Risk & Sustainable Management Group

Risk and Uncertainty Working Paper: R08#1

Research supported by an Australian Research Council Federation Fellowship
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Generalized Invariant Preferences: Two-parameter Representations of Preferences

By

Robert G. Chambers

and

John Quiggin

Australian Research Council Federation Fellow, University of Queensland

Schools of Economics and Political Science
University of Queensland
Brisbane, 4072
rsmg@uq.edu.au
http://www.uq.edu.au/economics/rsmg

THE UNIVERSITY OF QUEENSLAND AUSTRALIA

Generalized Invariant Preferences: Two-parameter Representations of Preferences[1]

R. G. Chambers[2] and J. Quiggin[3]

February 11, 2008

[1]We would like to thank Simon Grant for helpful discussions on the content of this paper.
[2]University of Maryland
[3]University of Queensland

Two-parameter representations of preferences under uncertainty, such as mean-variance preferences (Tobin 1958, Markowitz 1959), are naturally appealing. Most importantly, they correspond directly to the natural interpretation of choice under uncertainty as a trade-off between *risk* and *expected return*. Such preferences also yield relatively tractable comparative-static analysis and form the basis of the highly successful capital asset pricing model (CAPM) (Sharpe 1964, Lintner 1965).

Quiggin and Chambers (2004) derive conditions under which preferences can be characterized in terms of a preference function with two arguments, the mean and an index of risk that is sublinear (positively linearly homogeneous and subadditive) in deviations from that mean. The best-known example of such a risk index is the standard deviation, which gives rise to mean-variance preferences. However, more general risk indexes, incorporating, for example, skewness and higher moments may also be represented in this form. Thus, because investors seem to display a preference for skewness, this extended class of models potentially solves an important problem that is frequently associated with mean-variance preferences.

The central idea of Quiggin and Chambers (2004) is to replace concepts of constant absolute and relative risk aversion, defined over the stochastic return space, with the less demanding requirement that rankings of equal-mean random variables are unaffected by radial expansions and by translations in the direction of the constant act. The concepts of constant absolute risk aversion and constant relative risk aversion correspond exactly to the imposition of translation homotheticity and radial homotheticity, respectively, upon the preference functional over stochastic outcomes. Quiggin and Chambers (2004) generalize by imposing these properties only over equal-mean returns. They refer to such preferences as *invariant*, and show that they can be characterized by preferences over the mean and a risk index. However, because the probability measure determining the mean is exogenously given in this framework, invariant preferences necessarily presume the existence of a known (unique) probability measure.

In reality, however, there is considerable disagreement over the probabilities of relevant events, and many decisionmakers may not possess well-defined probability distributions over events. Evidence comes from both the market place, where, for example, experts can differ widely in their assessment of the probabilities of common market events such as recessions, and from the laboratory, where empirical evidence rather routinely questions the existence of probablistically sophisticated preferences. In particular, it appears doubtful that beliefs can be represented by probabilities derived from historical relative frequencies, as is standard in empirical applications of mean-variance theory. In the literature on choice under uncertainty, situations of this kind are commonly analyzed in terms of *ambiguity* (Ellsberg 1961).

There have been very many contributions to this literature. Maccheroni, Marinacci and Rustichini (2006) presented a model of variational preferences, in which preferences are characterized by a utility function u on outcomes and an ambiguity index c on the set of probabilities on the states of the world. Preferences of the kind described by Maccheroni, Marinacci and Rustichini (2006) can satisfy the translation invariance property of Quiggin and Chambers (2004). The relationship between the two models is explored further by Grant and Polak (2007).

In this paper, we generalize the model of Quiggin and Chambers (2004) to allow for ambiguity, and derive conditions, referred to as *generalized invariance,* under which a two-argument representation of preferences may be obtained independent of the existence of a unique probability measure. The first of these two arguments inherits the properties of standard means, namely, that they are upper semi-continuous, translatable and positively linearly homogeneous. But instead of being additive, these *generalized means* are superadditive. Superadditivity allows for means that are computed (conservatively) with respect to a set of prior probability measures rather than a singleton probability measure. The second argument of the preference structure is a further generalization of the risk index derived in Quiggin and Chambers (2004). It is sublinear in deviations from the generalized mean discussed above.

The paper is organized as follows. After setting up the notation, we state axioms that give rise to a generalized model of uncertainty with multiple priors. The axioms are weaker than those of Gilboa and Schmeidler (1989). Next, we show that these preference give rise to the generalized invariance property, and derive a representation theorem that yields a two-parameter representation of preferences. Finally, we show how the two-parameter representation can be used to provide tractable concepts of comparative aversion to uncertainty as well as a very tractable analytic framework upon which to base comparative-static analysis, even in the absence of well-defined subjective probabilities.

1 Notation and background

The stochastic setting is modelled by a measurable space, $\Omega = (S, \Sigma)$, where S represents the set of states of Nature and Σ represents an algebra of measurable events. Probability measures on Σ are denoted by $\pi \in \Delta$ where Δ is the probability simplex. Preferences are defined over acts, represented by measurable mappings from S to a space X of consequences, where $X \subseteq \mathbb{R}^M$. X is assumed to be an unbounded, closed, convex cone that contains the origin, $0 \in \mathbb{R}^M$.

A random variable, $\tilde{f}$, can be thought of as the element of X^S defined by

$$\tilde{f} = [f(s) : s \in S],$$

where $f : S \to X$ is the measurable map defining the random variable. We adopt the standard abuse of notation by which, for any $x \in X$, we also let x denote the constant act such that

$$x(s) = x, \qquad \forall s \in S.$$

The set of all measurable mappings $f : S \to X$ is denoted $\mathcal{F}$.

2 Axioms

We assume the existence of a preference ordering $\succeq$ on $\mathcal{F}$ with properties described by the axioms given below. The asymmetric and symmetric components of the preference ordering are denoted, respectively, $\succ$ and $\sim$. The preference ordering $\succeq$ on $\mathcal{F}$ induces an ordering on X, $\succeq_X$.(The subscript will be dropped except where ambiguity is likely.)

Denote

$$\Pi^*_{\succsim} = \left\{ \pi \in \Delta : \pi' \tilde{f} \succsim \tilde{f} \text{ for all } \tilde{f} \in \mathcal{F} \right\}.$$

$\Pi^*_{\succsim}$ is the set of priors for which the standard mean generated by the given prior for each act is at least weakly preferred to the act itself. Hence, $\Pi^*_{\succsim}$ corresponds intuitively to the set of priors (possibly empty) for which the decisionmaker is risk averse in its usual sense. For $\tilde{f} \in F$ define:

$$\mu\left(\tilde{f};\Pi^*_{\succsim}\right) = \left\{\pi^{o\prime}\tilde{f} : \pi^o \in cl\left(\Pi^*_{\succsim}\right), \pi'\tilde{f} \succeq \pi^{o\prime}\tilde{f} \text{ for all } \pi \in \Pi^*_{\succsim}\right\}$$

if $\Pi^*_{\succsim}$ is non-empty and $\underline{x}$ otherwise. Observe that $\mu\left(\tilde{f};\Pi^*_{\succsim}\right)$ may not be a singleton, but that if $x, x' \in \mu\left(\tilde{f};\Pi^*_{\succsim}\right)$, then necessarily $x \sim x'$. $\mu\left(\tilde{f};\Pi^*_{\succsim}\right)$ is the least preferred mean value of the act $\tilde{f}$ generated from $\Pi^*_{\succsim}$. Hence, it has a straightforward interpretation as a generalized mean value. If $\Pi^*_{\succsim}$ is nonempty, $\mu\left(\tilde{f};\Pi^*_{\succsim}\right) \succeq \tilde{f}$.

We impose the following properties on $\succeq$:

Axiom 1 *A.1 Weak order: If $\tilde{f}, \tilde{g}, \tilde{h} \in \mathcal{F}$ then*

(a) either $\tilde{f} \succeq \tilde{g}$ or $\tilde{g} \succeq \tilde{f}$ (completeness)

(b) $\tilde{f} \succeq \tilde{g}$ and $\tilde{g} \succeq \tilde{h} \Rightarrow \tilde{f} \succeq \tilde{h}$.

Axiom 2 *A.2 Continuity: If $\tilde{f}, \tilde{g}, \tilde{h} \in \mathcal{F}, x \in X$, the sets $\left\{\alpha \in [0,1] : \alpha\tilde{f} + (1-\alpha)x \succeq \alpha\tilde{g} + (1-\alpha)x\right\}$ and $\left\{\alpha \in [0,1] : \tilde{h} \succeq \alpha\tilde{g} + (1-\alpha)x\right\}$ are closed.*

Axiom 3 *A.3 Monotonicity: If $\tilde{f}, \tilde{g} \in \mathcal{F}$ and $f(s) \succeq_X g(s), \forall s \in S$, then $\tilde{f} \succeq \tilde{g}$. If $f(s) \succ_X g(s), \forall s \in S$, then $\tilde{f} \succ \tilde{g}$.*

Axiom 4 *A.4 Uncertainty aversion: If $\tilde{f}, \tilde{g} \in \mathcal{F}$ and $\alpha \in (0,1)$*

$$\tilde{f} \sim \tilde{g} \Rightarrow \alpha\tilde{f} + (1-\alpha)\tilde{g} \succeq \tilde{f}.$$

Axiom 5 *A.5 Non-degeneracy: $\tilde{f} \succ \tilde{g}$ for some $\tilde{f}, \tilde{g}$*

Axiom 6 *A.6 Equal-mean certainty independence: For $x \in X$, $\alpha \in (0,1)$ and $\tilde{f}, \tilde{g} \in \mathcal{F}$ such that $\mu\left(\tilde{f};\Pi^*_{\succsim}\right) \sim \mu\left(\tilde{g};\Pi^*_{\succsim}\right)$,*

$$\tilde{f} \succeq \tilde{g} \Leftrightarrow \alpha\tilde{f} + (1-\alpha)x \succeq \alpha\tilde{g} + (1-\alpha)x.$$

A.1 to A.6 ensure that $\Pi^*_{\succsim}$ is closed (A.2), convex (A.1 and A.4), and non-empty (otherwise A.6 would give rise to a contradiction). In what follows, for the sake of a compact notation, we shall write $\mu\left(\tilde{f};\Pi^*_{\succsim}\right)$ as $\mu^*\left(\tilde{f}\right)$.

