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A Matter of Interpretation: Bargaining over Ambiguous Contracts

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Abstract

We present a formal treatment of contracting in the face of ambiguity. The central idea is that boundedly rational individuals will not always interpret the same situation in the same way. More specifically, even with well defined contracts, the precise actions to be taken by each party to the contract might be disputable. Taking this potential for dispute into account, we analyze the effects of ambiguity on contracting. We find that risk averse agents will engage in ambiguous contracts for risk sharing reasons. We provide an application where ambiguity motivates the use of a liquidated damages contract.

JEL Classification: D80, D82

Key words: ambiguity, bounded rationality, expected uncertain utility, incomplete contracts, liquidated damages.

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1 Introduction

Language is a matter of interpretation, and interpretations will differ. This fact is of fundamental importance in the construction of contracts, which are written or verbal agreements that the parties act in particular ways under particular conditions. For any contract to be successfully implemented, the parties must agree on whether the relevant conditions apply. A contract that is ambiguous, in the sense that parties may differ in their interpretation of the conditions that apply, and therefore of the actions that are required, will lead to disputes and, ultimately, litigation.

To avoid disputes, parties to a contract may seek to avoid ambiguous terms, even when the resulting contract is incomplete, in the sense that opportunities for risk-sharing or productive cooperation are foregone. For example, parties may adopt a standard contract, in which the terms are well-defined as a result of established precedents, even if a variation on the standard contract could potentially yield a Pareto improvement.

Although the problems with ambiguous contracts have been much discussed in the legal literature, the central point that ambiguous contractual terms can lead to incomplete contracts has received relatively little attention from economists. This is because contracts are typically modelled as state-contingent acts, with incompleteness arising from the fact that some states may be non-contractible or from state-contingent preferences that are ambiguous, in the technical sense that there exists no well-defined probability distribution over the state space. The language in which contracts are written is either not specified or derived from the state space.

The idea that incompleteness in contracts arises from an inability to specify and contract on the state space is not new; it has been a standard argument at least since Williamson (1975, 1985) drew attention to the importance of transactions costs in determining contractual structures. These transactions costs are typically imputed to incompleteness of the state space. However, as Maskin and Tirole (1999) observe, incompleteness of the state space is not, in itself, sufficient to preclude the achievement of the first best contract. Provided that the optimal contract does not depend on welfare-irrelevant distinctions between states, Maskin and Tirole show that an optimal contract may be achieved that depends only on welfare outcomes and not on knowledge of the physical state space.

Segal (1999) has argued that in some complex environments, distinctions between complete and incomplete contracts might become trivial. Bernheim and Whinston (1998) showed that incomplete contracts might be chosen by agents who face strategic ambiguity. Spier (1992) has argued that incomplete contracts might be chosen as signalling devices. Mukerji (1998) and Mukerji and Talon (2001) discussed incomplete contracts in the presence of the decision-theoretic concept of 'ambiguity', which refers to a situation in which an agent's preferences cannot be rationalized by a specific probability distribution over a commonly known state space.

Board and Chung (2007, 2009) develop both syntactic and semantic descriptions of their object-based model of unawareness. They show how differences in the awareness of parties to a contract can tempt the more aware party to try to exploit her advantage by drafting ambiguous contingencies that in general would favor her given her greater awareness. Anticipating this, however, the less aware party may forgo contracting with this party for fear of being exploited. Thus certain legal doctrines that entail construing ambiguous terms against the drafter, might work to deter such opportunistic drafting by the more aware party and lead to simpler (albeit incomplete) contracts being agreed upon.

In this paper, we adopt an alternative approach, in which the language in which contractual terms are specified is taken as primitive. A contract is simply a set of conditional actions, built up using an '**if** t **then** a **else** a'' where t is a contractual term (or test) and a and a' are actions.

We then consider contracts between two parties, using the same contractual language but with possibly different interpretations of the tests specified in the contract. We define a test as being *conclusive* for one party relative to the other party, if whenever it is satisfied for the former it is also satisfied for the latter. If a test is conclusive for both parties then it follows there is no possibility of disagreement about whether the test is satisfied or not, and we denote such tests as *unambiguous*. Tests that are subject to any possibility of disagreement are described as ambiguous. Even though we assume both parties are mutually cognizant as to whether a test is conclusive for one party or not, and hence whether it is unambiguous or not, in situations where the parties disagree over the outcome of an ambiguous test, disputes may still arise.

It is natural, for a party to consider the range of outcomes that might arise given the ambiguity he or she perceives to be associated with the range of possible interpretations by the other party. We show how this can give rise to preferences that may be represented by a Gul and Pesendorfer (2009) expected uncertain utility maximizer. Thus, our approach establishes a connection between aversion to linguistic ambiguity (the sense in which the term 'ambiguity' is normally found in ordinary usage) and state-contingent ambiguity (the sense in which the term is commonly used in decision theory).

Given these preferences, we show that for a two-agent bargaining process over risk-sharing contracts, an individually rational and efficient contract involves a trade-off between risk and ambiguity. A finer contractual specification increases the gains from risk sharing when the contract is implemented successfully, but also increases the ambiguity of the contract and creates more possibilities for dispute. In this context, we find that risk aversion makes agents more likely to engage in contracts involving ambiguous terms and discuss the trade off between risk aversion and willingness to contract in the face of ambiguity.

In an application we consider a situation in which there exists the possibility of one party defaulting on performance. We show how ambiguity about the actual loss suffered by the injured party as a result of the default, may lead the parties to simplify specify in the contract liquidated damages, namely, a fixed payment for default without reference to the actual losses suffered by the injured party.

The paper is organized as follows. We begin with an illustrative example. In section 3, we set up the formal language in which contracts are specified. Next, in Section 4, we develop the concept of contractual ambiguity, and derive preferences over ambiguous contracts. In Section 6 we formulate the associated bargaining problem and characterize the set of individually rational and efficient contracts. Section 7, contains our application of our model to liquidated damages. In Section 8 we discuss the implications of our analysis and its relationship to the existing literature on incomplete contracts and bounded rationality.

2 An Illustrative example

In informal discussions of ambiguous contracts, it is common to refer to 'gray areas'. Some contracts, or contingencies specified in contracts, are seen as having gray areas, thereby giving rise to possibilities of disagreement and dispute, while others are seen as relatively clear-cut and unambiguous.

We develop these ideas in an example, specified as follows.[1] Suppose two individuals *Row* (Rowena) and *Col* (Colin) are contemplating entering into a risk-sharing contract. They will draw a card from a pack. The card may be all white, all black, all red or it may be white at the top and black at the bottom. From the viewpoint of an unboundedly rational observer there are four possible states of the world, one for each card.

Each player sees the world as white, black or red. However, *Row* always observes the bottom half of the card, while *Col* always observes the top half. Thus, if the card is white at the top and black at the bottom, *Row* will construe the card is black, while *Col* will construe it as white. The underlying state space and the two individuals' partitions of the black–white spectrum are summarized in the following table, where X denotes a pair of observations that is inconsistent with the problem description and therefore does not correspond to a state:

		Col's observation		
	Card drawn is:	*white (at top)*	*black (at top)*	*red (at top)*
	white (at bottom)	white white	X	X
Row's observation	*black (at bottom)*	white black	black black	X
	red (at bottom)	X	X	red red

Suppose the state-contingent endowments of the two individuals are given in the following bi-

[1] We are indebted to Bob Brito for suggesting this example.

matrix,

		Col's endowment		
	Card drawn is:	*white (at top)*	*black (at top)*	*red (at top)*
	white (at bottom)	2 2	X	X
Row's endowment	*black (at bottom)*	2 1	3 1	X
	red (at bottom)	X	X	1 3

Each individual faces a single source of uncertainty that is measurable with respect to his own partition of the state space. We assume that both players are risk-averse and view the three elements of their respective partitions as 'exchangeable' (Chew and Sagi 2006).[2] Hence both parties would prefer the non state-contingent allocation $(2, 2)$ in every state. So, ignoring (for the moment) any possibility of future disagreement and dispute, both would find it attractive to sign a risk-sharing contract of contingent transfers from *Col* to *Row* :

$$\tilde{c} = \begin{cases} 1 & \text{if the card drawn is black} \\ -1 & \text{if the card drawn is red} \\ 0 & \text{otherwise.} \end{cases} \cdot$$

In the formal framework developed below, if such a contract were signed, the presumption is that each party will assess which contingency has obtained according to her or his own semantics. For *Row*, this entails assessing that 'the card drawn is black' is true when she makes the observation the card drawn 'is black (at the bottom)', while for *Col*, this entails assessing that 'the card drawn is black' is true when he observes the card drawn 'is black (at the top)'.

[2] In this context, 'exchangeable' is equivalent to each individual being indifferent between betting on any element of his or her partition.

The card that is white at the top and black at the bottom creates a possibility for disagreement since *Row* will interpret this as 'black', and so believe that she is entitled to receive a payment of 1. *Col* will in the same situation interpret this as 'white', so he will expect no payment is required. Hence, a disagreement will ensue.

