Defining bad news: changes in return distributions that decrease risky asset demand*

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Abstract

We provide a random variable characterization of the necessary and sufficient conditions for a shift of the distribution of rate of return on the risky asset in the two asset portfolio problem to reduce demand for all risk–averse expected utility maximizing investors. We provide random variable characterizations of the shifts that reduce both demand and expected utility for all risk–averse investors and a random variable characterization of shifts in the payoff of the market portfolio that reduce the equilibrium price of the market portfolio and make all risk–investors worse off.
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1 Introduction

What changes in the distribution of risky asset returns causes investors to reduce their demand for risky assets? Besides understanding individual portfolio choice, understanding such changes is also helpful in understanding the equilibrium price of the market portfolio. The aggregate quantity of the market portfolio of risky assets is fixed in the short run: changes that reduces the demand for the risky asset in the two asset portfolio problem reduce the equilibrium price of the market portfolio. We study a von Neumann–Morgenstern expected–utility maximizing investor choosing an optimal portfolio consisting of one risky and one riskless asset. As a comparative statics exercise, we shift the rate of return distribution for the risky asset. What are the necessary and sufficient conditions on the shift in distribution for all risk–averse investors to reduce their demand for the risky asset? When does such a shift make investors worse off?

We provide a random variable characterization of the necessary and sufficient shifts in distribution showing explicitly how to take the initial random variable for the excess rate of return on the risky asset and construct a new random variable for the excess rate of return on the risky asset, such that all risk–averse investors reduce risky asset demand. We also provide conditions for such a shift to decrease risky asset demand and to decrease the expected utility of all risk-averse investors—our definition of bad news.

We show how to change the payoffs of the market portfolio in order to to reduce the equilibrium price of the market portfolio. We also provide conditions when the change in payoffs reduces the price of the market portfolio and makes all risk-averse investors worse off.

2 Literature Review

Gollier (1995) provides necessary and sufficient conditions to reduce risky asset demand in the form of conditions on all the partial expectations of the distribution functions of returns before and after the shift. Gollier and Schlesinger (2002) use the conditions to characterize the shifts than reduce the risky asset price in an endowment economy. Athey (2002) provides general characterizations of the necessary and sufficient conditions for monotone comparative statics properties in stochastic optimization problems, and applies the characterizations to the portfolio problem relating the conditions to a single–crossing–property on the change in distribution functions of the risky asset return.

First and second order stochastic dominance reductions in the risky asset's rate of return distribution do not always reduce risk asset demand for all risk–averse investors. One approach to derive comparative statics in the portfolio problem is to look for restrictions on the class of utility functions that lead to intuitive comparative static properties. For example, Rothschild and Stiglitz (1971), Kira and Ziemba (1980), and Hadar and Seo (1990) show that decreasing absolute risk aversion along with a coefficient of relative risk aversion bounded by one are sufficient conditions for a second order shift to reduce demand, and Fishburn and Porter (1976) show that similar conditions are needed for a first order shift to reduce demand.

Another approach to derive comparative statics in the portfolio problem is to impose conditions on the shifts that result in reductions in demand for all risk–averse utility functions. Examples include the Strong Increase in Risk introduced by Meyer and Ormiston (1985), Simple Increases in Risk introduced by Dionne and Gollier (1992), Relatively Strong Increases in Risk introduced by Black and Bulkey, and Relatively Weak Increases in Risk introduced Dionne, Eeckhoudt and Gollier (1993). All these shifts are sufficient to reduce demand for the risky asset in the two asset portfolio problem with a riskless asset for all risk–averse investors.

Landsberger and Meilijson (1990) define a Mean Preserving Increase in Risk Around ν: a mean–preserving spread where probability mass is shifted around intervals whose closure contains ν. For $\nu = 0$, the shifts satisfy the sufficient conditions to decrease risky asset demand for all risk–averse investors. We provide a random variable characterization of these shifts around $\nu = 0$ but allowing for the mean to decrease, and use the random variables as part of our construction of the random variables which characterize the necessary and sufficient conditions for all risk–averse investors to reduce risky asset demand. We also apply techniques developed by Gollier and Kimball (1996) to the shifts to develop the necessary and sufficient conditions to reduce risky asset demand.

3 Assumptions and notation

A risk–averse investor maximizes the expected value of a concave, increasing von Neumann-Morgenstern utility function $U : \mathcal{R} \rightarrow \mathcal{R}$, assumed to be everywhere at least once differentiable, with first derivative $U' : \mathcal{R} \rightarrow \mathcal{R}^+$. The investor has initial wealth W, and allocates X to a risky asset with the remainder, $W - X$, allocated to a riskless asset. The utility function and initial wealth satisfy

$$U'(W) < \infty \tag{1}$$

The risky asset has rate of return given by the random variable R^y. The riskless rate of return is, without loss of generality, set to zero: R^y is therefore interpreted as an excess return. The excess rate of return on the risky asset R^y has bounded support equal to $[-1, +1]$[1], and the cumulative probability distribution $F^y(r) \equiv \text{Prob}(R^y \leq r)$ satisfies

$$0 < F^y(0) < 1, \tag{2}$$

and the bounded support assumption implies

$$0 < E[R^y] < \infty. \tag{3}$$

Condition (2) rules out arbitrage opportunities, and conditions (1) and (3) ensure that any risk-averse investor holds a positive amount of the risky asset in the optimal portfolio (Arrow (1965)).

The investor's portfolio problem is

$$\max_X E\left[U(W + R^y X)\right]. \tag{4}$$

Denoting the optimal level of risky investment by X^*, the first-order condition, which is necessary

[1] We can also extend our results to the unbounded support case.

and sufficient for a solution to (4) is

$$\int_{-1}^{1} U'(W + rX^*) r dF^y(r) = 0. \tag{5}$$

Suppose that the random variable describing the excess rate of return on the risky asset changes from R^y to R^z, with the associated probability distribution changing from F^y to F^z. By the concavity of U, the optimal holding of the risky asset weakly decreases if and if only if the first order condition in equation (5) is less than or equal to zero when evaluated at X^*,

$$\int_{-1}^{1} U'(W + rX^*) r dF^z(r) \leq 0. \tag{6}$$

Gollier (1995) provides the necessary and sufficient conditions on the probability distribution of excess rate of return for such a shift to lead to reduced risky asset demand for all risk–averse investors.

Theorem 0 *[Proposition 2, Gollier (1995)] Let R^y and R^z be the random variables describing the excess rate of return on the risky asset before and after the shift with associated probability distributions F^y and F^z. The demand for the risky asset is reduced for all strictly risk–averse investors if and only if there is a constant m such that*

$$\int_{-1}^{t} r dF^z(r) \leq m \int_{-1}^{t} r dF^y(r), \ \ \forall t \in [-1, 1]. \tag{7}$$

We refer to $\int_{-1}^{t} r dF^y(r)$ as the partial expectations of the random rate of return R^y.

If conditions (7) hold with a value of $m \leq 0$ the expected excess return of the new distribution is negative, which trivially leads to a reduction in demand. Any distribution of the excess rate of return with a expected return is negative satisfies equation (7) for some $m \leq 0$. We will therefore only consider shifts such that the expected excess return of the risky asset is positive both before and after the shift. In this case, equation (7) must hold with a value of $m > 0$.

When condition (7) holds as an equality for all t for any $m > 0$, then risky asset demand is unchanged for all risk-averse investors. We can also determine the effects on the investors' expected utility when condition (7) holds as an equality. When the equalities hold for some $0 < m < 1$, all risk-averse investors expected utility decreases, and when the equalities hold for $m > 1$, all risk-averse investors expected utility increases. When condition (7) for any $m > 0$, all risk-averse investors' risky asset demand (weakly) decreases. When the condition holds with $0 < m < 1$, all risk-averse investors' expected utility decreases. Such a shift in distribution is our definition of bad news.

4 Demand classes

Since our focus is on changes in the return distribution that lead to a reduction in risky asset demand by every risk–averse investor, it is useful to define demand classes of return distributions such that every risk–averse investor's demand is the same for all return distributions in the same class. We first report the random variable characterization of changes in the return distribution that keep the random variables in the same demand class.

Definition 1 *The distributions F^y and F^z belong to the same demand class if and only if the optimal risky–asset demand for any risk–averse investor is the same under both distributions.*

Suppose the random rate of return R^y is replaced with a compound lottery composed of the excess return distribution, R^y with probability $0 < a \leq 1$ and a degenerate distribution at zero with probability $1 - a$. Let R^z denote the new random rate of return. Evaluating the first-order conditions at the original holding for R^y, X^*, but using the rate of return R^z:

$$\begin{aligned}\int_{-1}^{1} rU'(W + rX^*)dF^z(r) &= a\left(\int_{-1}^{1} rU'(W + rX^*)dF^y(r)\right) + (1-a)\left(0U'(W)\right) \\ &= a(0) + (1-a)(0) \\ &= 0,\end{aligned} \tag{8}$$

where the second lines follows from the optimality of X^* and condition (1). The optimal risky asset holding is therefore the same with the risky rate of returns R^y and R^z. Theorem 1 describes the necessary and sufficient shifts to keep demand the same for all investors, and provides a random variable construction of the associated shifts in distribution.

Theorem 1 *The following statements are equivalent.*

1. *The distributions F^y and F^z are in the same demand class.*

2. *Define $\pi_0^y \equiv Prob(R^y = 0)$. There exists a constant $0 < m \leq \frac{1}{1-\pi_0^y}$ such that*

$$\int_{-1}^{t} rdF^z(r) = m \int_{-1}^{t} rdF^y(r), \forall t \in [-1, 1]. \tag{9}$$

3. *Let $\overset{d}{=}$ denote 'distributed as.' There exists a random variable θ and a constant $0 < m \leq \frac{1}{1-\pi_0^y}$ such that*

$$R^z \overset{d}{=} R^y + \theta. \tag{10}$$

Let I be an indicator function. When $0 \leq m < 1$, the conditional distribution of θ is

$$F^\theta(s|R^y = r) = mI(s \geq 0) + (1-m)I(s \geq -r). \tag{11}$$

When $1 \leq m \leq \frac{1}{1-\pi_0^y}$, the conditional distribution of θ is

$$F^\theta(s|R^y = r) = \begin{cases} I(s \geq 0), & \textit{for } r \neq 0, \\ \frac{m-1}{\pi_0^y} F^y(s) + \left(1 - \frac{m-1}{\pi_0^y}\right) I(s \geq 0), & \textit{for } r = 0. \end{cases} \tag{12}$$

For $0 < m < 1$ and conditional on a realization of $R^y = r$, the random variable θ equals $-r$ with probability m, and equals 0 with probability $1 - m$. For $1 \leq m \leq \frac{1}{1-\pi_0^y}$ and conditional on $R^y \neq 0$, θ equals 0 with probability 1. For $1 \leq m \leq \frac{1}{1-\pi_0^y}$ and conditional on $R^y = 0$, the random variable θ is a mixture of two random variables: with probability $\frac{(m-1)(1-\pi_0^y)}{\pi_0^y}$ a random variable with the same distribution as R^y conditional on $R^y \neq 0$, and with probability $1 - \frac{(m-1)(1-\pi_0^y)}{\pi_0^y}$ a degenerate random variable equal to 0 with probability 1. In both cases, the partial expectations of $R^z = R^y + \theta$ equal m times the partial expectations of R^y.