Except for A.6., which we have labelled equal-mean certainty independence, these properties are the same as in Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006). Gilboa and Schmeidler's (1989) version of certainty independence requires for $x \in X$, $\alpha \in (0,1)$, and $\tilde{f}, \tilde{g} \in \mathcal{F}$

$$\tilde{f} \succeq \tilde{g} \Leftrightarrow \alpha\tilde{f} + (1-\alpha)x \succeq \alpha\tilde{g} + (1-\alpha)x$$

Thus, A.6 is strictly weaker than the corresponding axiom of Gilboa and Schmeidler (1989) by virtue of the restriction to $\tilde{f}, \tilde{g} \in \mathcal{F}$ such that $\mu^*\left(\tilde{f}\right) \sim \mu^*(\tilde{g})$. While A.6 is strictly weaker than the corresponding axiom of Gilboa and Schmeidler, it still has important consequences. For example,

Lemma 7 *If preferences satisfy A.1 and A.6, for $x, y, z \in X$, $\alpha \in (0,1)$,*

$$x \sim y \Leftrightarrow \alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z.$$

Lemma 7, which Gilboa and Schmeidler (1989) preferences also satisfy, has a number of implications. First, by taking $\alpha = \frac{1}{2}$ it establishes that, for constant acts, preferences satisfying our axioms also satisfy the Herstein and Milnor (1953) axioms for the existence of an affine utility structure that maps X to the reals. Second, by taking z to be either x or y, it implies that indifference surfaces are linear over constant acts. Moreover, because we explicitly assume $0 \in X$, Lemma 7 implies for $x, y \in X$, and $\beta > 0$ that

$$x \sim y \Leftrightarrow \beta x \sim \beta y.$$

Preferences over constant acts must be radially homothetic with linear indifference surfaces.

Maccheroni, Marinacci and Rustichini (2006) adopt an alternative weakening of the Gilboa–Schmeidler certainty-independence axiom, namely that for all $\tilde{f}, \tilde{g} \in \mathcal{F}$, $x, y \in X$ and $\alpha \in (0,1]$

$$\alpha\tilde{f} + (1-\alpha)x \succeq \alpha\tilde{g} + (1-\alpha)x \Rightarrow \alpha\tilde{f} + (1-\alpha)y \succeq \alpha\tilde{g} + (1-\alpha)y$$

It can be shown (Maccheroni, Marinacci and Rustichini 2006)) that Gilboa and Schmeidler's (1989) certainty-independence axiom can be recast as follows for all $x, y \in X$, $\alpha, \beta \in (0,1]$ and $\tilde{f}, \tilde{g} \in \mathcal{F}$:

$$\alpha\tilde{f} + (1-\alpha)x \succeq \alpha\tilde{g} + (1-\alpha)x \Rightarrow \beta\tilde{f} + (1-\beta)x \succeq \beta\tilde{g} + (1-\beta)x.$$

Hence, as Maccheroni, Marinacci, and Rustichini (2006) put it, the Gilboa and Schmeidler (1989) independence axiom requires two forms of preference independence: independence when mixing with constant acts, and independence with respect to the mixing parameter.[1] Their reformulation retains independence when mixing with constant acts, but not independence with respect to the mixing parameter. Our version of certainty independence requires independence when mixing random acts *possessing indifferent generalized means* with constant acts as well as independence with respect to the mixing parameter (again over indifferent generalized means).

2.1 Utility space

Under the stated assumptions, the analysis may be undertaken in utility space. We have

Lemma 8 *If preferences satisfy A.1–A.6, there exists a utility function* $u : X \to \mathbb{R}$ *such that:* $u(tx) = tu(x)$, $t > 0$; $u(x) \geq u(y) \Leftrightarrow x \succeq y$; *and* $u(x+y) = u(x) + u(y)$.

Note that this is slightly stronger than the corresponding result in Gilboa and Schmeidler (1989) because our assumptions ensure that X contains the origin, so that with an appropriate normalization, u is linear rather than merely affine.

Given a utility function $u : X \to \mathbb{R}$, representing $\succeq$, there is an induced mapping from $\mathcal{F}$ to $U \subseteq \mathbb{R}^S$, where

$$U = \left\{ \tilde{u} \in \mathbb{R}^S : \tilde{u} = u \circ \tilde{f},\ \tilde{f} \in \mathcal{F} \right\}.$$

In what follows, we use the shorthand notation

$$\tilde{u}\left(\tilde{f}\right) \equiv u \circ \tilde{f}$$

so that the realization of $\tilde{u}\left(\tilde{f}\right)$ in state s is $u\left(\tilde{f}(s)\right)$. Conversely, the original preference relation over $\mathcal{F}$ induces a unique preference relationship over U, which with a minor abuse of notation is again denoted $\succeq$, satisfying, for $\tilde{f}, \tilde{g} \in \mathcal{F}$

$$\tilde{u}\left(\tilde{f}\right) \succeq \tilde{u}(\tilde{g}) \Leftrightarrow \tilde{f} \succeq \tilde{g}.$$

[1] Ghirardato, Maccheroni, and Marinacci (2005) elaborate on the role that certainty independence plays in the separation of beliefs and tastes for general preference structures.

Thus, axiom A.2 has a natural interpretation in utility space. More generally, the properties A.1–A.6 are inherited by $\succeq$ over U. Given the existence of a well-defined utility function and a generalized mean, the analysis of the properties of preferences may be undertaken in utility space. Denote by $\tilde{1} \in \mathbb{R}^S$ the unit vector with all entries equal to 1.

The generalized mean $\mu^* : \mathcal{F} \to X$, by Lemma 8, induces a corresponding generalized mean on U in the obvious way.

$$\begin{aligned}
\mu^*\left(\tilde{u}\left(\tilde{f}\right)\right) &= \left\{\pi^{o\prime}\tilde{u}\left(\tilde{f}\right) : \pi^o \in \Pi^*_{\succsim}, \pi'\tilde{u}\left(\tilde{f}\right) \succeq \pi^{o\prime}\tilde{u}\left(\tilde{f}\right) \text{ for all } \pi \in \Pi^*_{\succsim}\right\} \\
&= \left\{\pi^{o\prime}\tilde{u}\left(\tilde{f}\right) : \pi^o \in \Pi^*_{\succsim}, \pi'\tilde{u}\left(\tilde{f}\right) \geq \pi^{o\prime}\tilde{u}\left(\tilde{f}\right) \text{ for all } \pi \in \Pi^*_{\succsim}\right\} \\
&= \min\left\{\pi'\tilde{u}\left(\tilde{f}\right) : \pi \in \Pi^*_{\succsim}\right\}.
\end{aligned}$$

Thus, $\mu^*\left(\tilde{u}\left(\tilde{f}\right)\right)$ can be recognized as the (lower) support function for the closed, convex set $\Pi^*_{\succsim}$. Hence, it is upper semi-continuous, positively linearly homogeneous, and superadditive in $\tilde{u}$. It also satisfies $\mu^*(\tilde{u} + \delta) = \mu^*(\tilde{u}) + \delta$.

3 Generalized invariance and multiple priors

In this section, we show that preferences satisfying Axioms A.1 to A.6 satisfy a generalized version of the Quiggin and Chambers (2004) notion of invariance, which we refer to as *generalized invariant preferences.* We characterize that notion geometrically in terms of the preference maps showing that it is equivalent to imposing both constant risk aversion and constant relative risk aversion across indifferent mean returns. This geometric discussion emphasizes the role that restrictions on preference maps over stochastic outcomes play in inducing small parameter representations of preferences. Define

$$V(\tilde{u}) = \{\tilde{u}' \in U : \tilde{u}' \succeq \tilde{u}\}.$$

Lemma 9 *Under Axioms A.1 to A.6. $V(\tilde{u})$ satisfies:*

1. *(a) Either $\tilde{u}' \in V(\tilde{u})$ or $\tilde{u} \in V(\tilde{u}')$, (b) $\tilde{u}' \succeq \tilde{u} \Leftrightarrow V(\tilde{u}') \subseteq V(\tilde{u})$;*
2. *$V(\tilde{u})$ is closed;*
3. *$\tilde{u}' \geq \tilde{u} \Rightarrow V(\tilde{u}') \subseteq V(\tilde{u})$;*

4. if $\tilde{u}' \sim \tilde{u}$, $\alpha\tilde{u}' + (1-\alpha)\,\tilde{u} \in V(\tilde{u})$ for $\alpha \in (0,1)$;

5. $V(\tilde{u})$ has non-empty interior for some $\tilde{u}$;

6. $\mu^*(\tilde{u}) \in V(\tilde{u})$ for all $\tilde{u} \in U$; and

7. for $\mu^*(\tilde{u}) \sim \mu^*(\tilde{u}')$, $\delta \in \mathbb{R}$, and $\alpha \in (0,1)$, $\tilde{u}' \in V(\tilde{u}) \Leftrightarrow \alpha\tilde{u}' + (1-\alpha)\,\delta \in V(\alpha\tilde{u} + (1-\alpha)\,\delta)$.