In this setup, boundedly rational players are unable (in the absence of some increase in effort) to formulate a state description sufficiently refined to encompass this possibility, allowing the contract to specify a resolution. However, they may nonetheless be aware that disputes are possible. Depending on the weight they place on this possibility, they may choose a contract which offers only partial hedging, or even no contract at all. This corresponds closely to the risk–uncertainty distinction of Knight (1921) whose main concern was with uncertainties that could not be hedged through market contracts such as insurance, and therefore reduced to manageable risk. Uncertainty of this kind was central to Knight's idea of entrepreneurship.

While we do not formally model the awareness and knowledge of the players using epistemic logic as in, for example. Fagin et al. (1995), some comment on their presumed awareness and knowledge is necessary. The awareness of the players includes a number of elements. First, each player is aware of their own state-contingent description of the world and of the information available to them. Second, given the description above, each is aware that the other may not have access to their model of the world. In this example, *Row* and *Col* are both aware that each of them is aware of the statements 'the card is black' and 'the card is red'. However, each is also conscious that their model and the model of the other individual may be incomplete. In particular there may exist other details about the world of which neither is currently aware, that lead to different interpretations by the two about the semantic content of those statements for the two players. But, as noted in the introduction, we further assume that both are mutually cognizant as to whether a test is conclusive or not for an individual.

Thus, the central feature of the example is that players are boundedly rational, but nonetheless sophisticated enough to reason about their own bounded rationality and that of others. This is consistent with the observation of Maskin and Tirole (1999, p. 106) that the central problem in contracting is not incompleteness *per se* but bounded rationality:

"if we are to explain 'simple institutions' such as property rights, authority (or more generally decision processes) short-term contracts and so forth a theory of bounded rationality is certainly an important, perhaps ultimately an essential ingredient."

The central concern of this paper is to develop a model of contracting between parties whose bounded rationality is embodied in the ambiguity of the language they use to describe the world. To this end, it is useful to relate propositional or syntactic descriptions of the world to an underlying state space when the parties involved are boundedly rational. We will follow the constructive decision theory approach of Blume, Easley and Halpern (2006) in which the propositional representation is taken as primitive, along with the set of actions on which contingent contracts can be written.

3 Formal languages and contracts.

We consider two parties $i = 1, 2$, and following the approach of Blume et al. (2006), we assume that both players have access to a non-empty set of primitive test propositions $T_0 = \{t_1, ..., t_K\}$ and a set of actions A_0. Let T denote the closure of T_0 under conjunction ($\wedge$) and negation ($\neg$).

A typical action $a \in A_0$ might be 'player i performs service z for player j in return for consideration w'. Formally, we take A_0 to be a compact and convex subset of a separable metric space.

We are interested in the set of contracts C, which are constructed inductively from the set of actions A_0 and the set of tests T by taking the closure under the 'if-then-else' construction. That is, we take each a in A_0 to be a contract, and then we require, for any pair of contracts c and c' and any test t in T, that the program '**if** t **then** c **else** c'' should be a contract in C. This contract requires the parties to follow the course of action as determined by contract c if test t is satisfied and follow the course of action as determined by contract c' otherwise.

Tests and contracts are simply strings of symbols with no inherent semantic content. The semantics will be derived from preferences of each individual rather than being given in advance. More precisely, we derive, for each player, a state space Σ^i. Although Blume et al. (2006) allow for the non-uniqueness of a state space, we adopt their canonical state space and set $\Sigma^i = \Sigma = 2^{T_0}$

for $i = 1, 2$ and refer to it as Σ hereafter.

Hence we may enumerate $\sigma_s \in \Sigma$ where $s = (s_1, ..., s_K)$ is a vector of zeros and ones (a binary number) and we use s_i to denote the i'th component of s. Letting 0 denote the number $(0, 0...0)$ and $S - 1 = 2^K - 1$ denote $(1, ..., 1)$, the indexes s range from 0 to $S - 1$, and the cardinality of the state space is S. Conversely, for any $\sigma \in \Sigma$, we denote the corresponding index by $s(\sigma) \in \{0, \dots, S-1\}$.

A test interpretation is a function $\pi : T \to 2^{\Sigma}$, where $\pi(t)$ is the set of states in which the test t is true. Notice that the state space induces a test interpretation constructed as follows. For each t_i in $T_0 = \{t_1, ..., t_K\}$, the set $\pi(t_k) = \{\sigma_s \in \Sigma : s_k = 1\}$. The test interpretation is then inductively extended to tests in T by the rule: for any $t, t' \in T$, $\pi(t \wedge t') = \pi(t) \cap \pi(t)$, and $\pi(\neg t) = \Sigma - \pi(t)$.

Each state $\sigma \in \Sigma$ can be identified with a test $t(\sigma) = t_1(\sigma) \wedge ... \wedge t_K(\sigma) \in T$ defined as follows. For each $k = 1, ..., K$ let:

$$t_k(\sigma) = \begin{cases} t_k \text{ if } s_k(\sigma) = 1; \\ \neg t_k \text{ if } s_k(\sigma) = 0. \end{cases}$$

By construction $\pi(t(\sigma)) = \{\sigma\}$ meaning the test $t(\sigma)$ is satisfied only at the state σ.

For any $a \in A_0$, f_a is the unconditional state-contingent action $f_a(\sigma) = a$ for all $\sigma \in \Sigma$. Fix a pair of contracts c and c' in C with associated state-contingent actions f_c and $f_{c'}$. Then for any test t in T, the state-contingent action associated with the contract $c'' =$ '**if** t **then** c **else** c'' is given by $f_{c''}(\sigma) = f_c(\sigma)$ if $\sigma \in \pi(t)$, and $f_{c''}(\sigma) = f_{c'}(\sigma)$ if $\sigma \notin \pi(t)$. Hence it follows from the inductive construction of the set of contracts above that for each c in C, there is an associated state-contingent action $f_c : \Sigma \to A_0$.

For any given state-contingent action f, there are many contracts that yield that state-contingent action. For a given state-contingent action f, we define the associated canonical contract c_f with an exhaustive specification given by

$$\textbf{if } \sigma_0 \textbf{ then } f(\sigma_0) \textbf{ else if } \sigma_1 \textbf{ then } f(\sigma_1) \textbf{ else ... else } f(\sigma_{S-1})$$

3.1 The Illustrative example continued

To illustrate the ideas and concepts we have introduced above, let us apply this framework to the example discussed in section 2.

Set $T_0 = \{t_1, t_2\}$, where t_1 corresponds to the test proposition, 'card drawn is red' and t_2 corresponds to test proposition, 'card drawn is black.' Formally, the state space Σ is given by $\{\sigma_{(0,0)}, \sigma_{(0,1)}, \sigma_{(1,0)}, \sigma_{(1,1)}\}$ or equivalently $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ but for ease of exposition we denote it by $\Sigma = \{\boldsymbol{W}, \boldsymbol{B}, \boldsymbol{R}, \boldsymbol{RB}\}$, derived from the table

	$\neg t_2$	t_2
$\neg t_1$	$\boldsymbol{W}$	$\boldsymbol{B}$
t_1	$\boldsymbol{R}$	$\boldsymbol{RB}$

.

The state $\boldsymbol{W}$ (respectively, $\boldsymbol{B}$, $\boldsymbol{R}$) corresponds to the state of the world in which the card drawn is 'red' (respectively, black, white), while the state $\boldsymbol{RB}$, is the 'impossible' state in which the card is both red and black.

The set of actions is the set of transfers from column to row, $A_0 = [-3, 3]$. The set of contracts can thus be characterized by state-contingent transfers $(A_0)^{\Sigma}$.

Without loss of generality, we take the endowment in the 'impossible' state $\boldsymbol{RB}$ to be $(0, 0)$. Hence the state-contingent endowments are given by

	State			
Ind.	$\boldsymbol{W}$	$\boldsymbol{B}$	$\boldsymbol{R}$	$\boldsymbol{RB}$
z_σ^{Row}	2	1	3	0
z_σ^{Col}	2	3	1	0

.

4 Introducing ambiguity

Because we have chosen formally identical state spaces for the players, the test-interpretation of each player and the language of each player are identical. The distinction and the source of disputes is thus purely semantic. Disputes arise from the players disagreeing about the state that has obtained, or, equivalently, which tests have been satisfied. In this section we first introduce ambiguity by way of ambiguous tests and show how this makes some contracts 'ambiguous'. We then develop a model of ambiguity averse decision-makers.