Table 1 provides numerical example of conditions 2 and 3 in Theorem 1 for discrete rate of return distributions with state space {-0.1, 0.0, 0.1, 0.2, 0.3}. The top half of each panel is the distribution function, the cumulative distribution function, and the partial expectations for the original risky rate of return R^y. The bottom half of each panel report the distribution function, the cumulative distribution function, and the partial expectations for the shifted random variables $R^{z1} \stackrel{d}{=} R^y + \theta^1$ and $R^{z2} \stackrel{d}{=} R^y + \theta^2$.

Panel A of Table 1 is an example of a shift toward zero satisfying equation (11). The shift in Panel A is generated by a random variable θ^1 satisfying equation (11) for $m = 0.8$. The conditional distribution of θ^1 is

$$\text{Prob}(\theta^1 = s | R^y = r) = \begin{cases} 0.2, & \text{for } s = -r, \\ 0.8, & \text{for } s = 0, \\ 0.0, & \text{else.} \end{cases} \tag{13}$$

The partial expectations of $R^z 1$ equal the partial 0.8 times the partial expectations of R^y.

Panel B is an example of a shift away from zero satisfying equation (12). The shift in Panel B is generated by a random variable θ^2 satisfying equation (12) for $m = 1.7$. The conditional distribution of θ^2 is

$$\text{Prob}(\theta^2 = s | R^y = 0) = \begin{cases} \frac{0.7}{0.44} \times 0.120, & \text{for } s = -0.1, \\ 1 - \frac{0.7}{0.440} \times 0.560, & \text{for } s = 0.0, \\ \frac{0.7}{0.44} \times 0.040, & \text{for } s = 0.1, \\ \frac{0.7}{0.44} \times 0.320, & \text{for } s = 0.2, \\ \frac{0.7}{0.44} \times 0.080, & \text{for } s = 0.3, \end{cases} \tag{14}$$

$$\text{Prob}(\theta^2 = 0 | R^y \neq 0) = 1.00.$$

The partial expectations of R^z equal the partial 1.7 times the partial expectations of R^y.

5 Reducing demand

Landsberger and Meilijson (1990) define a mean preserving increase in risk around ν. Such a shift is a mean–preserving spread in which the closure of the interval where probability is removed contains

ν. Landsberger and Meilijson prove that any shift in probabilities from F^y to F^z that satisfies

$$\int_{-1}^{t} dF(r)^z(r-\nu) \leq \int_{-1}^{t} dF(r)^y(r-\nu), \ \forall t \in [-1,1], \text{ and } E[R^y] = E[R^z]$$

must be generated by a sequence of mean preserving increases in risk around ν. We use a similar construction. In Lemma 1 we show that condition (7) holds for $m=1$ if and only if R^z is generated from R^y by a sequence of mean non-increasing increases in risk around zero. In such a shift, the mean cannot increase, and all mass that is shifted towards zero must be shifted to or past zero.

Lemma 1 *The following statements are equivalent.*

1.
$$\int_{-1}^{t} r dF^z(r) \leq \int_{-1}^{t} r dF^y(r), \forall t \in [-1,1]. \tag{15}$$

2. *There exists a sequence of random variables* $\{\epsilon^i\}_{i=1}^{\infty}$ *such that*

$$R^0 \stackrel{d}{=} R^y, \ R^i \stackrel{d}{=} R^{i-1} + \epsilon^i, \ R^z \stackrel{d}{=} \lim_{i\to\infty} R^i, \tag{16}$$

satisfying

$$E[\epsilon^i | R^{i-1}] \leq 0, \tag{17}$$

$$Prob(0 < \epsilon^i < -R^{i-1} | R^{i-1} < 0) = 0, \tag{18}$$

$$Prob(-R^{i-1} < \epsilon^i < 0 | R^{i-1} \geq 0) = 0. \tag{19}$$

The shifts described in Lemma 1 are sufficient to reduce risky asset demand for all risk-averse investors.

Lemma 2 *Suppose that there exists a sequence of random variables* $\{\epsilon^i\}_{i=1}^{\infty}$ *such that*

$$R^0 \stackrel{d}{=} R^y, \ R^i \stackrel{d}{=} R^{i-1} + \epsilon^i, \ R^z \stackrel{d}{=} \lim_{i\to\infty} R^i, \tag{20}$$

satisfying

$$E[\epsilon^i | R^{i-1}] \leq 0, \tag{21}$$

$$Prob(0 < \epsilon^i < -R^{i-1} | R^{i-1} < 0) = 0, \tag{22}$$

$$Prob(-R^{i-1} < \epsilon^i < 0 | R^{i-1} \geq 0) = 0. \tag{23}$$

Then demand for the risky asset for all risk-averse investors is lower with risky asset rate of return R^z than with risky asset rate of return R^y.

In order to reduce demand, the original risky rate of return is transformed by adding a sequence of random variables with a non-positive mean, such that the probability mass shifted towards zero is shifted to or past zero.

Table 2 is an example of a shift satisfying the conditions in Lemma 2 for a one step sequence in which conditions (21) through (23) hold for ϵ^1. The conditional distribution of ϵ^1 is

$$\text{Prob}(\epsilon^1 = s | R^y = 0.10) = \begin{cases} 0.25, & \text{for } s = -0.20, \\ 0.25, & \text{for } s = 0.00, \\ 0.50 & \text{for } s = 0.10, \\ 0.00, & \text{else,} \end{cases} \tag{24}$$
$$\text{Prob}(\epsilon^1 = 0.00 | R^y \neq 0.10) = 1.00.$$

Here, $E[\epsilon^1 | R^y] \leq 0$ and probability mass moves from 0.10 to -0.10 and 0.20; all the probability mass that is shifted toward zero is shifted all the way to zero. Conditions (21) through (23) hold, and therefore conditions (7) hold for $m = 1$.

Table 3 is an example of a second order shift in distribution that does not reduce demand. The original distribution R^y is that same as in Table 2. But now the distribution of ϵ^1 is

$$\text{Prob}(\epsilon^1 = s | R^y = 0.20) = \begin{cases} 0.25, & \text{for } s = -0.10, \\ 0.50, & \text{for } s = 0.00, \\ 0.25 & \text{for } s = 0.10, \\ 0, & \text{else,} \end{cases} \tag{25}$$
$$\text{Prob}(\epsilon^1 = 0 | R^y \neq 0.20) = 1.00.$$

Here, $E[\epsilon^1|R^y] \leq 0$, but probability mass moves from 0.20 to 0.10 and 0.30; all the probability mass that is shifted toward zero is not shifted all the way to zero or beyond. Condition (23) does not hold and therefore conditions (7) do not hold with $m = 1$ for $r = 0.10$ and $r = 0.20$.

Combing Theorem 1 and Lemmas 1 and 2 provides a complete characterization of the necessary and sufficient conditions to reduce risky asset demand.

Theorem 2 *The following statements are equivalent.*

1. *Demand for the risky asset for all risk-averse investors is lower with risky asset rate of return R^z than with risky asset rate of return R^y.*

2. *There exists a constant $m > 0$ such that*

$$\int_{-1}^{t} rdF^z(r) \leq m \int_{-1}^{t} rdF^y(r), \forall t \in [-1, 1]. \tag{26}$$

3. *There exists random variables $\theta^1, \{\epsilon^i\}_{i=1}^{\infty}, \theta^2$ such that*

$$R^0 \overset{d}{=} R^y + \theta^1, \tag{27}$$

$$R^i \overset{d}{=} R^{i-1} + \epsilon^i, \;\; R^\infty \overset{d}{=} \lim_{i\to\infty} R^i, \tag{28}$$

$$R^z \overset{d}{=} R^\infty + \theta^2, \tag{29}$$

satisfying

$$F^{\theta^1}(s|R^y = r) = kI(s \geq 0) + (1-k)I(s \geq -r), \; \textit{for some } 0 \leq k < 1, \tag{30}$$

$$E[\epsilon^i|R^{i-1}] \leq 0, \tag{31}$$

$$Prob(0 < \epsilon^i < -R^{i-1}|R^{i-1} < 0) = 0, \tag{32}$$

$$Prob(-R^{i-1} < \epsilon^i < 0|R^{i-1} \geq 0) = 0. \tag{33}$$

and

$$F^{\theta^2}(s|R^\infty = r) = \begin{cases} I(s \geq 0) & \mathit{for}\ r \neq 0 \\ \frac{l-1}{\pi_0^\infty}F^\infty(s) + \left(1 - \frac{l-1}{\pi_0^\infty}\right) I(s \geq 0), & \mathit{for}\ r = 0, \end{cases} \tag{34}$$

where

$$\pi_0^\infty = Prob\,(R^\infty = 0) \tag{35}$$

$$1 < l \leq \frac{1}{1 - \pi_0^\infty}, \tag{36}$$

and

$$k \times l = m. \tag{37}$$

Table 4 in an example of the shifts defined in Theorem 2. Panel A reports the initial rate of return distribution for the risky rate of return. Panel B reports the effect of a shift between distributions in the same demand class; here with $m = 0.8$. The shift is the same as in Panel A in Table 1; the partial expectations with the new risky rate of return, R^0 are 0.8 times the partial expectations under the original risky rate of return.

Panel C of the table is a shift that reduces the partial expectations of the random rate of return R^0. Comparing 0.8 times the partial expectations of the original random rate of return, R^y, with the partial expectations for the transformed rate of return, R^∞, condition (7) is satisfied for $m = 0.8$. The resulting random rate of return has the same distribution as the original distribution in Panel B in Table 1. Panel D reports the effect of a shift between distributions in the same demand class to the random rate of return R^∞ with $m = 1.7$ The shift is the same as in Panel B in Table 1. The final risky rate of return R^z satisfies condition (7) with $m = 0.8 \times 1.7 = 1.36$.

6 Does a demand reducing shift make all investors worse off?

The shifts described in Lemma 1 are a second order stochastic dominance worsening in the risky–asset return distribution (Landsberger and Meilijson (1990))—the shifts therefore make all risk-averse investors worse off. The expected utility effects of a demand reducing shift in the rate of return distribution depend on the expected utility effects of a shift between distributions in the same demand class. Lemma 3 shows that a shift between distributions R^y to R^z in the same demand class with $m < 1$ decreases the expected utility of all risk-averse investors.

