throughout that bond prices are set so that households never wish to borrow. This assumption does not restrict the set of competitive allocations; it ensures non-negative interest rates at all times.

[8] The latter interpretation follows from $N_t(s^t) = P_{s,t}(s^{t-1})/P_t(s^t)$, $Q^k_t(s^t) = [U_{2t}(s^t)/U_{1t}(s^t)]D^k_{t+1}(s^t)$ and

$$A_t(s^{t-1})/P_t(s^t) + \sum_{k=1}^{K-1} B^{k+1}_t(s^{t-1})/P_t(s^t) Q^k_t(s^t) = P_{s,t}(s^{t-1})/P_t(s^t) \left\{ a_t(s^{t-1}) + \sum_{k=1}^{K-1} b^{k+1}_t(s^{t-1}) Q^k_t(s^t) \right\}.$$

Although the portfolio weights $\{a_t, \{b_t^{k+1}\}_{k=1}^{K-1}\}$ are s^{t-1}-measurable, variations in the price level (i.e. in $N_t(s^t)$) or the nominal term structure (i.e. $\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_{t+1}^k(s^t)$) allow the government to adjust the value of its portfolio in response to and, hence, hedge contemporaneous shocks. However, this hedging comes at a cost, $N_t(s^t)$ and $\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_{t+1}^k(s^t)$ terms also capture the distortions associated with innovations to the price level or term structure. If events at t induce flexible price firms to alter their prices relative to their previously expected level, then $N_t(s^t)$ departs from 1 and an inefficient allocation of production across firms will occur. If the price of the k-th maturity outstanding bonds falls, then $\{\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_{t+1}^k(s^t)\}_{k=1}^K$ also departs from 1 and the short run nominal interest rate must exceed zero either now or in some future state. This results in a misallocation of consumption across cash and credit goods as households seek to economize on their use of cash.

B. Comparison with Existing Models

The key difference between our model and others becomes apparent in the measurability constraints. We contrast our version of these conditions with those in the less restrictive environment of Lucas and Stokey (1983) and the more restrictive ones of Siu (2004) and Aiyagari et al (2002).

In the model of Lucas and Stokey, the government can trade real state contingent debt, and so the analogue of (8) is:

$$\xi_t(s^t)/U_{1t}(s^t) = a_t(s^t). \tag{9}$$

The portfolio weight a_t is s^t-measurable so that (9), unlike (8), does not represent a collection of cross state restrictions. Except at date 0, when $a_0(s^0)$ is fixed, the constraints in (9) are redundant.

On the other hand, in Siu (2004), nominal debt of only one period is traded and (8) reduces to:

$$\xi_t(s^t)/U_{1t}(s^t) = N_t(s^t) a_t(s^{t-1}). \tag{10}$$

Thus, $\frac{\xi_t(s^t)}{U_{1t}(s^t)}$ can be varied across states s_t only through contemporaneous inflation shocks $N_t(s^t)$. Since there is no long term debt, there is clearly no opportunity to devalue this debt through increases in future nominal interest rates.

Finally, in Aiyagari et al (2002), only real debt of one period is traded, which implies:

$$\xi_t(s^t)/U_{1t}(s^t) = a_t(s^{t-1}). \tag{11}$$

In this case, the debt value $a_t(s^{t-1})$ cannot be altered contemporaneously.

C. No Lending and Nominal Wealth-in-Advance Constraints

The *no lending constraints* ensure that the household's bond holdings are non-negative. For maturities $k > 1$, we have $b_t^k \geq 0$, while for one period nominal liabilities $a_t \geq 0$. Finally, we have a sequence of *nominal wealth-in-advance constraints.* Households must use some fraction of their nominal wealth in the liquidity trading round to obtain the money necessary for cash good consumption. Consequently, their total nominal wealth restricts their cash good consumption. Using the measurability constraints (8), this restriction can be expressed as:

$$\xi_t(s^t) \geq U_{1t}(s^t)c_{1t}(s^t). \tag{12}$$

Proposition 2 formally characterizes competitive allocations; its proof is in the Appendix.

Proposition 2. *$e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^\infty$ is a competitive allocation at $\{P_{s,0}, M_0, \{B_0^k\}_{k=1}^K\}$ if there exists a sequence of portfolio weights $\{a_t, \{b_t^{k+1}\}_{k=1}^{K-1}\}_{t=0}^\infty$ with $a_0 = \frac{M_0+B_0^1}{P_{s0}}$ and $b_0^{k+1} = \frac{B_0^{k+1}}{P_{s0}}$, such that the portfolio weight sequence and e^∞ satisfy $\forall\ i, t, s^t$, $c_{it}(s^t) > 0$, $(1-\rho)L_{ft}(s^t) + \rho L_{st}(s^t) \in (0, T)$, (8), no lending: $\forall k, t, s^{t-1}$, $b_t^k(s^{t-1}) \geq 0$ and $a_t(s^{t-1}) \geq 0$, (12) and*

1. *(Transactions)*[9] *for all t, s^t,*

$$U_{1t}(s^t)/U_{2t}(s^t) \geq 1; \tag{13}$$

2. *(Resource) for all t, s^t,*

$$G(s_t) + c_{1t}(s^t) + c_{2t}(s^t) = \theta(s_t)[(1-\rho)L_{ft}^{\frac{\alpha}{\mu}}(s^t) + \rho L_{st}^{\frac{\alpha}{\mu}}(s^t)]^\mu; \tag{14}$$

3. *(Sticky price optimality) for all $t > 0$, s^{t-1},*

$$\sum_{s^t|s^{t-1}} \pi(s^t|s^{t-1})U_{lt}(s^t)[L_{ft}(s^t)^{1-\frac{\alpha}{\mu}}L_{st}(s^t)^{\frac{\alpha}{\mu}} - L_{st}(s^t)] = 0. \tag{15}$$

If $e^\infty = \{c_{1t}, c_{2t}, L_{ft}, L_{st}\}_{t=0}^\infty$ is a competitive allocation at $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$ with each $c_{it} > 0$ and $(1-\rho)L_{ft} + \rho L_{st} \in (0, T)$, then e^∞ satisfies (8), no lending, (12)-(15) for some sequence of portfolio weights $\{a_t, \{b_t^{k+1}\}_{k=1}^{K-1}\}_{t=0}^\infty$, with $a_0 = \frac{M_0+B_0^1}{P_{s0}}$ and $b_0^{k+1} = \frac{B_0^{k+1}}{P_{s0}}$.

[9] This restriction is sometimes referred to as a no arbitrage constraint, since it implies non-negative nominal interest rates. In our model, arbitrage between money and bonds is automatically precluded by the no borrowing restriction on households. However, the cash-in-advance constraint continues to ensure that the marginal utility of cash goods exceeds that of credit goods. Our "transactions" label indicates the source of the restriction.

IV. Ramsey problems for incomplete and complete markets economies

The Ramsey problem with non-contingent nominal debt Given Proposition 2, the optimal policy problem in an economy with initial triple $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$ can be formulated as:

$$\textbf{Problem 1:} \quad \sup_{\{c_{1t}, c_{2t}, L_{ft}, L_{st}, a_t, \{b_t^k\}_{k=2}^K\}_{t=0}^\infty} E\left[\sum_{t=0}^\infty \beta^t U(c_{1t}, c_{2t}, (1-\rho)L_{f,t} + \rho L_{s,t})\right] \tag{RP}$$

subject to $a_0 = \frac{M_0 + B_0^1}{P_{s0}}$ and $b_0^k = \frac{B_0^k}{P_{s0}}$, for all t, $c_{1t}, c_{2t} \geq 0$ and $(1-\rho)L_{ft} + \rho L_{st} \in [0, T]$, (8), no lending and (12)-(15).

The Ramsey problem with complete markets Before analyzing (RP) in detail, we briefly turn to the benchmark complete markets economy. In this the government faces no restrictions on lending and can trade contingent claims. The corresponding Ramsey problem is essentially that considered by Siu (2004) and others. We merely state two key properties of its solution.

Proposition 3. *Under our assumed preferences, after period 0: 1) the Friedman rule holds and $U_{1t} = U_{2t}$, 2) flexible price firms set their prices equal to those of sticky price firms and $N_t = 1$.*

Thus, when markets are complete, optimal policy implies nominal yields equal to zero at all maturities and an absence of inflation surprises, i.e. P_t/P_{t-1} is s^{t-1}-measurable, $t \geq 1$.