Upon defining,

$$\begin{aligned} M_\mu(\alpha) &\equiv \{\tilde{u} \in \mathcal{F} : \mu^*(\tilde{u}) \sim \alpha\}, \\ K_\mu(\tilde{u}) &\equiv V(\tilde{u}) \cap M_\mu(\mu^*(\tilde{u})), \end{aligned}$$

Lemma 9 immediately gives

Proposition 10 *Under Axioms A.1 to A.6, preferences satisfy:*

i) generalized radial invariance

$$K_\mu(t\tilde{u}) = tK_\mu(\tilde{u}), t > 0;$$

ii) generalized multiplicative spread invariance

$$K_\mu(\alpha\tilde{u} + (1-\alpha)\mu^*(\tilde{u})) = K_\mu(\alpha\tilde{u}) + (1-\alpha)\mu^*(\tilde{u})\tilde{1}, \quad \alpha \in (0,1); \text{ and}$$

iii) generalized translation invariance

$$K_\mu(\tilde{u} + \delta) = K_\mu(\tilde{u}) + \delta\tilde{1}, \qquad \delta \in \Re.$$

Proposition 10 demonstrates that preferences satisfying Axioms A.1 to A.6 satisfy the natural extension of Quiggin and Chambers (2004) concept of invariant preferences that replaces a mean calculated with a singleton prior with μ^*. Thus, we shall refer to preferences satisfying the conditions in Proposition 10 as being *generalized invariant* in what follows.

4 Representation theorem

We now show that generalized invariant preferences admit a simple two-parameter representation. The primary argument in favor of two-parameter representations of preferences, such as the mean-variance framework, is the analytic tractability offered by a simple decomposition of preferences into a pure wealth component (the mean) and a pure risk component (the variance). A similar degree of tractability only became available in expected-utility models when notions of constant absolute risk aversion and constant relative risk aversion were developed.

Geometrically, both constant absolute risk aversion and constant relative risk aversion can be visualized as imposing different types of homotheticity on preference maps over stochastic outcomes. Hence, basic consumer and producer theory teach us that, even in the absence of the strong separability assumptions implied by expected utility theory, these restrictions offer a tremendous amount of analytic tractability (Chambers and Quiggin, 2000). When imposed jointly, constant absolute risk aversion and constant relative risk aversion require that preferences over stochastic outcomes assume the Gilboa–Schmeidler (1989) form with a linear utility structure (Safra and Segal, 1998; Quiggin and Chambers, 1998).

Choose $x, y \in X$ so that, for $\tilde{f} \in F$ and $s \in S$, $x \succeq \tilde{f}(s) \succeq y$. By $A.2$, a standard argument reveals there exists an $\alpha \in [0,1]$ such that

$$\alpha x + (1-\alpha) y \sim \tilde{f}.$$

Hence, for any act $\tilde{f}$, there exists a certainty equivalent $c_f \in X$ yielding a certainty-equivalent utility of $u(c_f)$.

We now characterize that certainty-equivalent utility. Define the benefit function (Luenberger, 1992)

$$B(\tilde{u}; V) = \max\{\beta \in \mathbb{R} : \tilde{u} - \beta \in V\},$$

if there is some $\beta \in \mathbb{R}$ such that $\tilde{u} - \beta \in V$ and ∞ otherwise. $B(\tilde{u}; V)$ is a complete function representation of preferences in the sense that

$$B\left(\tilde{u}; V\left(\tilde{u}^0\right)\right) \geq 0 \Leftrightarrow \tilde{u} \in V\left(\tilde{u}^0\right) \Leftrightarrow \tilde{u} \succeq \tilde{u}^0.$$

That $\tilde{u} \in V(\tilde{u}^0) \Rightarrow B(\tilde{u}; V(\tilde{u}^0)) \geq 0$ follows by construction, while Lemma 9.3 implies $\tilde{u} \succeq \tilde{u} - B(\tilde{u}; V(\tilde{u}^0)) \in V(\tilde{u}^0)$ if $B(\tilde{u}; V(\tilde{u}^0)) \geq 0$.[2]

[2]One can also show, for example, that B is concave and nondecreasing in $\tilde{u}$ under our axioms.

By construction $V' \subseteq V$ implies $B(\tilde{u}; V) \geq B(\tilde{u}; V')$ because $\tilde{u} - B(\tilde{u}; V') \in V' \subseteq V$. The benefit function thus gives the certainty equivalent utility as a special case. To obtain the certainty equivalent utility, set

$$e(\tilde{u}) \equiv = -B(0; V(\tilde{u})).$$

By the fact that $B(0; V(\tilde{u}))$ is a complete function representation of preferences we obtain

$$\tilde{u}' \succeq \tilde{u} \Leftrightarrow e(\tilde{u}') \geq e(\tilde{u}),$$

so that we can equivalently write

$$V(\tilde{u}) = \{\tilde{u}' : e(\tilde{u}') \geq e(\tilde{u})\}.$$

Finally, notice that Lemma 9.3 implies for $\tilde{u}' \geq \tilde{u}$ that

$$e(\tilde{u}') \geq e(\tilde{u}).$$

Applying Proposition 10 allows us to demonstrate that when restricted to comparisons across indifferent mean sets, preferences consistent with our axioms are translatable and radially homothetic:

Proposition 11 *Under Axioms A.1 to A.6, whenever* $\mu^*(\tilde{u}) = \mu^*(\tilde{u}')$:

(i) $e(\tilde{u}) \geq e(\tilde{u}') \Rightarrow e(\tilde{u} + \delta) \geq e(\tilde{u}' + \delta)$, $\delta \in \Re$;

ii) $e(\tilde{u}) \geq e(\tilde{u}') \Rightarrow e(t\tilde{u}) \geq e(t\tilde{u}')$, $t > 0$; *and*

(iii) $e(\tilde{u}) \geq e(\tilde{u}') \Rightarrow e(\alpha\tilde{u} + (1-\alpha)\mu^*(\tilde{u})) \geq e(\alpha\tilde{u}' + (1-\alpha)\mu^*(\tilde{u}'))$.

To obtain a two-parameter representation of generalized invariant preferences, we first define for given μ^* a risk index as a function of the form:

$$\rho(\tilde{u} - \mu^*(\tilde{u}); P^*) = \sup_p \{p'(\tilde{u} - \mu^*(\tilde{u})) : p \in P^*\},$$

where P^* is a closed convex set containing the origin. $\rho(\tilde{u} - \mu^*(\tilde{u}); P^*)$, obviously, is the support function for P^*. By the properties of support functions it is lower semi-continuous and sublinear (positively linearly homogeneous and subadditive) in $\tilde{u} - \mu^*(\tilde{u})$ and nonegative (because P^* contains the origin). This definition leads to our main representation result.

Theorem 12 *Under Axioms A.1–5, preferences are generalized invariant if and only if there exists a linear utility function* $u : X \to \mathbb{R}_+$*, a risk index* ρ*, and a mapping* $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that: (i) for* $\tilde{u} \in U$*,* $e(\tilde{u}) = \phi(\mu^*(\tilde{u}), \rho(\tilde{u} - \mu^*(\tilde{u}); P^*))$*; (ii)* ϕ *is increasing in its first argument and decreasing in its second, and (iii)* $\tilde{u}' \geq \tilde{u} \Rightarrow \phi(\mu^*(\tilde{u}'), \rho(\tilde{u}'; P^*)) \geq \phi(\mu^*(\tilde{u}), \rho(\tilde{u} - \mu^*(\tilde{u}); P^*))$.

A detailed proof of Theorem 12 is contained in an appendix. The argument follows Quiggin and Chambers (2004), Results 4 and 5. However, the use of a generalized mean creates some minor technical difficulties not encountered in Quiggin and Chambers (2004). Namely, $K_\mu(\tilde{u})$ is not necessarily convex. It is useful, therefore, to summarize the main steps in the argument, and observe how the extension to generalized invariance is undertaken. It is obvious that preferences of the form specified by (*i*) and (*ii*) satisfy generalized invariance. For the converse, the argument goes as follows:

By Lemma 9.6, $\mu^*(\tilde{u}) \in K_\mu(\tilde{u})$, and hence the translated set, $K_0(\tilde{u}) = K_\mu(\tilde{u}) - \mu^*(\tilde{u})\tilde{1}$, contains the origin. Thus, $co\{K_0(\tilde{u})\}$, where $co\{A\}$ is the convex hull of the set A, is a closed convex set containing the origin which is fully characterized by its sublinear and lower semi-continuous (in $\tilde{u}'$) gauge function:

$$d(\tilde{u}', co\{K_0(\tilde{u})\}) = \inf\{t > 0 : \tilde{u}' \in tco\{K_0(\tilde{u})\}\},$$

if there is t such that $\tilde{u}' \in tco\{K_0(\tilde{u})\}$ and ∞ otherwise. Defining

$$r(\tilde{u} - \mu^*(\tilde{u}); \tilde{u}^0) = d(\tilde{u} - \mu^*(\tilde{u}), co\{K_0(\tilde{u})\}),$$

then yields a sublinear, nonegative, and lower semi-continuous function of $\tilde{u} - \mu^*(\tilde{u})$. The Hahn–Banach theorem thus implies that $r(\tilde{u} - \mu^*(\tilde{u}); \tilde{u}^0)$ can then always be written in the form

$$r(\tilde{u} - \mu^*(\tilde{u}); \tilde{u}^0) = \sup_p \left\{p'(\tilde{u} - \mu^*(\tilde{u})) : p \in P^*\left(\tilde{u}^0\right)\right\},$$

where $P^*(\tilde{u}^0) \subseteq \mathbb{R}^S$ is a closed convex set that contains the origin. As can be shown, generalized invariance implies that $r(\tilde{u} - \mu^*(\tilde{u}); \tilde{u}^0)$ and $P^*(\tilde{u}^0)$ are both homogeneous of degree minus one in $\tilde{u}^0$ and invariant to translations of $\tilde{u}^0$. Hence, with no true no loss of generality, by imposing an appropriate normalization, such as the requirement that $||P^*|| = 1$

(where $||\bullet||$ is the Euclidean volume), we may write

$$\rho\left(\tilde{u}-\mu^*\left(\tilde{u}\right);P^*\right)=\sup_p\left\{p'\left(\tilde{u}-\mu^*\left(\tilde{u}\right)\right):p\in P^*\right\},$$

as required.

