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CORRELATED EQUILIBRIUM IN GAMES WITH INCOMPLETE INFORMATION

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Correlated Equilibrium in Games with Incomplete Information*

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Abstract

We define a notion of correlated equilibrium for games with incomplete information in a general setting with finite players, finite actions, and finite states, which we call Bayes correlated equilibrium. The set of Bayes correlated equilibria of a fixed incomplete information game equals the set of probability distributions over actions, states and types that might arise in any Bayes Nash equilibrium where players observed additional information. We show that more information always shrinks the set of Bayes correlated equilibria.

KEYWORDS: Correlated equilibrium, incomplete information, robust predictions, information structure.

JEL CLASSIFICATION: C72, D82, D83.

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1 Introduction

We present a notion of correlated equilibrium in games with incomplete information. Aumann (1974), (1987) introduced the notion of correlated equilibrium in games with complete information. A number of definitions of correlated equilibrium in games with incomplete information have been suggested, notably in Forges (1993). Our definition is driven by a different motivation from the earlier literature; we seek the solution concept which characterizes the set of Bayes Nash equilibria which can be sustained by some information structure in a fixed economic setting. This leads us to suggest an equilibrium notion - Bayes correlated equilibrium - which is weaker than the weakest definition of incomplete information correlated equilibrium (the Bayesian solution) in Forges (1993), because it allows play to be correlated with states that are not known by any player.

We distinguish between the "basic game" or "payoff environment" from the "information structure" or "belief environment" in the definition of the game. By payoff environment, we refer to the set of actions, the set of payoff states, the utility functions of the agents, and the common prior over the payoff states. By belief environment, we refer to the type space of the game, which is generated by a mapping from the payoff states to a probability distribution over types. The separation between payoff and belief environment enables us to ask how changes in the belief environment affect the equilibrium set for a given and fixed payoff environment. We introduce a natural partial order on information structures that captures when one information structure contains more information than another. This partial order is a variation on a many player generalization of the ordering of Blackwell (1953) introduced by Lehrer, Rosenberg, and Shmaya (2010), (2011). We show that the set of Bayes correlated equilibria shrinks as the informativeness of the information structure increases.

The present definition of Bayes correlated equilibrium is used prominently in the analysis of our companion paper, "Robust Predictions in Games with Incomplete Information", Bergemann and Morris (2011b). In the companion paper, we analyze how much can be said about the joint distribution of actions and states on the basis of the knowledge of the payoff environment alone. There we refer to “robust predictions” as those predictions which can be made with the knowledge of the payoff environment alone, and without any assumption about the belief environment. In the companion paper, the analysis was confined to an environment with quadratic and symmetric payoff functions, a continuum of agents and normally distributed uncertainty about the common payoff state. But this tractable class of models enabled us to offer robust predictions in terms of restrictions on the first and second moments of the joint distribution over actions and state. By contrast, here we present the definition of the Bayes correlated equilibrium in a canonical game theoretic framework with a finite number of agents, a finite set of pure actions and a finite set of payoff states. After we introduce the relevant notions, we show in Section 7 how the present results translate into the setting with a continuum of anonymous agents that we considered in Bergemann and Morris (2011b).

A number of papers have considered alternative definitions of correlated equilibrium in games with incomplete information, most notably Forges (1993), (2006). In this paper, we document the relationship between our version of correlated equilibrium and the various definitions in the literature. In the discussion of the various definitions of correlated equilibrium, we will find it is useful to divide restrictions that the various solution concepts impose on the joint distribution over actions, states and types into two classes: *feasibility* constraints on the distributions of action-type-state profiles, which are required to hold independently of the payoff functions, and *incentive* constraints which reflect the rationality of players' choices. The only feasibility condition that we impose in defining the Bayes correlated equilibrium is a consistency requirement that demands that the action-type-state distribution of the equilibrium implies the distribution on the exogenous variables, namely the common prior on the payoff states and types. In contrast, in many of the existing solution concepts, the feasibility conditions are intended to capture the outcome of some form of communication among the agents with an *uninformed mediator*. It is then natural to impose additional restriction on the action-state type distribution in equilibrium which have to hold *conditional* on the agents' types. For example, the "Bayesian solution", the weakest of Forges' five definitions, imposes the restriction, referred to here as join feasibility, that the distribution over states conditional on agents' types is not changed conditional on the mediator's recommendations. Our notion of Bayes correlated equilibrium is closest to the "Bayesian solution" but is strictly weaker than the Bayesian solution, because we do not insist on join feasibility. Our notion is equivalent to the Bayesian solution if we add a "dummy player" who observes nature's move perfectly but does not take any actions.