Lemma 3 *The following statements are equivalent.*

1. *The distributions F^y and F^z are in the same demand class, and the expected utility of all risk-averse investors are lower with risky asset rate of return F^z than with F^y.*

2. *There exists a constant $0 < m < 1$ such that*

$$\int_{-1}^{t} r dF^z(r) = m \int_{-1}^{t} r dF^y(r), \text{ for all } t \in [-1, 1]. \tag{38}$$

3. *There exists a random variable θ and a constant $0 < m \leq 1$ such that*

$$R^z \overset{d}{=} R^y + \theta, \tag{39}$$

where the conditional distribution of θ satisfies

$$F^\theta(s|R^y = r) = mI(s \geq 0) + (1 - m)I(s \geq -r). \tag{40}$$

Combining Theorem 2 and Lemma 3 provides a sufficient condition to reduce demand and expected utility for all risk-averse investors—our definition of bad news.

Theorem 3 *Suppose that there exists a constant $0 \leq m < 1$ such that*

$$\int_{-1}^{t} r dF^z(r) \leq m \int_{-1}^{t} r dF^y(r), \, \forall t \in [-1, 1]. \tag{41}$$

Then demand for the risky asset and expected utility for all risk-averse investors is lower with risky asset rate of return R^z than with risk asset rate of return R^y.

But there are demand reducing shifts that increase the expected utility of some risk-averse investor.

Theorem 4 *Suppose that demand is reduced for all risk-averse investors and expected utility is increasing for some risk-averse investor. Then there exists a constant $m > 1$ such that*

$$\int_{-1}^{t} r dF^z(r) \leq m \int_{-1}^{t} r dF^y(r),\ \forall t \in [-1, 1]. \tag{42}$$

Our results also provide random variable characterizations of the shifts that reduced demand and decrease expected utility for all risk-averse investors, and the shifts that reduce demand for all risk–averse investors and increase expected utility for some risk-averse investor.

As an example of a demand reducing shifts in Theorem 3 and Theorem 4, consider an investor with an initial wealth of 1 and the CARA expected utility function

$$U(C) = e^{-10C}, \tag{43}$$

and the random variables reported in Table 4. Figure 1 plots the expected utility for the investor against the risky asset holding for the investor under the distributions of the risky rate of return in Table 4. The solid line (–) is the expected utility with the original risky rate of return distribution in Panel A of the Table, R^y, and the optimal risky asset holding is approximately 0.85. The dashed line (- -) is the expected utility with the rate of return distribution obtained by a shift within the same demand class with $m = 0.8$ reported in Panel B, R^0. Consistent with Lemma 3, the investor's optimal risky asset holding stays the same as with the original rate of return, but the investor's expected utility drops.

The dotted line (..) is the expected utility after the demand decreasing shift in Panel C, from R^0 to R^∞. Since the shift is a second order shift, the expected utility decreases, and consistent with Lemma 1, demand decreases from approximately 0.85 to approximately 0.65. The shift from the original rate of return, R^y to R^∞ satisfies condition (7) with $m = 0.8$. The shift therefore satisfies the conditions in Theorem 3 and comparing the solid line and the dotted line, both the investor's expected utility and risky-asset demand decreases.

The dashed-dotted line (–.–) is the expected utility after the shift within the same demand class with $m = 1.7$ reported in Panel D, from R^∞ to R^z. The equalities

$$\int_{-1}^{t} r dF^z(r) = 1.7 \int_{-1}^{t} r dF^\infty(r), \text{ for all } t \in [-1, 1] \tag{44}$$

imply that

$$\int_{-1}^{t} r dF^\infty(r) = \frac{1}{1.7} \int_{-1}^{t} r dF^z(r), \text{ for all } t \in [-1, 1]. \tag{45}$$

Since $\frac{1}{1.7} < 1$, Lemma 3 implies that the expected utility increases and the risky asset demand stays the same relative to the demand and expected utility with the risky rate of return R^∞.

The overall shift from R^y to R^z satisfies conditions (7) for $m = 0.8 \times 1.7 = 1.36$. The shift therefore does not satisfy the conditions in Theorem 3. Comparing the solid line to the dash-dotted line in the Figure, risky asset demand has decreased and expected utility has risen for the investor, consistent with Theorem 4.

7 Equilibrium prices

We analyze equilibrium prices in a two-period endowment economy, based on Lucas (1978). Aggregate output in the first period is normalized to one and aggregate output in the second period is the random variable Δ^y, with bounded support, $[L, H]$ with $0 < L < H$ and distribution function F^y. There is a representative agent with time-separable utility function

$$V(C_1) + EU(C_2), \tag{46}$$

with $V : \mathcal{R} \to \mathcal{R}$ and $U : \mathcal{R} \to \mathcal{R}$ concave, increasing functions with first derivatives. Let P^y_{mkt} be the equilibrium price of the market portfolio and let P^y_b be the equilibrium riskless bond price, using first period consumption as the numeraire.

The equilibrium price of the market portfolio is

$$P^y_{mkt} = \frac{\int_L^H U'(\delta)\delta dF^y(\delta)}{V'(1)}, \tag{47}$$

and the equilibrium riskless bond price is

$$P_b^y = \frac{\int_L^H U'(\delta) dF^y(\delta)}{V'(1)}. \tag{48}$$

underline Suppose that we change the cumulative distribution of second period aggregate output from F^y to F^z. When will the equilibrium risky asset price decrease? Gollier and Schlesinger (2002) present the conditions in terms of changes in the distributions in terms of conditions on the partial expectations of the payoffs. We provide the random variable characterization of the shifts.

We prove the results using the bond price as the numeraire. Equations (47) and (48) together imply

$$\int_L^H U'(\delta) \left(\delta - P_{mkt}^y / P_b^y\right) dF^y(\delta) = 0. \tag{49}$$

Since $U' > 0$, if equation (49) is negative when evaluated using F^z, i.e.

$$\int_L^H U'(\delta) \left(\delta - P_{mkt}^y / P_b^y\right) dF^z(\delta) < 0, \tag{50}$$

at the original market portfolio price and riskless bond price, then the equilibrium price of the market portfolio relative to the riskless bond price will fall.

Condition (50) is the same form as the condition to reduce risky asset demand in equation (6), with the random variables R^y replaced with the random variable $\Delta^y - P_{mkt}^y / P_b^y$. Our results can therefore be used to characterize the shifts in the payoff of the risky asset that decrease the price, that decrease the price and make all investors worse off and the shifts that decrease the price and make at least one investor better off. We report the random variable condition of bad news in the next theorem — shifts that reduce the price of the market portfolio and make all investors worse off.

Theorem 5 *A change in the payoff on market portfolio from Δ^y with probability distribution F^y to Δ^z with probability distribution F^z leads to a decrease in the equilibrium price of the market portfolio relative to the riskless bond price by every risk averse investor and makes all risk-averse* *investors worse off if and only if there exist random variables* $\theta^1, \{\epsilon^i\}_{i=1}^{\infty}$ *such that*

$$\Delta^0 \stackrel{d}{=} \Delta^y + \theta^1, \tag{51}$$

$$\Delta^i \stackrel{d}{=} \Delta^{i-1} + \epsilon^i, \;\; \Delta^z \stackrel{d}{=} \lim_{i\to\infty} \Delta^i, \tag{52}$$

satisfying

$$F^{\theta^1}(s|\Delta^y = \delta) = mI(s \geq 0) + (1-m)I(s \geq -(\delta - P^y_{mkt}/P^y_b)), \; for \; 0 \leq m \leq 1, \tag{53}$$

$$E[\epsilon^i|\Delta^{i-1}] \leq 0, \tag{54}$$

$$Prob(P^y_{mkt}/P^y_b < \epsilon^i < -\Delta^{i-1}|\Delta^{i-1} < P^y_{mkt}/P^y_b) = 0, \tag{55}$$

$$Prob(-\Delta^{i-1} < \epsilon^i < P^y_{mkt}/P^y_b)|\Delta^{i-1} \geq P^y_{mkt}/P^y_b) = 0. \tag{56}$$

8 Conclusions

We provide a random variable characterization of necessary and sufficient conditions for a transformation of the risky asset return to induce a decrease in the optimal risky asset holding for every risk–averse investor. We characterize the shifts that keep risky asset holding constant and the shifts necessary to reduce risky asset demand.

The overall change in the distribution of the risky asset return to reduce demand consists of three types of shifts. The first shift keeps risky asset holding constant, and is a concentration of the distribution toward zero. The second shift is a necessary condition to reduce risk asset holding, and is a sequence of shifts, in each of which any probability shifted toward zero is shifted at least to or past zero. The third shift again keeps risky asset holding constant, and is an expansion of the distribution away from zero.

Our contribution is to provide a relatively simple random variable characterization of each type of shift that involves adding a particular random variable to the risky rate of return. Any change in the risky asset return distribution that reduces every risk–averse investor's demand for the risky asset can be analyzed as a combination of relatively simple transformations of the risky return. We provide a random variable characterization of bad news that leads to a decrease in expected utility for all risk–averse investors, and a drop in demand.

The characterization of the necessary and sufficient conditions for a decrease in risky asset demand by every risk–averse investor is of interest at more than one level. Aside from the obvious relevance to the theory of investor portfolio optimization, such a characterization when applied to the market–wide level to the representative investor—who is forced to hold the fixed supply of the risky assets—gives necessary and sufficient conditions for public news about the market's payoffs to cause a drop in the price of the market portfolio.

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Proofs

Proof of Theorem 1

Proof that statement 1 implies statement 2.

We choose utility functions and an m that imply that statement 2 must hold for arbitrary choices of t. Suppose that F^y is the original distribution and F^z is the new distribution. Define

$$m \equiv \frac{\int_{-1}^{1} r dF^z(r)}{\int_{-1}^{1} r dF^y(r)}. \tag{A1}$$

By assumption $\int_{-1}^{1} r dF^y(r) > 0$, so than m is well-defined.

Let t be such that $\int_{-1}^{t} r dF^y(r) \leq 0$. Define the constant $\alpha \geq 0$ as

$$\alpha \equiv \frac{-\int_{-1}^{t} r dF^y(r)}{\int_{-1}^{1} r dF^y(r)}. \tag{A2}$$

Let $X > 0$ and $W > 0$ be arbitrary positive constants and define

$$\hat{C} = W + Xt. \tag{A3}$$

Define the increasing concave utility function

$$U(C) = \begin{cases} (1+\alpha)C, & C \leq \hat{C}, \\ \hat{C} + \alpha C, & C > \hat{C}. \end{cases} \tag{A4}$$

The first-order condition for an investor with utility function given in equation (A4) and initial wealth W is:

$$0 = \int_{-1}^{\frac{\hat{C}-W}{X^*}} r dF^y(r) + \alpha \int_{-1}^{1} r dF^y(r), \tag{A5}$$

with X^* the optimal risky-asset holding. From equations (A2) and (A3), $X^* = X$.