5 Comparisons of uncertainty aversion and comparative statics

Although Axioms 1–6 imply quite weak restrictions on preferences, much of the standard comparative static analysis for mean-variance models remains applicable.

5.1 Measures of risk aversion

If for $u\in\mathbb{R}$

$$\max\left\{\beta:0-\beta\in V\left(u\right)\right\}=-u,$$

then $e\left(\tilde{u}\right)$ satisfies Aczél's (1990) agreement property, $e\left(u\right)=u$, and

$$\begin{aligned} e\left(u\right) &= \phi\left(\mu^*\left(u\right),\rho\left(u;P^*\right)\right) \\ &= \phi\left(\mu^*\left(u\right),0\right) \\ &= \phi\left(u,0\right) \\ &= u, \quad u\in\mathbb{R}. \end{aligned}$$

Given a generalized invariant preference structure whose certainty equivalent satisfies the agreement property and a probability measure π, one might define the *uncertainty premium* (for that measure) as the difference between the mean for that probability measure and the certainty equivalent

$$D\left(\tilde{u};\pi\right)=\pi'\tilde{u}-\phi\left(\mu^*\left(\tilde{u}\right),\rho\left(\tilde{u}-\mu^*\left(\tilde{u}\right);P^*\right)\right).$$

If $\pi\in\Pi^*_{\succeq}$, then

$$\begin{aligned} \pi'\tilde{u} &\geq \mu^*\left(\tilde{u}\right) \\ &= \phi\left(\mu^*\left(\tilde{u}\right),0\right) \\ &\geq \phi\left(\mu^*\left(\tilde{u}\right),\rho\left(\tilde{u}-\mu^*\left(\tilde{u}\right);P^*\right)\right), \end{aligned}$$

and so $D(\tilde{u};\pi) \geq 0$. Observe that $D(\tilde{u};\pi)$ can be partitioned as follows

$$D(\tilde{u};\pi) = (\pi'\tilde{u} - \mu^*(\tilde{u})) + R(\mu^*(\tilde{u}), \rho(\tilde{u} - \mu^*(\tilde{u}); P^*))$$

where

$$\begin{aligned} R(\mu,\rho) &= \mu - \phi(\mu,\rho) \\ &= \phi(\mu,0) - \phi(\mu,\rho) \end{aligned}$$

The components of the decomposition may be interpreted as follows: $(\pi'\tilde{u} - \mu^*(\tilde{u}))$ is an *ambiguity/pessimism premium* relative to $\pi \in \Delta$ that emerges from the individual's conservative evaluation of mean returns; and $R(\mu^*,\rho)$ measures the effect of shifting from a riskless situation with mean μ^* to a risky situation with mean μ^* and risk index ρ. Hence, one can think of it as the *risk premium* associated with the generalized mean, μ, and the generalized invariant certainty equivalent. If $\pi \in \Pi^*_{\succeq}$, then both the ambiguity premium and the risk premium are positive.

Suppose individuals A and B have generalized invariant preferences and share the same u, $\Pi^*_{\succeq}$, and P^*. Then differences in preferences are entirely determined by ϕ, which in the decomposition given above, is taken to reflect risk attitudes. So in this case, comparing uncertainty aversion reduces to comparing risk aversion in the sense described above. Define:

$$L(\mu,\rho) = \{(\hat{\mu},\hat{\rho}) \geq (\mu,\rho) : \phi(\hat{\mu},\hat{\rho}) \geq \phi(\mu,\rho)\},$$

where the first inequality is taken in vector terms. $L(\mu,\rho)$ is the set of generalized means and risk indexes which dominate (μ,ρ) and for which the increase in the generalized means from μ at least compensates for the associated increase in risk exposure.

We say that A *is more risk averse than* B *if* $L^A(\mu,\rho) \subseteq L^B(\mu,\rho)$ *for all* (μ,ρ). That is, A is more risk averse than B if any combination of an increase in mean μ and riskiness ρ that is acceptable to A is also acceptable to B. Visually, A is more risk averse than B if for all (μ,ρ), $A's$ indifference curve passing through (μ,ρ) is more steeply sloped than $B's$. In the case where ϕ is smooth, it follows immediately that

$$L^A(\mu,\rho) \subset L^B(\mu,\rho) \text{ for all } (\mu,\rho) \Rightarrow -\phi_2^A(\mu,\rho)/\phi_1^A(\mu,\rho) \geq -\phi_2^B(\mu,\rho)/\phi_1^B(\mu,\rho) \text{ for all } (\mu,\rho).$$

This definition of 'more risk averse' is consistent with the traditional definition that has gained currency in the literature on mean-variance preferences. Suppose that preferences can be expressed in the form $v(\mu, \sigma^2)$ where, with a slight abuse of notation, μ is now understood to be the mean for a fixed probability measure, σ is the associated standard deviation, and v is smooth. Then, taking σ as ρ, according to our terminology, A is more risk averse than B if for all (μ, σ)

$$-2v_2^A\left(\mu, \sigma^2\right)/v_1^A\left(\mu, \sigma^2\right) \geq -2v_2^B\left(\mu, \sigma^2\right)/v_1^B\left(\mu, \sigma^2\right).$$

For mean-variance preferences, $-2v_2(\mu, \sigma^2)/v_1(\mu, \sigma^2)$ coincides exactly with Epstein's (1985) approximation to the Arrow–Pratt risk-aversion measure for general preferences and admits the interpretation of "...twice the risk premium per unit of variance for a small gamble" (Epstein, 1985, p.949). Following this terminology, we shall refer to $-\phi_2(\mu, \rho)/\phi_1(\mu, \rho)$ as the *generalized Arrow–Pratt risk aversion measure* in what follows.

5.2 Choice problems and comparative statics

Choice problems for individuals with generalized invariant preferences can always be decomposed analytically. First isolate an 'efficient frontier' and then from that efficient frontier pick an optimal risk exposure as characterized by ρ. Consider the general choice problem in which $C(\theta) \subseteq U$ is a closed, convex choice set parametrized by $\theta \in \Theta \subset \mathbb{R}^K$. An individual with generalized invariant preferences facing this choice set solves:

$$\max_{\tilde{u}} \left\{\phi\left(\mu^*(\tilde{u}), \rho\left(\tilde{u} - \mu^*(\tilde{u}); P^*\right)\right) : \tilde{u} \in C(\theta)\right\}.$$

Assume that a well defined solution exists to this problem.

Because ϕ is increasing in μ and decreasing in ρ, this optimization problem can be rewritten as

$$\max_{\mu^*} \left\{\phi\left(\mu^*, \hat{\rho}\left(\mu^*; P^*; C(\theta)\right)\right)\right\},$$

where

$$\hat{\rho}\left(\mu^*; P^*; C(\theta)\right) = \min_{\tilde{u}} \left\{\rho\left(\tilde{u} - \mu^*; P^*\right) : \tilde{u} \in C(\theta), \mu^*(\tilde{u}) = \mu^*\right\}.$$

Because $\rho(\tilde{u} - \mu^*; P^*)$ is sublinear, $\mu^*(\tilde{u})$ is superlinear, and $C(\theta)$ is convex, this first-stage programming problem can be handled by standard convex programming tools.

Denote the subdifferential of $\rho(\tilde{u}-\mu^*;P^*)$ by $\partial\rho(\tilde{u}-\mu^*;P^*)$ and that of $\mu^*(\tilde{u})$ by $\partial\mu^*(\tilde{u})$, and note that the properties of support functions guarantee that, respectively, $\partial\rho(\tilde{u}-\mu^*;P^*)\subset P^*$ and $\partial\mu^*(\tilde{u})\subset\Pi^*_{\succeq}$. The convexity of ρ (a consequence of sublinearity) and the concavity of μ^* (a consequence of superlinearity) ensure that each possesses a well-defined one-sided directional derivative so long as they are proper. Denote the one-sided directional derivative of $\rho(\tilde{u}-\mu^*(\tilde{u});P^*)$ in the direction of $\tilde{u}^0$ by $d_+\rho(\tilde{u}-\mu^*(\tilde{u});P^*;\tilde{u}^0)$ and that of $\mu^*(\tilde{u})$ by $d_+\mu^*(\tilde{u};\tilde{u}^0)$ and note that they are, respectively, sublinear and superlinear and satisfy $d_+\rho(\tilde{u}-\mu^*(\tilde{u});P^*;\tilde{u}^0)\geq p'\tilde{u}^0$ for all $p\in\partial\rho(\tilde{u}-\mu^*;P^*)$ and $d_+\mu^*(\tilde{u};\tilde{u}^0)\leq\pi'\tilde{u}^0$ for all $\pi\in\partial\mu^*(\tilde{u})\subset\Pi^*_{\succeq}$. Hence, $d_+\mu^*(\tilde{u};\tilde{u}^0)$ has a natural interpretation as a generalized mean computed with respect to a subset of $\Pi^*_{\succeq}$.

First-order necessary and sufficient conditions for a solution, therefore, require for any small movement in a feasible direction $\tilde{u}^0$ ($(\tilde{u}+\lambda\tilde{u}^0)\in C(\theta)$) that

$$d_+\rho\left(\tilde{u}-\mu^*;P^*;\tilde{u}^0\right)-\psi d_+\mu^*\left(\tilde{u};\tilde{u}^0\right)\geq 0,$$

where ψ is a Lagrangean multiplier whose optimal value in the smooth case equals $\left[\frac{\partial\hat{\rho}(\mu^*;P^*;C(\theta))}{\partial\mu^*}\right]$. In what follows, we shall assume that $C(\theta)$ is *regular* in the sense that it admits a solution to this problem.

Once this first-stage is solved, one can then use $\hat{\rho}(\mu^*;P^*;C(\theta))$ to define *efficient frontiers* of the choice set $C(\theta)$ as given by, for example,

$$\{(\mu^*,\rho):\rho=\hat{\rho}(\mu^*;P^*;C(\theta))\},$$

which traces out the minimal risk, as measured by $\rho(\tilde{u}-\mu^*;P^*)$, consistent with the choice set $C(\theta)$ and a generalized mean return of μ^*.