4.1 Conclusive and ambiguous tests and contracts

In this section we introduce the notion of ambiguous tests. This notion will be based on a primitive notion of conclusiveness of a test. The idea of conclusiveness of a test t for an individual i with respect to individual $(3-i)$ is that if she finds herself in a position where she assesses that t is satisfied, then she is sure that individual $(3-i)$ will assess t as satisfied also. The set of conclusive tests for individual i will be denoted by T_C^i. We presume that the individuals are mutually cognizant of T_C^1 and T_C^2. The test t is *unambiguous* if it is conclusive for both individuals. The set of unambiguous tests for individuals 1 and 2 is denoted $T_U = T_C^1 \cap T_C^2$.[3]

To ensure that the sets of conclusive tests match our intuition, we assume that T_C^1 and T_C^2 exhibit the following properties.

Properties of Conclusive Tests: For any pair of tests t and t' in T :

(i) the test $t \vee \neg t$ is in T_C^i (that is, all tautologies are conclusive);

(ii) if the test t is in T_C^i then the test $\neg t$ is in $T_C^{(3-i)}$(that is, the negation $\neg t$ is conclusive for the individual $(3-i)$ with respect to i);

(iii) if the tests t and t' are in T_C^i, then the test $t \vee t'$ is in T_C^i (that is, T_C^i is closed under disjunction);

(iv) if $\pi(t) = \pi(t')$ and the test t is in T_C^i, then the test t' is also in T_C^i.

In a specific example, there will typically be some subset of tests that we would like to be conclusive. Recall that in the illustrative example, the primitive tests are $T_0 = \{t_1, t_2\}$, where t_1 is the test proposition 'card drawn is red' and t_2 is test proposition 'card drawn is black'. To capture this example, we would like a test that proves the card is Red $(t_1 \wedge \neg t_2)$ to be conclusive

[3] In a model with more than two individuals, it would be necessary to use the notation $T_U^{1,2}$, since the set of unambiguous tests is specific to the given pair $(1, 2)$. In the two-player model presented here, this is unnecessary and superscripts are dropped for simplicity.

for each player, a test that proves the card is white ($\neg t_1 \wedge \neg t_2$) to be conclusive for Row, and a test that proves the card is black ($\neg t_1 \wedge t_2$) to be conclusive for Col. This implies that $\{(t_1 \wedge \neg t_2), (\neg t_1 \wedge \neg t_2)\} \subseteq T_C^{Row}$ and that $\{(t_1 \wedge \neg t_2), (\neg t_1 \wedge t_2)\} \subseteq T_C^{Col}$. To ensure that the properties (i) - (iv) of conclusive tests are satisfied, we need to extend the sets. We will show that there is a unique smallest extension which satisfies the four properties.

For this, let (T_1, T_2) be a pair of test sets satisfying $T_i \subseteq T$ for $i = 1, 2$. We say that (T_1', T_2') is conclusive extension of (T_1, T_2) iff $T_i \subseteq T_i' \subseteq T$ for $i = 1, 2$ and (T_1', T_2') satisfies properties (i) to (iv) of conclusiveness. Let (T_1', T_2') and (T_1'', T_2'') be two conclusive extensions of (T_1, T_2). We say that (T_1', T_2') is smaller than (T_1'', T_2'') iff $T_i' \subseteq T_i''$ for $i = 1, 2$.

Let $\mathcal{T}(T_1, T_2)$ denote the set of all conclusive extensions of the pair (T_1, T_2). Define the pair (T_1^*, T_2^*) by $T_i^* = \bigcap_{(T_1', T_2') \in \mathcal{T}(T_1, T_2)} T_i'$ for $i = 1, 2$.

Proposition 1 *Let (T_1, T_2) be a pair of test sets satisfying $T_i \subseteq T$ for $i = 1, 2$. The pair (T_1^*, T_2^*) is a conclusive extension of (T_1, T_2), and it is smaller than any other conclusive extension of (T_1, T_2).*

Proof. First we show that (T_1^*, T_2^*) is a conclusive extension of (T_1, T_2). For this, we need to show that $T_i \subseteq T_i^* \subseteq T$ and that (T_1^*, T_2^*) satisfies (i) to (iv). Clearly $T_i^* \subseteq T$ since T_i^* is the intersection of sets that are all subsets of T. Fix i. Since each conclusive extension $(T_1', T_2') \in \mathcal{T}(T_1, T_2)$ satisfies $T_i \subseteq T_i'$, T_i^* will be a superset of T_i provided $\mathcal{T}(T_1, T_2)$ is non-empty. Since the pair (T, T) is a conclusive extension of any given pair of test sets (T_1, T_2), we have non-emptiness of $\mathcal{T}(T_1, T_2)$. Thus, $T_i \subseteq T_i^* \subseteq T$.

Next, observe that the properties (i) to (iv) of conclusiveness of (T_1^*, T_2^*) follow from the respective properties on each conclusive extension $(T_1', T_2') \in \mathcal{T}(T_1, T_2)$.

Finally, consider any $(T_1', T_2') \in \mathcal{T}(T_1, T_2)$ and fix i, and $t \in T_i^* = \bigcap_{(T_1'', T_2'') \in \mathcal{T}(T_1, T_2)} T_i''$. Hence, $t \in T_i'$. Hence, (T_1^*, T_2^*) is smaller than any conclusive extension of (T_1, T_2). ■

In what follows, we will always presume that we are using the smallest conclusive extension for a given application.

The next proposition shows that the properties of conclusive tests guarantee that any test which is satisfied in every state or in no state is unambiguous and that the set of unambiguous tests is closed under negation and conjunction.

Proposition 2 *Fix* T_C^1 *and* T_C^2. *If* T_C^1 *and* T_C^2 *satisfy the properties of conclusive tests then for each pair of tests* t *and* t' *in* T*:*

(i) *if* $\pi(t) = \Sigma$ *or* $\pi(t) = \varnothing$ *then* $t \in T_U$*;*

(ii) *if* t,$t' \in T_U$, *then* (a) $\neg t \in T_U$ *and* (b) $t \wedge t' \in T_U$

Proof. (i) First, let $\pi(t) = \Sigma$. By property *(i)*, the test $t \vee \neg t$ is in T_C^i for $i = 1, 2$. Since $\pi(t \vee \neg t) = \Sigma = \pi(t)$, it follows by property *(iv)* and the definition of an unambiguous test that $t \in T_U$. Next, let $\pi(t) = \varnothing$. Then, $\pi(\neg t) = \Sigma$, so as just shown above using properties *(i)* and *(iv)*, the test $\neg t$ is in T_U. Then, by property *(ii)*, the test $\neg\neg t$ is in T_C^i for $i = 1, 2$, and so by the definition of an unambiguous test, the test $\neg\neg t \in T_U$. Noting that $\pi(t) = \pi(\neg\neg t)$, it follows from property *(iv)* that $t \in T_U$.

(ii) Let t,$t' \in T_U$. Then, t,$t' \in T_C^i$ for $i = 1, 2$. (a) Consider $\neg t$. By property *(ii)* and the definition of an unambiguous test, $\neg t \in T_U$; (b) Consider $t \wedge t'$. Observe that $\pi(t \wedge t') = \pi(\neg(\neg t \vee \neg t'))$. By properties *(ii)* and *(iii)* and the definition of an unambiguous test, the test $\neg(\neg t \vee \neg t') \in T_U$. Thus applying property *(iv)*, $t \wedge t' \in T_U$. ■

Given that the two individuals are mutually cognizant of T_C^1 and T_C^2 and that they satisfy the four properties listed above, it follows that for any contract of the form 'if t then a else a',' if t is an unambiguous test then both individuals anticipate that they will agree whether or not test t has been satisfied and thus they will agree whether or not the contract calls for action a or for action a'. If, however, the test is conclusive only for individual i and is not conclusive for individual $(3 - i)$, then although i anticipates that when she has assessed test t is satisfied individual $(3 - i)$ will agree the contract calls for action a, individual $(3 - i)$ believes when he has assessed test t is satisfied, there may be a disagreement with i about whether the contract calls for action a or a'. But it follows from property (ii) that individual $(3 - i)$ anticipates that when he has assessed test t is not satisfied, individual i will also have assessed that test t is not satisfied and so will agree that the contract calls for action a'. Individual i, on the other hand, anticipates that when she has assessed that test t is not satisfied there may be a disagreement with individual $(3-i)$ about whether the contract calls for action a or a'.

We can use the test interpretation to derive the set of unambiguous events.

Definition 1 *The set of* unambiguous events $\mathcal{E}_U \subseteq 2^{\Sigma}$ *is given by:*

$$\mathcal{E}_U = \{E \subseteq \Sigma : \pi(t) = E \textit{ for some } t \in T_U\} \ .$$

The set of ambiguous events $\mathcal{E}_A = 2^{\Sigma} - \mathcal{E}_U$.