Evaluating the first-order condition using the distribution F^z,

$$0 = \int_{-1}^{\frac{\hat{C}-W}{X^*}} r dF^z(r) + \alpha \int_{-1}^{1} r dF^z(r), \tag{A6}$$

or

$$\begin{aligned}
\int_{-1}^{t} r dF^z(r) &= -\alpha \int_{-1}^{1} r dF^z(r) \\
&= \left(\frac{\int_{-1}^{t} r dF^y(r)}{\int_{-1}^{1} r dF^y(r)} \right) \times \int_{-1}^{1} r dF^z(r) \\
&= \left(\frac{\int_{-1}^{1} r dF^z(r)}{\int_{-1}^{1} r dF^y(r)} \right) \times \int_{-1}^{t} r dF^y(r) \\
&= m \int_{-1}^{t} r dF^y(r), \qquad \text{(A7)}
\end{aligned}$$

where the second line follows from the definition of α, and the fourth line follows from the definition of m. We have therefore shown that for all t where $\int_{-1}^{t} r dF^y(r) \leq 0$, statement 1 implies statement 2.

We now show that the result holds for a t such that $\int_{-1}^{t} r dF^y(r) > 0$. Define the constant β

$$\beta = \frac{\int_{-1}^{t} r dF^y(r) + \int_{-1}^{1} r dF^y(r)}{-\int_{-1}^{0} r dF^y(r)}. \qquad \text{(A8)}$$

Since $-\int_{-1}^{0} r dF^y(r) > 0$, $\beta > 0$. Define the increasing, concave utility function

$$U(C) = \begin{cases} (2+\beta)C, & C \leq W, \\ \beta + 2C, & W < C \leq \hat{C} \\ \beta + \hat{C} + C, & C > \hat{C}. \end{cases}, \qquad \text{(A9)}$$

with $\hat{C}$ given in equation (A3). The first-order condition for the optimal risky asset holding for an investor with wealth of W is:

$$0 = \beta \int_{-1}^{0} r dF^y(r) + \int_{-1}^{\frac{\hat{C}-W}{X^*}} r dF^y(r) + \int_{-1}^{1} r dF^y(r), \qquad \text{(A10)}$$

which using the definition of β is satisfied for $X^* = X$.

Evaluating the first-order condition using the distribution F^z,

$$0 = \beta \int_{-1}^{0} r dF^z(r) + \int_{-1}^{\frac{\hat{C}-W}{X^*}} r dF^z(r) + \int_{-1}^{1} r dF^z(r), \tag{A11}$$

or

$$\begin{aligned}
\int_{-1}^{t} r dF^z(r) &= -\beta \int_{-1}^{0} r dF^z(r) - \int_{-1}^{1} r dF^z(r) \\
&= \left(\frac{\int_{-1}^{t} r dF^y(r) + \int_{-1}^{1} r dF^y(r)}{\int_{-1}^{0} r dF^y(r)} \right) \int_{-1}^{0} r dF^z(r) - \int_{-1}^{1} r dF^z(r) \\
&= \left(\frac{\int_{-1}^{t} r dF^y(r) + \int_{-1}^{1} r dF^y(r)}{\int_{-1}^{0} r dF^y(r)} \right) \times m \int_{-1}^{0} r dF^y(r) - m \int_{-1}^{1} r dF^y(r) \\
&= m \int_{-1}^{t} r dF^y(r), \qquad \text{(A12)}
\end{aligned}$$

where the second line follows from the definition of β and the third line follows from the definition of m and equation (A7) since $\int_{-1}^{0} r dF^y(r) < 0$. We have therefore shown that statement 1 implies statement 2.

Proof that statement 2 implies statement 3.

Differentiating equation (9) with respect to t, $t dF^z(t) = m t dF^y(t)$, implying that

$$dF^z(t) = m dF^y(t), \ \ \forall t \in [-1, 1], t \neq 0. \tag{A13}$$

Letting $\pi_0^y \equiv \text{Prob}(R^y = 0)$ and $\pi_0^z \equiv \text{Prob}(R^z = 0)$, equation (A13) implies that

$$\begin{aligned}
\pi_0^z &= 1 - \int_{r \neq 0} dF^z(r) \\
&= 1 - m \int_{r \neq 0} dF^y(r) \\
&= 1 - m \left(1 - \pi_0^y\right) \\
&= (1 - m) + m \pi_0^y. \qquad \text{(A14)}
\end{aligned}$$

The requirement that $0 \leq \pi_0^z \leq 0$ implies that $0 \leq m \leq \frac{1}{1-\pi_0^y}$. Conditions (A13) and (A14) imply

$$F^z(r) = mF^y(r) + (1-m)I(r \geq 0). \tag{A15}$$

We now show how to construct the random variable θ. There are two cases to consider.

Case 1: $0 < m \leq 1$.

Equation (A14) implies that $\pi_0^z > 0$. Define the discrete random variable θ by

$$\begin{aligned} \text{Prob}(\theta = -r | R^y = r) &= 1 - m \\ \text{Prob}(\theta = 0 | R^y = r) &= m \end{aligned} \tag{A16}$$

The conditional cdf of θ satsifies equation (11). The cdf of $R^y + \theta$ is

$$\begin{aligned} \text{Prob}(R^y + \theta \leq r) &= \int_{-1}^{1} \text{Prob}(\theta \leq r - s | R^y = s) dF^y(s) \\ &= \int_{-1}^{1} \left[mI(r - s \geq 0) + (1-m)I(r - s \geq -s) \right] dF^y(s) \\ &= mF^y(r) + (1-m)I(r \geq 0). \end{aligned} \tag{A17}$$

The second line follows from the definition of θ in equation (A16), and the third line follows because $r - s \geq 0$ only when $s \leq r$, and because $r - s \geq -s$ only when $r \geq 0$.

Case 2: $m > 1$.

Rewriting equation (A14), $\pi_0^y = (1 - \frac{1}{m}) + \frac{1}{m}\pi_0^z$, and since $m > 1$, $\pi_0^y \geq (1 - \frac{1}{m}) > 0$. Define the random variable θ with conditional distribution,

$$\begin{aligned} \text{Prob}(\theta = 0 | R^y \neq 0) &= 1, \\ \text{Prob}(\theta \leq s | R^y = 0) &= \frac{m-1}{\pi_0^y} F^y(s) + \left(1 - \frac{m-1}{\pi_0^y}\right) I(s \geq 0). \end{aligned} \tag{A18}$$

The conditional cdf of θ satsifies equation (12). The cdf of $R^y + \theta$ is

$$\begin{aligned}
\text{Prob}(R^y + \theta \leq r) &= \int_{-1}^{1} \text{Prob}(\theta \leq r - s | R^y = s) dF^y(s) \\
&= \int_{-1, s\neq 0}^{r} dF^y(s) + \left[\frac{m-1}{\pi_0^y} \int_{-1}^{r} dF^y(s) + \left(1 - \frac{m-1}{\pi_0^y}\right) I(r \geq 0) \right] \pi_0^y \\
&= \int_{-1}^{r} dF^y(s) + (m-1) \left[F^y(r) - I(r \geq 0) \right] \\
&= mF^y(r) + (1-m) I(r \geq 0), \qquad \text{(A19)}
\end{aligned}$$

where the second line follows from the conditional distribution of θ, and the third line follows since $\int_{-1,s\neq 0}^{r} dF^y(s) + I(r \geq 0)\pi_0^y = \int_{-1,}^{r} dF^y(s)$.

Proof that statement 3 implies statement 1.

Using equations (A13) and (A14), the first-order condition with R^z evaluated at the optimal risky asset demand with R^y, X^*, for an investor with utility function U and initial wealth W is

$$\begin{aligned}
\int_{-1}^{1} rU'(W + X^* r) dR^z(r) &= \int_{-1, r\neq 0}^{1} rU'(W + X^* r) dF^z + \pi_0^z \left(0U'(W)\right) \\
&= \int_{-1, r\neq 0}^{1} rU'(W + X^* r) m dF^y(r) + \left(0U'(W + X^* r)\right) \pi^y m \\
&= m \left(\int_{-1}^{1} rU'(W + X^* r) dF^y(r) \right) \\
&= 0, \qquad \text{(A20)}
\end{aligned}$$

where the second line follows from condition (1), and the fourth line follows from the optimality of X^*. ■

We use Lemma A1 and Lemma A2 in the proof of Lemma 1.

Lemma A1 *Suppose that the random variable* R^f *with cumulative distribution function* F*, continuous density* f *on support* $[-1, 1]$ *and the random variable* R^g *with cumulative distribution* G*, continuous density* g *and support* $[-1, 1]$ *satisfy*

$$\int_{-1}^{t} rf(r)dr \leq \int_{-1}^{t} rg(r)dr, \ -1 \leq t \leq 1. \qquad \text{(A21)}$$

For $n = 1, ...,$ divide $[-1, 1]$ into $2n$ equally spaced intervals. Define $r_i^n = \frac{i}{n}$ for $i = -n, -n+1, ..., n$, and the sequence of supports $S_n = \{r_i^n\}_{i=1}^{2n}$ for $n = 1, 2, ...,$ with $\lim_{n\uparrow\infty} S_n = [-1, 1]$.

Then there exists a two sequences of discrete random variables R^{f_n} and R^{g_n} on supports S_n for $n = 1, 2, ..., \infty$ with distributions f^n and g^n and cumulative distributions F^n and G^n satisfying for all n

$$\sum_{i=-n}^{t} r_i^n f_i^n \leq \sum_{i=-n}^{t} r_i^n g_i^n, \text{ for } t = 1, 2, ...2n, \tag{A22}$$

with

$$\lim_{n\to\infty} \int_{-1}^{1} |F(r) - F^n(r)|\, dr + \int_{-1}^{1} |G(r) - G^n(r)|\, dr = 0. \tag{A23}$$

Proof of Lemma A1

For $i = -n, -n+1, ..., -1$, define the discrete distribution

$$f_i^n = \int_{r_i^n}^{r_{i+1}^n} \left(\frac{r}{r_i^n}\right) f(r)dr, \tag{A24}$$

and similarly for ${g^n}_i$. For $i = -n, -n+1, ..., -1$ and $r_i^n \leq r \leq r_{i+1}^n$, $0 \leq \frac{r}{r_i^n} \leq 1$, implying that

$$0 \leq f_i^n \leq \int_{r_i^n}^{r_{i+1}^n} f(r)dr, \text{ and } 0 \leq g_i^n \leq \int_{r_i^n}^{r_{i+1}^n} g(r)dr. \tag{A25}$$

For $i = 1, 2, ..., n$, define

$$f_i^n = \int_{r_{i-1}^n}^{r_i^n} \left(\frac{r}{r_i^n}\right) f(r)dr, \tag{A26}$$

and similarly for ${g^n}_i$. For $i = 1, 2, ..., n$ and $0 \leq r_{i-1}^n \leq r \leq r_i^n$, $0 \leq \frac{r}{{r_i}n} \leq 1$, implying that

$$0 \leq f_i^n \leq \int_{r_{i-1}^n}^{r_i^n} f(r)dr, \text{ and } 0 \leq g_i^n \leq \int_{r_{i-1}^n}^{r_i^n} g(r)dr \tag{A27}$$

Define

$$f_i^n = 1 - \sum_{i\neq 0} f_i^n, \tag{A28}$$

and similarly for g_0^n. Equations (A24) through (A28) imply that $0 \leq f_0^n \leq 1$, and similarly for g_0^n.