$\hat{\rho}(\mu^*;P^*;C(\theta))$ is common across all individuals with generalized invariant preferences who share a common u, $\Pi^*_{\succeq}$, and P^*. Thus, for such individuals, where they locate on the efficient frontier is determined by their risk preferences as characterized by $L(\mu,\rho)$. With a slight abuse of notation, denote

$$\mu(\theta)\in\arg\max_{\mu^*}\{\phi(\mu^*,\hat{\rho}(\mu^*;P^*;C(\theta)))\}.$$

In the smooth case, the solution to this generalized choice problem is, therefore, characterized by the first-order condition

$$-\frac{\phi_2(\mu,\rho)}{\phi_1(\mu,\rho)} = \left[\frac{\partial\hat{\rho}(\mu^*;P^*;C(\theta))}{\partial\mu^*}\right]^{-1},$$

which involves equating the generalized Arrow–Pratt measure of risk aversion to the slope of the efficient frontier.

On the basis of this formulation of the choice problem and our definition of 'more risk averse', we are led to the general result:

Proposition 13 *Suppose individuals A and B share the same u, $\Pi^*_{\succeq}$, P^*, and $C(\theta)$. If A is more risk averse than B and $\hat{\mu}(\rho;\Pi^*_{\succeq};C(\theta))$ is increasing in ρ:*

$$\begin{aligned}\hat{\rho}^B\left(\mu^B(\theta);P^*;C(\theta)\right) &\geq \hat{\rho}^A\left(\mu^A(\theta);P^*;C(\theta)\right),\\ \mu^B(\theta) &\geq \mu^A(\theta).\end{aligned}$$

Thus, our definition of 'more-risk averse' ensures that, for two individuals sharing the same Π and P and a positive risk–return trade off, the more risk averse of the two will expose himself or herself to more risk in return for a higher general mean return.

There are several other observations to be drawn from this discussion of the general choice problem for an individual with generalized invariant preferences. First, and perhaps most importantly, even in the absence of uniquely defined probability measures or a reliance upon the standard deviation to measure risk, it is possible to construct simple two-parameter decision models for generalized invariant preferences that assume a form almost identical to those that have long been analyzed in mean-variance analysis. Hence, results obtained in a mean-variance framework, which do not depend critically upon the presumption of a single probability measure, will translate directly and easily to our framework. Thus, both the intuition and the analysis of mean-variance analysis is more robust than perhaps generally thought. Epstein (1985) has shown that a mean-variance preference functional will characterize rational choice given the assumption of a fixed probability measure and a seemingly innocuous assumption on systematic changes in risk aversion. Here we show that generalized invariance allows one to extend the general intuition of the mean-variance framework to decision frameworks where well-defined subjective probabilities may not exist and where, even if they did, higher-order moments can influence choice.

Second, by identifying an efficient frontier, that two-parameter choice problem can always be collapsed into a single dimensional choice problem that only involves determining the individual's optimal exposure to 'risk' as defined above. Thus, comparative static analysis for such general choice problems is a relatively simple matter of determining the result of the interplay between risk attitudes, as characterized by ϕ, and changes in the efficient frontier in response to changes in θ. In many interesting instances, therefore, comparative-static analysis will reduce to determining the supermodularity or submodularity properties of $\hat{\rho}(\mu^*; P^*; C(\theta))$ in (μ^*, θ). A case in point is the standard asset allocation problem.

Example 14 *Consider the standard portfolio allocation problem. There are J risky assets whose (gross) returns are given by $\tilde{R}_j$, $j = 1, ..., J$ and a riskless asset whose return is given by $(1 + r)$. Denote the net (excess) return on the jth risky asset by*

$$\tilde{N}_j = \tilde{R}_j - (1 + r).$$

If initial wealth is w, then returns, in monetary units, from investing α_j percent of initial wealth to each of the J risky assets is

$$\tilde{u} = w(1 + r) + w \sum_j \alpha_j \tilde{N}_j.$$

An individual with generalized invariant preferences, therefore, solves

$$\max_{\alpha} \left\{ \phi \left(w(1 + r) + w\mu^* \left(\sum_j \alpha_j \tilde{N}_j \right), w\rho \left(\sum_j \alpha_j \tilde{N}_j - \mu^* \left(\sum_j \alpha_j \tilde{N}_j \right); P^* \right) \right) \right\},$$

where we have used the homogeneity and invariance properties of μ^* and ρ. This asset allocation problem can be conveniently rewritten as

$$\max_{\alpha,\mu^0} \left\{ \phi \left(w(1 + r) + w\mu^0, w \min \left\{ \rho \left(\sum_j \alpha_j \tilde{N}_j - \mu^0; P^* \right) : \mu^0 = \mu^* \left(\sum_j \alpha_j \tilde{N}_j \right) \right\} \right) \right\}.$$

Let

$$\rho^* \left(\mu^0 \right) = \min \left\{ \rho \left(\sum_j \alpha_j \tilde{N}_j - \mu^0; P^* \right) : \mu^0 = \mu^* \left(\sum_j \alpha_j \tilde{N}_j \right) \right\},$$

and notice that the homogeneity properties of ρ and μ^* imply for $\mu^0 > 0$ that

$$\begin{aligned} \rho^* \left(\mu^0 \right) &= \mu^0 \min_{\frac{\alpha}{\mu^o}} \left\{ \rho \left(\sum_j \frac{\alpha_j}{\mu^0} \tilde{N}_j - 1; P^* \right) : 1 = \mu^* \left(\sum_j \frac{\alpha_j}{\mu^0} \tilde{N}_j \right) \right\} \\ &= \mu^0 \rho^* (1). \end{aligned}$$

Thus, the asset allocation problem in the presence of a riskless asset can be reduced to one of choosing the mix of holdings between the riskless asset and the holding, μ^0*, of a single risky derivative asset constructed from the* J *marketed risky assets with generalized mean return normalized to one and risk exposure given by* $\rho(1)$*. Hence, the individual investor solves:*

$$\max_{\mu^0} \left\{ \phi\left(w(1+r) + w\mu^0, \mu^0 \rho^*(1)\right)\right\}.$$

The efficient portfolio for the risky derivative asset with generalized mean normalized to one is given by the solution to

$$\min_{\alpha} \left\{ \rho\left(\sum_j \alpha_j \tilde{N}_j - 1; P^*\right) : 1 = \mu^*\left(\sum_j \alpha_j \tilde{N}_j\right)\right\}.$$

Necessary and sufficient conditions for an interior solution to the portfolio-choice problem for the risky derivative asset include

$$d_+\rho\left(\sum_j \alpha_j \tilde{N}_j - 1; P^*; \tilde{N}_j\right) = \psi d_+\mu^*\left(\sum_j \alpha_j \tilde{N}_j; \tilde{N}_j\right), \qquad j = 1, .., J$$

where ψ *is a nonnegative Lagrangean multiplier whose optimal value equals* $\rho^*(1)$*. Thus, the optimal portfolio satisfies*

$$d_+\mu^*\left(\sum_j \alpha_j \tilde{N}_j; \tilde{N}_j\right) = \frac{d_+\rho\left(\sum_j \alpha_j \tilde{N}_j - 1; P^*; \tilde{N}_j\right)}{d_+\rho\left(\sum_j \alpha_j \tilde{N}_j - 1; P^*; \tilde{N}_k\right)} d_+\mu^*\left(\sum_j \alpha_j \tilde{N}_j; \tilde{N}_k\right)$$

for all k *and* j*. In a natural generalization of the mean-variance portfolio problem, the generalized means of net returns for all the risky assets can be related to one another directly through the generalized risk measure.*

Hence, just as in mean-variance analysis, there is two-fund separation. The efficient frontier for this two-fund portfolio problem is affine with intercept given by $w(1+r)$ *which corresponds to the mean return with zero risk and all wealth allocated to the riskless asset. The slope of the efficient frontier is given by* $\frac{1}{\rho^*(1)}$*. Thus, it follows that in the smooth case, an individual with generalized invariant preferences chooses his or her generalized mean return so that*

$$-\frac{\phi_2\left(w(1+r) + w\mu^0, w\mu^0\rho^*(1)\right)}{\phi_1\left(w(1+r) + w\mu^0, w\mu^0\rho^*(1)\right)} = \frac{1}{\rho^*(1)}.$$

More generally, an immediate corollary to Proposition 13 is that more risk averse individuals will purchase less of the risky derivative asset than less risk averse individuals.

5.3 Monotonicity

In practical applications, one must always be aware that arbitrary choices of $\phi(\mu, \rho)$ can be problematic and can yield the resulting generalized invariant model inconsistent with our axioms and with Theorem 12. In particular, arbitrary two-parameter representations may not display global monotonicity.