Lemma 3 *The set of unambiguous events $\mathcal{E}_U$ is an algebra of subsets of Σ, that is, it is non-empty, and closed under taking complements and intersection.*

Proof. Assertion *(i)* of Proposition 2 implies that $\mathcal{E}_U$ is non-empty. Consider any pair of unambiguous events E and E' in $\mathcal{E}_U$. Since they are unambiguous events, there must exist tests t and t' in T_U, such that $\pi(t) = E$ and $\pi(t') = E'$. Assertion *(ii)* of Proposition 2 states that T_U is closed under negation and conjunction, so the tests $\neg t$ and $t \wedge t'$ are also in T_U. Since $\pi(\neg t) = \Sigma - E$ and $\pi(t \wedge t') = E \cap E'$, the events $\Sigma - E$ and $E \cap E'$ are unambiguous. ■

For each $\sigma \in \Sigma$, and for each individual i, we can derive from the set of unambiguous tests for individual i, the collection of possible states the other individual ($(3-i)$ may have determined as having obtained as follows.

Definition 2 (*Possibilty of Dispute Set for* i) *Suppose $T_C^i \subset T$, is the set of conclusive tests for individual i. For each σ in Σ, define the* possibility-of-dispute set for i *associated with state σ to be:*

$$D^i(\sigma) := \{\sigma' \in \Sigma : \textit{for each } t \in T_C^i,\ \sigma \in \pi(t) \Rightarrow \sigma' \in \pi(t)\}.$$

By construction, the set $D^i(\sigma)$ comprises those states that cannot be distinguished from σ by a conclusive test for i being satisfied. Clearly, $\sigma \in D^i(\sigma)$ for each $\sigma \in \Sigma$, so $D^i(\sigma) \neq \varnothing$ for each $\sigma \in \Sigma$. We will refer to $\{D^i(\sigma)\}_{\sigma \in \Sigma}$ as the *possibility of disputes for i.*

For each $\sigma \in \Sigma$ we can define the smallest unambiguous event $E(\sigma)$ containing σ by $E(\sigma) := \bigcap_{E \in \{F \in \mathcal{E}_U : \sigma \in F\}} E$. We have the following facts which shows that coarsest common-refinement of

$\left\{D^1(\sigma)\right\}_{\sigma\in\Sigma} \cup \left\{D^2(\sigma)\right\}_{\sigma\in\Sigma}$ is the finest unambiguous partition of Σ. More specifically, for each state σ, the possibility-of-dispute set for i, $D^i(\sigma)$, is a subset of $E(\sigma)$ with equality, if and only if $D^1(\sigma) = D^2(\sigma)$, and $D^i(\sigma)$ is a singleton if and only if the test $t(\sigma)$ associated with the state σ is an conclusive test for i.

Lemma 4 *For each $\sigma \in \Sigma$: (a) $D^i(\sigma) \subseteq E(\sigma)$ and $D^1(\sigma) = D^2(\sigma) \Rightarrow D^i(\sigma) = E(\sigma)$; (b) $D^i(\sigma) = \{\sigma\}$ if and only if $t(\sigma) \in T_C^i$.*

Proof. (a) First we show $D^i(\sigma) \subseteq E(\sigma)$. Suppose that $\sigma' \in D^i(\sigma)$, but $\sigma' \notin E(\sigma)$. Observe that $E(\sigma) \neq \varnothing$. Hence, there must be some $E \in \{F \in \mathcal{E}_U : \sigma \in F\}$, and $\sigma' \notin E$. Since $E \in \mathcal{E}_U$, there is a test $t \in T_U$ such that $\pi(t) = E$. Also, $\sigma \in E(\sigma)$. Since $\sigma' \in D^i(\sigma)$, it follows from the definition of $D^i(\sigma)$ that $\sigma' \in \pi(t) = E$, which is a contradiction. Hence, we conclude that $D^i(\sigma) \subseteq E(\sigma)$.

Next we show that $E(\sigma) \subseteq D^i(\sigma)$ whenever $D^1(\sigma) = D^2(\sigma)$. Suppose that $\sigma' \in E(\sigma)$, but $\sigma' \notin D^1(\sigma) = D^2(\sigma)$. Then there is some test $t \in T_U$ such that $\sigma \in \pi(t)$ but $\sigma' \notin \pi(t)$. Then $\pi(t)$ is an unambiguous event containing σ but not containing σ'. Hence $E(\sigma) \subseteq \pi(t)$,and $\sigma' \notin E(\sigma)$, which is again a contradiction. Hence we conclude that $E(\sigma) \subseteq D^i(\sigma)$.

(b) (If) Clearly, $\{\sigma\} \subseteq D^i(\sigma)$ from the definition of $D^i(\sigma)$. Next, since $t(\sigma) \in T_C^i$ and $\pi(t(\sigma)) = \{\sigma\}$, it follows that if $\sigma' \neq \sigma$, then $\sigma' \notin D^i(\sigma)$, that is, $D^i(\sigma) \subseteq \{\sigma\}$.

(Only-if) Since $D^i(\sigma) = \{\sigma\}$, it follows that for each $\sigma' \neq \sigma$, there is a test $t' \in T_U^i$ such that $\sigma \in \pi(t')$ and $\sigma' \notin \pi(t')$. Since T_C^i is closed under conjunction by assertion *(ii)* of Proposition 2, we can take the conjunction of these tests over $\Sigma - \{\sigma\}$ to obtain a conclusive test for i, $t^* \in T_C^i$ that excludes everything but σ, that is, $\pi(t^*) = \{\sigma\}$. Since $\pi(t(\sigma)) = \{\sigma\} = \pi(t^*)$, it follows from property (iv) of the conclusive test set T_C^i that $t(\sigma) \in T_C^i$. ■

Notice that if a contract is measurable with respect to the *unambiguous partition*, $\left\{E^i(\sigma)\right\}_{\sigma\in\Sigma}$ although the individuals might disagree about the actual state that has obtained, they will never disagree about which action the contract prescribes. Hence such contracts are viewed as *unambiguous.*

Definition 3 *A contract is* unambiguous *if for all for all $\sigma, \sigma' \in \Sigma$, $E(\sigma) = E(\sigma') \Rightarrow f_c(\sigma) =$*

$f_c(\sigma')$. We denote by C_U the set of unambiguous contracts.

5 Preferences over contracts.

To model the individual's preferences over contracts, we adopt the *expected uncertain utility* model of Gul and Pesendorfer (2009). To employ their model, we take the 'state-space' to be the *product* space of the individual state spaces $\Sigma \times \Sigma$. Preferences $\succsim$ are assumed to be defined over acts, that is functions $\mathbf{f}$ from $\Sigma \times \Sigma$ to the set of outcomes, which following Gul and Pesendorfer, we take to be a interval of finite length $[m, M]$ of monetary outcomes. An expected *separable* uncertain-utility (ESUU) decision maker is characterized by a preference scaling function $v : [m, M] \to R$, an ambiguity attitude parameter $\alpha \in [0, 1]$ and a 'prior' $(\mathcal{E}, \mu)$, where $\mathcal{E}$ is a sigma-algebra of subsets of $\Sigma \times \Sigma$, and μ is a probability measure on $\mathcal{E}$.

The decision maker evaluates each act $\mathbf{f}$ according to its expected interval-utility, where the interval-utility of the interval $[x, y]$, $x \leq y$, is given by the separable interval-utility function $u(x, y) = \alpha v(x) + (1 - \alpha) v(y)$.

To compute $V(f)$, we find $\mathbf{f}_-$, the largest $\mathcal{E}-$measurable function satisfying $\mathbf{f}_- \leq \mathbf{f}$, and $\mathbf{f}_+$, the smallest $\mathcal{E}-$measurable function satisfying $\mathbf{f}_+ \geq \mathbf{f}$. Gul & Pesendorfer (2009) refer to $(\mathbf{f}_-, \mathbf{f}_+)$ as the $\mathcal{E}-$measurable *envelope* of the act $\mathbf{f}$. For a given prior $(\mathcal{E}, \mu)$ they show for each act there is a unique (up to a set of $\mu-$measure 0) $\mathcal{E}-$measurable envelope.[4] The expected separable uncertain-utility of $\mathbf{f}$ is thus given by:

$$\begin{aligned} U(\mathbf{f}) &= \int_{\Sigma \times \Sigma} u(\mathbf{f}_-, \mathbf{f}_+) \, d\mu \\ &= \alpha \int_{\Sigma \times \Sigma} v(\mathbf{f}_-) \, d\mu + (1 - \alpha) \int_{\Sigma \times \Sigma} v(\mathbf{f}_+) \, d\mu. \end{aligned} \tag{1}$$

Notice that for any act $\mathbf{f}$ that is $\mathcal{E}-$measurable, $\mathbf{f}_- = \mathbf{f} = \mathbf{f}_+$, and so $U(\mathbf{f}) = \int_{\Sigma \times \Sigma} v(\mathbf{f}) \, d\mu$ is the standard subjective expected utility.

We presume that each individual i is an ESUU maximizer, so her preferences over acts can be represented by the 4-tuple $\langle \mathcal{E}^i, \mu^i, v^i, \alpha^i \rangle$.