Comparing policy in incomplete and complete markets economies It is convenient to rewrite the measurability constraints (8) in matrix form. Given an allocation, let $\Xi_t(s^{t-1})$ be the $N \times 1$ vector with n-th element $\xi_t(s^{t-1}, \hat{s}_n)/U_{1t}(s^{t-1}, \hat{s}_n)$; let $\Psi_t(s^{t-1})$ be the $N \times K$ matrix with $(n, 1)$-th element $\psi_{1,t}(s^{t-1}, \hat{s}_n) = N_t(s^{t-1}, \hat{s}_n)$ and (n, k)-th element, $k > 1$, $\psi_{k,t}(s^{t-1}, \hat{s}_n) = N_t(s^{t-1}, \hat{s}_n)\frac{U_{2t}(s^{t-1}, \hat{s}_n)}{U_{1t}(s^{t-1}, \hat{s}_n)} D_{t+1}^k(s^{t-1}, \hat{s}_n)$. The measurability constraints may be restated as, $\forall\ t$, s^{t-1},

$$\Xi_t(s^{t-1}) \in \text{Span}(\Psi_t(s^{t-1})). \tag{16}$$

It follows from Proposition 3 that when markets are complete, the optimal continuation allocation sets each $\Psi_t(s^{t-1})$, $t \geq 1$, equal to the unit matrix. By (16), implementation of this allocation in the nominal debt economy is only possible if its surplus values $\xi_t(s^t)/U_{1t}(s^t)$ are s^{t-1}-measurable. Typically, this is not true and the optimal complete markets allocation *cannot* be implemented with non-contingent nominal debt. The logic is simple: this allocation usually requires that fiscal shocks be hedged and liability values varied, yet it also precludes state-contingent variations in interest rates and inflation. The latter are precisely the means by which hedging is attained in an economy with non-contingent nominal debt. At the optimal complete markets allocation all nominal assets (regardless of maturity) have the same risk free return and no fiscal hedging is possible.

Although, the optimal complete markets allocation does not usually satisfy the measurability constraints, there are arbitrarily small perturbations of it that do. Small, state-specific substitutions of cash for credit good consumption at the optimal complete markets allocation can be used to perturb the matrices $\Psi_t(s^{t-1})$ so that (16) holds. However, while allocations very close to the optimal complete markets one can be implemented in non-contingent nominal debt economies, their implementation usually requires very large negative (and positive) asset positions at some maturities. Such positions are needed to obtain sufficient hedging off of the small variations in interest rates and inflation implied by the perturbation. Clearly, restrictions on short selling in general, and our no lending constraints in particular, prevent the government from obtaining large negative asset positions. In doing so they usually preclude allocations in a neighborhood of the optimal complete markets one.

Comparing policy in economies with real and nominal incompleteness Buera and Nicolini (2004) and Angeletos (2002) consider economies in which the government can trade real non-contingent claims of various maturities. Angeletos shows that generically the optimal complete markets allocation can be implemented with non-contingent real debt if the number of maturities traded exceeds the number of states. However, the calibrated examples of Buera and Nicolini suggest that the government may need to take large debt positions to achieve this implementation. Buera and Nicolini regard this as a problem. In contrast, with non-contingent *nominal* debt, the optimal complete markets allocation can not usually be implemented since, as noted, it implies that all nominal assets offer the same riskless return. Moreover, since the implementation of allocations close to the optimal complete markets one typically requires extreme asset market positions, the problem identified by Buera and Nicolini is more severe in an economy with nominal debt.

V. Optimal hedging with nominal incompleteness

This section uses an analytically tractable example to isolate a key motive for issuing long term debt. Such debt permits the postponement of distortions to private consumption decisions.

General Case Let $-\omega_0$ and $-\omega_t$, $t > 0$, denote respectively the Lagrange multipliers on the implementability and t-th period measurability constraints in (RP). Define the cumulative multiplier $\psi_t := -\sum_{j=0}^{t} \omega_j$; ψ_t is the shadow value of the government's continuation primary surplus stream. ω_t represents a shock to ψ_t and may be interpreted as a measure of the government's additional desire for funds in period t. We leave the underlying source of shocks unspecified; they may perturb government spending, productivity or both. In the subsequent discussion, the essential requirement is that shocks induce variation in the shadow value of the primary surplus stream across states. Their exact source is irrelevant.

The first order conditions from Problem (RP) reveal the basic cost-hedging tradeoff that influences the choice of maturity structure. These conditions deliver the asset pricing equation:

$$\frac{Q_t^k}{P_t} = \beta E_t \left[\frac{Q_{t+1}^{k-1}}{P_{t+1}} \frac{U_{1t+1}\psi_{t+1}}{U_{1t}\psi_t} \right] + \kappa_t^k, \tag{17}$$

where κ_t^k is a normalized multiplier from (k,t)-th no no lending constraint. $\mathcal{M}_{t+1} = \beta \frac{U_{1t+1}\psi_{t+1}}{U_{1t}\psi_t}$ has a natural interpretation as the government's stochastic discount factor. Expanding (17) yields the CAPM-like equation:

$$E_t[R_{t+1}^k] - R_{t+1}^f = -R_{t+1}^f \mathrm{Cov}_t \left(\frac{Q_{t+1}^{k-1}}{P_{t+1}}, \beta \frac{U_{1t+1}}{U_{1t}} \frac{\psi_{t+1}}{\psi_t} \right) + \hat{\kappa}_t^k \tag{18}$$

Here R_{t+1}^k is the gross return on the k-th period nominal debt and R_{t+1}^f is the riskless rate implied by the government's stochastic discount factor (i.e. $1/E_t[\mathcal{M}_{t+1}]$). The CAPM equation (18) formalizes the tradeoff between the expected cost and the hedging benefits of borrowing at a specific maturity. Larger hedging benefits (as captured by the covariance term in (18)) are associated with either a larger cost premium ($E_t[R_{t+1}^k] - R_{t+1}^f$) or a reduced no lending shadow price $\hat{\kappa}_t^k$.

Suppose that $U(c_1, c_2, l) = (1-\gamma)\log(c_1) + \gamma \log(c_2) + v(l)$ for some smooth, concave, decreasing function v. This functional form simplifies the analysis by rendering the primary surplus values in the measurability constraints independent of the consumption allocation. We can then focus on the implications of this allocation for liability values and the hedging properties of nominal debt. The first order conditions for $c_{it}(s^t)$, $i = 1, 2$ may now be combined to give:

$$[U_{1t}(s^t) - U_{2t}(s^t)] = -\eta_t(s^t) \left[U_{11t}(s^t) + U_{22t}(s^t) \right] + \sum_{j=0}^{K-2} \beta^{-j} \delta_{t,t-j}(s^{t-1}) \omega_{t-j}(s^{t-j}) \tag{19}$$

where $\eta_t(s^t)$ is the multiplier on the s^t-th transactions constraint, $\delta_{t,t-j}(s^{t-1}) = \frac{\partial \Gamma_{t-j}}{\partial c_{1t}}(s^{t-1}) - \frac{\partial \Gamma_{t-j}}{\partial c_{2t}}(s^{t-1}) > 0$, for $t-j \geq 0$, $\Gamma_{t-j} = N_{t-j}U_{1t-j}\left[a_{t-j} + \sum_{k=1}^{K-1}\gamma_{t-j,k}b_{t-j}^{k+1}\right]$ is the $t-j$-th period value of the government's liabilities and for $t-j<0$, $\Gamma_{t-j} := 0$. Equation (19) describes the costs and benefits of a small substitution of cash for credit goods at date t. The term on the left hand side gives the utility cost of the substitution, while those terms on the right hand side capture the shadow benefits from relaxing the transactions constraint and from hedging. Note that the marginal hedging benefit term, the final term on the right hand side, is a weighted sum of the measurability shadow prices $\{\omega_{t-j}\}_{j=0}^{K-2}$, where the weights describe the effect of the cash-credit substitution on government's liability values. If $\sum_{j=0}^{K-2}\beta^{-j}\delta_{t,t-j}(s^{t-1})\omega_{t-j}(s^{t-j}) > 0$, then $U_{1t}(s^t) - U_{2t}(s^t) > 0$ and the s^t-nominal interest is positive, otherwise, the nominal interest rate is 0.

Two features of optimal interest rate policy become apparent in (19). First, nominal interest rates in period t are used to adjust liability values and hedge shocks in multiple past periods as well as the present. In this way, the effect of a shock on nominal interest rates is propagated over time. Second, the weight attached to ω_{t-j} is scaled by β^{-j}. Other things equal, this scaling implies that current nominal interest rates are more sensitive to funding need shocks further back in the past (but within the maturity of the government's portfolio). The logic behind this is straightforward. A perturbation to the cash-credit allocation at date $t+k$, confers a hedging benefit at t. Since the nominal debt price is given by the (distorted) conditional expectation of the product of cash-credit marginal rates of substitution (MRS's) over the term of the debt, a perturbation at $t+k$ is, potentially, as effective at altering the price of outstanding debt (with maturity in excess of $k+2$) as a perturbation at t. However, the utility cost of the former perturbation is not born for k periods and is correspondingly discounted. This is a force for the optimal postponement of the nominal interest rate adjustments used in the hedging of shocks; the relative scaling in (19) captures this. It is also a force for the use of longer term debt; such debt permits greater postponement. To see this more explicitly, consider the following example.