The well-known case of Tobin–Markowitz mean-variance preferences illustrates the problem. To place these preferences within the invariant framework, take $\Pi^*_{\succeq} = \{\pi\}$ and define $\langle x, y\rangle_\pi$ as the expectations inner product for this measure π, with the corresponding norm denoted by $|x|_\pi$, and let $\frac{x}{y}$ represent elementwise division. Then we have, by the properties of support functions, that, for the standard deviation:

$$\begin{aligned} P^*_\sigma &= \left\{p : p'y \leq \left(\pi'y^2\right)^{\frac{1}{2}} \text{ for all } y\right\} \\ &= \{p : p'y \leq |y|_\pi \text{ for all } y\} \\ &= \left\{p : \left\langle \frac{p}{\pi}, y\right\rangle_\pi \leq |y|_\pi \text{ for all } y\right\}, \end{aligned}$$

which using the Cauchy–Schwarz inequality for generalized norms gives:

$$P^*_\sigma = \left\{p : \left|\frac{p}{\pi}\right|_\pi \leq 1\right\}.$$

Now consider the standard Tobin–Markowitz setup:

$$\phi\left(\pi'\tilde{u}, \rho\left(\tilde{u} - \pi'\tilde{u}; P^*_\sigma\right)\right) = \pi'\tilde{u} - \gamma\left[\rho\left(\tilde{u} - \pi'\tilde{u}; P^*_\sigma\right)\right]^2,$$

where $2\gamma > 0$ is the generalized Arrow–Pratt measure of risk aversion. Then for a small but arbitrary perturbation of $\tilde{u}$ to $\tilde{u} + \lambda\tilde{u}^0$ with $\lambda > 0$ and $\tilde{u}^0 \in \mathbb{R}^S_+$, monotonicity requires nonnegativity of the directional derivative

$$\pi'\tilde{u}^0 - 2\gamma d_+\rho\left(\tilde{u} - \mu^*(\tilde{u}); P^*; \tilde{u}^0 - \pi'\tilde{u}^0\right) \geq 0,$$

or, more compactly,

$$\left[-\frac{1}{2\gamma}\right] \geq d_+\rho\left(\tilde{u} - \mu^*(\tilde{u}); P^*; \tilde{u}^0 - \pi'\tilde{u}^0\right). \tag{1}$$

Hence, as the generalized Arrow–Pratt measure of risk aversion gets arbitrarily large, the domain over which Tobin–Markowitz preferences fail monotonicity grows.

Taking $u^0(s) = 1$ and $u^0(k) = 0$ for all $k \in S$, $k \neq s$ expression (1) and the properties of directional derivatives allows us to infer that

$$\left[-\frac{1}{2\gamma}\right] \geq \max\left\{p_s \frac{(1-\pi_s)}{\pi_s} : p \in \partial\rho\left(\tilde{u} - \mu^*(\tilde{u}); P^*\right)\right\}.$$

for arbitrary $s \in S$. Hence, a necessary and sufficient condition for monotonicity in the neighborhood of $\tilde{u}$ is:

$$\left[-\frac{1}{2\gamma}\right] \geq \max\left\{p_s \frac{(1-\pi_s)}{\pi_s} : p \in \partial\rho\left(\tilde{u} - \mu^*(\tilde{u}); P^*\right), s \in S\right\}.$$

6 Concluding comments

The mean-standard deviation model of preferences has proved powerful in a range of contexts, most notably in the analysis of asset pricing. However, it embodies restrictions that render it implausible as a model of individual behavior. First, while the mean and standard deviation are useful concepts in statistics, there is no particular reason why investor preferences over distributions of returns should be expressed in terms of these two moments alone. Moreover, there is substantial empirical evidence suggesting that neither the standard deviation nor the variance successfully capture individual perceptions of risk. Second, like the expected utility model, the mean-standard deviation model assumes that individuals have well defined subjective probabilities over events, even though there is significant empirical and theoretical evidence that this assumption is not valid in general.

In this paper, it has been shown that both of these assumptions may be relaxed without a substantial loss of analytical power. The mean may be replaced by a generalized mean allowing for a range of probabilities, while the standard deviation may be replaced by an invariant risk index while still generating a tractable representation of asset demand and sharp comparative static results. This paper also illustrates the powerful role that notions familiar from standard consumer and producer theory, once provided with a behavioral foundation, can play in developing tractable analytical models of preferences over uncertain outcomes.

7 References

Aczél, J. (1990), 'Determining merged relative scores', *Journal of Mathematical Analysis and Applications*, 150, 20–40.

Aliprantis, C. D., and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide.* Berlin: Springer-Verlag, 1999.

Anscombe, F. and Aumann, R. (1963), 'A definition of subjective probability', *Annals of Mathematical Statistics*, 49, 453–67.

Aubin, J.-P., and I. Ekeland. *Applied Nonlinear Analysis.* Mineola, NY: Dover Publications, 2007.

Chambers, R. G. and J. Quiggin. *Uncertainty, Production, Choice and Agency: The State-Contingent Approach.* New York: Cambridge University Press, 2000.

Cochrane, J. (2001) *Asset Pricing*, Princeton University Press, Princeton.

Ellsberg, D. (1961), 'Risk, ambiguity and the Savage axioms', *Quarterly Journal of Economics*, 75(4), 643–69.

Epstein, L. G. (1985) 'Decreasing risk aversion and mean-variance analysis', *Econometrica* 53, 945-962.

Gilboa, I. and Schmeidler, D. (1989), 'Maxmin expected utility with non-unique prior', *Journal of Mathematical Economics*, 18(2), 141–53.

Ghirardato, P., Maccheroni, F., and Marinacci, M. (2005), 'Certainty independence and the separation of utility and beliefs', *Journal of Economic Theory* 120, 129-136.

Grant, S. and Polak, B. (2007), 'Generalized variational preferences', paper presented at Australian Economic Theory Workshop, Canberra, February.

Herstein, I. N. and Milnor, J. (1953) 'An axiomatic approach to measurable utility', *Econometrica* 21, 291-297.

Lintner, J. (1965), 'The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets', *Review of Economics and Statistics*, 47(1), 13–37.

Luenberger, D.G. (1992), 'Benefit functions and duality', *Journal of Mathematical Economics*, 21, 461–81.

Maccheroni, F., Marinacci, M. and Rustichini, A. (2006), 'Ambiguity Aversion, Robustness, and the Variational Representation of Preferences', *Econometrica*, 74, 1447–98.

Markowitz, H. (1959) *Portfolio Selection: The Efficient Diversification of Investments*, Yale University Press, New Haven.

Quiggin, J. and Chambers, R.G. (2004), 'Invariant risk attitudes', *Journal of Economic Theory*, 117(1), 96–118.

Quiggin, J. and Chambers, R.G. (1998), 'Risk Premiums and Benefit Measures for Generalized Expected Utility Theories' *Journal of Risk and Uncertainty 17: 121-38.*.

Safra, Z. and Segal, U. (1998), 'Constant risk aversion', *Journal of Economic Theory*, 83(1), 19–42.

Segal, U. and Spivak, A. (1990), 'First-order versus second-order risk-aversion', *Journal of Economic Theory*, 51(1), 111–25.

Sharpe, W. (1964), 'Capital asset prices: A theory of market equilibrium under conditions of risk', *Journal of Finance*, 19, 425–42.

Tobin, J. (1958), 'Liquidity preference as behavior toward risk', *Review of Economic Studies*, 25(1), 65–86.

Yaari, M. (1987), 'The dual theory of choice under risk', *Econometrica*, 55(1), 95–115.

8 Appendix

Proof of Lemma 7: For $x, y, z \in X$ with $x \sim y$, $\mu^*(x) \sim \mu^*(y)$ because for constant acts $x = \mu^*(x)$. By A.6, therefore, for $\alpha \in (0,1)$

$$x \succeq y \Leftrightarrow \alpha x + (1-\alpha) z \succeq \alpha y + (1-\alpha) z$$

Symmetrically,

$$y \succeq x \Leftrightarrow \alpha y + (1-\alpha) x \succeq \alpha x + (1-\alpha) z.$$

which establishes the result.

Proof of Lemma 8: Axioms A.1 and A.2 correspond to Axioms 1 and 2 of Herstein and Milnor (1953), and Lemma 7 yields their Axiom 3 by taking $\alpha = \frac{1}{2}$. Hence the existence of an affine utility representation follows from Theorem 8 of Herstein and Milnor (1953). Since X is an unbounded, closed, convex cone with $0 \in X$, we can set $u(0) = 0$ and the result follows.

Proof of Lemma 9:

1) a) Let $\tilde{u}' = u \circ \tilde{f}'$ and $\tilde{u} = u \circ \tilde{f}$ for $\tilde{f}, \tilde{f}' \in \mathcal{F}$. Then by $A.1$ and Lemma 8 either $\tilde{u} \succeq \tilde{u}'$ or $\tilde{u}' \succeq \tilde{u}$ and the conclusion follows from the definition of V. b) is trivial.

2) Closedness follows from Lemma 8 and $A.2$.

3) By Lemma 8, for $x', x \in X$ if $u(x') \geq u(x)$, then $u(x') \succeq u(x)$. Hence, by the first part of $A.3$, for $\tilde{u}', \tilde{u} \in \mathcal{F}$, if $\tilde{u}' \geq \tilde{u}$, then $\tilde{u}' \succeq \tilde{u}$, so that $\tilde{u}' \in V(\tilde{u})$.

4) Choose $\tilde{f}, \tilde{g} \in \mathcal{F}$, so that $\tilde{f} \sim \tilde{g}$. Then $\tilde{u}\left(\tilde{f}\right) \sim \tilde{u}(\tilde{g})$. By $A.4$ for $\alpha \in (0,1)$, $\alpha \tilde{f} + (1-\alpha)\tilde{g} \succeq \tilde{f}$ so that

$$\tilde{u}\left(\alpha \tilde{f} + (1-\alpha)\tilde{g}\right) \succeq \tilde{u}\left(\tilde{f}\right)$$

and Lemma 8 then implies

$$a\tilde{u}\left(\tilde{f}\right) + (1-\alpha)\tilde{u}(\tilde{g}) \succeq \tilde{u}\left(\tilde{f}\right).$$

5) Follows by Lemma 8 and $A.5$.

6) Follows from the definition of μ^* and Lemma 8.