A number of papers - notably Gossner (2000) and Lehrer, Rosenberg, and Shmaya (2010), (2011) - have examined comparative statics of how changing the information structure effects the set of predictions that can be made about players' actions, under Bayes Nash equilibrium or alternative solution concepts. We review these results and their relation to our new result on the comparative statics of the information structure. We discussed above that as the agents become more informed, where information is encoded in their type, the set of possible predictions must be reduced. As the agents have more private information, the incentive constraints, here referred to as obedience constraints, will become tighter. The role of the private information in refining the equilibrium prediction is important in our "Robust Prediction" agenda. We will formalize this result here in the general framework of the current paper rather than in the specific environment of quadratic payoff functions and normally distributed uncertainty of Bergemann and Morris (2011b).

We say that two information structures are informationally equivalent to each other if each is more informed than the other. We also show that two information structures are informationally equivalent if and only if they generate the same probability distributions over players' beliefs and higher order beliefs. Thus it is a corollary of our comparative result that two information structures support the same set of Bayes correlated equilibria if and only if there are equivalent in terms of the higher order beliefs they generate.

We also illustrate the notion of Bayes correlated equilibrium and the resulting robust predictions in a single player two action two state game (decision problem), which is closely related to the sender receiver problem studied by Kamenica and Gentzkow (forthcoming), where the sender is allowed to commit to a communication strategy.

In a series of papers collected in Bergemann and Morris (2012) we have studied "robust mechanism design" (see Bergemann and Morris (2011a) for an introductory essay). In this earlier setting, the agents knew their own "payoff types", and while there was common knowledge of how utilities depended on the profile of payoff types, the agents were allowed to have *any* beliefs and higher order beliefs about others' payoff types. We then defined a mechanism to be *robust* if the social choice function or correspondence could be truthfully implemented in the direct mechanism as a Bayes Nash equilibrium for *any* beliefs and higher order beliefs about others' payoff types. In Bergemann and Morris (2007), we discussed the game theoretic framework underlying the analysis in the mechanism design environment. The notion of Bayes correlated equilibrium is motivated by the same concern for robustness but it encodes a less demanding notion of robustness. The Bayes correlated equilibrium insists that the common prior over the state and type distribution is preserved, and in the case of the "null information structure" that the common prior over the state alone is preserved, but all additional correlation due to unobserved communication or information among the agents is permitted.

We proceed as follows. In Section 2, we describe a general incomplete information game and compare Bayes Nash equilibrium with a solution concept which we call Bayes correlated equilibrium. In Section 3, we describe our robust predictions approach and explain the key role played by an "epistemic" result: the set of Bayes correlated equilibrium probability distributions over actions, types and payoff-relevant variables equals the set of probability distributions of actions, types and payoff-relevant variables that might arise in a Bayes Nash equilibrium if players were able to observe additional information signals beyond their original types.

In Section 4, we explain how the solution concept we dub "Bayes correlated equilibrium" relates to the literature, in particular Forges (1993), (2006). In Section 5, we report results on comparing information structures. In Section 6, we review special cases in order to illustrate the robust predictions agenda more broadly. In Section 7, we describe analogues of our results for continuum anonymous player games, which apply to our work in "Robust Predictions in Games with Incomplete Information". Section 8 concludes and contains a discussion of the relation to the signed covariance result of Chwe (2006) and the "payoff types" environments of Bergemann and Morris (2007).