Example We make two additional simplifying assumptions. First, we suppose that there is a single shock s at date 1 drawn from $S = \{\widehat{s}_1, \widehat{s}_2\}$. In the remainder of this section all variables dated $t \geq 1$ will be indexed by this shock. Again, the exact source of the shock is not important, we merely require that it introduces stochastic variation into the measurability constraint shadow prices. Since there are no shocks drawn in later periods, $t > 1$, $\omega_t = 0$. Second, we suppose that there is no debt of maturity greater than one outstanding in period 0 so that for all $t = 1, \ldots, K-2$,

$\delta_{t,0}(s^{t-1}) = 0$. These assumptions disentangle the effects of multiple shocks, they imply that variations in nominal interest rates in periods $t \geq 1$ are used solely to hedge the period 1 shock and that their variation has no implications for asset prices prior to period 1. We retain the utility function $U(c_1, c_2, l) = (1-\gamma)\log c_1 + \gamma \log c_2 + v(l)$.

Using the formulas for debt prices, the first order condition for the k-th maturity portfolio weight evaluated in period 1 is:

$$0 = -\sum_{s \in S} \left[N_1(s) \prod_{j=1}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)} \right] U_{11}(s)\omega_1(s)\pi(s) + \kappa^k. \tag{20}$$

The first term on the right hand side of (20) is the expected product of the normalized period 1 debt price (the bracketed term) and $U_{11}(s)\omega_1(s)$. This term incorporates the period 1 hedging benefits of k maturity debt. Apart from knife-edge cases, $\omega_1(s') > 0 > \omega_1(s'')$ for some pair s', s'' and there is stochastic variation in the government's ex post desire for funds. Manipulation of (20) and the first order conditions for consumption reveals that it is *weakly* optimal to use *some* of the longest term debt and that for each date $t = 1, \cdots, K-1$, $\frac{U_{2t}(s')}{U_{1t}(s')} > 1 = \frac{U_{2t}(s'')}{U_{1t}(s'')}$. Intuitively, longer term debt gives the government greater flexibility in using nominal interest rates to hedge the shock s. Furthermore, this hedging is achieved through increases in nominal interest rates over the term of the outstanding debt in period 1 when the government's ex post desire for funds is high. It is then immediate that the first term on the right hand side of (20) is increasing in k and so, in fact, it is *strictly* optimal to use *only* the longest term debt.

We now consider the pattern of nominal interest rates in state s'. In the current example, equation (19) reduces to:

$$U_{1t}(s) - U_{2t}(s) = -\eta_t(s)\left[U_{11t}(s) + U_{22t}(s)\right] - \beta^{-(t-1)}\delta_{t,1}(s)\omega_1(s), \tag{21}$$

with $\delta_{t,1}(s) = F(s)\left[\frac{U_{11t}(s)}{U_{1t}(s)} + \frac{U_{22t}(s)}{U_{2t}(s)}\right]$ and $F(s) = N_1(s)\prod_{j=1}^{K-1}\frac{U_{2j}(s)}{U_{1j}(s)}U_{11}(s)b_1^K$. This in turn implies:

$$\frac{U_{1t}(s)}{U_{2t}(s)} = H\left(\max(0, \beta^{-(t-1)}F(s)\omega_1(s))\right), \tag{22}$$

for some increasing and convex function H with $H(0) = 1$. It follows that when $\omega_1(s) > 0$, nominal interest rates are increasing in t, for $t = 2, \cdots, K$. Otherwise, they are set to 0. Put differently, when the government receives a shock to the shadow value of its primary surplus stream, the increases in cash-credit MRS's necessary to devalue its liabilities are postponed until later in the term of the debt.

The following results formalize these arguments; their proofs are supplied in the appendix.

Lemma 4. *(Use of long term debt) Suppose that the optimal multipliers satisfy* $\omega_0 > 0$ *and* $\omega_1(s) > 0 > \omega_1(s')$*, some pair* $s, s' \in S$*, then the government uses only the longest term debt. When* $\omega_1(s) > 0$*, the nominal interest rate is greater than zero in periods* $t = 2, \cdots, K-1$*.*

Lemma 5. *(Postponement effect) Suppose that the optimal multipliers satisfy* $\omega_0 > 0$ *and* $\omega_1(s) > 0 > \omega_1(s')$*. In the state* s *such that* $\omega_1(s) > 0$*,* $Q^1_{t+1}(s) < Q^1_t(s)$*,* $k = 1, \cdots, K-1$*. For* $t > K-1$*,* $Q^1_t(s) = 1$*. In the state* s *such that* $\omega_1(s) < 0$*,* $Q^1_t(s) = 1$ *for all* t*.*

Implications for the yield curve and term premia Lemma 5 has immediate implications for the yield curve. Since all uncertainty is resolved at date 1, the expectations hypothesis holds from that date onwards and $Q^k_1 = \prod_{j=1}^k Q^1_{1+j}$. It follows that if $\omega_1(s) \leq 0$, then the date 1 yield curve remains at zero. On the other hand, if $\omega_1(s) > 0$, the yield curve rises and steepens over the horizon $k = 1, \cdots, K-1$. Yields at maturities greater than $K-1$ asymptote towards zero as the maturity increases. Thus, the yield curve is hump shaped, with the hump occurring at maturity $K-1$. As time passes and the debt outstanding at the time of the shock matures, the hump occurs at a progressively shorter maturity, before disappearing at date K.

Lemma 5 also implies that the proportional variation in the period 1 debt price $\frac{Q_1^{k-1}(s)}{E_0[Q_1^{k-1}]}$ across states is increasing in the debt's maturity k, until $k = K-1$. This contributes to a date 0 term premium that is increasing in the maturity of the debt until $k = K-1$. Despite the relative cost of K-maturity debt, the government uses it because it is able to postpone the distortions associated with fiscal hedging.

VI. A recursive formulation

We now look for a recursive formulation of (RP). This formulation must ensure that continuation choices attain the primary surplus and liability values implied by past implementability/measurability constraints. More formally, the measurability constraint may be rewritten as:

$$[U_{1t}(s^t)a_{t-1}(s^{t-1}) + U_{2t}(s^t)\sum_{k=1}^{K-1} b_{t-1}^{k+1}(s^{t-1})D_{t+1}^k(s^t)]N_t(s^t) = \Lambda_t(s^t) + \beta\phi_{t+1}(s^t), \tag{23}$$

where $\phi_{t+1}(s^t) = E_{s^t}\left[\xi_{t+1}(s^{t+1})\right]$. Additionally, (12) can be recast in terms of ϕ_{t+1} as:

$$\Lambda_t(s^t) + \beta\phi_{t+1}(s^t) \geq U_{1t}(s^t)c_{1t}(s^t). \tag{24}$$

The tuple $\{\phi_{t+1}, \{D^k_{t+1}\}_{k=1}^{K-1}\}$ may be interpreted as a list of implicit "promises" made by the government at t concerning the value of its primary surplus stream and of specific bonds within its portfolio. Satisfaction of (23) requires that future choices implement these promises. Our earlier definitions imply that the ϕ_t and D^k_t variables evolve recursively according to:

$$\phi_t(s^{t-1}) = E_{s^{t-1}}\left[\Lambda_t(s^t) + \beta\phi_{t+1}(s^t)\right], \tag{25}$$

$D^1_t := 1$ and for $k = 2, \cdots, K-1$,

$$D^k_t(s^{t-1}) = E_{s^{t-1}}\left[\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D^{k-1}_{t+1}(s^t) \frac{N_t(s^t)U_1(s^t)}{E_{s^{t-1}}[N_t(s^t)U_{1t}(s^t)]}\right]. \tag{26}$$

Our recursive approach to (RP) treats the variables $x_t = \{s_{t-1}, \phi_t, \{D^k_t\}_{k=1}^{K-1}\}$ as state variables that summarize relevant aspects of the past history of the economy and ensure that past constraints are satisfied. As with most Ramsey problems, the initial period of (RP) differs from subsequent ones. In the initial period, the government faces a fixed vector of portfolio weights $\{a_0, \{b_0^{k+1}\}_{k=1}^{K-1}\}$ rather than a fixed vector of state variables x_0. In later periods, this is reversed: the government can be modeled as entering period $t \geq 1$ with a state vector x_t and choosing portfolio weights $\{a_t, \{b_t^{k+1}\}_{k=1}^{K-1}\}$ along with a current allocation $\{c_{1t}, c_{2t}, L_{ft}, L_{st}\}$ and a continuation state vector x_{t+1}. Thus, the continuation of the Ramsey problem is recursive in the state variables $\{x_t\}$. In the remainder of this section, we formally state the associated dynamic programming problem. The policy functions that solve this problem can be used to generate an optimal continuation allocation along with corresponding optimal policies.[10]

Let $\mathbf{X}$ denote the set of tuples $\{s, \phi, \{D^k\}_{k=2}^{K-1}\}$ that are attained by some continuation competitive allocation in its initial period. We collect recursive versions of the constraints that define a competitive allocation into a correspondence Γ. Given an inherited tuple of state variables, these constraints ensure that a current consumption-labor allocation and tuple of future state variables are consistent with the requirements of a competitive allocation.