7) By A.6, for $x \in X$, $\alpha \in (0,1)$ and $\tilde{f}, \tilde{g} \in \mathcal{F}$ such that $\mu^*\left(\tilde{f}\right) \sim \mu^*(\tilde{g})$

$$\tilde{f} \succeq \tilde{g} \Longleftrightarrow \alpha \tilde{f} + (1-\alpha) x \succeq \alpha \tilde{g} + (1-\alpha) x.$$

By Lemma 8 $\mu^*\left(\tilde{u}\left(\tilde{f}\right)\right) = u\left(\mu^*\left(\tilde{f}\right)\right)$, and thus $\mu^*\left(\tilde{f}\right) \sim \mu^*(\tilde{g}) \Rightarrow \mu^*\left(\tilde{u}\left(\tilde{f}\right)\right) \sim \mu^*(\tilde{u}(\tilde{g}))$. Thus, for $x \in X$, $\alpha \in (0,1)$ and $\tilde{f}, \tilde{g} \in \mathcal{F}$ such that $\mu^*\left(\tilde{f}\right) \sim \mu^*(\tilde{g})$, A.3, A.6, and Lemma 8 require

$$\tilde{u}\left(\tilde{f}\right) \succeq \tilde{u}(\tilde{g}) \Longleftrightarrow \alpha\tilde{u}\left(\tilde{f}\right) + (1-\alpha)\,u(x) \succeq \alpha\tilde{u}(\tilde{g}) + (1-\alpha)\,u(x).$$

Set $\tilde{u} = \tilde{u}(\tilde{g})$, $\tilde{u}' = \tilde{u}\left(\tilde{f}\right)$, and $\delta = u(x)$ to obtain the desired result.

Proof of Proposition 10 Note first that $\mu^*(t\tilde{u}) = t\mu^*(\tilde{u})$ for $t > 0$ and $\mu^*(\tilde{u}+\delta) = \mu^*(\tilde{u}) + \delta$. Hence, by Lemma 8 $\mu^*(t\tilde{u}) \sim t\mu^*(\tilde{u})$ and $\mu^*(\tilde{u}+\delta) \sim \mu^*(\tilde{u}) + \delta$. Therefore to establish that for $t > 0$

$$K_\mu(t\tilde{u}) = tK_\mu(\tilde{u}),$$

we only need establish that for $\mu^*(\tilde{u}) \sim \mu(\tilde{u}')$, $\tilde{u}' \in V(\tilde{u}) \Leftrightarrow t\tilde{u}' \in V(t\tilde{u})$ for $t > 0$. That it follows for $t < 1$ is immediate from Lemma 9.7 by setting $\delta = 0$. For $t > 1$, define $\tilde{u} = t\tilde{u}^0$ and $\tilde{u}' = t\tilde{u}^+$ to obtain

$$t\tilde{u}^{+\prime} \in V\left(t\tilde{u}^0\right) \Leftrightarrow \alpha t\tilde{u}^+ \in V\left(\alpha t\tilde{u}^0\right),$$

and then take $\alpha = \frac{1}{t}$.

That

$$K_\mu(\alpha\tilde{u} + (1-\alpha)\mu^*(\tilde{u})) = K_\mu(\alpha\tilde{u}) + (1-\alpha)\mu^*(\tilde{u})\tilde{1},$$

follows from $K_\mu(t\tilde{u}) = tK_\mu(\tilde{u})$ for $t > 0$ and Lemma 9.7 by taking $\delta = \mu^*(\tilde{u})$. That

$$K_\mu(\tilde{u}+\delta) = K_\mu(\tilde{u}) + \delta\tilde{1}, \qquad \delta \in \Re,$$

follows from the first two parts of the Proposition.

Proof of Proposition 11 The conditions $\mu^*(\tilde{u}) = \mu^*(\tilde{u}')$, $e(\tilde{u}) \geq e(\tilde{u}')$ ensure $K_\mu(\tilde{u}) \subseteq K_\mu(\tilde{u}')$ and the result now follows from 10 and the definition of the certainty equivalent.

Proof of Theorem 12 With several changes, the proof strategy follows Quiggin and Chambers (2004).

Sufficiency: Suppose $\mu^*(\tilde{u}) = \mu^*(\tilde{u}') = \mu$ and $e(\tilde{u}) \geq e(\tilde{u}')$. Then, under the stated hypothesis,

$$\phi(\mu(\tilde{u}), \rho(\tilde{u}; P^*)) \geq \phi(\mu(\tilde{u}'), \rho(\tilde{u}'; P^*)),$$

and hence, since $\mu^*(\tilde{u}) = \mu^*(\tilde{u}') = \mu$, $\rho(\tilde{u}; P^*) \leq \rho(\tilde{u}'; P^*)$. Hence, by the agreement property of μ^* and the translation invariance of $\rho\left(\tilde{u} - \mu^*\left(\tilde{u}\right); P^*\right)$

$$\begin{aligned} \phi(\mu(\tilde{u} + \delta\tilde{1}), \rho(\tilde{u} + \delta\tilde{1}; P^*)) &= \phi(\mu(\tilde{u}) + \delta, \rho(\tilde{u}; P^*)) \\ &\geq \phi(\mu(\tilde{u}') + \delta, \rho(\tilde{u}'; P^*)) \\ &= \phi(\mu(\tilde{u}' + \delta\tilde{1}), \rho(\tilde{u}' + \delta\tilde{1}; P^*)) \end{aligned}$$

The argument for radial invariance is similar.

Necessity. The existence and properties of the utility function u and the mean μ^*, have already been established. The proof of the proposition is in three parts.

First, given generalized invariance, we establish the existence of a risk index ρ with the desired properties.

Second, we show that any two of the conditions $\mu(\tilde{u}) = \mu(\tilde{u}'), \rho(\tilde{u}; P^*) = \rho(\tilde{u}'; P^*)$ and $e\left(\tilde{u}\right) = e\left(\tilde{u}'\right)$ implies the third and hence that preferences may be represented in the form $e\left(\tilde{u}\right) = \phi(\mu(\tilde{u}), \rho(\tilde{u}; P^*))$

Third, we show that ϕ has the desired properties

First part: Existence of a risk index We begin with a preliminary claim.

Claim 15 *: If preferences are generalized invariant, for $\tilde{u} \neq \mu^*\left(\tilde{u}\right)$, then, for any $\tilde{u}^0$ there is an index $r\left(\tilde{u} - \mu^*\left(\tilde{u}\right), \tilde{u}^0\right)$ that is nonnegative, lower semicontinuous and sublinear in $\tilde{u} - \mu^*\left(\tilde{u}\right)$, and that satisfies*

$$r\left(\tilde{u} - \mu^*\left(\tilde{u}\right), \tilde{u}^0 + \delta\right) = r\left(\tilde{u} - \mu^*\left(\tilde{u}\right), \tilde{u}^0\right),$$

for $\delta \in \mathbb{R}$, and

$$r\left(\tilde{u} - \mu^*\left(\tilde{u}\right), \lambda\tilde{u}^0\right) = r\left(\frac{\tilde{u} - \mu^*\left(\tilde{u}\right)}{\lambda}, \tilde{u}^0\right),$$

for $\lambda > 0$. Further, if $e\left(\tilde{u}^0\right) = e\left(\tilde{u}^1\right), \mu^*\left(\tilde{u}^0\right) = \mu^*\left(\tilde{u}^1\right)$, then, for all $\tilde{u}$

$$r\left(\tilde{u} - \mu^*\left(\tilde{u}\right), \lambda\tilde{u}^0\right) = r\left(\tilde{u} - \mu^*\left(\tilde{u}\right), \lambda\tilde{u}^1\right)$$

Proof of Claim 15: Consider the gauge function of the closed convex set A defined by

$$d\left(\tilde{u}, A\right) = \inf\left\{t > 0 : \tilde{u} \in tA\right\},$$

if there is a t such that $\tilde{u} \in tA$, and ∞ otherwise. d is positively linearly homogeneous and subadditive (sublinear) in $\tilde{u}$ (Aliprantis and Border [2, Lemma 5.36]). It is lower semicontinuous in $\tilde{u}$ if and only if A contains zero (Aliprantis and Border [2, Theorem 5.39]). Moreover,

$$A = \{\tilde{u} : d(\tilde{u}, A) \leq 1\}. \tag{2}$$

Take

$$r\left(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0\right) = d\left(\tilde{u} - \mu^*(\tilde{u}), co\left\{K_\mu\left(\tilde{u}^0\right) - \mu^*\left(\tilde{u}^0\right)\tilde{1}\right\}\right).$$

By Lemma 9, $K_\mu(\tilde{u}^0) - \mu^*(\tilde{u}^0)\tilde{1}$ is a closed convex set that contains the origin. Moreover, generalized translation invariance implies

$$K_\mu\left(\tilde{u}^0\right) - \mu^*\left(\tilde{u}^0\right)\tilde{1} = K_\mu\left(\tilde{u}^0 - \mu^*\left(\tilde{u}^0\right)\right),$$

so that

$$r\left(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0\right) = d\left(\tilde{u} - \mu^*(\tilde{u}), co\left\{K_\mu\left(\tilde{u}^0 - \mu^*\left(\tilde{u}^0\right)\right)\right\}\right),$$

is the gauge function for a closed convex set containing the origin. Hence, it is nonnegative, and lower semicontinuous and sublinear in $\tilde{u} - \mu^*(\tilde{u})$ as required.