For each individual i, we use her set of conclusive tests T_C^i to generate the algebra $\mathcal{E}^i$, and we

[4] See Gul & Pesendorf (2009, Lemma 2, p4).

present a method for translating each contract c in C into an act $\mathbf{f}_c^i : \Sigma \times \Sigma \to [m, M]$. Thus her preferences over acts induce a preference ordering over contracts given by $V^i(c) = U^i\left(\mathbf{f}_c^i\right)$.

The algebras $\mathcal{E}^1$ and $\mathcal{E}^2$ are uniquely determined from a given contracting problem using the set of primitive propositions T_0 and the set of conclusive test sets T_1^C and T_2^C in the following way. First, we generate the canonical state space Σ from the set of test propositions T_0. Next, $\mathcal{E}^1$ is defined to be the algebra generated from the partition $\left\{\{\sigma\} \times D^1(\sigma) : \sigma \in \Sigma\right\} \cup \{\{\sigma\} \times \Sigma \backslash D^1(\sigma) : \sigma \in \Sigma\}$; and $\mathcal{E}^2$ is the algebra generated from the partition $\left\{D^2(\sigma) \times \{\sigma\} : \sigma \in \Sigma\right\} \cup \left\{\Sigma \backslash D^2(\sigma) \times \{\sigma\} : \sigma \in \Sigma\right\}$.

Since each party presumes the other will see something within her or his own dispute set, we require that the measures μ^1 and μ^2 defined on $\mathcal{E}^1$ and $\mathcal{E}^2$, respectively, satisfy the following condition: for all σ in Σ,

$$\mu^1\left(\{\sigma\} \times \Sigma \backslash D^1(\sigma)\right) = \mu^2\left(\Sigma \backslash D^2(\sigma) \times \{\sigma\}\right) = 0.$$

The other parameters $v^1, v^2, \alpha^1, \alpha^2$ are as in Gul-Pesendorfer (2009).

It still remains to describe how a given contract c is translated into an act for each individual i. For this, let Σ^i denote the i'th component of $\Sigma \times \Sigma$, and let $y^i : A \times \Sigma^i \to [m, M]$ be the outcome function where $y^i(a, \sigma)$ is the outcome in $[m, M]$ that results for individual i when action a from A is undertaken and she observes σ in Σ.

We presume that individual 1 associates the contract c with the 'act' $\mathbf{f}_c^1 : \Sigma \times \Sigma \to [m, M]$, where,

$$\mathbf{f}_c^1(\sigma, \sigma') = y^1\left(f_c(\sigma'), \sigma\right),$$

and individual 2, associates the contract c with the 'act' $\mathbf{f}_c^2 : \Sigma \times \Sigma \to [m, M]$, where

$$\mathbf{f}_c^2(\sigma', \sigma) = y^2\left(f_c(\sigma'), \sigma\right).$$

Thus, in her evaluation of a contract it is as if each individual presumes that the contract c will be implemented according to what the other party sees.[5]

[5] In situations where, the individuals' interpretations are opposed, ex post, specifying the act this way could be viewed as a reduced form of the (certainty-equivalent) equilibrium outcome the individual expects to receive from a dispute. For example, suppose that individuals anticipate a dispute will lead to a 'war of attrition' game (Maynard Smith, 1974) in which each player's equilibrium payoff is equal to their security level. In this case, that corresponds to the utility of the outcome associated with the other player's interpretation. We thank Roger Myerson for the suggestion that disputes might be viewed as wars of attrition.

Notice that any mapping from $\Sigma \times \Sigma$ to A that is measurable with respect to Σ^i is measurable with respect to $\mathcal{E}^i$. It is therefore convenient for each $i = 1, 2$, to denote by $\mu^i_{\Sigma^i}$ the marginal distribution of μ^i on Σ^i.

Putting this all together we have that an ESUU decision maker $\left\langle \mathcal{E}^i, \mu^i, v^i, \alpha^i \right\rangle$, with output function y^i will evaluate a contract c according to the function:

$$V^i(c) = \sum_{\sigma \in \Sigma} \mu^i_{\Sigma^i}(\sigma) \left[\alpha^i \min_{\sigma' \in D^i(\sigma)} v^i\left(y^i\left(f_c(\sigma'), \sigma\right)\right) + \left(1 - \alpha^i\right) \max_{\sigma' \in D^i(\sigma)} v^i\left(y^i\left(f_c(\sigma'), \sigma\right)\right) \right]$$

5.1 The illustrative example continued

Returning to our illustrative example, recall $T_0 = \{t_1, t_2\}$, where t_1 is the test proposition 'card drawn is red' and t_2 is test proposition 'card drawn is black'.

As Row only sees the bottom of a card and Col only sees the top, it follows as remarked earlier that $T^{Row} = \{(t_1 \wedge \neg t_2), (\neg t_1 \wedge \neg t_2)\} \subseteq T^{Row}_C$ and that $T^{Col} = \{(t_1 \wedge \neg t_2), (\neg t_1 \wedge t_2)\} \subseteq T^{Col}_C$. We take (T^{Row}_C, T^{Col}_C) to be the smallest conclusive extension of (T^{Row}, T^{Col}).

It follows that the possibility of disputes sets for Row and Col are given by: $D^{Row}(\boldsymbol{W}) = \{\boldsymbol{W}\}$, $D^{Row}(\boldsymbol{B}) = \{\boldsymbol{B},\ \boldsymbol{W}\}$, $D^{Row}(\boldsymbol{R}) = \{\boldsymbol{R}\}$, $D^{Row}(\boldsymbol{B}) = \{\boldsymbol{B},\ \boldsymbol{W}\}$ and $D^{Row}(\boldsymbol{RB}) = \{\boldsymbol{RB}\}$; and $D^{Col}(\boldsymbol{W}) = \{\boldsymbol{B},\ \boldsymbol{W}\}$, $D^{Col}(\boldsymbol{B}) = \{\boldsymbol{B}\}$, $D^{Col}(\boldsymbol{R}) = \{\boldsymbol{B}\}$, and $D^{Col}(\boldsymbol{RB}) = \{\boldsymbol{RB}\}$.

The prior $\left(\mathcal{E}^{Row}, \mu^{Row}\right)$ is admissible if $\mathcal{E}^{Row}$ is the algebra generated from the partition:

$$\begin{aligned} &\{\{(\boldsymbol{W},\boldsymbol{W})\}, \{(\boldsymbol{W},\boldsymbol{B}), (\boldsymbol{W},\boldsymbol{R}), (\boldsymbol{W},\boldsymbol{RB})\}, \{(\boldsymbol{B},\boldsymbol{W}), (\boldsymbol{B},\boldsymbol{B})\}, \{(\boldsymbol{B},\boldsymbol{R}), (\boldsymbol{B},\boldsymbol{RB})\} \\ &, \{(\boldsymbol{R},\boldsymbol{R})\}, \{(\boldsymbol{R},\boldsymbol{W}), (\boldsymbol{R},\boldsymbol{B}), (\boldsymbol{R},\boldsymbol{RB})\}, \{(\boldsymbol{RB},\boldsymbol{RB})\}, \{(\boldsymbol{RB},\boldsymbol{W}), (\boldsymbol{RB},\boldsymbol{B}), (\boldsymbol{RB},\boldsymbol{R})\}\} \end{aligned}$$

and

$$\begin{aligned} & \mu^{Row}(\{(\boldsymbol{W},\boldsymbol{B}), (\boldsymbol{W},\boldsymbol{R}), (\boldsymbol{W},\boldsymbol{RB})\}) = \mu^{Row}(\{(\boldsymbol{B},\boldsymbol{R}), (\boldsymbol{B},\boldsymbol{RB})\}) \\ = \ & \mu^{Row}(\{(\boldsymbol{R},\boldsymbol{W}), (\boldsymbol{R},\boldsymbol{B}), (\boldsymbol{R},\boldsymbol{RB})\}) = \mu^{Row}(\{(\boldsymbol{RB},\boldsymbol{W}), (\boldsymbol{RB},\boldsymbol{B}), (\boldsymbol{RB},\boldsymbol{R})\}) = 0. \end{aligned}$$