2 Bayes Nash and Bayes Correlated Equilibrium

Throughout the paper, we will fix a finite set of players and a finite set of payoff states of the world. There are I players, $1, 2, ..., I$, and we write i for a typical player. We write Θ for the payoff states of the world and θ for a typical element of Θ.

A "basic game" G consists of (1) for each player i, a finite set of actions A_i and a utility function $u_i : A \times \Theta \rightarrow \mathbb{R}$; and (2) a full support prior $\psi \in \Delta(\Theta)$, where we write $A = A_1 \times \cdots \times A_I$. Thus $G = \left((A_i, u_i)_{i=1}^{I}, \psi\right)$. An "information structure" S consists of (1) for each player i, a finite set of types or "signals" T_i; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(T)$, where we write $T = T_1 \times \cdots \times T_I$. Thus $S = \left((T_i)_{i=1}^{I}, \pi\right)$.

Together, the "payoff environment" or "basic game" G and the "belief environment" or "information structure" S define a standard "incomplete information game". While we use different notation, this division of an incomplete information game into the "basic game" and the "information structure" is a standard one in the literature, see, for example, Lehrer, Rosenberg, and Shmaya (2010).

A (behavioral) strategy for player i in the incomplete information game game (G, S) is $\beta_i : T_i \rightarrow \Delta(A_i)$. Write $\widehat{B}_i$ for the set of strategies of player i in the game (G, S). The following is the standard definition of Bayes Nash equilibrium in this setting.

Definition 1 *A strategy profile β is a Bayes Nash equilibrium (BNE) of (G, S) if for each $i = 1, 2, ..., I$, $t_i \in T_i$ and $a_i \in A_i$ with $\beta_i(a_i|t_i) > 0$, we have*

$$\sum_{a_{-i} \in A_{-i}} \sum_{t_{-i} \in T_{-i}} \sum_{\theta \in \Theta} u_i((a_i, a_{-i}), \theta) \psi(\theta) \left(\prod_{j \neq i} \beta_j(a_j|t_j)\right) \pi((t_i, t_{-i})|\theta) \tag{1}$$
$$\geq \sum_{a_{-i} \in A_{-i}} \sum_{t_{-i} \in T_{-i}} \sum_{\theta \in \Theta} u_i\left((a_i', a_{-i}), \theta\right) \psi(\theta) \left(\prod_{j \neq i} \beta_j(a_j|t_j)\right) \pi((t_i, t_{-i})|\theta),$$

for each $a_i' \in A_i$.

For notational ease, we shall henceforth use the convention of describing a multiple sum through a single summation symbol:

$$\sum_{a_{-i}\in A_{-i}}\sum_{t_{-i}\in T_{-i}}\sum_{\theta\in\Theta}u_i\left(\left(a_i,a_{-i}\right),\theta\right)\psi\left(\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j\right)\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right)$$
$$\triangleq\sum_{a_{-i},t_{-i},\theta}u_i\left(\left(a_i,a_{-i}\right),\theta\right)\psi\left(\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j\right)\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right).$$

The relevant space of uncertainty in the incomplete information game (G,S) is $A\times T\times\Theta$, and we will write ν for a typical element of $\Delta\left(A\times T\times\Theta\right)$. There are two kinds of constraints imposed in defining alternative versions of incomplete information correlated equilibrium: "feasibility" constraints and "incentive" constraints. Our preferred definition will impose one feasibility condition, "consistency", which simply says that the marginal of distribution ν on the exogenous variables T and Θ is consistent with the description of the game (G,S).