That $r(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0 + \delta) = r(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0)$ for $\delta \in \mathbb{R}$ follows from the definition of $r(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0)$ and the agreement property of generalized means. For $\lambda > 0$,

$$\begin{aligned} r\left(\tilde{u} - \mu^*(\tilde{u}), \lambda\tilde{u}^0\right) &= d\left(\tilde{u} - \mu^*(\tilde{u}), co\left\{K_\mu\left(\lambda\tilde{u}^0 - \mu^*\left(\lambda\tilde{u}^0\right)\right)\right\}\right) \\ &= d\left(\tilde{u} - \mu^*(\tilde{u}), co\left\{K_\mu\left(\lambda\tilde{u}^0 - \lambda\mu^*\left(\tilde{u}^0\right)\right)\right\}\right) \\ &= d\left(\tilde{u} - \mu^*(\tilde{u}), \lambda co\left\{K_\mu\left(\tilde{u}^0 - \mu^*\left(\tilde{u}^0\right)\right)\right\}\right) \\ &= \frac{1}{\lambda} r\left(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0\right), \end{aligned}$$

where the second equality follows by the sublinearity of μ^*, the third equality by generalized radial invariance.□

By the fact that: $r(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0)$ is lower semicontinuous and sublinear in $\tilde{u} - \mu^*(\tilde{u})$, it can always be written in the form (e.g., Aubin and Ekeland, Theorem 1.5.8, p. 30)

$$r\left(\tilde{u} - \mu^*(\tilde{u}), \tilde{u}^0\right) = \sup_p \left\{p'(\tilde{u} - \mu^*(\tilde{u})) : p \in P\left(\tilde{u}^0\right)\right\},$$

where the fact that $r(\tilde{u}-\mu^*(\tilde{u}),\tilde{u}^0)$ is nonnegative implies $P(\tilde{u}^0)$ is a closed convex set that contains the origin, and

$$P\left(\tilde{u}^0\right)=\left\{\tilde{u}-\mu^*(\tilde{u}):p'(\tilde{u}-\mu^*(\tilde{u}))\leq r\left(\tilde{u}-\mu^*(\tilde{u}),\tilde{u}^0\right)\right\}.$$

Hence, by Claim 15 and the properties of $r(\tilde{u}-\mu^*(\tilde{u}),\tilde{u}^0)$ for $\delta\in X$

$$P\left(\tilde{u}^0+\delta\right)=P\left(\tilde{u}^0\right),$$

and for $\lambda>0$

$$P\left(\lambda\tilde{u}^0\right)=\frac{1}{\lambda}P\left(\tilde{u}^0\right).$$

Thus, with virtually no true loss in generality, we may choose an arbitrary $\tilde{u}^0$ and define

$$P^*=P\left(\tilde{u}^0\right).$$

The analysis above shows that ρ has the desired properties.

Second part: Representation by ϕ Suppose $e(\tilde{u})=e(\tilde{u}')$, then

$$e(\tilde{u})\geq e(\tilde{u}'),$$

and

$$e(\tilde{u}')\geq e(\tilde{u}).$$

The first inequality can only be true if $V(\tilde{u})\subseteq V(\tilde{u}')$, and the second can only be true if $V(\tilde{u}')\subseteq V(\tilde{u})$, whence $V(\tilde{u}')=V(\tilde{u})$. Also suppose that $\mu^*(\tilde{u})=\mu^*(\tilde{u}')=\mu$, then $K_\mu(\tilde{u})=K_\mu(\tilde{u}')$, and generalized translation invariance implies $K_\mu(\tilde{u}'-\mu)=K_\mu(\tilde{u}-\mu)$. Thus,

$$\tilde{u}-\mu\sim\tilde{u}'-\mu,$$

Generalized radial invariance implies

$$\frac{\tilde{u}-\mu}{r(\tilde{u}'-\mu,\tilde{u}^0)}\sim\frac{\tilde{u}'-\mu}{r(\tilde{u}'-\mu,\tilde{u}^0)}\sim\tilde{u}^0-\mu^*\left(\tilde{u}^0\right),$$

whence

$$r\left(\frac{\tilde{u}-\mu}{r(\tilde{u}'-\mu,\tilde{u}^0)},\tilde{u}^0\right)=1,$$

and thus by sublinearity

$$r\left(\tilde{u}-\mu,\tilde{u}^{0}\right)=r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right).$$

Suppose $e(\tilde{u})=e(\tilde{u}')$ and $r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)=r\left(\tilde{u}'-\mu^{*}(\tilde{u})',\tilde{u}^{0}\right)$, then it must be true that

$$\frac{\tilde{u}-\mu^{*}(\tilde{u})}{r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)}\sim\frac{\tilde{u}'-\mu^{*}(\tilde{u}')}{r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)}\sim\tilde{u}^{0}-\mu^{*}\left(\tilde{u}^{0}\right),$$

and generalized radial invariance implies that

$$\tilde{u}-\mu^{*}(\tilde{u})\sim\tilde{u}'-\mu^{*}(\tilde{u}'),$$

which can only be true when $e(\tilde{u})=e(\tilde{u}')$ if $\mu^{*}(\tilde{u})=\mu^{*}(\tilde{u}')$.

Now suppose that $r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)=r\left(\tilde{u}'-\mu^{*}(\tilde{u}'),\tilde{u}^{0}\right)$ and $\mu^{*}(\tilde{u})=\mu^{*}(\tilde{u}')$. Because $r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)=r\left(\tilde{u}'-\mu^{*}(\tilde{u}'),\tilde{u}^{0}\right)$, it must true that

$$\frac{\tilde{u}-\mu^{*}(\tilde{u})}{r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)}\sim\frac{\tilde{u}'-\mu^{*}(\tilde{u}')}{r\left(\tilde{u}-\mu^{*}(\tilde{u}),\tilde{u}^{0}\right)}\sim\tilde{u}^{0}-\mu^{*}\left(\tilde{u}^{0}\right),$$

and generalized radial invariance implies

$$\tilde{u}-\mu^{*}(\tilde{u})\sim\tilde{u}'-\mu^{*}(\tilde{u}').$$

When $\mu^{*}(\tilde{u})=\mu^{*}(\tilde{u}')$, generalized translation invariance implies

$$\tilde{u}\sim\tilde{u}',$$

whence $e(\tilde{u})=e(\tilde{u}')$. Hence e may be written in the form

$$e(\tilde{u})=\phi\left(\mu^{*}(\tilde{u}),\rho\left(\tilde{u}-\mu^{*}(\tilde{u});P^{*}\right)\right)$$

as claimed.

Third part: Properties of ϕ

It remains to be shown that ϕ is increasing in its first argument and decreasing in its second. Suppose $\mu^{*}(\tilde{u})=\mu^{*}(\tilde{u}')=\mu$, then $e(\tilde{u})\geq e(\tilde{u}')$ if and only if $K_{\mu}(\tilde{u})\subseteq K_{\mu}(\tilde{u}')$. By generalized translation invariance, then $e(\tilde{u})\geq e(\tilde{u}')$ only if $\tilde{u}-\mu\in K_{\mu}(\tilde{u}'-\mu)$, whence

$$r\left(\tilde{u}-\mu,\tilde{u}'\right)\leq 1. \tag{3}$$

By definition

$$\frac{\tilde{u}-\mu}{r\left(\tilde{u}-\mu,\tilde{u}'\right)} \succeq \tilde{u}'-\mu,$$

and generalized radial invariance implies that

$$\frac{\tilde{u}-\mu}{r\left(\tilde{u}-\mu,\tilde{u}'\right)r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right)} \succeq \frac{\tilde{u}'-\mu}{r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right)} \in K_{\mu}\left(\tilde{u}^{0}-\mu^{*}\left(\tilde{u}^{0}\right)\right).$$

Thus, by the properties of gauge functions

$$r\left(\frac{\tilde{u}-\mu}{r\left(\tilde{u}-\mu,\tilde{u}'\right)r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right)},u^{0}\right) \leq 1,$$

from which sublinearity yields

$$r\left(\tilde{u}-\mu,u^{0}\right) \leq r\left(\tilde{u}-\mu,\tilde{u}'\right)r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right),$$

which with (3) gives

$$r\left(\tilde{u}-\mu,u^{0}\right) \leq r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right).$$

This establishes that if $\mu^{*}\left(\tilde{u}\right)=\mu^{*}\left(\tilde{u}'\right)$, then

$$\left[e\left(\tilde{u}\right)-e\left(\tilde{u}'\right)\right]\left[r\left(\tilde{u}-\mu,u^{0}\right)-r\left(\tilde{u}'-\mu,\tilde{u}^{0}\right)\right] \leq 0.$$

For $\delta>0$, the agreement property of the generalized means and Lemma 9.3 imply that

$$\begin{aligned} e\left(\tilde{u}+\delta\tilde{1}\right) &= \phi\left(\mu^{*}\left(\tilde{u}\right)+\delta,\sup_{p}\left\{p'\left(\tilde{u}-\mu^{*}\left(\tilde{u}\right)\right):p\in P^{*}\right\}\right) \\ &\geq e\left(\tilde{u}\right) \\ &= \phi\left(\mu^{*}\left(\tilde{u}\right),\sup_{p}\left\{p'\left(\tilde{u}-\mu^{*}\left(\tilde{u}\right)\right):p\in P^{*}\right\}\right), \end{aligned}$$

which establishes that ϕ is nondecreasing in it first argument. Finally, Lemma 9.3 requires that $\tilde{u}'\geq\tilde{u}\Rightarrow\phi\left(\mu^{*}\left(\tilde{u}'\right),\rho\left(\tilde{u}'-\mu^{*}\left(\tilde{u}'\right);P^{*}\right)\right)\geq\phi\left(\mu^{*}\left(\tilde{u}\right),\rho\left(\tilde{u}-\mu^{*}\left(\tilde{u}\right);P^{*}\right)\right)$.

Proof of 13 Consider any $(\hat{\mu},\hat{\rho})\in L^{A}\left(\mu^{B}\left(\theta\right),\rho^{B}\left(\theta\right)\right)$, where $\rho^{B}\left(\theta\right)$ denotes the optimal risk exposure for B. Thus, we consider any increase in risk and return, relative to the optimum for B, that is acceptable to A. By the definition of greater risk aversion,

$$L^{A}\left(\mu^{B}\left(\theta\right),\rho^{B}\left(\theta\right)\right) \subseteq L^{B}\left(\mu^{B}\left(\theta\right),\rho^{B}\left(\theta\right)\right)$$

so $(\hat{\mu},\hat{\rho})\in L^{B}\left(\mu^{B}\left(\theta\right),\rho^{B}\left(\theta\right)\right)$ and the optimality of $\left(\mu^{B}\left(\theta\right),\rho^{B}\left(\theta\right)\right)$ for B implies that either $\hat{\mu}=\mu^{B}\left(\theta\right),\hat{\rho}=\rho^{B}\left(\theta\right)$ or$(\hat{\mu},\hat{\rho})\notin C\left(\theta\right)$.

Hence, $\rho^{A}\left(\theta\right)\leq\rho^{B}\left(\theta\right)$, $\mu^{A}\left(\theta\right)\leq\mu^{B}\left(\theta\right)$ as required.