On the Relationship Between Determinate and MSV Solutions in Linear RE Models

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ABSTRACT This paper considers the possibility that, in linear rational expectations (RE) models, all determinate (uniquely non-explosive) solutions coincide with the minimum state variable (MSV) solution, which is unique by construction. In univariate specifications of the form $y_t = AE_t y_{t+1} + Cy_{t-1} + u_t$ that result holds: if a RE solution is unique and non-explosive, then it is the same as the MSV solution. Also, this result holds for multivariate versions if the A and C matrices commute and a regularity condition holds. More generally, however, there are models of this form that possess unique non-explosive solutions that differ from their MSV solutions. Examples are provided and a procedure for easily constructing others is described.

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1. Introduction

Much recent research in economics, especially monetary economics, has emphasized the concept of determinacy of rational expectation solutions—i.e., the property of a solution being the only non-explosive solution. It is well known that in linear rational expectations (RE) models a necessary and sufficient condition for determinacy to prevail is that the number of eigenvalues of the system's matrix pencil that exceed 1.0 in modulus equals the number of non-predetermined endogenous variables.[1] In various prominent cases, this condition does not obtain so there is no unique non-explosive solution.

Some researchers[2] have focused attention on the minimum state variable (MSV) solution, defined and promoted in McCallum (1983, 1999), which is by construction unique but possibly explosive, and exists if the model has any real (non-imaginary) solution.[3] It is obviously the case that some MSV solutions are not determinate, but it is unclear whether there are models in which a determinate solution exists but is not the MSV solution. That possibility has been hinted at by McCallum (1983, 1998) and Uhlig (1999), but examples have not been examined. There are some reasons, perhaps, to suspect that it might be true that all unique stable solutions are MSV solutions. Such a situation is easily seen to prevail in univariate models of the form $y_t = AE_t y_{t+1} + Cy_{t-1} + u_t$ and, as is shown below, also holds for multivariate versions if the A and C matrices commute and a regularity condition due to Binder and Pesaran (1995) obtains.

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[1] See, e.g., Blanchard and Kahn (1980), Binder and Pesaran (1995), King and Watson (1998), among many others.

[2] Examples include Bullard and Mitra (2002), Barro (1989), Faust and Svensson (2001), and Leitemo (2003).

[3] It is important to note that the term "minimum state variable" is here being used in the manner of McCallum (1983, 1999) or Evans (1986), rather than that of Evans and Honkapohja (2001) or Gauthier (2003), which permits more than one MSV solution. See Section 2 below.

Furthermore, there are recent results by Gauthier (2002, 2003) and Desgranges and Gauthier (2003) showing, among other things, that the same result holds in univariate perfect foresight models with additional lagged terms and/or expected future values.

It transpires, nevertheless, that with or without commuting A and C matrices, there can exist unique stable solutions that differ from the MSV solution. This will be demonstrated below, in Section 4. In addition, the paper discusses (in Section 5 and elsewhere) various aspects of the two types of solutions and their implied criteria for selection of a RE solution. Section 2 outlines the specification to be utilized and provides preliminary results, while Section 3 mentions conditions under which unique stable solutions will invariably be MSV solutions. Finally, Section 6 provides a very short conclusion.

2. Preliminaries

Because our main result consists of a counterexample, it will not be necessary to utilize a framework with full generality. Instead, it will be convenient to consider the specification treated by McCallum (1983, pp. 164-166). With y_t denoting a m×1 vector of endogenous variables, the system is

(1) $y_t = A\, E_t y_{t+1} + C\, y_{t-1} + u_t,$

where $u_t = R\, u_{t-1} + \varepsilon_t$, with R a stable m×m matrix and ε_t a white noise vector.[4] Also, it is assumed that A is nonsingular. That is a strong assumption, which renders the formulation (1) highly inconvenient from a practical perspective, but is acceptable for the purposes at hand. Furthermore, for other purposes the implied case can provide a useful precursor for a more general analysis, as is illustrated in McCallum (2003).

[4] A stable matrix has all its eigenvalues less than 1 in modulus. In (1), constant terms have been suppressed for notational simplicity while A and C are of dimension m×m.