Definition 6. *Let $\Gamma(s, \phi, \{D^k\}_{k=2}^{K-1})$ equal all tuples $\{a, \{b^k\}_{k=2}^{K}, c_1,\ c_2,\ L_f,\ L_s, \phi',\ \{D^{k\prime}\}_{k=2}^{K-1}\}$ that*

[10] LSY (2006) formally demonstrate the recursivity of the continuation Ramsey problem in these state variables.

satisfy for each s', $c_i(s') \geq 0$, $i = 1, 2$, $(1-\rho)L_f(s') + \rho L_s(s') \in [0, L]$, (13)-(15), $a \geq 0$, $b^k \geq 0$, (23)-(26) and for each s' $(s', \phi'(s'), \{D^{k\prime}(s')\}_{k=2}^{K-1}) \in \mathbf{X}$.

The correspondence Γ provides the constraint set for our dynamic programming problem:

$$V(s, \phi, \{D^k\}_{k=2}^{K-1}) = \sup_{\Gamma(s,\phi,\{D^k\}_{k=2}^{K-1})} E_s[U(c_1, c_2, (1-\rho)L_f + \rho L_s) + \beta V(s', \phi', \{D^{k\prime}\}_{k=2}^{K-1})]. \tag{27}$$

Problem (27) can be solved numerically and its computed policy functions used to obtain the optimal continuation Ramsey allocation along with the supporting optimal fiscal and monetary policies at each initial state vector $\{s_0, \phi_1, \{D_1^k\}_{k=2}^{K-1}\}$. We pursue this approach in Section VII. Before doing so, we provide an example that permits an analytical solution and that builds intuition.

VII. A Calibrated Example

A. Numerical method and parameter values

Numerical method We solve the dynamic programming problem (27) numerically and then back out the implied optimal policies. The state space $\mathbf{X}$ for these problems is endogenous and of dimension K. In our calculations we restrict the state space to be a K-dimensional rectangular set $\tilde{X}$ and check that enlarging $\tilde{X}$ does not significantly alter the numerical results we report. The dynamic programming problem is solved by a value iteration. The main computational difficulty concerns the dimension of the state space which is increasing in the maximal debt maturity K. To enable us to solve problems with a maturity structure of reasonable length, we use Smolyak's algorithm to approximate the government's value function on a sparse grid fitted to $\tilde{X}$. For further details on Smolyak's algorithm see Krueger and Kubler (2004).

Calibration To permit comparability of our results to those in Siu (2004) and Chari et al (1991), we compute a baseline case with parameter values that are close to theirs. In this baseline case, we assume preferences of the form:

$$U(c_1, c_2, l) = \log\left[(1-\gamma)c_1^{\phi} + \gamma c_2^{\phi}\right]^{\frac{1}{\phi}} + \psi \log(T - l). \tag{28}$$

The preference parameters γ, ϕ and β are set to 0.58, 0.79 and 0.96; ψ is chosen so that approximately 30% of an agent's time is spent working. The values of γ and ϕ are similar to those used by Siu (2004) and Chari et al (1991). We follow Siu (2004) and set the production parameters α, μ and ρ to 1.0, 1.05 and 0.08 respectively. Government spending takes on two values $\underline{G}$ and $\overline{G}$. The government spending process has a mean of around 20% of GDP in a complete markets model with a debt to GDP ratio of 60%. We set the standard deviation of this process to be 6.7% and its autocorrelation coefficient to 0.95. These values are close to those estimated from the data and conform to the values used in Siu.[11]

Our baseline case sets the maximal maturity K to 7. We contrast this with variations of the model in which K is less than 7 and conjecture that all of the effects we identify would be quantitatively reinforced if K were raised above 7.

B. Results

B.1. Maturity structure

All numerical calculations confirm that the government uses only the longest maturity debt available. In each hedging trading round, it funds its deficit and refinances its portfolio with debt of maturity K. In the remainder of this section, we focus on the implications of optimal policy for nominal interest rates, inflation and debt holding returns. We illustrate these implications with short simulations that highlight the consequences of particular sequences of shocks and with sample moments from long simulations.

B.2. Short simulations

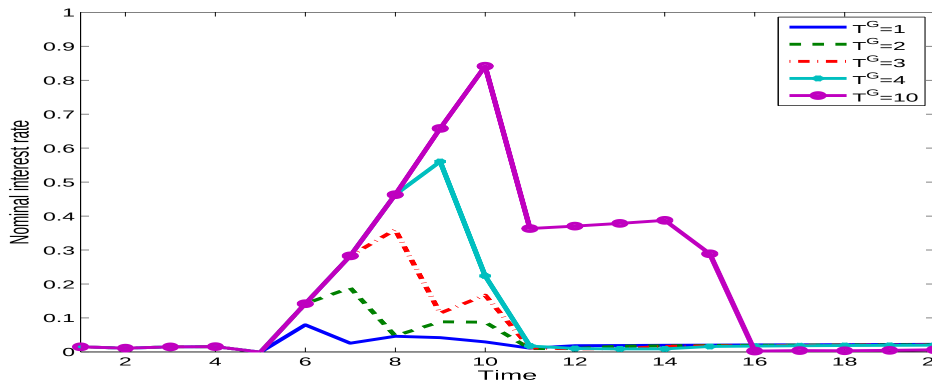
This section presents short simulations. In each, low spending shocks are drawn until period 4, high spending shocks from period 5 until $4 + T^G$ and low spending shocks thereafter. The government has an initial debt to output ratio of about 40%.

Nominal interest rates Figure 1 shows several sample paths for one period nominal interest rates. The figure indicates that a transition from low to high spending at date t is associated with an accumulation of nominal interest rates until $\min(t + T^G, t + K - 1)$, i.e. until the spell of high spending shocks ends or until the maturity date of debt outstanding at t is reached. In the first case, the nominal interest rate quickly falls back to 0; in the second (see the $T^G = 10$ line in the

[11] Given space constraints, we report only results from an economy with government spending shocks. We have obtained similar quantitative results in an economy with productivity shocks.

figure), it falls to a positive number, falling back to 0 only when the high spending spell finally comes to an end.

Figure 1. One period nominal interest rates



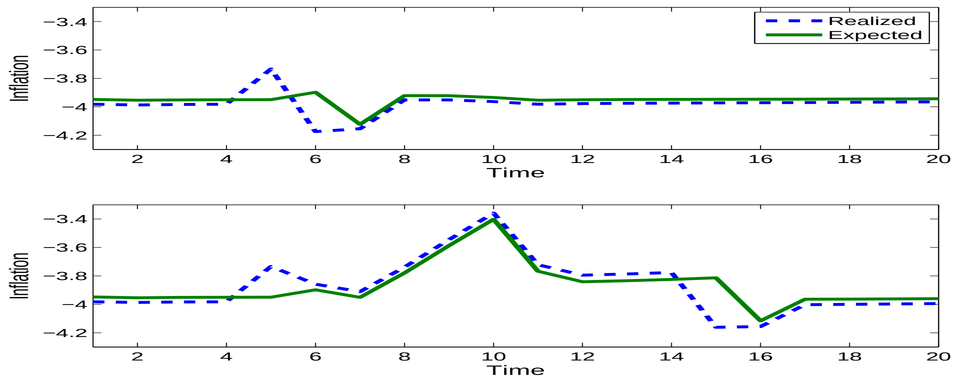
Higher future interest rates after a transition from low to high spending deliver capital losses to long term debt holders and contribute to fiscal hedging. The particular pattern of interest rates across time and states reflects postponement and efforts to hedge shocks in multiple periods. The highest nominal interest rates occur later in the term of the debt so postponing their costs and after several consecutive high spending shocks when they can contribute to hedging in multiple periods.

The qualitative effects described above occur in economies with shorter maximal debt maturities $K = 3, \cdots, 6$, but they are quantitatively damped. For example, when $K = 3$ (and $T^G = 10$), the nominal interest rate peaks at 0.3% rather than 0.83%.