Similarly, the prior $\left(\mathcal{E}^{Col}, \mu^{Col}\right)$ is admissible if $\mathcal{E}^{Col}$ is the algebra generated from the partition:

$$\begin{aligned} &\{\{(\boldsymbol{W},\boldsymbol{W}), (\boldsymbol{B},\boldsymbol{W})\}, \{(\boldsymbol{R},\boldsymbol{W}), (\boldsymbol{RB},\boldsymbol{W})\}, \{(\boldsymbol{B},\boldsymbol{B})\}, \{(\boldsymbol{W},\boldsymbol{B}), (\boldsymbol{R},\boldsymbol{B}), (\boldsymbol{RB},\boldsymbol{B})\} \\ &, \{(\boldsymbol{R},\boldsymbol{R})\}, \{(\boldsymbol{W},\boldsymbol{R}), (\boldsymbol{B},\boldsymbol{R}), (\boldsymbol{RB},\boldsymbol{R})\}, \{(\boldsymbol{RB},\boldsymbol{RB})\}, \{(\boldsymbol{W},\boldsymbol{RB}), (\boldsymbol{B},\boldsymbol{RB}), (\boldsymbol{R},\boldsymbol{RB})\}\} \end{aligned}$$

and

$$\mu^{Col}\left(\left\{\left(\boldsymbol{R},\boldsymbol{W}\right),\left(\boldsymbol{RB},\boldsymbol{W}\right)\right\}\right)=\mu^{Col}\left(\left\{\left(\boldsymbol{W},\boldsymbol{B}\right),\left(\boldsymbol{R},\boldsymbol{B}\right),\left(\boldsymbol{RB},\boldsymbol{B}\right)\right\}\right)$$

$$=\quad\mu^{Col}\left(\left\{\left(\boldsymbol{W},\boldsymbol{R}\right),\left(\boldsymbol{B},\boldsymbol{R}\right),\left(\boldsymbol{RB},\boldsymbol{R}\right)\right\}\right)=\mu^{Col}\left(\left\{\left(\boldsymbol{W},\boldsymbol{RB}\right),\left(\boldsymbol{B},\boldsymbol{RB}\right),\left(\boldsymbol{R},\boldsymbol{RB}\right)\right\}\right)=0.$$

For concreteness, further suppose that

$$\begin{aligned}\mu^{Row}_{\Sigma^1}\left(\boldsymbol{W}\right) &= \mu^{Row}_{\Sigma^1}\left(\boldsymbol{B}\right)=\mu^{Row}_{\Sigma^1}\left(\boldsymbol{R}\right)=\frac{1}{3},\ \mu^{Row}_{\Sigma^1}\left(\boldsymbol{RB}\right)=0,\text{ and}\\ \mu^{Col}_{\Sigma^2}\left(\boldsymbol{W}\right) &= \mu^{Col}_{\Sigma^2}\left(\boldsymbol{B}\right)=\mu^{Col}_{\Sigma^2}\left(\boldsymbol{R}\right)=\frac{1}{3},\ \mu^{Col}_{\Sigma^2}\left(\boldsymbol{RB}\right)=0.\end{aligned}$$

We shall also assume that for both individuals, $v(.)$ is the common continuous, strictly concave and strictly increasing utility function over final wealth.

Finally if we take the ambiguity attitude parameter $\alpha^i=\alpha>0$ to be the same for both players, then the expected separable uncertain-utility of a contract for Row and Col, are generated, respectively, by the functionals:

$$\begin{aligned}V^{Row}(c) &= \frac{1}{3}v\left(f_c\left(\boldsymbol{W}\right)+2\right)+\frac{1}{3}\left[(1-\alpha)\max_{\sigma\in\{\boldsymbol{B},\boldsymbol{W}\}}v\left(f_c(\sigma)+1\right)+\alpha\min_{\sigma\in\{\boldsymbol{B},\boldsymbol{W}\}}v\left(f_c(\sigma)+1\right)\right]\\ &\quad+\frac{1}{3}v\left(f_c\left(\boldsymbol{R}\right)+3\right) \qquad (2)\\ V^{Col}(c) &= \frac{1}{3}\left[(1-\alpha)\max_{\sigma\in\{\boldsymbol{B},\boldsymbol{W}\}}v\left(-f_c(\sigma)+2\right)+\alpha\min_{\sigma\in\{\boldsymbol{B},\boldsymbol{W}\}}u\left(-f_c(\sigma)+2\right)\right]\\ &\quad+\frac{1}{3}v\left(-f_c(B)+3\right)+\frac{1}{3}v\left(-f_c\left(\boldsymbol{R}\right)+1\right). \qquad (3)\end{aligned}$$

Some aspects of these preferences are noteworthy. The players' (*ex ante*) preference for signing a given hedging contract will be stronger the more risk-averse they are, that is, the stronger their preference for the non-stochastic allocation over the original endowment. Their preference for signing a hedging contract will be less the more weight they place on the possibility of different interpretations giving rise to disputes. Thus risk and ambiguity work in opposite directions. This result applies generally to problems involving ambiguous risk sharing contracts.

6 The bargaining problem

We consider now the bargaining the two individuals can engage in, where the set of alternatives over which bargaining is to be conducted is taken to be some subset of the set of contracts C, characterized in section 3. For the bulk of the paper we use the entire set of contracts C. However, in Section 7, we use a subset of C in an application of our theory to liquidated damages. For ease of exposition, however, we give the results of this section assuming the full set of contracts C. We further assume that there is a designated contract $c_0 \in C$, which we take to be the disagreement action that will result should the bargaining process break down and no agreement is reached. We restrict our attention in what follows to individuals with *ESUU* preferences, that is, preferences admitting a expected utility representation of the form (1).

As C and c_0 will be fixed throughout, we shall identify the bargaining problem with the pair of preferences relations of the bargainers over C. Thus a bargaining problem in our set-up can be identified by a tuple $\left(\left\langle \mathcal{E}^1, \mu^1, v^1, \alpha^1 \right\rangle, \left\langle \mathcal{E}^2, \mu^2, v^2, \alpha^2 \right\rangle\right)$ with associated utility representations V^1 and V^2. So that the problem is not vacuous, we assume there exists a contract $\hat{c}$ in C such that $V^i(\hat{c}) > V^i(c_0)$, $i = 1, 2$. That is, $\hat{c}$ (strictly) Pareto dominates c_0.

Denote by $\mathcal{B}$ the class of Bargaining problems for the analysis. To aid the analysis, it is convenient to define the *cardinal bargaining problem* induced by the preferences of the two bargainers in the following way.

Definition 4 *Fix a bargaining problem* $\left(\left\langle \mathcal{E}^1, \mu^1, v^1, \alpha^1 \right\rangle, \left\langle \mathcal{E}^2, \mu^2, v^2, \alpha^2 \right\rangle\right)$ *in* $\mathcal{B}$*. The* cardinal bargaining problem *associated with this bargaining problem is the set* $B \subset \mathbb{R}_2$*, given by*

$$B = \left\{(v_1, v_2) : \exists c \in C, (v_1, v_2) \leq \left(V^1(c), V^2(c)\right)\right\}.$$

Notice that B is comprehensive by construction. Since A_0 is compact, it follows that C is compact as well, and hence it follows by the construction of B that it is closed. To allow for a simple and convenient characterization of the set of individually rational and efficient contracts in a given bargaining problem we assume the bargaining problem exhibits the following property introduced by Grant and Kajii (1995).

Definition 5 (C-Convexity) *A bargaining problem* $\left(\left\langle \mathcal{E}^1, \mu^1, v^1, \alpha^1 \right\rangle, \left\langle \mathcal{E}^2, \mu^2, v^2, \alpha^2 \right\rangle\right)$ *in* $\mathcal{B}$ *exhibits* C-convexity *if for any pair of contracts* c *and* c' *in* C*, there exists a contract* c'' *in* C *such that*

$$V^i(c'') \geq \frac{1}{2}V^i(c) + \frac{1}{2}V^i(c'), \; i = 1, 2.$$

A sufficient condition for a bargaining problem to exhibit C-convexity, is for the (state-dependent) utility functions $v^i \circ \left(y^i(\cdot,\sigma)\right): A \to \mathbb{R}$ to be concave in a.[6]

As the name suggests, a bargaining problem that exhibits C-convexity has associated with it a *convex* cardinal bargaining problem.

Lemma 5 *If $\left(\left\langle \mathcal{E}^1,\mu^1,v^1,\alpha^1\right\rangle,\left\langle \mathcal{E}^2,\mu^2,v^2,\alpha^2\right\rangle\right)$ in $\mathcal{B}$ is a C-convex bargaining problem then the associated cardinal bargaining problem B is convex.*

Proof. Fix an arbitrary pair (v_1,v_2) and (v_1',v_2') in B. To establish that B is convex it is sufficient to show that $\left(\frac{1}{2}v_1+\frac{1}{2}v_1',\frac{1}{2}v_2+\frac{1}{2}v_2'\right)$ is also in B. Since (v_1,v_2) and (v_1',v_2') are both in B, it follows from the definition of B that there exists contracts c and c' in C, such that $\left(V^1(c),V^2(c)\right)\geq(v_1,v_2)$ and $\left(V^1(c'),V^2(c')\right)\geq(v_1',v_2')$. By C-convexity there exists a contract c'' in C, such that $V^i(c'')\geq\frac{1}{2}V^i(c)+\frac{1}{2}V^i(c')$, $i=1,2$. Hence,

$$\begin{aligned}\left(V^1(c''),V^2(c'')\right) &\geq \left(\frac{1}{2}V^1(c)+\frac{1}{2}V^1(c'),\frac{1}{2}V^2(c)+\frac{1}{2}V^2(c')\right)\\ &\geq \left(\frac{1}{2}v_1+\frac{1}{2}v_1',\frac{1}{2}v_2+\frac{1}{2}v_2'\right),\end{aligned}$$

as required. ■

If all bargaining problems in $\mathcal{B}$ are convex, then for each problem we have a simple characterization of the set of individually rational and efficient contracts: they are the contracts that are at least as good for both individuals as the disagreeement contract c_0 and maximize a weighted utilitarian social welfare function, for some set of (normalized) non-negative weights.