Definition 2 *Distribution $\nu\in\Delta\left(A\times T\times\Theta\right)$ is consistent for (G,S) if, for all $t\in T$ and $\theta\in\Theta$, we have*

$$\sum_{a\in A}\nu\left(a,t,\theta\right)=\psi\left(\theta\right)\pi\left(t|\theta\right). \tag{2}$$

We will also impose the weakest natural incentive constraint, "obedience", that says that a player i who knows his type t_i, his recommended action a_i and the distribution ν only has an incentive to follow that recommendation.

Definition 3 *Distribution $\nu\in\Delta\left(A\times T\times\Theta\right)$ is obedient for (G,S) if, for each $i=1,...,I$, $t_i\in T_i$ and $a_i\in A_i$, we have*

$$\sum_{a_{-i},t_{-i},\theta}u_i\left(\left(a_i,a_{-i}\right),\theta\right)\nu\left(\left(a_i,a_{-i}\right),\left(t_i,t_{-i}\right),\theta\right)$$
$$\geq\sum_{a_{-i},t_{-i},\theta}u_i\left(\left(a_i',a_{-i}\right),\theta\right)\nu\left(\left(a_i,a_{-i}\right),\left(t_i,t_{-i}\right),\theta\right); \tag{3}$$

for all $a_i'\in A_i$.

Now our leading definition of correlated equilibrium for incomplete information games will be:

Definition 4 *A probability distribution $\nu\in\Delta\left(A\times T\times\Theta\right)$ is a Bayes correlated equilibrium (BCE) of (G,S) if it is consistent and obedient.*

As we will discuss in detail below in Section 4, this is a weakening of the definition of "Bayesian solution" in Forges (1993), with the difference that we work with a incomplete information game description that does not integrate out payoff states and thus allows the mediator to make action recommendations that depend on a payoff state that is observed by nobody. We will discuss in the next section why this definition is interesting for our robust predictions agenda.

A Bayes Nash equilibrium β is a strategy profile in $\widehat{B}$. A Bayes correlated equilibrium ν is an element of $\Delta(A \times T \times \Theta)$ and thus a distribution over action-type-state profiles. To compare the two solution concepts, we would like to discuss the distribution of action-type-state profiles generated by a BNE.

Definition 5 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is induced by strategy profile $\beta \in \widehat{B}$ if, for each $a \in A$, $t \in T$ and $\theta \in \Theta$, we have*

$$\nu(a,t,\theta) = \psi(\theta)\,\pi(t|\theta)\prod_{i=1}^{I}\beta_i(a_i|t_i). \tag{4}$$

Distribution $\nu \in \Delta(A \times T \times \Theta)$ is Bayes Nash equilibrium action-type-state distribution of (G,S) if there exists a Bayesian Nash equilibrium β of (G,S) that induces it.

We also have the following important straightforward observation:

Lemma 1 *Every Bayes Nash equilibrium action-type-state distribution of (G,S) is a Bayes correlated equilibrium of (G,S).*

Proof. Consistency follows immediately by summing across action profiles in equation (4) in the definition of a Bayes Nash equilibrium action-type-state distribution. Now if ν is induced by BNE β, then

$$\begin{aligned}
& \sum_{a_{-i},t_{-i},\theta} u_i((a_i,a_{-i}),\theta)\,\nu((a_i,a_{-i}),(t_i,t_{-i}),\theta) \\
= & \sum_{a_{-i},t_{-i},\theta} u_i((a_i,a_{-i}),\theta)\,\psi(\theta)\,\pi(t|\theta)\prod_{j=1}^{I}\beta_j(a_j|t_j), \text{ by (4)} \\
= & \ \beta_i(a_i|t_i)\sum_{a_{-i},t_{-i},\theta} u_i((a_i,a_{-i}),\theta)\,\psi(\theta)\,\pi(t|\theta)\prod_{j\neq i}\beta_j(a_j|t_j) \\
\geq & \ \beta_i(a_i|t_i)\sum_{a_{-i},t_{-i},\theta} u_i\left((a_i',a_{-i}),\theta\right)\psi(\theta)\,\pi(t|\theta)\prod_{j\neq i}\beta_j(a_j|t_j), \text{ by (1)} \\
= & \sum_{a_{-i},t_{-i},\theta} u_i\left((a_i',a_{-i}),\theta\right)\psi(\theta)\,\pi(t|\theta)\prod_{j=1}^{I}\beta_j(a_j|t_j) \\
= & \sum_{a_{-i},t_{-i},\theta} u_i\left((a_i',a_{-i}),\theta\right)\nu((a_i,a_{-i}),(t_i,t_{-i}),\theta),
\end{aligned}$$