In this setting, the MSV solution will be of the form

(2) $y_t = \Omega\, y_{t-1} + \Gamma\, u_t$.

Accordingly, $E_t y_{t+1} = \Omega(\Omega y_{t-1} + \Gamma u_t) + \Gamma R u_t$ and straightforward undetermined-coefficient reasoning yields the requirement that the solution for Ω satisfies

(3) $A\Omega^2 - \Omega + C = 0$,

where all of the matrices are of order m×m. There are other implications, of course, but the occurrence of multiple solutions arises entirely because of the nonlinear nature of (3); for a given Ω, Γ is determined uniquely. In this setting, the MSV concept requires that $\Omega = 0$ if $C = 0$, since otherwise the solution would in that case include extraneous variables, and the MSV solution is defined as the one whose expression for Ω continuously approaches 0 as C approaches a zero matrix.

With A invertible, the matrix quadratic equation (3) can, as is well known, be expressed in a first-order manner as

(4) $$\begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}C & A^{-1} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix}.$$

Let M denote the 2m×2m matrix M in (4) and assume, without significant loss of generality, that it is diagonalizable. Then it follows that $M = P^{-1}\Lambda P$, with Λ a diagonal matrix with the eigenvalues of M on its diagonal. Then we can premultiply (4) by P, where $P^{-1} = H$ is the matrix of (right) eigenvectors of M, to obtain

(5) $$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix},$$

where the P_{ij} are submatrices of P and where the diagonals of Λ_1 and Λ_2 contain the eigenvalues of M.

To obtain the MSV solution, McCallum (1983) orders or groups the eigenvalues (and associated eigenvectors) so that Λ_1 includes those that approach 0 as C approaches 0.[5] Then the MSV expression for Ω is implied by the second row of (5) to be[6]

(6) $\Omega = -\, P_{22}^{-1}\, P_{21}$.

Further, since PM = ΛP, we have (from the lower left submatrix) that $-P_{22}A^{-1}C = \Lambda_2 P_{21}$ so if the inverse of Λ_2 exists, (6) gives the solution

(7) $\Omega = P_{22}^{-1}\Lambda_2^{-1}\, P_{22}A^{-1}C$,

for which Ω approaches 0 as C approaches 0. For this conclusion, it needs to be shown that Λ_2^{-1} exists in the limit. But the eigenvalues of M are obtained from det[M − λI] = 0, and using a result on the determinant of a partitioned matrix,[7] we have that

(8) $$\det[M - \lambda I] \;=\; \det\begin{bmatrix} -\lambda I & I \\ -A^{-1}C & A^{-1} - \lambda I \end{bmatrix}$$

$$=\; \det[A^{-1} - \lambda I]\, \det[-\lambda I + I(A^{-1} - \lambda I)^{-1}A^{-1}C].$$

From the latter we see that for any ordering of the eigenvalues, half of them will (continuously) approach zero and the other half will approach the eigenvalues of A^{-1} as C approaches 0. Thus with the MSV definition of Λ_1, it is implied that the eigenvalues of Λ_2 approach those of A^{-1}, which are all non-zero.

At this point it should be emphasized that expression (7) gives different solutions for different groupings of eigenvalues into Λ_1 and Λ_2. Since M is 2m×2m, there are $(2m)!/(m!)^2$ different groupings, each of which provides a solution given by (6). There is

[5] The identification of this grouping is based on the continuity of eigenvalues with respect to the elements of the underlying matrix (M, in this case). We let C approach a zero matrix by replacing C with Cα in all relevant expressions and letting the real scalar α vary continuously from 1.0 to 0.
[6] This row can be written as $(P_{21} + P_{22}\Omega)\Omega = \Lambda_2(P_{21} + P_{22}\Omega)$.
[7] See Johnston (1972, p. 95).

only one for which (7) is well defined in the limit as C approaches 0, however, since (8) implies that all others feature Λ_2 matrices that are not invertible when C = 0.