Inflation Figure 2 shows paths of realized and conditional expected inflation ($E_{t-1}\left[\frac{P_t}{P_{t-1}}\right]$) for short $T^G = 1$ and long $T^G = 10$ spells of high spending shocks. When $T^G = 1$, a small positive inflation innovation in period 5 (of about 0.2%), is followed by a small negative innovation in the next period. These innovations contribute to fiscal hedging by devaluing and then revaluing the government's portfolio as the high spending spell begins and ends. When $T^G = 10$, a similar positive innovation in period 5, is followed by an increase in both realized *and* expected inflation over periods 7-10. The latter increase is associated with the corresponding rise in nominal interest rates at this time.[12] As the rise in nominal interest rates is attenuated after period 11, so too is that in expected and realized inflation. When the high spending spell comes to an end then, as in the $T^G = 1$ case, there is a small negative inflation innovation.

[12] A risk-adjusted Fisher equation holds: $\beta E_t[U_{1t+1}/U_{1t}]\ E_t[P_t/P_{t+1}]\ Q_t^1 + \beta\ \mathrm{Cov}_t[U_{1t+1}/U_{1t}, P_t/P_{t+1}] = 1$.

Figure 2. Inflation, $T^G = 1$, $T^G = 10$



Holding returns Collectively, these changes in nominal interest rates and innovations in inflation deliver real capital gains and losses to households. In doing so they alter realized real holding returns and allow the government to hedge shocks. Figure 3 shows paths for realized and conditional expected real holding returns on the government's debt portfolio for $T^G = 1$ and $T^G = 10$. When a low government spending shock persists realized real holding returns are slightly below their expected level, when a high shock persists realized real holding returns are slightly above this level, indicating a moderate degree of hedging at these times. More significant adjustments in realized holding returns coincide with transitions from low to high and high to low government spending shock states. Both when $T^G = 1$ and when it equals 10, the realized holding return falls from 4.1% to 2.4% at the onset of the high government spending spell. The effect of this is to reduce the real value of the government's liabilities by about 0.7% of GDP. In the $T^G = 1$ (resp. $T^G = 10$) case, the realized real holding return jumps to 5.3% (resp. 5.8%) on the reverse transition.

Yield curves Figure 4 plots the evolution of the yield curve as the economy is hit by a series of high spending shocks. Initially, at $t = 4$, government spending is low and the yield curve (dotted line) is fairly flat and close to zero. With the realization of the first high spending shock at $t = 5$, nominal interest rates rise at all maturities (solid line). Consistent with the pattern of short run nominal interest rates and our earlier simple example, the increase is greatest at the longest outstanding maturity $K - 1 = 6$. Thus, the yield curve is hump shaped, tilting upwards over maturities $k = 1$ to $K - 1$ and downwards from $K - 1$ onwards. Over periods 6 to 10, the hump rises and passes to lower maturities. Once all initially outstanding debt has matured in period 11, the yield curve falls back to a lower level and adopts a flatter shape (solid-circle line).

Figure 3. Debt holding returns, $T^G = 1$, $T^G = 10$

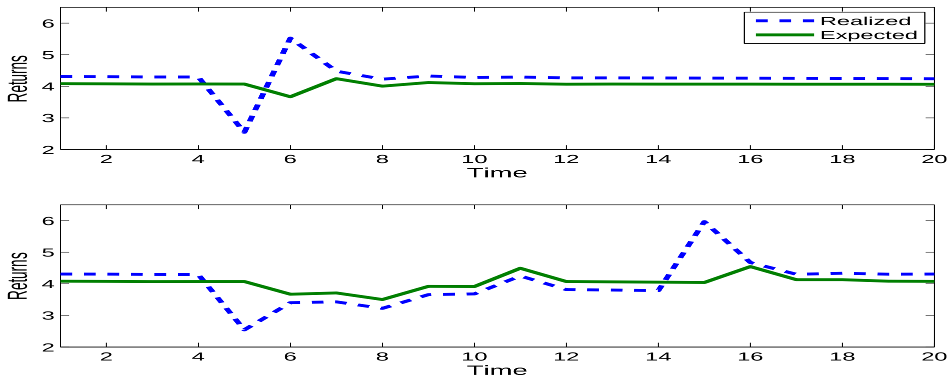
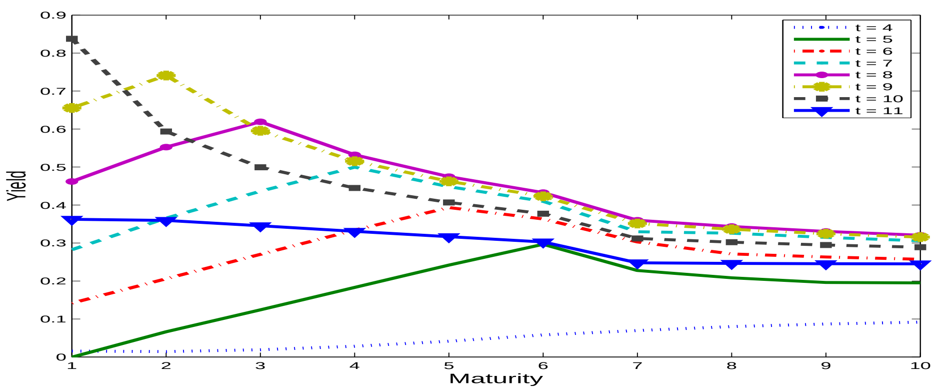


Figure 4. Yield curves



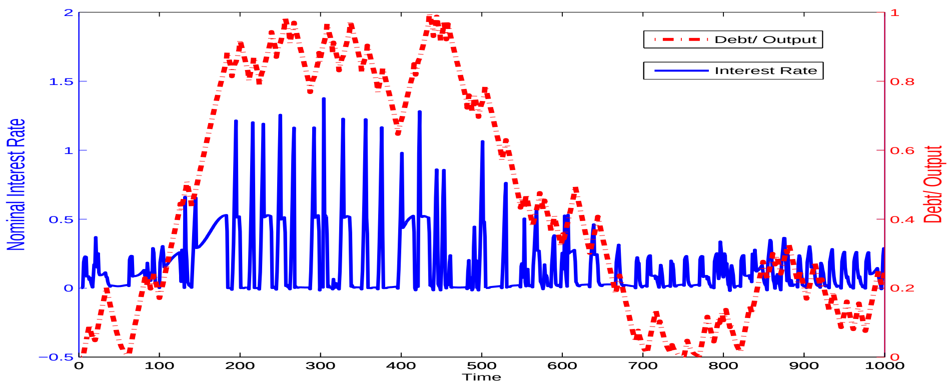
B.3. Long simulations

In this section we report results from long simulations of various economies. Each simulation is of length $T_{\text{sample}} = 20,000$.

Interest rate variation and debt levels Figure 5 shows simulated values for the debt to output ratio (dashed line) and nominal interest rates (solid line) for the first 1,000 periods of the baseline economy with $K = 7$. The figure shows that the volatility of nominal interest rates is increasing in the debt level. At high debt levels a given interest rate volatility induces greater absolute variation in the government's total liability value and provides a more effective hedge against fiscal shocks.

The extent of fiscal hedging Let ΔVB_t denote the variation in the real value of the government's portfolio across shock states at date t, $\Delta VB_t = \sum_{k=1}^{K} \left[\frac{Q_t^{k-1}(\underline{G})}{P_t(\underline{G})} - \frac{Q_t^{k-1}(\overline{G})}{P_t(\overline{G})} \right] B_t^k$. Table 1

Figure 5. Debt levels and nominal interest rates



shows long simulation sample averages of ΔVB_t for economies with $K = 1$ and $K = 7$ under two different normalizations. The first normalization provides a measure of the extent of fiscal hedging relative to the size of the economy, variations in portfolio values are divided by the conditional expectation of output. The associated sample moments, $\Delta VB/Y := \sum_{t=0}^{T_{\text{sample}}} \frac{\Delta VB_t}{E_{t-1}[Y_t]}$, are given in the first row of the table with the next two rows breaking these variations down into components that come from nominal capital losses and from contemporaneous inflations. The second normalization gives a measure of the extent of fiscal hedging relative to the optimal amount in a complete markets economy with the same primary surplus value.[13] We denote sample moments under this normalization by $\Delta VB/\Delta VB^C$.

The results in Table 1 may be summarized as follows. First, our measures of fiscal hedging increase 5-fold as the maximal debt maturity rises from 1 to 7. When $K = 1$, variations in portfolio values are, on average, about 0.4% of GDP; when $K = 7$, these variations average about 2.1% of GDP. Second, as K rises the extent to which fiscal hedging is obtained from movements in debt prices rather than contemporaneous inflations increases. When $K = 1$, all hedging must necessarily come from contemporaneous innovations in the price level; when K rises to 7, over 80% of the variation in average portfolio values comes from changes in debt prices. Finally, the amount of fiscal hedging relative to the complete markets economy is quite small when $K = 1$ ($\triangle VB/\triangle VB^C = 4.7\%$), but is significantly greater when $K = 7$, ($\triangle VB/\triangle VB^C = 24.4\%$).