which establishes the result. ■

We will be interested in what can be said about actions and states if types are not observed. This suggests the following definitions.

Definition 6 *Action-state distribution* $\mu \in \Delta(A \times \Theta)$ *is induced by* $\nu \in \Delta(A \times T \times \Theta)$ *if it is the marginal of* ν *on* $A \times \Theta$. *Action-state distribution* $\mu \in \Delta(A \times \Theta)$ *is a BNE action-state distribution of* (G, S) *if it is induced by a BNE action-type-state distribution of* (G, S). *Action-state distribution* $\mu \in \Delta(A \times \Theta)$ *is a BCE action-state distribution of* (G, S) *if it is induced by a Bayes correlated equilibrium of* (G, S).

An important special information structure is the "null" information with players knowing nothing about the states. Formally, the null information structure $S_0 = \left(\left(\{t_i^0\}\right)_{i=1}^{I}, \pi^0\right)$, where t_i^0 is the singleton type of player i and $\pi^0\left(t^0|\theta\right) = 1$ for each $\theta \in \Theta$. We will abbreviate the (degenerate) incomplete information game (G, S_0) to G. Observe that in the special case of a null information structure, the space $A \times T \times \Theta$ reduces to $A \times \Theta$ and the consistency condition (2) on $\mu \in \Delta(A \times \Theta)$ becomes

$$\sum_{a \in A} \mu(a, \theta) = \psi(\theta) \tag{5}$$

for all $\theta \in \Theta$; and the obedience constraint (3) reduces to

$$\sum_{a_{-i}, \theta} u_i\left((a_i, a_{-i}), \theta\right) \mu\left((a_i, a_{-i}), \theta\right) \geq \sum_{a_{-i}, \theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right) \mu\left((a_i, a_{-i}), \theta\right); \tag{6}$$

for each $i = 1, ..., I$, $a_i \in A_i$ and $a_i' \in A_i$. Now the following definition is a special case of definition 4:

Definition 7 *A probability distribution* $\mu \in \Delta(A \times \Theta)$ *is a Bayes correlated equilibrium (BCE) of a basic game* G *if it is consistent (satisfying condition (5)) and obedient (satisfying condition (6)).*

Another important and straightforward observation we will use is that if an action-type-state distribution ν is a BCE of an incomplete information game (G, S), then the action-state distribution induced by ν is a BCE of the basic game:

Lemma 2 *If* $\mu \in \Delta(A \times \Theta)$ *is induced by a BCE* $\nu \in \Delta(A \times T \times \Theta)$, *then* μ *is a BCE of* G.

Proof. Summing consistency condition (2) across types gives consistency condition (5). Summing obedience condition (3) across types gives obedience condition (6). ■

As we will discuss in detail below, this result is in the spirit of Proposition 4 of Forges (1993), which shows that "any" correlated equilibrium solution concept of (G, S) generates an equilibrium of the basic game G.

3 Robust Predictions

Consider an analyst who knows that

1. G describes actions, payoff functions depending on fundamental states, and a prior distribution on fundamental states.

2. Players have observed at least information structure S.

3. The full, common prior, information structure is common certainty among the players.

4. The players' actions follow a Bayes Nash equilibrium.

What can she deduce about the joint distribution of actions, types in the "information structure" S and states? In this section, we will formalize this question and show that all she can deduce is that the distribution will be a BCE distribution of (G, S).