Let us now consider the particular solution given by (7) when the eigenvalues are instead arranged so that Λ_1 includes those that are smallest (in modulus). Also recall that H denotes the eigenvectors of M so that PH = I. The latter implies that $P_{21}H_{11} + P_{22}H_{21} = 0$ and from the upper left-hand submatrix of $MH = H\Lambda$ we have that $H_{21} = H_{11}\Lambda_1$. Therefore $\Omega = -P_{22}^{-1}P_{21} = P_{22}^{-1}P_{22}H_{21}H_{11}^{-1} = H_{21}H_{11}^{-1} = H_{11}\Lambda_1 H_{11}^{-1}$. But the latter has the same eigenvalues as Λ_1, which under present assumptions are the m smallest eigenvalues of M. If there is a unique stable (determinate) solution, it will of course feature an Ω whose eigenvalues are the m smallest eigenvalues of M. Therefore, if there is a unique stable RE solution, it will be given by expression (7) with Λ_1 including the smallest eigenvalues of M.

Is it likely that the unique stable solution and the MSV solution will coincide, if the former exists? Clearly, if the entries in C are all small, so that C is close to a zero matrix, they will coincide since the MSV solution for Ω will have near-zero eigenvalues—and these will then tend to be the smallest of M's eigenvalues, which are those that appear in Λ_1 for the unique stable solution. Thus there is a distinct tendency for unique stable and MSV solutions to coincide. Indeed, they must coincide unless the set of eigenvalues, which includes only the m smallest, changes in composition as α goes from 1 to 0. For if it does not, then (7) will apply to the unique stable solution in the limit, making it correspond to the MSV solution.

3. Special Cases

Let us now briefly consider the special cases mentioned above in which it is true that the unique stable and MSV solutions coincide. The simplest example is that in which m = 1, i.e., the model (1) is univariate. Let us write the quadratic (3) for that case as $a\phi^2 - \phi + c = 0$, where we use ϕ in place of Ω. In this case we have roots for ϕ equal to $\Lambda_1 = (1-\sqrt{1-4ac})/2a$ and $\Lambda_2 = (1+\sqrt{1-4ac})/2a$. The first of these approaches 0 as c approaches 0, the second approaching a^{-1}, so it gives the MSV solution. In addition, however, the first has the smaller modulus since $+\sqrt{1-4ac}$ has the same sign as 1. Consequently, the stated coincidence obtains quite generally.

Somewhat less familiar is the result, of which the former is a special case, that obtains when A and C in (1) commute. It is a standard result in matrix analysis that if two matrices commute (and are diagonalizable) then they can be diagonalized by the same matrix. That implies that the same matrix, say T, diagonalizes both A and C and also sums and products of those two matrices. Therefore it follows, as shown by Binder and Pesaran (1995, p. 158), that the terms in (1) can be diagonalized to yield

(9) $$\Lambda_A\Lambda_\Omega^2 - \Lambda_\Lambda + \Lambda_C = 0,$$

where each Λ matrix includes the eigenvalues of the designated matrix on its diagonal and zeros elsewhere. But (9) implies m distinct scalar equations of the quadratic form considered in the previous paragraph. If the model has a unique stable solution, these m equations must have m roots (eigenvalues of M) with modulus less than 1.0 and m with modulus greater than 1.0. The MSV solution will assign one root—the one for which λ_Ω approaches 0 as λ_C approaches 0—from each of these quadratic equations to the matrix Λ_1, so it will coincide with the unique stable solution if and only if each of the quadratics has one root greater (and one root smaller) than 1.0 in modulus. Binder and Pesaran (1995, p. 157) describe a regularity condition that guarantees such a configuration, but it seems to be of limited interest in the present context.

4. Examples

We now turn to a numerical specification that provides a counterexample to the conjecture that all unique stable solutions to models of form (1) are also MSV solutions. It is given by equation (1) with the following A and C matrices:

$$(10) \qquad A = \begin{bmatrix} -1.5 & 1.2 \\ 0.5 & -1.3 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 1.2 & 0.5 \\ 0.5 & 1.6 \end{bmatrix}$$

For simplicity, we can take the R matrix to contain only zeros; that assumption does not affect the solutions for Ω. The magnitudes relevant for our issues of concern are the eigenvalues of M in the problem as just specified, i.e., with $\alpha = 1$, and for other values of α on the interval [0, 1]. In Table 1, the eigenvalues are reported for α equal to 0.8, 0.6, 0.4, 0.2, and 0.0. They are reported in order of decreasing modulus in each case. The