[13] More precisely, the long simulation of a nominal debt model generates a sequence of expected primary surplus values $\{\phi_t\}_{t=1}^{T_{\text{sample}}}$. These values serve as state variables in recursive formulations of both the nominal debt and the complete markets models. We use the policy functions from the latter to compute complete markets variations in real portfolio values at each ϕ_t generated along the sample path of a nominal debt economy. These portfolio variations are then used to normalize those from the nominal debt economy.

Table 1: Financing Government Spending

	$K = 1$	$K = 7$
$\triangle VB/Y$	**0.393**	**2.17**
change in inflation	**0.393**	**0.386**
change in price of debt	**0.000**	**1.78**
$\triangle VB/\triangle VB^C$	**4.70**	**24.4**

Variability and persistence of inflation and interest rates Table 2 reports standard deviations, autocorrelations and correlations with government spending shocks for inflation and nominal interest rates from long simulations of economies with $K = 1$, 3 and 7. The table indicates that all of these statistics are increasing in the maximal debt maturity. Thus, fiscal hedging in economies with higher maximal debt maturities leads to inflation and interest rate processes that are more volatile, persistent and correlated with spending shocks.

Table 2: Statistics from Long Simulations

	$K = 1$	$K = 3$	$K = 7$
inflation			
st. deviation	**0.227**	**0.290**	**0.334**
autocorrelation	**0.315**	**0.430**	**0.655**
correlation with G-shock	**0.167**	**0.369**	**0.589**
1-period nom. interest rate			
st. deviation	**0.168**	**0.181**	**0.364**
autocorrelation	**0.081**	**0.152**	**0.713**
correlation with G-shock	**-0.396**	**0.420**	**0.509**

Welfare Increasing the maximal feasible debt maturity K provides positive, but small increases in welfare. Let (a_0, s) be an initial state for the Ramsey problem (RP) with $K = 1$ and denote the corresponding optimal household allocation by $\{c_{1t}(a_0, s), c_{2t}(a_0, s), l_t(a_0, s)\}_{t=0}^{\infty}$. Define $W(a_0, s)$ to be the optimal payoff to household in an economy with $K = 7$ if the initial state is (a_0, s) and let $\Delta(a_0, s)$ be such that $W(a_0, s) = E_s \sum_{t=0}^{\infty} \beta^t U((1 + \Delta(a_0, s))c_{1t}(a_0, s), (1 + \Delta(a_0, s))c_{2t}(a_0, s), l_t(a_0, s))$, i.e. $\Delta(a_0, s)$ is the proportional increase in the $K = 1$ optimal consumption allocation (at (a_0, s)) necessary to yield the same payoff as in the $K = 7$ economy. $\Delta \times 100\%$ varies between 0.02% and 0.1% with larger values at higher initial debt values.

C. Sensitivity Analysis

Here, we briefly describe the sensitivity of our results to changes in some key parameters. All changes in sample moments that we report are from long simulations and are relative to the baseline long simulation. Throughout, we assume the government can trade debt of up to 7 periods maturity.

Volatility of shocks When government spending shocks are more volatile the value of fiscal hedging is enhanced. A doubling of the standard deviation of government spending shocks to 14% causes a near doubling of the standard deviations of inflation and the one period nominal interest rates in the long simulation. Their standard deviations rise from 0.334 to 0.668 and 0.36 to 0.71 respectively. The sample correlation coefficients for inflation and government spending shocks rises from 0.589 to 0.63, that for one period interest rates and the shocks rises from 0.51 to 0.57.

Cash-credit good elasticity of substitution Our baseline preferences assume a fairly high elasticity of substitution between cash and credit goods of about 4.8. When this is reduced, distortions to the cash-credit good margin are less costly and the government is prepared to distort this margin more. For example, when preferences are log-log in cash and credit goods (and so have a unit elasticity of substitution) the standard deviation of the one period nominal interest rate in the long simulation rises is 0.70 (versus 0.36 in the baseline case); the correlation of one period nominal interest rates with government spending shocks is 0.62 (versus a baseline value of 0.51).

VIII. Conclusion

We have explored optimal debt management and taxation when the government is restricted to using non-contingent nominal debt of various maturities and is limited in its ability to lend. We identify a postponement motive for using long term debt and find that the government relies almost exclusively on the longest term debt available in calibrated examples. Other contributors have argued that the use of long term debt may raise the government's financing costs or expose it to unnecessary risk. Their arguments have implicitly treated inflation and/ or the yield curve as parameters. In our model, which endogenizes all prices, the holding return on long term nominal debt is more volatile than that on short term debt. However, this volatility is deliberate and is used to hedge fiscal shocks. Higher risk premia on long term debt are the analogues of insurance premia paid by the government and are not per se a rationale for shortening the maturity structure.

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Appendix

Proof of Proposition 1

Necessity Suppose $\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^{\infty}$ is an interior competitive allocation with no government lending at $\{P_{s0}, A_0, \{B_0^{k+1}\}_{k=1}^{K-1}\}$. We show that it satisfies the conditions in the proposition.

There is no loss of generality in assuming that at the equilibrium bond prices households have no desire to borrow. The interiority of the competitive allocation implies that the constraints $\forall i,\ t,\ c_{it} \geq 0$ and

$(1-\rho)L_{ft}+\rho L_{st} \in [0,T]$ are non-binding. We assume the existence of optimal Lagrange multipliers on the households' constraints. Let $\mu_t(s^t)$ denote the multiplier on the household's period t cash-in-advance constraint. Similarly, let $\lambda_t(s^t)$ and $\widetilde{\lambda}_t(s^t)$ denote, respectively, the multipliers on the liquidity and hedging round budget constraints. The transversality condition: $\lim_{t\to\infty} \beta^t\ E[\lambda_t(s^t)\ \{\widetilde{A}_t(s^t)+\sum_{k=0}^{K-1} Q_t^k(s^t)\widetilde{B}_t^{k+1}(s^t)\}] = 0$. The first order conditions for consumption and labor supply are:

$$c_{1t}: \quad \{\widetilde{\lambda}_t(s^t)+\mu_t(s^t)\}P_t(s^t) = U_{1t}(s^t) \tag{A1}$$

$$c_{2t}: \quad \widetilde{\lambda}_t(s^t)P_t(s^t) = U_{2t}(s^t) \tag{A2}$$

$$l_t: \quad \widetilde{\lambda}_t(s^t)(1-\tau_t(s^t))\frac{\partial I_t}{\partial l_t}(s^t) = -U_{lt}(s^t). \tag{A3}$$

The first order conditions for each of money and bonds are:

$$\widetilde{M}_t: \quad \lambda_t(s^t) = \mu_t(s^t)+\widetilde{\lambda}_t(s^t) \tag{A4}$$

$$M_{t+1}: \quad \widetilde{\lambda}_t(s^t) = \beta E_{s^t}[\lambda_{t+1}] \tag{A5}$$

$$\widetilde{B}_t^k: \quad Q_t^k(s^t)\lambda_t(s^t) = \widetilde{Q}_t^k(s^t)\widetilde{\lambda}_t(s^t) \tag{A6}$$

$$B_{t+1}^k: \quad \widetilde{Q}_t^k(s^t)\widetilde{\lambda}_t(s^t) = \beta E_{s^t}[Q_{t+1}^{k-1}\lambda_{t+1}] \tag{A7}$$

Combining (A1) and (A2), we obtain: $\frac{U_{1t}}{U_{2t}} = \frac{\widetilde{\lambda}_t+\mu_t}{\widetilde{\lambda}_t} \geq 1$. This establishes (13). Adding the household's and the government's hedging round budget constraints and using the definition of firm profits gives (14).

The first order condition from the final goods firm implies $\frac{P_{it}}{P_t} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{\mu-1}{\mu}}$, $i=f,s$. Thus, we have

$$P_{st}(s^{t-1}) = \left(Y_t(s^t)/Y_{st}(s^t)\right)^{\frac{\mu-1}{\mu}} P_t(s^t) = \left(Y_{ft}(s^t)/Y_{st}(s^t)\right)^{\frac{\mu-1}{\mu}} P_{ft}(s^t). \tag{A8}$$

The flexible price firm's first order conditions gives $P_{f,t}(s^t) = \frac{\mu}{\alpha}\frac{W_t(s^t)}{\theta(s_t)}L_{f,t}(s^t)^{1-\alpha}$. Combining this with (A8) we obtain $P_{st}(s^{t-1}) = \left(\frac{Y_{ft}(s^t)}{Y_{st}(s^t)}\right)^{\frac{\mu-1}{\mu}} \frac{\mu}{\alpha}\frac{W_t(s^t)}{\theta(s_t)}L_{ft}(s^t)^{1-\alpha}$ As in Siu (2004), this last expression, the first order condition from the sticky price firm's problem and the household's first order conditions imply (15).