We now show that conditon (A22) holds for all n. For $t = -n, -n+1, ..., -1$

$$\begin{aligned} \sum_{i=-n}^{t} r_i^n f_i^n &= \sum_{i=-n}^{t} r_i^n \int_{r_i^n}^{r_{i+1}^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\ &= \int_{r_{-n}^n}^{r_t^n} r f(r)dr \\ &\leq \int_{r_{-n}^n}^{r_t^n} r g(r)dr \\ &= \sum_{i=-n}^{t} r_i^n \int_{r_i^n}^{r_{i+1}^n} \left(\frac{r}{r_i^n}\right) g(r)dr \\ &= \sum_{i=-n}^{t} r_i^n g_i^n. \end{aligned} \tag{A29}$$

For $t = 0, 1, ..., n$,

$$\begin{aligned} \sum_{i=-n}^{t} r_i^n r f_i^n &= \sum_{i=-n}^{-1} r_i^n r \int_{r_i^n}^{r_{i+1}^n} \left(\frac{r}{r_i^n}\right) f(r)dr + 0 f_0^n + \sum_{i=1}^{t} r_i^n r \int_{r_{i-1}^n}^{r_i^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\ &= \int_{r_{-n}^n}^{r_t^n} r f(r)dr \\ &\leq \int_{r_{-n}^n}^{r_t^n} r g(r)dr \\ &= \sum_{i=-n}^{t} r_i^n g_i^n, \end{aligned} \tag{A30}$$

where the second line follows because $r_0^n = 0$, the third line follows from equation (A21) and the final line follow from the algebra on the first line applied to g^n and g. We have therefore shown that condition (A22) holds for all n.

We now show that the distribution functions converge. For $r_j^n \leq r < r_{i+1}^n$, the cumulative distribution function is

$$F^n(r) = \sum_{i=-n}^{j} f_i^n = 1 - \sum_{i=j+1}^{n} f_i^n, \tag{A31}$$

with $G^n(r)$ computed similarly.

For $j \leq 0$,

$$\begin{aligned}
\sum_{i=-n}^{j-1} f_i^n &= \sum_{i=-n}^{j} \int_r^{r_i^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\
&> \sum_{i=-n}^{j-1} \int_{r_i^n}^{r_{i+1}^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\
&= \sum_{i=-n}^{j-1} \left(\frac{(i+1)/n}{i/n}\right) \left[F(r_{i+1}^n) - F(r_i^n)\right] \\
&= \sum_{i=-n}^{j-1} (1+1/i) \left[F(r_{i+1}^n) - F(r_i^n)\right] \\
&= F(r_j^n) + \sum_{i=-n}^{j-1} (1/i) \left[F(r_{i+1}^n) - F(r_i^n)\right], \qquad \text{(A32)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=-n}^{j-1} f_i^n &= \sum_{i=-n}^{j} \int_r^{r_i^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\
&< \sum_{i=-n}^{j-1} \int_{r_{i+1}^n}^{r_i^n} f(r)dr \\
&= F(r_j^n). \qquad \text{(A33)}
\end{aligned}$$

For $j > 0$,

$$\begin{aligned}
\sum_{i=j}^{n} f_i^n &= \sum_{i=j}^{n} \int_{r_{i-1}^n}^{r_i^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\
&> \sum_{i=j}^{n} \int_{r_{i-1}^n}^{r_i^n} \left(\frac{r_{i-1}^n}{r_i^n}\right) f(r)dr \sum_{i=j}^{n} \int_{r_{i-1}^n}^{r_i^n} \left(\frac{(i-1)/n}{i/n}\right) \left[F(r_i^n - F(r_{i-1}^n)\right] \\
&= 1 - F(r_{j-1}^n) - \sum_{i=j}^{n} (1/i) \left[F(r_i^n - F(r_{i-1}^n)\right], \qquad \text{(A34)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=j}^{n} f_i^n &= \sum_{i=j}^{n} \int_{r_{i-1}^n}^{r_i^n} \left(\frac{r}{r_i^n}\right) f(r)dr \\
&< \sum_{i=j}^{n} \int_{r_{i-1}^n}^{r_i^n} f(r)dr \\
&= \sum_{i=j}^{n} \left[F(r_i^n - F(r_{i-1}^n)\right] \\
&= 1 - F(r_{j-1}^n). \qquad \text{(A35)}
\end{aligned}$$

By the continuity of f, there is an A such that $f(r) < A < \infty$ for all $-r \in [-1, 1]$. Then since $r_i^n - r_{i-1}^n = \frac{1}{n}$,

$$F(r_i^n) - F(r_{i-1}^n) < \frac{A}{n}. \qquad \text{(A36)}$$

For $r_{j-1}^n \leq r \leq r_j^n \leq 0$, $F(r_{j-1}^n) < F(r) \leq F(r_j^n)$ and

$$\sum_{i=-n}^{j-1} (1/i) \left[F(r_i^n - F(r_{i-1}^n)\right] \geq \sum_{i=-n}^{j-1} (1/i) \left(\frac{A}{n}\right), \qquad \text{(A37)}$$

implying that

$$\sum_{i=-n}^{j-1} f_i^n > F(r_j^m) + \sum_{i=-n}^{j-1} (1/i) \frac{A}{n}, \qquad \text{(A38)}$$

and

$$|F(r) - F^n(r)| \leq \max\left\{F(r_j^n) - F(r_{j-1}^n), \sum_{i=-n}^{j-1} (-1/i) \left(\frac{A}{n}\right)\right\}. \qquad \text{(A39)}$$

Therefore,

$$\int_{-1}^{0} |F(r) - F^n(r)|\, dr \leq \sum_{i=-n+1}^{0} [F(r_i^n) - F(r_{i-1}^n)]\frac{1}{n} + \sum_{j=-n+1}^{0} \left[\sum_{i=-n}^{j-1} (-1/i) \left(\frac{A}{n}\right)\right]. \qquad \text{(A40)}$$

Similarly,

$$\int_{0}^{1} |F(r) - F^n(r)|\, dr \leq \sum_{i=1}^{n} [F(r_i^n) - F(r_{i-1}^n)]\frac{1}{n} + \sum_{j=1}^{n} \left[\sum_{i=-n}^{j-1} (1/i) \left(\frac{A}{n}\right)\right], \qquad \text{(A41)}$$

and therefore

$$
\begin{aligned}
&\int_{-1}^{1} |F(r) - F^n(r)|\, dr \\
&\leq \sum_{i=-n+1}^{n} [F(r_i^n) - F(r_{i-1}^n)]\frac{1}{n} + \sum_{j=-n+1}^{0} \left[\sum_{i=-n}^{j-1} (-1/i)\left(\frac{A}{n}\right)\right] + \sum_{j=1}^{n} \left[\sum_{i=-n}^{j-1} (1/i)\left(\frac{A}{n}\right)\right] \\
&= [F(r_n^n) - F(r_{-n}^n)]\frac{1}{n} + \left(\sum_{j=-n+1}^{0} \left[\sum_{i=-n}^{j-1} (1/i)\right] + \sum_{j=1}^{n} \left[\sum_{i=-n}^{j-1} (-1/i)\right]\right)\frac{A}{n} \\
&= \frac{1}{n} + \frac{2A}{n},
\end{aligned}
\tag{A42}
$$

where the final line follows from evaluating the double summations. Equation (A42) implies

$$
\lim_{n\to\infty} \int_{-1}^{1} |F(r) - F^n(r)|\, dr = 0, \tag{A43}
$$

and similar algebra shows that

$$
\lim_{n\to\infty} \int_{-1}^{1} |G(r) - G^n(r)|\, dr = 0. \tag{A44}
$$

■

Lemma A2 *Let R^a and R^b be two discrete random variables with state spaces r_s, $s = -M, ...N$ and discrete probabilities $\pi_s^a = Prob(R^a = r_s)$ and $\pi_s^b = Prob(R^b = r_s)$ satisfying:*

$$
\sum_{-M}^{n} r_s \pi_s^b \leq \sum_{-M}^{n} r_s \pi_s^a, \ \ \forall n. \tag{A45}
$$

Then, there exists a finite sequence of random variables $\{\epsilon^i\}_{i=1}^{I}$ $I < \infty$ such that:

$$
R^0 \stackrel{d}{=} R^a, \ \ R^i \stackrel{d}{=} R^{i-1} + \epsilon^i, R^b \stackrel{d}{=} R^I, \tag{A46}
$$

$$
E[\epsilon^i | R^{i-1}] \leq 0, \tag{A47}
$$

$$
Prob(0 < \epsilon^i < -R^{i-1} | R^{i-1} < 0) = 0, \tag{A48}
$$

$$Prob(-R^{i-1} < \epsilon^i < 0 | R^{i-1} \geq 0) = 0. \tag{A49}$$

Proof of Lemma A2

Given $\sum_{-M}^{n} r_s \pi_s^b \leq \sum_{-M}^{n} r_s \pi_s^a$, define $\pi_s^0 \equiv \pi_s^a$. We now describe the iterations to construct the sequence of random variables $\{\epsilon^i\}_{i=1}^I$.

Iteration i: Define

$$l_i \equiv \min \left\{ s \left| \pi_s^b > \pi_s^{i-1} \right. \right\}, \;\; p_i \equiv \min \left\{ s \left| \pi_s^b < \pi_s^{i-1} \right. \right\}. \tag{A50}$$

It cannot be the case that $p_i \leq l_i \leq 0$ because that would imply that $\sum_{-m}^{p_i} r_s \pi_s^b > \sum_{-m}^{p_i} r_s \pi_s^{i-1}$, which violates equation (A45). We consider two cases.

Case 1: $l_i < p_i < 0$.