Table 1

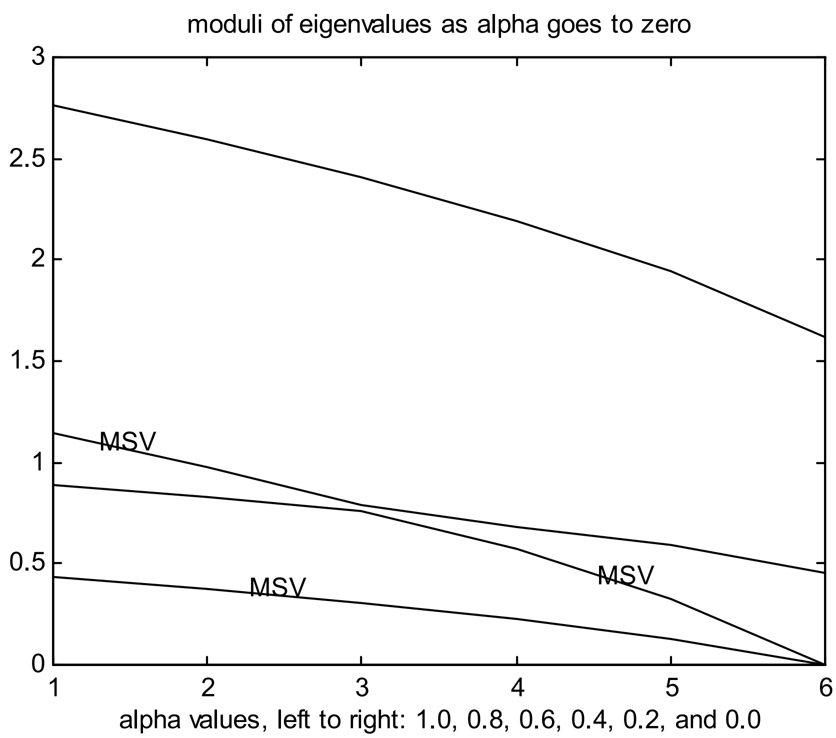
Eigenvalues of M for various values of α

$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.0$
–2.7022	–2.5402	–2.3615	–2.1593	–1.9211	–1.6156
1.0887	0.9267	–0.7961	–0.7108	–0.6066	–0.4585
–0.9365	–0.8702	0.7479	0.5456	0.3070	0.0000
0.4759	0.4096	0.3357	0.2505	0.1466	0.0000

results for the actual problem at hand are given in the first column. It is readily seen to be one in which there is a unique stable solution, since there are two eigenvalues with modulus greater than 1.0. Thus –0.9365 and 0.4795 are the diagonal elements of the Λ_1

matrix if the latter is defined as relevant for seeking a determinate solution, i.e., as including the smallest (modulus) eigenvalues. But what is the MSV solution for the model? Since eigenvalues are continuous functions of the model parameters, it is clear that the second-listed eigenvalue in the first two columns is the “same” as the third eigenvalue in the remaining columns.[8] Thus the composition of Λ_1 relevant for the MSV solution includes 1.0887 and 0.4759. The MSV solution differs from the unique stable solution; indeed, the MSV solution is dynamically explosive. A plot of the modulus of the four eigenvalues against α is shown in Figure 1. There the Matlab diagram

Figure 1



does not accurately show the crossing of eigenvalue moduli that occurs at a value of α slightly above 0.6 but, because one eigenvalue is positive and the other is negative, the

[8] Here “same” is used in the following sense: for each specific eigenvalue, its value is a continuous function of each of the elements of the M matrix.

numbers in Table 1 make the situation clear. (The two curves pertaining to the MSV solution are so labeled in the figure.)