Proposition 6 *Suppose the bargaining problem $\left(\left\langle \mathcal{E}^1,\mu^1,v^1,\alpha^1\right\rangle,\left\langle \mathcal{E}^2,\mu^2,v^2,\alpha^2\right\rangle\right)$ in $\mathcal{B}$ is C-convex. Then the contract c^* is individually rational and efficient if and only if*

$$\min\left\{\left[V^1(c)-V^1(c_0)\right],\left[V^2(c)-V^2(c_0)\right]\right\}\geq 0$$

and

$$c^*\in\operatorname{argmax}_{c\in C}\lambda V^1(c)+(1-\lambda)V^2(c)\,,\ \textit{for some}\ \lambda\ \textit{in}\ [0,1]\,. \tag{4}$$

[6] This holds naturally for risk-sharing contracts in which the action $a\in A_0\subset\mathbb{R}$ corresponds to a transfer of size a from bargainer 2 to bargainer 1, and $v^i\circ\left(y^i(\cdot,\sigma)\right)=\mu^i_{\Sigma^i}(\sigma)\,v\left(a\times(-1)^{i-1}+z^i_s\right)$ is the probability weighted utility of bargainer i's final wealth in state s after the transfer has been made. Concavity of $v^i\circ\left(y^i(\cdot,\sigma)\right)$ then follows naturally from risk aversion (that is, concavity of v the utility index over wealth).

The proof of Proposition 4 follows from the application of a standard separating hyperplane theorem for convex sets and so is omitted.

6.1 The illustrative example continued.

Taking $c_0 = 0$ (that is, no transfer is made), and denoting $V^{Row}(c_0) = V^{Col}(c_0) = \bar{u}$, an inidividually rational and efficient contract c for the illustrative example developed in section 5.1 is one for which $V^{Row}(c) \geq \bar{u}$, $V^{Col}(c) \geq \bar{u}$ and c is a solution to the maximization problem,

$$\max_{\langle (f_c(\boldsymbol{W}), f_c(\boldsymbol{B}), f_c(\boldsymbol{R})) \in [-3,3]^3 \rangle} \lambda V^{Row}(c) + (1-\lambda) V^{Col}(c), \text{ for some } \lambda \text{ in } [0,1].$$

For given λ in $[0,1]$ the solution $c^*(\lambda)$ with associated state-contingent transfers $(0, f_{c^*}(\boldsymbol{R}), f_{c^*}(\boldsymbol{B}), f_{c^*}(\boldsymbol{W}))$ (and assuming an interior solution with $f_{c^*}(\boldsymbol{R}) \leq f_{c^*}(\boldsymbol{W}) \leq f_{c^*}(\boldsymbol{B})$), satisfies the first-order conditions:

$$\begin{aligned}
f_{c^*}(\boldsymbol{W}) &: \lambda [v'(f_c(\boldsymbol{W})+2) + \alpha v'(f_c(\boldsymbol{W})+1)] - (1-\lambda)[(1-\alpha) v'(-f_c(\boldsymbol{W})+2)] = 0. \\
f_{c^*}(\boldsymbol{B}) &: \lambda (1-\alpha) v'(f_c(\boldsymbol{B})+1) - (1-\lambda)[\alpha v'(-f_c(\boldsymbol{B})+1) + v'(-f_c(\boldsymbol{B})+3)] = 0 \\
f_{c^*}(\boldsymbol{R}) &: \lambda v'(f_c(\boldsymbol{R})+3) - (1-\lambda) v'(-f_c(\boldsymbol{R})+1) = 0
\end{aligned}$$

Or rearranging, we obtain:

$$\begin{aligned}
& \frac{v'(f_{c^*}(\boldsymbol{W})+2) + \alpha v'(f_{c^*}(\boldsymbol{W})+1)}{(1-\alpha) v'(-f_{c^*}(\boldsymbol{W})+2)} \\
= & \frac{(1-\alpha) v'(f_{c^*}(\boldsymbol{B})+1)}{v'(-f_{c^*}(\boldsymbol{B})+3) + \alpha v'(-f_c(\boldsymbol{B})+2)} = \frac{v'(f_{c^*}(\boldsymbol{R})+3)}{v'(-f_{c^*}(\boldsymbol{R})+1)} = \frac{(1-\lambda)}{\lambda}. \qquad (5)
\end{aligned}$$

For the symmetric weighted utilitarian social welfare function (that is, $\lambda = 1/2$), we see immediately from (5) that for $\alpha = 0$, the solution is $(0, -1, 1, 0)$. That is, when the decision places all the weight from the possbility of dispute on his own interpretation being implemented, the symmetric solution is the full risk-sharing contract described in section 2.

To see what happens for $\lambda = 1/2$ and $\alpha > 0$, we have from (5) that $f_{c^*}(\boldsymbol{R}) = 1$ and furthermore whenever $f_{c^*}(\boldsymbol{B}) > 1/2 > f_{c^*}(\boldsymbol{W})$ holds, $f_{c^*}(\boldsymbol{B})$ and $f_{c^*}(\boldsymbol{W})$ are the unique solutions to:

$$\frac{1}{(1-\alpha)} \frac{v'(-f_{c^*}(\boldsymbol{B})+3)}{v'(f_{c^*}(\boldsymbol{B})+1)} + \frac{\alpha}{(1-\alpha)} \frac{v'(-f_{c^*}(\boldsymbol{B})+2)}{v'(f_{c^*}(\boldsymbol{B})+1)} = 1 \qquad (6)$$

$$\frac{1}{(1-\alpha)} \frac{v'(f_{c^*}(\boldsymbol{W})+2)}{v'(-f_{c^*}(\boldsymbol{W})+2)} + \frac{\alpha}{(1-\alpha)} \frac{v'(f_{c^*}(\boldsymbol{W})+1)}{v'(-f_{c^*}(\boldsymbol{W})+2)} = 1. \qquad (7)$$

respectively. Notice that the LHS of (6) is increasing in $f_{c^*}(\boldsymbol{B})$ and the LHS of (7) is decreasing in $f_{c^*}(\boldsymbol{W})$, so while $f_{c^*}(\boldsymbol{B}) > \frac{1}{2} > f_{c^*}(\boldsymbol{W})$ the solution is well-defined for each corresponding α.

For the critical value $\hat{\alpha}$ for which $f_{c^*}(\boldsymbol{B}) = f_{c^*}(\boldsymbol{W}) = \frac{1}{2}$, that is, $\hat{\alpha} = \left[1 - \left(v'\left(\frac{5}{2}\right)/v'\left(\frac{3}{2}\right)\right)\right]/2$, the optimal symmetric contract is the *unambiguous contract:* '**if** t_1 **then** -1 **else** $\frac{1}{2}$,' that is, if the card drawn is red then Row pays Col 1, and otherwise Col pays Row $\frac{1}{2}$. This remains the optimal contract for any $\alpha > \hat{\alpha}$, since the state-contingent act associated with this contract $\left(-1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ satisfies

$$\frac{v'\left(f_{c^*}(\boldsymbol{R}) + 3\right)}{v'\left(-f_{c^*}(\boldsymbol{R}) + 1\right)} = \frac{v'\left(f_{c^*}(\boldsymbol{B}) + 1\right) + v'\left(f_{c^*}(\boldsymbol{W}) + 2\right)}{v'\left(-f_{c^*}(\boldsymbol{B}) + 3\right) + v'\left(-f_{c^*}(\boldsymbol{W}) + 2\right)} = 1,$$

the first-order conditions for the optimal *unambiguous* contract.

7 An application: Liquidated Damages

To be effective, a contract must specify some sanction to be applied if one or other party fails to perform their obligations. In some cases, this is a relatively simple matter: failure to perform may be held to nullify the contract.

In other cases, however, failure by one party to perform an obligation may cause damage to the other.

For concreteness, let us consider an example where a supplier contracts with a builder to deliver materials on a given date. However, under certain conditions (expressed as tests), the supplier may be unable to deliver, and may default, declaring force majeure. Failure to deliver on time may force the builder to source the supplies elsewhere at high cost, or to delay the project. Thus, neither nullifying the contract nor requiring (delayed) performance is an adequate remedy. The costs of failure will depend on a variety of factors, which may be represented by tests. For example, rainy weather might halt construction with the result that the supplier's default causes no additional cost. In other cases, the default may occur at a crucial point in the project, creating unusually large damages.

In the absence of bounds on rationality, the parties could agree on a contract that listed all possible default states, and specified a payment to be made in each case. The bargaining solution in this case, derived from the state-dependent preferences of both parties, will be referred to as the first best contract. However, with ambiguity arising from bounded rationality, the first-best contract may no longer be chosen.