Case 1a. Suppose that $\pi_{l_i}^b - \pi_{l_i}^{i-1} > \frac{p_i}{l_i}\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right) > 0$. Define the random variable ϵ^i:

$$\begin{aligned}
\left[\epsilon^i \left| R^{i-1} = r_{p_i} \right.\right] &= \begin{cases} l_i - p_i, \text{ with probability } \left(\frac{p_i}{l_i}\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi_{p_i}^{i-1}, \\ -p_i \text{ with probability } \left(1 - \frac{p_i}{l_i}\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi_{p_i}^{i-1}, \\ 0 \text{ with probability } 1 - \left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi_{p_i}^{i-1}, \end{cases} && \text{(A51)} \\
\left[\epsilon^i \left| R^{i-1} \neq r_{p_i} \right.\right] &= 0 \text{ with probability } 1. && \text{(A52)}
\end{aligned}$$

This gives

$$\begin{aligned}
\pi_{l_i}^i &= \pi_{l_i}^{i-1} + \left(\frac{p_i}{l_i}\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right), \;\rightarrow \pi_{l_i}^{i-1} < \pi_{l_i}^i < \pi_{l_i}^b && \text{(A53)} \\
\pi_{p_i}^i &= \pi_{p_i}^{i-1}\left[1 - \left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi_{p_i}^{i-1}\right] = \pi_{p_i}^b, && \text{(A54)} \\
\pi_0^i &= \pi_0^{i-1} + \left(1 - \frac{p_i}{l_i}\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right) > \pi_0^{i-1}. && \text{(A55)}
\end{aligned}$$

We now compute

$$\sum_{-M}^{l_i} r_s\pi_s^i = \sum_{-M}^{l_i} r_s\pi_s^{i-1} + p_i\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right) < \sum_{-M}^{l_i} r_s\pi_s^{i-1}, \quad \text{(A56)}$$

$$\sum_{-M}^{p_i} r_s\pi_s^i = \sum_{-M}^{p_i} r_s\pi_s^{i-1} + p_i\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right) - p_i\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right) = \sum_{-M}^{p_i} r_s\pi_s^{i-1}, \quad \text{(A57)}$$

$$\sum_{-M}^{0} r_s\pi_s^i = \sum_{-M}^{0} r_s\pi_s^{i-1} + 0\left(1 - \frac{p_i}{l_i}\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right) = \sum_{-M}^{0} r_s\pi_s^{i-1}. \quad \text{(A58)}$$

This ends case 1a.

Case 1b: Suppose that $0 < \pi_{l_i}^b - \pi_{l_i}^{i-1} < \frac{p_i}{l_i}\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)$. Define the random variable ϵ^i:

$$\left[\epsilon^i \,\middle|\, R^{i-1} = r_{p_i}\right] = \begin{cases} l_i - p_i, \text{ with probability } \left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi^{i-1}p_i, \\ -p_i \text{ with probability } \left(\frac{l_i}{p_i} - 1\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi^{i-1}p_i, \\ 0 \text{ with probability } 1 - \left(\frac{l_i}{p_i}\right)\left(\pi_{p_i}^{i-1} - \pi_{p_i}^b\right)/\pi_{p_i}^{i-1}, \end{cases} \quad \text{(A59)}$$

$$\left[\epsilon^i \,\middle|\, R^{i-1} \neq r_{p_i}\right] = 0 \text{ with probability 1.} \quad \text{(A60)}$$

Straightforward algebra shows that $E\left[\epsilon^i \,\middle|\, R^{i-1} \neq r_{p_i}\right] = E\left[\epsilon^i \,\middle|\, R^{i-1} = r_{p_i}\right] = 0$, and

$$\pi_{l_i}^i = \pi_{l_i}^b, \quad \text{(A61)}$$

$$\pi_{p_i}^i = \pi_{p_i}^{i-1}\left[1 - \left(\frac{l_i}{p_i}\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right)/\pi^{i-1}p_i\right] < \pi_{p_i}^{i-1}, \quad \text{(A62)}$$

$$\pi_0^i = \pi_0^{i-1} + \left(\frac{l_i}{p_i} - 1\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) > \pi_0^{i-1}, \quad \text{(A63)}$$

and

$$\sum_{-M}^{l_i} r_s\pi_s^i = \sum_{-M}^{l_i} r_s\pi_s^{i-1} + l_i\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) < \sum_{-M}^{l_i} r_s\pi_s^{i-1}, \quad \text{(A64)}$$

$$\sum_{-M}^{p_i} r_s\pi_s^i = \sum_{-M}^{p_i} r_s\pi_s^{i-1} + l_i\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) - p_i\left(\frac{l_i}{p_i}\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) = \sum_{-M}^{p_i} r_s\pi_s^{i-1}, \quad \text{(A65)}$$

$$\sum_{-M}^{0} r_s\pi_s^i = \sum_{-M}^{0} r_s\pi_s^{i-1} + 0\left(\frac{l_i}{p_i} - 1\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) = \sum_{-M}^{0} r_s\pi_s^{i-1}. \quad \text{(A66)}$$

This ends case 1b.

The effect of each iteration is that $\pi^i_{l_i}$ increases with i, implying that l_i increases with i and $\pi^i_{p_i}$ decreases with i implying that π_i increases with i. In a finite number of iterations $p_i = 0$ which implies that $\pi^b_s \geq \pi^{i-1}_s \; \forall s < 0$.

Next, define

$$n_i \equiv \max \left\{ s \left| \pi^b_s > \pi^{i-1}_s \right. \right\}, \;\; q_i \equiv \max \left\{ s \left| \pi^b_s < \pi^{i-1}_s \right. \right\}. \tag{A67}$$

It cannot be the case that $0 < n_i < q_i$ because that would imply $\sum_{q_i}^N r_s \pi^b_s < \sum_{q_i}^N r_s \pi^{i-1}_s$, and, since $\sum_{-M}^N r_s \pi^b_s = \sum_{-M}^n r_s \pi^{i-1}_s$, that would imply $\sum_{-M}^{q_{i-1}} r_s \pi^b_s > \sum_{-M}^{q_{i-1}} r_s \pi^{i-1}_s$, which violates equation (A45).

Case 2: $0 < q_i < n_i$.

Case 2a. Suppose that $\pi^b_{n_i} - \pi^{i-1}_{n_i} > \frac{q_i}{n_i} \left(\pi^{i-1}_{q_i} - \pi^b_{q_i} \right) > 0$. Define the random variable ϵ^i:

$$\left[\epsilon^i \left| R^{i-1} = r_{q_i} \right. \right] = \begin{cases} n_i - q_i, \text{ with probability } \left(\frac{q_i}{n_i} \right) \left(\pi^{i-1}_{q_i} - \pi^b_{q_i} \right) / \pi^{i-1}_{q_i}, \\ -q_i \text{ with probability } \left(1 - \frac{q_i}{n_i} \right) \left(\pi^{i-1}_{p_i} - \pi^b_{p_i} \right) / \pi^{i-1}_{q_i}, \\ 0 \text{ with probability } 1 - \left(\pi^{i-1}_{p_i} - \pi^b_{p_i} \right) / \pi^{i-1}_{p_i}, \end{cases} \tag{A68}$$

$$\left[\epsilon^i \left| R^{i-1} \neq r_{q_i} \right. \right] = 0 \text{ with probability } 1. \tag{A69}$$

Straightforward algebra shows that

$$E \left[\epsilon^i \left| s = q_i \right. \right] = E \left[\epsilon^i \left| s \neq q_i \right. \right] = 0, \tag{A70}$$

and

$$\pi^i_{n_i} = \pi^b_{n_i}, \tag{A71}$$

$$\pi^i_{q_i} < \pi^{i-1}_{q_i}, \tag{A72}$$

$$\pi^i_0 > \pi^{i-1}_0, \tag{A73}$$

$$\sum_{-M}^{l_i} r_s\pi_s^i = \sum_{-M}^{l_i} r_s\pi_s^{i-1} + l_i\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) < \sum_{-M}^{l_i} r_s\pi_s^{i-1}, \tag{A74}$$

$$\sum_{-M}^{p_i} r_s\pi_s^i = \sum_{-M}^{p_i} r_s\pi_s^{i-1} + l_i\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) - p_i\left(\frac{l_i}{p_i}\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) = \sum_{-M}^{p_i} r_s\pi_s^{i-1}, \tag{A75}$$

$$\sum_{-M}^{0} r_s\pi_s^i = \sum_{-M}^{0} r_s\pi_s^{i-1} + 0\left(\frac{l_i}{p_i} - 1\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) = \sum_{-M}^{0} r_s\pi_s^{i-1}. \tag{A76}$$

This ends case 2a.

Case 2b: $0 < \pi_{n_i}^b - \pi_{n_i}^{i-1} < \frac{q_i}{n_i}(\pi_{n_i}^{i-1} - \pi_{n_i}^b)$.

Define the random variable ϵ^i

$$\left[\epsilon^i \left| R^{i-1} = r_{q_i}\right.\right] = \begin{cases} n_i - q_i, \text{ with probability } \left(\pi_{q_i}^b - \pi_{q_i}^{i-1}\right)/\pi_{q_i}^{i-1}, \\ -q_i \text{ with probability } \left(\frac{n_i}{q_i} - 1\right)\left(\pi_{p_i}^b - \pi_{p_i}^{i-1}\right)/\pi_{q_i}^{i-1}, \\ 0 \text{ with probability } 1 - \left(\frac{n_i}{q_i}\right)\left(\pi_{p_i}^b - \pi_{p_i}^{i-1}\right)/\pi_{p_i}^{i-1}, \end{cases} \tag{A77}$$

$$\left[\epsilon^i \left| R^{i-1} \neq r_{q_i}\right.\right] = 0 \text{ with probability 1.} \tag{A78}$$

Straightforward algebra shows that

$$E\left[\epsilon^i \left| R^{i-1} = r_{q_i}\right.\right] = E\left[\epsilon^i \left| s \neq q_i\right.\right] = 0, \tag{A79}$$

$$\pi_{n_i}^i = \pi_{n_i}^b, \tag{A80}$$

$$\pi_0^i > \pi_0^{i-1}, \tag{A81}$$

$$\pi_{q_i}^i < \pi_{q_i}^{i-1}, \tag{A82}$$

$$\sum_{-M}^{l_i} r_s\pi_s^i = \sum_{-M}^{l_i} r_s\pi_s^{i-1} + l_i\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) < \sum_{-M}^{l_i} r_s\pi_s^{i-1}, \tag{A83}$$

$$\sum_{-M}^{p_i} r_s\pi_s^i = \sum_{-M}^{p_i} r_s\pi_s^{i-1} + l_i\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) - p_i\left(\frac{l_i}{p_i}\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) = \sum_{-M}^{p_i} r_s\pi_s^{i-1}, \tag{A84}$$

$$\sum_{-M}^{0} r_s\pi_s^i = \sum_{-M}^{0} r_s\pi_s^{i-1} + 0\left(\frac{l_i}{p_i} - 1\right)\left(\pi_{l_i}^b - \pi_{l_i}^{i-1}\right) = \sum_{-M}^{0} r_s\pi_s^{i-1}. \tag{A85}$$

This ends case 2b.