Reflection indicates that there is a much simpler way of generating an example with an eigenvalue crossing that keeps a determinate solution from being the MSV solution. Consider the m = 2 case, and suppose that the two rows of (1) represent separate univariate models. One of these can be specified so as to imply an explosive univariate solution (one in which both eigenvalues exceed 1.0 in modulus) and the other to imply multiple stable solutions (both eigenvalues are less than 1.0 in modulus). A pair of such models does not constitute a legitimate bivariate model, and will not permit RE solutions with some software.[9] But by simply adding a very small non-zero value as one or more of the off-diagonal elements of the A or C matrix, a valid bivariate model of form (1) can be obtained. Yet with a very small value for this off-diagonal element, the eigenvalues for this bivariate model will be approximately the same as for the two univariate models taken together. Accordingly, they will include two stable and two explosive eigenvalues. The bivariate system will therefore be determinate; it will have one stable solution. The MSV solution will, however, involve one eigenvalue from each of the univariate models and will therefore differ from the unique stable solution.[10]

An example of the type just described is provided by the univariate models defined by $a_{11} = -0.4$, $c_{11} = 1.5$ and $a_{22} = -1.5$, $c_{22} = 0.2$ with zeros elsewhere. The first of these has two explosive roots (–3.5549 and 1.0549) and the second has two stable roots (–0.8277 and 0.1611). To create a non-degenerate bivariate model we change a_{12},

[9] A necessary rank condition is not satisfied. See, e.g., King and Watson (1998) or McCallum (1998).

[10] Note that if both univariate systems have unique stable solutions, the MSV and unique stable bivariate solutions will coincide. If both are explosive, both solution criteria will indicate an explosive solution. If both have multiple solutions, so will the bivariate model—but one solution will be selected by the MSV criterion.

a_{21}, c_{12}, and c_{21} from 0.0 to the values 0.01, 0.02, 0.02, and 0.01, respectively. Then the resulting eigenvalues for various values of α are as reported in Table 2. As in the example of Table 1, there is a unique stable solution for the model (i.e., with $\alpha = 1$) but it differs from the MSV solution.

Table 2

Eigenvalues of M for various values of α

$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.0$
−3.5563	−3.3873	−3.2038	−3.0012	−2.7719	−2.5011
1.0551	0.8862	−0.7703	−0.7387	−0.7044	−0.6666
−0.8275	−0.7998	0.7027	0.5001	0.2707	0.0000
0.1610	0.1332	0.1038	0.0721	0.0378	0.0000

5. Discussion

In light of the last type of example, it is interesting to note that whenever one combines one explosive and one indeterminate univariate model and adds very small off-diagonal elements, the resulting MSV solution will be explosive; one of its Λ_1 eigenvalues will be greater than 1.0 in modulus. There will, nevertheless, be a unique stable solution. Consider, then, the two approaches or criteria for designation of the relevant RE solution, one being to adopt only unique stable solutions and the other being to adopt the MSV solution. Clearly these approaches will lead to different predictions about the dynamic behavior of the bivariate model in the type of case being considered. This observation leads naturally to the question: Which outcome would actually prevail in an economic setting that combines, with very weak interaction, an explosive sector and one that has two stable solutions? One promising possibility is to determine, as in the work of Evans and Honkapohja (2001), which (if either) of the solutions is E-stable, a property that is closely related to the least-squares learnability of the solution. (For some relevant results, see McCallum (2003).)

Continuing in this vein, it is worthy of note that the continuity properties of the MSV and smallest-eigenvalue solution concepts are very different. By construction, the MSV solution's Ω matrix will vary continuously with the model's parameters (i.e., the elements of A and C). The smallest-eigenvalue criterion permits changes in the group of eigenvalues included in Λ_1, however, which are likely to result in major discontinuities in Ω and to involve changes in the existence or absence of a unique stable solution.

6. Conclusions

We close with a brief summary. The foregoing pages have considered the possibility that, in linear rational expectations models, all determinate (uniquely non-explosive) solutions coincide with the minimum state variable (MSV) solution, which exists and is unique by construction whenever a model has a real solution. In univariate specifications of the form $y_t = AE_ty_{t+1} + Cy_{t-1} + u_t$, with u_t autoregressive of order one, that result holds: if a RE solution is unique and non-explosive, then it is the same as the MSV solution. Also, this result holds for multivariate versions of that specification if the A and C matrices commute and a regularity condition, mentioned by Binder and Pesaran (1995), holds. More generally, however, there are models of this form that possess unique non-explosive solutions that differ from their MSV solutions. Examples are provided, a strategy for easily constructing such examples is outlined, and the sharply contrasting continuity properties of MSV and unique stable solutions are described.

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