Next take the household's hedging round budget constraint at t, multiply it by $\widetilde{\lambda}_t(s^t)$, add $\mu_t(s^t)\widetilde{M}_t(s^t)$ and use the household's first order conditions to obtain:

$$\begin{aligned}\widetilde{\lambda}_t(s^t)\sum_{k=1}^{K}\widetilde{Q}_t^k(s^t)\widetilde{B}_t^k(s^t) + \{\widetilde{\lambda}_t(s^t)+\mu_t(s^t)\}\widetilde{M}_t(s^t) &= U_{1t}(s^t)c_{1t}(s^t)+U_{2t}(s^t)c_{2t}(s^t)+U_{lt}(s^t)I_t(s^t)/W_t(s^t)\\ &+\beta E_{s^t}\left[\widetilde{\lambda}_{t+1}(s^{t+1})\sum_{k=1}^{K}\widetilde{Q}_{t+1}^k(s^{t+1})\widetilde{B}_{t+1}^k(s^{t+1}) + \{\widetilde{\lambda}_{t+1}(s^{t+1})+\mu_{t+1}(s^{t+1})\}\widetilde{M}_{t+1}(s^{t+1})\right].\end{aligned} \tag{A9}$$

Using the expressions for profits from the intermediate goods firms problems, $I_t(s^t)/W_t(s^t) = \Upsilon_t(s^t)$. Iterating on (A9) and using the household's first order and transversality conditions gives: $U_{1t}(s^t)\ [\sum_{k=1}^{K} Q_t^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)}$

$+ \frac{\widetilde{M}_t(s^t)}{P_t(s^t)}] = \xi_t(s^t)$, where $\xi_t(s^t) \equiv E_{s^t}[\sum_{j=0}^{\infty} \beta^{t+j}\{U_{1t+j}c_{1t+j}(s^{t+j}) + U_{2t+j}c_{2t+j}(s^{t+j}) + U_{lt+j}\Upsilon_{t+j}(s^{t+j})\}]$. The household's liquidity round budget constraint at t and the last equation imply:

$$A_t(s^{t-1})/P_t(s^t) + \sum_{k=1}^{K-1} Q_t^k(s^t)B_t^{k+1}(s^{t-1})/P_t(s^t) = \xi_t(s^t)/U_{1t}(s^t). \tag{A10}$$

Combining (A8), the definition of N_t and the household's first order conditions gives $\frac{P_t}{P_{t+1}} = \frac{1}{\beta}\frac{N_{t+1}U_{2t}}{E_t[N_{t+1}U_{1t+1}]}$. Using this and the household's first order conditions again, we obtain:

$$Q_t^k = E_t\left[\frac{U_{2t+k-1}}{U_{1t}}\prod_{j=0}^{k-2}\left\{\frac{N_{t+j+1}U_{2t+j}}{E_{t+j}[N_{t+j+1}U_{1t+j+1}]}\right\}\right] = \frac{U_{2t}}{U_{1t}}D_{t+1}^k. \tag{A11}$$

Combining (A8), (A10), (A11) and the definitions $a_t(s^{t-1}) = A_t(s^{t-1})/P_{st}(s^{t-1})$ and $b_t^k(s^{t-1}) = B_t^k(s^{t-1})/P_{st}(s^{t-1})$, we have the implementability/ measurability constraints (8). The definitions of b_t^k and a_t and the fact that $B_t^k \geq 0$ and $A_t \geq 0$ gives the no lending constraints. Finally, from (8), the non-negativity constraints on debt and the cash-in-advance constraint, we obtain: $\frac{\xi_t(s^t)}{U_{1t}(s^t)} = \sum_{k=1}^{K} Q_t^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)} + \frac{\widetilde{M}_t(s^t)}{P_t(s^t)} \geq c_{1t}(s^t)$, and, hence, (12).

Sufficiency We construct a candidate competitive equilibrium from an allocation and a portfolio weight sequence satisfying the conditions in the proposition. First we set prices. At date 0, $P_{s,0}$ is a parameter, while P_0 and P_{f0} are set to $P_0(s^0) = P_{s0}\left(Y_{s0}(s^0)/Y_0(s^0)\right)^{\frac{\mu-1}{\mu}}$ and $P_{f0}(s^0) = \left(Y_{s0}(s^0)/Y_{f0}(s^0)\right)^{\frac{\mu-1}{\mu}}$ respectively. For $t > 0$, set the relative sticky price to:

$$P_{st}/P_{t-1} = \beta/U_{2t-1}E_{t-1}[(Y_t/Y_{st})^{\frac{\mu-1}{\mu}}U_{1t}], \tag{A12}$$

the gross (final goods) rate of inflation to:

$$P_t(s^t)/P_{t-1}(s^{t-1}) = P_{s,t}(s^{t-1})/P_{t-1}(s^{t-1})\left(Y_{s,t}(s^t)/Y_t(s^t)\right)^{\frac{\mu-1}{\mu}}. \tag{A13}$$

and the flexible price to $P_{ft}(s^t) = P_{st}(s^{t-1})\left(Y_{st}(s^t)/Y_{ft}(s^t)\right)^{\frac{\mu-1}{\mu}}$. These conditions allow us to recursively recover all goods prices. For $k > 0$ and $t \geq 0$, set the asset prices Q_t^k from the period t liquidity round budget constraint to:

$$Q_t^k(s^t) = \frac{U_{2t}(s^t)}{U_{1t}(s^t)}D_{t+1}^k(s^t). \tag{A14}$$

Also, for $k > 0$ and $t \geq 0$, set the asset prices from the period t hedging round budget constraint to be $\widetilde{Q}_t^k(s^t) = D_{t+1}^k(s^t)$. For $t > 0$, we set the portfolios purchased by households in the hedging round as follows. The level of debt of $k > 1$ maturity is fixed at $B_t^k(s^{t-1}) = b_t^k(s^{t-1})P_{st}(s^{t-1})$. Using the no lending constraint, $B_t^k(s^{t-1}) \geq 0$. Also by this constraint, $a_t(s^{t-1}) \geq 0$, and we can choose $M_t(s^{t-1}) \geq 0$ and $B_t^1(s^{t-1}) \geq 0$

so that $M_t(s^{t-1}) + B_t^1(s^{t-1}) = a_t(s^{t-1})P_{st}(s^{t-1})$. Let $A_t(s^{t-1}) = a_t(s^{t-1})P_{st}(s^{t-1})$. Next we turn to the portfolios purchased in the liquidity round. For $t \geq 0$, the money supply is set to $\widetilde{M}_t(s^t) = P_t(s^t)c_{1t}(s^t)$. From the measurability constraints (8), the above definitions of goods prices, asset prices and portfolios and the nominal wealth-in-advance constraints (12), we have: $\frac{A_t(s^{t-1})}{P_t(s^t)} + \sum_{k=1}^{K-1} Q_t^k(s^t) \frac{B_t^{k+1}(s^{t-1})}{P_t(s^t)} = \frac{\xi_t(s^t)}{U_{1t}(s^t)} \geq c_{1t}(s^t) = \frac{\widetilde{M}_t(s^t)}{P_t(s^t)}$. It follows that, at each date t, we can choose a non-negative debt portfolio $\{\widetilde{B}_t^k(s^t)\}_{k=1}^K \in \mathbb{R}_+^K$ so that the liquidity round budget constraints hold with equality. Hence, the no lending, liquidity round budget and cash-in-advance constraints are satisfied. The government's debt holdings are set equal to the household's holdings of bonds.

We verify the household's first order conditions. Set the real wage to $\frac{W_t}{P_t} = \frac{\alpha}{\mu}\theta_t {L_{ft}}^{\alpha-1}\left(\frac{Y_t}{Y_{ft}}\right)^{\frac{\mu-1}{\mu}}$, the income tax rate to $(1-\tau_t) = -\frac{U_{lt}}{U_{2t}}\frac{P_t}{W_t}$ and the Lagrange multipliers to $\lambda_t P_t = U_{1t} \geq 0$, $\widetilde{\lambda}_t P_t = U_{2t} \geq 0$ and $\mu_t P_t = U_{1t} - U_{2t} \geq 0$. It is then immediate that $\lambda_t = \mu_t + \widetilde{\lambda}_t$, $\{\widetilde{\lambda}_t + \mu_t\}P_t = U_{1t}$, $\widetilde{\lambda}_t P_t = U_{2t}$ and $\widetilde{\lambda}_t(1-\tau_t)\partial I_t/\partial l_t = -U_{lt}$. Also, (A12) and (A13) imply $U_{2t} = \beta E_t\left(\frac{P_t}{P_{t+1}}U_{1t+1}\right)$, so that $\lambda_t = \beta E_{s^t}[\lambda_{t+1}]$. Finally, the definitions of Q_t^k, $\widetilde{Q}_{t+1}^k$ and the multipliers gives $Q_t^k\lambda_t = \widetilde{Q}_{t+1}^k\widetilde{\lambda}_t$ and $\widetilde{Q}_{t+1}^k\widetilde{\lambda}_t = \beta E_{s^t}[Q_{t+1}^{k-1}\lambda_{t+1}]$.