The effect each iteration is that $\pi_{n_i}^i$ increase with i implying that n_i increases with i and $\pi_{q_i}^i$ decreases with i implying that q_i decreases with i. Since there is a finite state space, $q_i = 0$ in a finite number of iterations, which implies that $\pi_s^b \geq \pi_s^{i-1}$ for all $s > 0$ at that iteration.

Combining cases 1 and 2, we have $\pi_s^b \geq \pi_s^{i-1}$ for all $s \neq 0$ and $\pi_0^b \leq \pi_0^{i-1}$. If $\pi_0^b = \pi_0^{i-1}$, then $\pi_s^b = \pi_s^{i-1}$ for all s, and we are done. If If $\pi_0^b < \pi_0^{i-1}$, then $\pi_s^b \geq \pi_s^{i-1}$ for all s. Define the random variable ϵ^i:

$$\begin{aligned}\left[\epsilon^i \,\middle|\, R^{i-1} = 0\right] &= \begin{cases} s \text{ with probability } \frac{\left(\pi_s^b - \pi_s^{i-1}\right)}{\pi_0^{i-1}}, \text{ for } s = -M, -M+1, ..., -1, 1, 2, ...N, \\ 0 \text{ with probability } 1 - \frac{\left(\pi_0^{i-1} - \pi_0^b\right)}{\pi_0^{i-1}}, \end{cases} && \text{(A86)} \\ \left[\epsilon^i \,\middle|\, R^{i-1} \neq 0\right] &= 0 \text{ with probability } 1. && \text{(A87)}\end{aligned}$$

This gives

$$\begin{aligned}\pi_s^i &= \pi_s^{i-1} + \left(\pi_s^b - \pi_s^{i-1}\right) = \pi_s^b, && \text{(A88)} \\ \pi_0^i &= \pi_0^b. && \text{(A89)}\end{aligned}$$

We have therefore constructed a sequence of random variables $\{\epsilon^i\}_{i=1}^I$ satisfying the required conditions. ■

Proof of Lemma 1

Proof that statement 1 implies statement 2.

If F^y and F^z are discrete distributions, then Lemma A2 implies the result. If F^y and F^z are continuous distributions, then by Lemma A1, we can approximate R^y and R^z with sequences R^{y_n} and R^{z_n} with discrete distributions f^{y_n} and f^{z_n} such that

$$\sum_{i=-n}^{t} r_i^n f_i^{z_n} \leq \sum_{i=-n}^{t} r_i^n f_i^{y_n}, \text{ for } t = 1, 2, ...2n, \tag{A90}$$

with

$$\lim_{n\to\infty} \int_{-1}^{1} |F^y(r) - F^{y_n}(r)|\, dr + \int_{-1}^{1} |F^z(r) - F^{z_n}(r)|\, dr = 0. \tag{A91}$$

By Lemma A2, the random variable R^{y_n} can be transformed to the random variable R^{z_n} by adding a finite sequence of random variables ϵ_i^n satisfying the required conditions.

If F^y and F^z are mixed discrete-continous distributions, then we approximate the continuous part as above, and use Lemma A2 to construct the appropriate finite sequence of random variables.

Proof that statement 2 implies statement 1.

Suppose $R^z \stackrel{d}{=} R^y + \epsilon^1$ with the conditional distribution of ϵ^1 satisfying conditions (17) through (19) for $i = 1$, with $R^0 \stackrel{d}{=} R^y$. We show that conditions (15) hold. The condition therefore also holds for a sequence $\{\epsilon^i\}_{i=1}^{\infty}$, each satisfying the conditions reported in the lemma.

Using iterated expectations,

$$\begin{aligned}\int_{-1}^{t} r dF^x(r) &= E\left[I(R^z \leq t)R^z\right] \\ &= E\left[E\left[I(R^y + \epsilon^1 \leq t)(R^y + \epsilon^1)\middle| R^y = r\right]\right].\end{aligned} \quad . \tag{A92}$$

We show that

$$E\left[I(R^y + \epsilon^1 \leq t)(R^y + \epsilon^1)\middle| R^y = r\right] \leq I(r \leq t)r \tag{A93}$$

which implies the result.

Case 1: $R^y = r > 0$.

$$\begin{aligned}&E\left[I(R^y + \epsilon^1 \leq t)(R^y + \epsilon^1)\middle| R^y = r\right] \\ &= \int_{-1}^{-r} I(r + e \leq t)(r + e)dF^1(e|R^y = r) + \int_{0}^{1} I(r + e \leq t)(r + e)dF^1(e|R^y = r) \\ &\leq \int_{-1}^{-r} I(r \leq t)(r + e)dF^1(e|R^y = r) + \int_{0}^{1} I(r \leq t)(r + e)dF^1(e|R^y = r) \\ &= I(r \leq t)\left(r + E\left[\epsilon^1\middle| R^y = r\right]\right) \\ &\leq I(r \leq t)r.\end{aligned} \tag{A94}$$

The first line follows from condition (19), the second line follows because $I(x \leq t)$ is a non-negative, non-increasing function of x, and the third line follows from condition (17).

Case 2: $R^y = r \leq 0$.

$$
\begin{aligned}
&E\left[I(R^y + \epsilon^1 \leq t)(R^y + \epsilon^1)\middle|\, R^y = r\right] \\
&= \int_{-1}^{r} I(r + e \leq t)(r + e)dF^1(e|R^y = r) + \int_{-r}^{1} I(r + e \leq t)(r + e)dF^1(e|R^y = r) \\
&\leq \int_{-1}^{r} I(r \leq t)(r + e)dF^1(e|R^y = r) + \int_{-r}^{1} I(r \leq t)(r + e)dF^1(e|R^y = r) \\
&= I(r \leq t)\left(r + E\left[\epsilon^1\middle|\, R^y = r\right]\right) \\
&\leq I(rt)r.
\end{aligned} \tag{A95}
$$

The first line follows from condition (18), the second line follows because $I(x \leq t)$ is a non-negative non-increasing function of x, and the third line follows from condition (17). ∎

Proof of Lemma 2

Suppose $R^1 \stackrel{d}{=} R^y + \epsilon^1$ with the conditional distribution of ϵ^1 satisfying conditions (17) through (19) for $i = 1$, with $R^0 \stackrel{d}{=} R^y$. We show that for U' a positive non-increasing function,

$$
E\left[U'(W + (R^y + \epsilon^1)X^*)(R^y + \epsilon^1)\middle|\, R^y = r\right] \leq U'(W + rX^*)r, \text{ for all } r \in [-1, 1], \tag{A96}
$$

which implies the result by iterated expectations. Define $U'(x) \equiv U'(W + X^*x)$.

Case 1: $R^y = r > 0$.

$$
\begin{aligned}
&E\left[U'(R^y + \epsilon^1)(R^y + \epsilon^1)\middle|\, R^y = r\right] \\
&= \int_{-1}^{-r} U'(r + e)(r + e)dF^1(e|R^y = r) + \int_{0}^{1} U'(r + e)(r + e)dF^1(e|R^y = r) \\
&\leq \int_{-1}^{-r} U'(r)(r + e)dF^1(e|R^y = r) + \int_{0}^{1} U'(r)(r + e)dF^1(e|R^y = r) \\
&= U'(r)\left(r + E\left[\epsilon^1\middle|\, R^y = r\right]\right) \\
&\leq U'(r)r.
\end{aligned} \tag{A97}
$$

The first line follows from condition (19), the second line follows because U' is a positive, non-increasing function, and the third line follows from condition (17).

Case 2: $R^y = r \leq 0$.

$$\begin{aligned}
&E\left[U'(R^y+\epsilon^1)(R^y+\epsilon^1)\middle|\, R^y = r\right] \\
&= \int_{-1}^{r} U'(r+e)(r+e)dF^1(e|R^y=r) + \int_{-r}^{1} U'(r+e)(r+e)dF^1(e|R^y=r) \\
&\leq \int_{-1}^{r} U'(r)(r+e)dF^1(e|R^y=r) + \int_{-r}^{1} U'(r)(r+e)dF^1(e|R^y=r) \\
&= U'(r)\left(r + E\left[\epsilon^1\middle|\, R^y=r\right]\right) \\
&\leq U'(r)r.
\end{aligned} \tag{A98}$$

The first line follows from condition (18), the second line follows because U' is a positive non-increasing function, and the third line follows from condition (17). ■

Proof of Theorem 2

The equivalence of statement 1 and statement 2 follows from Theorem 0. The equivalence of statement 3 and statement 1 follows from combining Theorem 1, Lemma 1 and Lemma 2. ■

Proof of Lemma 3

Proof that statement 2 implies statement 1.

From Theorem 1, risky asset demand is the same for both distributions for all risk-averse investors only if condition (9) holds for some m. We now show that utility is reduced for $0 < m < 1$.

Suppose that the partial expectations of the random rate of return R^z equal the m times the partial expectations of the random rate of return R^y with $0 < m < 1$, Then, R^z is distributed as a mixture of R^y with probability m and a degenerate random variable at 0 with probability $1-m$. Let U be an arbitrary concave utility function and normalize utility so that $U(W) = 0$. Since expected utility is invariant to affine transformations, such a normalization does not change the investor's preferences. Let $X^* > 0$ be the investor's optimal demand with risky asset rate of returns R^y and R^z. Since $X^* > 0$ is optimal,

$$\left(\int_{-1}^{1} U(W + X^* r)dF^y(r)\right) > U(W) = 0. \tag{A99}$$

The investor's expected utility with distribution R^z is

$$\begin{aligned}
\int_{-1}^{1} U(W+X^*r)dF^z(r) &= m\left(\int_{-1}^{1} U(W+X^*r)dF^y(r)\right) + (1-m)\left(U(W+X^*0)\right) \\
&= m\left(\int_{-1}^{1} U(W+X^*r)dF^y(r)\right) \\
&< \left(\int_{-1}^{1} U(W+X^*r)dF^y(r)\right), \qquad \text{(A100)}
\end{aligned}$$

where the final line follows from inequality (A99) and $0 < m < 1$.

Proof that statement 1 implies statement 2. Suppose that condition (38) holds for some $m > 1$. Then

$$\int_{-1}^{t} rdF^y(r) = \frac{1}{m}\int_{-1}^{t} rdF^y(r), \text{ for all } t \in [-1,1]. \qquad \text{(A101)}$$

Since $m > 1$, $\frac{1}{m} < 1$ the proof that statement 1 implies statement 2 implies

$$\int_{-1}^{1} U(W+X^*r)dF^y(r) < \int_{-1}^{1} U(W+X^*r)dF^x(r), \qquad \text{(A102)}$$

which implies the result.

Proof of the equivalence of statement 2 and statement 3.