One solution, in the presence of ambiguity, is for the contract to specify that the defaulting party should compensate the other party an amount depending only on the amount of their loss. In the event of a dispute over the magnitude of the loss, a court or other external arbiter will determine the payment. If the loss amounts are unambiguous, then this type of contract might be chosen in this setting.

Another possibility is that of *liquidated damages*, in which the payment for a specific breach is fixed in the contract, without reference to the actual losses suffered by the injured party. This type of contract might result when the loss amounts are ambiguous and contracting on them would result in excessive disputes.

We now consider how these three alternative contracts may be represented. Here, we restrict attention for our Nash bargaining to a subset of the full set of contracts C characterized in section 3. We begin by assuming that the set of primitive tests T_0 includes: (a) (default test) an unambiguous test t_d, interpreted as 'the test that party 1 must default'; (b) (set of non-default tests) a set $\hat{T}$ of unambiguous non-default tests which apply only in the non-default state. That is, the set of tests $\hat{T} \subset T$, are also unambiguous. Thus, the only potential disputes relate to the consequences of default, and not to the question of whether party 1 has in fact defaulted.

The action set $A = \hat{A} \times A_M$ is the Cartesian product a set of actions $\hat{A}$ relevant to the performance of the contract and a set of payment actions $A_M = \{a_m : m \in [-M, M]\}$. The set $\hat{A}$ is assumed to include the default action a_0. Actions $a_m \in A_M$ are interpreted as 'party 1 (2) pays m dollars to party 2 (1)' for positive (negative) values of m. Actions $\hat{a} \in \hat{A}/a_o$ are unavailable or prohibitively costly to party 1 in the event $\pi(t_d)$, the default event.

Thus, any feasible contract c must satisfy

$$f_c(\sigma) \in \hat{A} \times A_M \qquad \sigma \in \pi(\neg t_d)$$

$$f_c(\sigma) \in \{a_0\} \times A_M \qquad \sigma \in \pi(t_d)$$

That is, the contract specifies a set of actions to be performed, and payments to be made, in the absence of default and a set of payments to be made in the presence of default. Payments made in the presence of default are referred to as damages payments and are assumed to be positive, that is, the defaulting party 1 makes a payment to party 2.

We assume that, for each party, the outcome function y^i takes the separable form

$$y^i\left(\left(\hat{a}, a_m\right), \sigma\right) = w^i_\sigma\left(\hat{a}\right) + m$$

where $w^i_\sigma\left(\hat{a}\right)$ may be interpreted as the monetary value to party i of the action $\hat{a}$ performed in state σ. Hence, for any default state $\sigma \in \pi\left(t_d\right)$, $-w^i_\sigma\left(a_0\right)$ may be interpreted as the loss incurred by party i in state σ, consequent on default. For simplicity, we also presume that the disagreement contract gives each party zero utility and that the default amounts that can occur are given by the finite set $Z \subset \mathbb{R}_+$.[7]

In the absence of ambiguity, the Nash bargaining solution contract must satisfy the risk-sharing condition that, for any σ, $\sigma' \in \pi\left(t_d\right)$.

$$\frac{\mu^1_{\Sigma^1}\left(\sigma\right)\left(v^1\right)'\left(w^1_\sigma\left(a_0\right) - a_m\left(\sigma\right)\right)}{\mu^1_{\Sigma^1}\left(\sigma'\right)\left(v^1\right)'\left(w^1_{\sigma'}\left(a_0\right) - a_m\left(\sigma'\right)\right)} = \frac{\mu^2_{\Sigma^2}\left(\sigma\right)\left(v^2\right)'\left(w^2_\sigma\left(a_0\right) + a_m\left(\sigma\right)\right)}{\mu^2_{\Sigma^2}\left(\sigma'\right)\left(v^2\right)'\left(w^2_{\sigma'}\left(a_0\right) + a_m\left(\sigma'\right)\right)}$$

Since the contract is unambiguous in the absence of default, this contract will, in general, be unambiguous if and only if the set of unambiguous tests is rich enough to distinguish any pair of states $\sigma, \sigma' \in \pi\left(t_d\right)$ such that either $w^1_\sigma\left(a_0\right) \neq w^1_{\sigma'}\left(a_0\right)$ or $w^2_\sigma\left(a_0\right) \neq w^2_{\sigma'}\left(a_0\right)$.

Suppose, however, that tests relevant to the effects of default on the welfare of party 1 (the defaulting party) are ambiguous.

In this case, we may, consider the case of a contract with damages dependent on losses to party 2. For any $\delta \in Z$, let us suppose there exists a test $t_\delta \in T$ that is satisfied if and only if default occurs, and the associated loss is δ, that is, on the event $\pi\left(t_\delta\right) = \pi\left(t_d\right) \cap \left\{\sigma : w^2_\sigma\left(a_0\right) = \delta\right\}$. The members of the set of events $\left\{\pi\left(t_\delta\right) : \delta \in Z\right\} \cup \left\{\pi\left(\neg t_d\right)\right\}$, are mutually exclusive and exhaustive, and therefore constitute a partition of the state space. A loss-dependent damages contract c^*,

[7] Note that we can restrict ourselves to a finite set of loss values, rather than specifying the loss as a real number. This treatment is both more realistic (it is hard to define an irrational number of dollars) and, given our setup, more tractable.

restricted to be measurable with respect to this partition satisfies:

$$\begin{array}{ll} f_{c^*}(\sigma) \in \hat{A} \times A_M & \sigma \in \pi(\neg t_d) \\ f_{c^*}(\sigma) = (a_0, g(\delta)) & \sigma \in \pi(t_\delta) \end{array},$$

where $g : Z \rightarrow A_M$ is a function relating the loss borne by party 2 to the associated damages payment from party 1. Note that we do not require $g(\delta) = \delta$. That is, the damages payment from party 1 to party 2 need not be equal to the loss incurred by party 2. Depending on the risk-sharing properties of the contract and on the state-dependent preferences of party 1, the damages payment to party 2, $g(\delta)$, may be less than, equal to or greater than the loss δ incurred by party 2.[8]

The tests t_δ may still be ambiguous. For example, the parties may disagree over what items should be counted as losses arising from default and how they should be valued. Thus, such contracts are likely to, and regularly do, produce disputes.

If losses are ambiguous, and dispute costs are high, parties may prefer a liquidated damages contract, with a specified payment $\hat{m}$. The required test set is then the minimal set $T = \{t_d\}$ and a liquidated damages contract c^{**} specifying payment $\hat{m}$ satisfies

$$\begin{array}{ll} f_{c^{**}}(\sigma) \in \hat{A} \times A_M & \sigma \in \pi(\neg t_d) \\ f_{c^{**}}(\sigma) = (a_0, a_{\hat{m}}) & \sigma \in \pi(t_d) \end{array}$$

That is, either the contract applicable in the absence of default is implemented or party 2 pays the liquidated damage sum $\hat{m}$. As long as the test t_d is unambiguous, so is the liquidated damages contract.

8 Concluding comments

We have provided a formal model for incorporating ambiguity into decision making. The ambiguity in our model arises from the bounded rationality of the players which is expressed as limited abilities to perform tests over the possible contingencies. This limitation results in each player having a limited individual description of the world.

[8] In general, risk-sharing would imply that the damages payment should be less than the loss. In the model presented here, losses are force majeure not discretionary, so there is no incentive-based reason for exemplary or punitive damages. However, consideration of the state-contingent preferences of party 1 suggests instances where risk-sharing may imply a payment larger than the loss. Suppose that high-losses to party 2 occur when the good is in high demand and subject to constrained supply. Then party 1, having defaulted as a result of inability to supply on time may be able to sell the good at a high price and therefore (involuntarily) benefit from default.

Contracting was restricted in this context to the types of test based contingent plans described in Blume et al. (2006). In this context we were able to show how ambiguity can affect incentives for risk sharing, and the desirability of contracts.

The representation of ambiguity proposed here suggests new approaches to a range of issues in contract theory. Some of these issues have proved difficult to address using approaches based on unbounded rationality, or on arbitrary constraints on rationality. In the case of risk sharing, we have shown that ambiguity may lead players to prefer incomplete risk sharing to possibly ambiguous contracts. In the application of liquidated damages we saw that the presence of ambiguity about the damages suffered by the party injured by default can lead to a contract that stipulates a simple penalty payment to be made in the event one party unilaterally defaults on performance.

The analysis of liquidated damages suggests the possibility of broader applications in agency theory. The standard principal-agent problem is one where contracting is limited to some observable unambiguous characteristics like output, rather than a full set of characteristics including effort levels which may be ambiguous. The framework developed here suggests the possibility of an endogenous choice between contracts over different characteristics, where the choice of the contractual variables chosen depends on the level of ambiguity and potential gains from risk sharing. While this application would require overcoming some new technical details involving the appropriate treatment of tests, the benefit would be the development of a theory of contracting in which the terms of the contract, over which the parties actually bargain, plays the central role.

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