Next we verify the household's hedging round budget constraints. Combining (8), (A13) and (A14) gives:

$$\xi_t(s^t) = \frac{U_{1t}(s^t)}{P_t(s^t)}\left[A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t)B_t^{k+1}(s^{t-1})\right]. \tag{A15}$$

Hence, using the liquidity round budget constraint and the definitions of $Q_t^k(s^t)$ and $\widetilde{Q}_t^k(s^t)$ and dividing by $U_{2t}(s^t)$, we have $\frac{\xi_t(s^t)}{U_{2t}(s^t)} = \frac{U_{1t}(s^t)}{U_{2t}(s^t)}\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^K \widetilde{Q}_t^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)}$. Adding $\frac{U_{2t}(s^t)-U_{1t}(s^t)}{U_{2t}(s^t)}\frac{\widetilde{M}_t(s^t)}{P_t(s^t)}$ to each side of this equation, using the definition of $\xi_t(s^t)$ and $\tau_t(s^t)$ and $\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} = c_{1t}(s^t)$ yields $\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^K \widetilde{Q}_t^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)}$ $= c_{1t}(s^t) + c_{2t}(s^t) - (1-\tau_t(s^t))I_t(s^t) + \frac{\beta}{U_{2t}(s^t)}E_{s^t}[\xi_{t+1}(s^{t+1})]$. Then, using (A15) at $t+1$, the definitions of Q_{t+1}^k, $\widetilde{Q}_t^k$and P_{t+1} and the condition $U_{2t} - \beta E_{t+1}[\frac{P_t}{P_{t+1}}U_{1t+1}] = 0$, we obtain:

$$\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^K \widetilde{Q}_t^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)} = c_{1t}(s^t) + c_{2t}(s^t) - (1-\tau_t(s^t))I_t(s^t) + \frac{A_{t+1}(s^t)}{P_t(s^t)} + \sum_{k=2}^{K-1} \widetilde{Q}_t^k(s^t)\frac{B_{t+1}^k(s^t)}{P_t(s^t)} \tag{A16}$$

The hedging round budget constraint at t then follows from (A16) and the definition of $\widetilde{A}_t(s^t)$.

By (8) and the interiority of the allocation, $\xi_0$is finite. Using the definition of ξ_t, we have for all T, $E[\xi_0] = E[\sum_{t=0}^\infty \beta^t\{U_{1t}c_{1t} + U_{2t}c_{2t} + U_{lt}\Upsilon_t\}] = E[\sum_{t=0}^T \beta^t\{U_{1t}c_{1t} + U_{2t}c_{2t} + U_{lt}\Upsilon_t\}] + \beta^{T+1}E[\xi_{T+1}]$. Taking limits and using the period $T+1$ measurability constraint then gives: $\lim_{T\to\infty}\beta^{T+1}E[\xi_{T+1}] = \lim_{T\to\infty}\beta^{T+1}$ $E[U_{1T+1}(\frac{A_{T+1}}{P_{T+1}} + \sum_{k=1}^{K-1} Q_{T+1}^k\frac{B_{T+1}^{k+1}}{P_{T+1}})] = 0$ which confirms the transversality condition. Hence, the allocation is feasible and optimal for households at the derived prices and tax rates. The household's budget constraints, the resource constraint and the definitions of A_{gt} and B_{gt} ensure that the government's budget constraints are satisfied. It is easy to verify that the derived choices of firms satisfy their first order conditions and are optimal. ■

Proof of Lemma 4: For a proof that a solution to the government's problem exists, see LSY (2006). Let $\{a_1^*, \{b_1^{k*}\}_{k=2}^K\}$ denote an optimal portfolio. Since $\phi_1 > 0$, either $a_1^* > 0$ or $b_1^{k*} > 0$ for some k. Let $\hat{k}$ denote the smallest k such that for all $k > \hat{k}$, $b_1^{k*} = 0$. Suppose $\hat{k} < K$. Then, for $t \geq \max\{2, \hat{k}\}$, the first order condition for c_{it} reduces to $0 = U_{it} + \eta_t [U_{1it} - U_{2it}] - \chi_t$. If $U_{1t} > U_{2t}$, then $\eta_t = 0$ and this first order condition implies that each $U_{it} = \chi_t$. We deduce that in fact $U_{1t} = U_{2t}$. It then follows from the measurability constraint that the optimal allocation can be implemented with a portfolio in which either $b_1^K = b_1^{\hat{k}*}$ and $b_1^{\hat{k}} = 0$ or, if $\hat{k} = 1$, $b_1^K = a_1^*$ and $a_1 = 0$. All other portfolio weights remain the same.

The first order condition for b_1^{k+1} is:

$$0 = -\sum_{s \in S} \omega_1(s) N_1(s) U_{21}(s) \prod_{j=2}^{k} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) + \kappa^{k+1}, \tag{A17}$$

where $\kappa^{k+1} \geq 0$ is the Lagrange multiplier on the corresponding no lending constraint. Wlog assume $b_1^{K*} > 0$, so that from (A17) that either A) $\omega_1(s) = 0$ for each s or B) $\omega_1(s) > 0 > \omega_1(s')$ for some pair s, s'. We assume the latter. (In fact, Case A holds only in knife edge cases). Now, suppose $b_1^{k*} > 0$ for $k < K$, then $-\sum_{s \in S} \omega_1(s) N_1(s) U_{21}(s) \prod_{j=2}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) = 0$. The combined first order condition for c_{1t} and c_{2t}, $t \in \{2, \cdots, K-1\}$, (21) implies that $U_{1t} - U_{2t} > 0$ if and only if $\omega_1 > 0$, and $U_{1t} - U_{2t} = 0$ otherwise. Hence, $-\sum_{s \in S} \omega_1(s)\ N_1(s)\ U_{21}(s) \prod_{j=2}^{k-1} \frac{U_{2j}(s)}{U_{1j}(s)} \prod_{j=k}^{K-1} \frac{U_{2j}(s)}{U_{1j}(s)}\ \pi(s) > 0$. But this contradicts the first order condition (A17) at $k + 1 = K$. Thus, $b_1^{k*} = 0$ for $k < K$. By a similar argument, using the relevant first order condition, $a_1^* = 0$ as well. The lemma is proven. ■

Proof of Lemma 5 It follows from the proof of Lemma 4, that if $\omega_1(s) \leq 0$ or $t \geq K$, then $U_{1t}(s) = U_{2t}(s)$ and $Q_t^1(s) = 1$; if $\omega_1(s) > 0$ and $t = 2, \cdots, K-1$, then $U_{1t}(s) - U_{2t}(s) > 0$ and $Q_t^1(s) < 1$. Also, $a_1, b_1^k = 0$ at the optimal allocation. Using the first order conditions for c_{1t} and c_{2t}, when $\omega_1(s) > 0$ and $t = 2, \cdots, K-1$ and the fact that $U_{11t}/U_{1t} = -U_{1t}/(1-\gamma)$, $U_{22t}/U_{2t} = -U_{2t}/\gamma$, we derive

$$\frac{\frac{U_{1t}}{U_{2t}} - 1}{\frac{\gamma}{1-\gamma} \frac{U_{1t}}{U_{2t}} + 1} = \beta^{-(t-1)} \frac{\omega_1}{\gamma} b_1^K \prod_{j=1}^{K-1} \left[\frac{U_{2j}}{U_{1j}}\right] U_{11} N_1. \tag{A18}$$

Since $\beta \in (0,1)$ and $\omega_1 > 0$, we deduce that for $t = 2, \cdots, K-2$, $\frac{U_{1t+1}}{U_{2t+1}} > \frac{U_{1t}}{U_{2t}}$. Hence, $1 > Q_t^1 > Q_{t+1}^1$, for $t = 2, \cdots, K-2$. If $Q_1^1 = 1$, we are finished. If not, the first order conditions for c_{11} and c_{22} and (A18) give:

$$\frac{\frac{U_{11}}{U_{21}} - 1}{\frac{\gamma}{1-\gamma} \frac{U_{11}}{U_{21}} + 1} < \frac{\omega_1}{\gamma} b_1^K \prod_{j=1}^{K-1} \left[\frac{U_{2j}}{U_{1j}}\right] U_{11} N_1 < \frac{\frac{U_{1t}}{U_{2t}} - 1}{\frac{\gamma}{1-\gamma} \frac{U_{1t}}{U_{2t}} + 1}, \qquad t = 2, \cdots, K-2. \tag{A19}$$

Thus, $Q_{t+1}^1 < Q_t^1$ for $t = 1, \cdots, K-2$. ■