The result follows from Theorem 1. ■

Proof of Theorem 3

The result follows from Lemma 3 and Theorem 2. ■

Proof of Theorem 4

By Theorem 2, a reduction in demand implies that equation (42) holds for some m. Since expected utility is increasing for some investor, Theorem 3 implies that equation (42) cannot hold for any $m < 1$. ■

Proof of Theorem 5 The result follows by recognizing that equation (50) is of the same form as equation (6), and applying Theorem 2 and Theorem 3 to the random variables $\Delta^y - p^y_{mkt}/p^y_b$ and $\Delta^z - p^y_{mkt}/p^y_b$. ■

Table 1: Shifting within a demand class

t	-0.10	0.00	0.10	0.20	0.30
Panel A: Shifting towards zero: $m = 0.8$					
Original distribution, R^y					
$\text{Prob}(R^y = t)$	0.100	0.300	0.200	0.300	0.100
$F^y(t)$	0.100	0.400	0.600	0.900	1.000
$\sum_{r \leq t} r\text{Prob}(R^y = r)$	-0.010	-0.010	0.010	0.070	0.100
Shifted distribution, $R^{z1} \overset{d}{=} R^y + \theta^1$					
$\text{Prob}(R^y = t)$	0.080	0.440	0.160	0.240	0.080
$F^y(t)$	0.080	0.520	0.680	0.920	1.000
$\sum_{r \leq t} r\text{Prob}(R^y = r)$	-0.008	-0.008	0.008	0.056	0.080
Panel B: Shifting away from zero: $m = 1.7$					
Original distribution, R^y					
$\text{Prob}(R^y = t)$	0.120	0.440	0.040	0.320	0.080
$F^y(t)$	0.120	0.560	0.600	0.920	1.000
$\sum_{r \leq t} r\text{Prob}(R^y = r)$	-0.012	-0.012	-0.008	0.056	0.080
Shifted distribution, $R^{z2} \overset{d}{=} R^y + \theta^2$					
$\text{Prob}(R^y = t)$	0.204	0.048	0.068	0.544	0.136
$F^y(t)$	0.204	0.252	0.320	0.864	1.000
$\sum_{r \leq t} r\text{Prob}(R^y = r)$	-0.020	-0.020	-0.014	0.095	0.136

The table reports the probability distribution function, the cumulative probability distribution function, and the partial expectations for the random variables R^y, R^{z1} and R^{z2}. The random variable $R^{z1} \overset{d}{=} R^y + \theta^1$ with the conditional distribution functions for θ^1 given in equation (13), and the random variable $R^{z2} \overset{d}{=} R^y + \theta^2$ with the distribution functions for θ^2 given in equation (14).

Table 2: A shift satisfying the sufficient conditions to reduce demand

t	-0.10	0.00	0.10	0.20	0.30
Original distribution, R^y					
$\text{Prob}(R^y = t)$	0.080	0.440	0.160	0.240	0.080
$F^y(t)$	0.080	0.520	0.680	0.920	1.000
$\sum_{r \leq t} r\text{Prob}(R^y = r)$	-0.008	-0.008	0.008	0.056	0.080
Shifted distribution, $R^z \stackrel{d}{=} R^y + \epsilon^1$					
$\text{Prob}(R^z = t)$	0.120	0.440	0.040	0.320	0.080
$F^z(t)$	0.120	0.560	0.600	0.920	1.000
$\sum_{r \leq t} r\text{Prob}(R^z = r)$	-0.012	-0.012	-0.008	0.056	0.080

The table reports the probability distribution function, the cumulative probability distribution function, and the partial expectations for the random variables R^y, and R^z. The random variable $R^z \stackrel{d}{=} R^y + \epsilon^1$ with the conditional distribution functions for ϵ^1 given in equation (24).

Table 3: A shift that does not reduce demand

t	-0.10	0.00	0.10	0.20	0.30
		Original distribution, R^y			
$\text{Prob}(R^y = t)$	0.080	0.440	0.160	0.240	0.080
$F^y(t)$	0.080	0.520	0.680	0.920	1.000
$\sum_{r \le t} r\text{Prob}(R^y = r)$	-0.008	-0.008	0.008	0.056	0.080
		Shifted distribution, $R^z \overset{d}{=} R^y + \epsilon^1$			
$\text{Prob}(R^z = t)$	0.080	0.440	0.220	0.120	0.140
$F^z(t)$	0.080	0.520	0.740	0.860	1.000
$\sum_{r \le t} r\text{Prob}(R^z = r)$	-0.008	-0.008	0.014	0.038	0.080

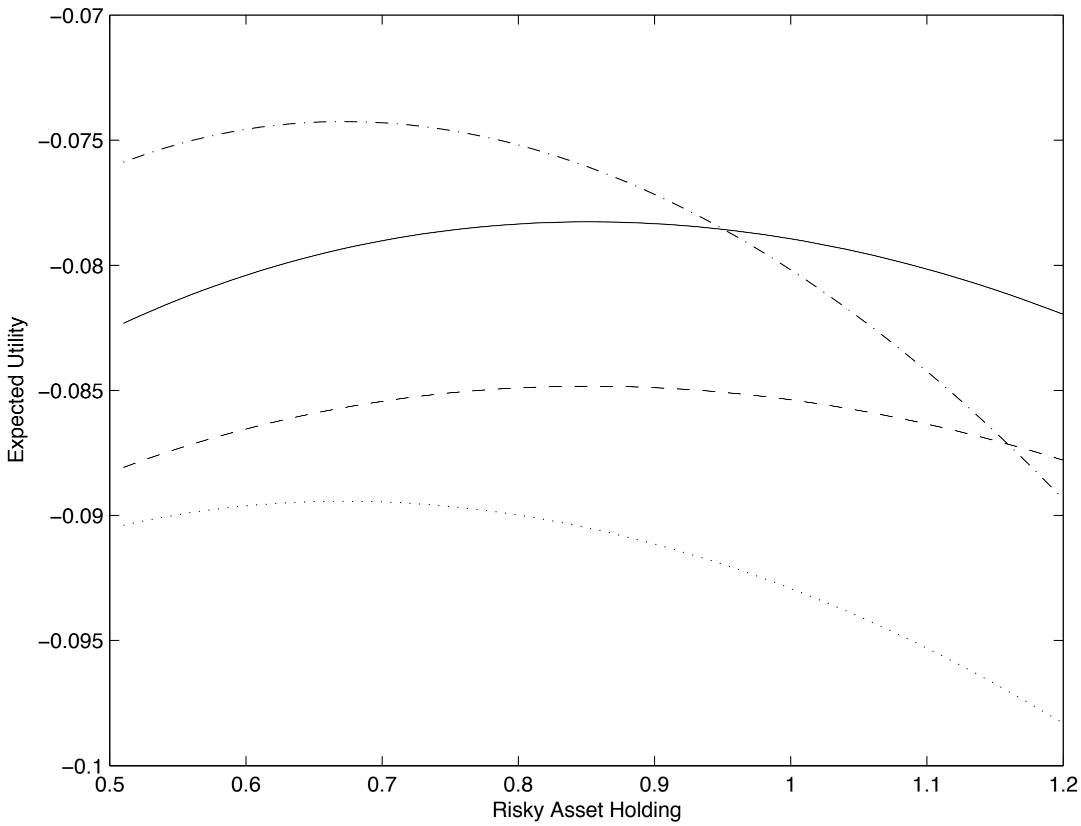
The table reports the probability distribution function, the cumulative probability distribution function, and the partial expectations for the random variables R^y, and R^z. The random variable $R^z \overset{d}{=} R^y + \epsilon^1$ with the conditional distribution function for ϵ^1 given in equation (25).

Table 4: A shift satisfying the necessary and sufficient conditions to reduce demand

t	-0.10	0.00	0.10	0.20	0.30
Panel A: Original distribution, R^y					
$\text{Prob}(R^y = t)$	0.100	0.300	0.200	0.300	0.100
$F^y(t)$	0.100	0.400	0.600	0.900	1.000
$\sum_{r \le t} r\text{Prob}(R^y = r)$	-0.010	-0.010	0.010	0.070	0.100
Panel B: Shifted in with $m = 0.8$: $R^0 \overset{d}{=} R^y + \theta^1$					
$\text{Prob}(R^0 = t)$	0.080	0.440	0.160	0.240	0.080
$F^0(t)$	0.080	0.520	0.680	0.920	1.000
$\sum_{r \le t} r\text{Prob}(R^0 = r)$	-0.008	-0.008	0.008	0.056	0.080
Panel C: Demand reducing shift: $R^\infty \overset{d}{=} R^0 + \epsilon^1$					
$\text{Prob}(R^\infty = t)$	0.120	0.440	0.040	0.320	0.080
$F^\infty(t)$	0.120	0.560	0.600	0.920	1.000
$\sum_{r \le t} r\text{Prob}(R^\infty = r)$	-0.012	-0.012	-0.008	0.056	0.080
$0.8 \times \sum_{r \le t} r\text{Prob}(R^y = r)$	-0.008	-0.008	0.008	0.056	0.080
Panel D: Final distribution, shifted out with $m = 1.7$, $R^z \overset{d}{=} R^\infty + \theta^2$					
$\text{Prob}(R^z = t)$	0.204	0.048	0.068	0.544	0.136
$F^z(t)$	0.204	0.252	0.320	0.864	1.000
$\sum_{r \le t} r\text{Prob}(R^z = r)$	-0.020	-0.020	-0.014	0.095	0.136
$1.36 \times \sum_{r \le t} r\text{Prob}(R^y = r)$	-0.014	-0.014	0.014	0.095	0.136

The table reports the probability distribution function, the cumulative probability distribution function, and the partial expectations for the random variables R^y, R^0, R^∞ and R^z. The random variable $R^0 \overset{d}{=} R^y + \theta^1$ with the distribution function for θ^1 conditional on R^y given in equation (13), the random variable $R^\infty \overset{d}{=} R^0 + \epsilon^1$ with the distribution function for ϵ^2 conditional on R^0 the same as the distribution function of ϵ^1 conditional on R^y given in equation (24), and the random variable $R^z \overset{d}{=} R^\infty + \theta^2$ with the distribution function for θ^2 conditional on R^∞ the same as the distribution function of θ^2 conditional on R^y given in equation (14).

Figure 1: Expected utility for different risky rate of return distributions



The figure plots the expected utility against the risky asset holding for an investor with initial wealth of 1 and CARA utility function $U(C) = e^{-10C}$ with the risky rate of return distributions reported in Table 4. The solid line (—) is the expected utility with the original risky rate of return distribution in Panel A, R^y; the dashed line (- - -) is the expected utility with the rate of return distribution obtained by a shift within the same demand class with $m = 0.8$ in Panel B, R^0; the dotted line (...) is the expected utility after the demand decreasing shift in Panel C, R^∞; and the dashed-dotted line (–.–) is the expected utility after the shift within the same demand class with $m = 1.7$ reported in Panel D.