To formalize this, let $\widetilde{S} = \left((Z_i)_{i=1}^{I}, \phi\right)$ be a supplementary information structure, over and above S, and suppose each agent i observes a supplementary signal $z_i \in Z_i$, where $\phi : \Theta \times T \to Z$ describes the distribution of supplementary signals. Now $\left(G, S, \widetilde{S}\right)$ is an "augmented incomplete information game". Write $\beta_i : T_i \times Z_i \to \Delta(A_i)$ for a behavior strategy of player i in the augmented incomplete information game.

Definition 8 *A strategy profile β is a Bayes Nash equilibrium of the augmented game $\left(G, S, \widetilde{S}\right)$ if, for each $i = 1, 2, ..., I$, $t_i \in T_i$, $z_i \in Z_i$ and $a_i \in A_i$ with $\beta_i(a_i|(t_i, z_i)) > 0$, we have*

$$\sum_{a_{-i}, t_{-i}, z_{-i}, \theta} u_i((a_i, a_{-i}), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \left(\prod_{j \neq i} \beta_j(a_j|t_j, z_j)\right) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta)$$

$$\geq \sum_{a_{-i}, t_{-i}, z_{-i}, \theta} u_i\left((a_i', a_{-i}), \theta\right) \psi(\theta) \pi((t_i, t_{-i})|\theta) \left(\prod_{j \neq i} \beta_j(a_j|t_j, z_j)\right) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta).$$

for each $a_i' \in A_i$.

Write ν_β for the probability distribution over $A \times T \times \Theta$ generated by strategy profile β, so

$$\nu_\beta(a, t, \theta) = \psi(\theta) \pi(t|\theta) \sum_{z \in Z} \phi(z|t, \theta) \left(\prod_{i=1}^{I} \beta_i(a_i|t_i, z_i)\right).$$

Definition 9 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a BNE action-type-state distribution of $\left(G, S, \widetilde{S}\right)$ if there exists a BNE β of $\left(G, S, \widetilde{S}\right)$ such that $\nu = \nu_\beta$.*

Theorem 1 *A probability distribution* $\nu \in \Delta(A \times T \times \Theta)$ *is a Bayes correlated equilibrium of* (G,S) *if and only if it is a BNE action-type-state distribution of* $\left(G,S,\widetilde{S}\right)$ *for some augmented information structure* $\widetilde{S}$.

Proof. Suppose that ν is a correlated equilibrium of (G,S). Thus

$$\begin{aligned} & \sum_{a_{-i},t_{-i},\theta} u_i\left(\left(a_i,a_{-i}\right),\theta\right)\nu\left(\left(a_i,a_{-i}\right),\left(t_i,t_{-i}\right),\theta\right) \\ \geq & \sum_{a_{-i},t_{-i},\theta} u_i\left(\left(a_i',a_{-i}\right),\theta\right)\nu\left(\left(a_i,a_{-i}\right),\left(t_i,t_{-i}\right),\theta\right); \end{aligned}$$

for each i, $t_i \in T_i$, $a_i \in A_i$ and $a_i' \in A_i$; and

$$\sum_{a\in A} \nu\left(a,t,\theta\right) = \psi\left(\theta\right)\pi\left(t|\theta\right)$$

for all $t \in T$ and $\theta \in \Theta$. Construct an augmented information structure $\widetilde{S} = \left(\left(Z_i\right)_{i=1}^{I},\phi\right)$ with each $Z_i = A_i$ and

$$\phi\left(a|t,\theta\right) = \nu\left(a|\theta,t\right).$$

Now in the augmented incomplete information game $\left(G,S,\widetilde{S}\right)$, consider the "truthful" strategy profile β with $\beta_i\left(a_i|t_i,a_i\right) = 1$ for all i, t_i and a_i. Clearly, we have $\nu_\beta = \nu$. Now

$$\begin{aligned} & \sum_{a_{-i},t_{-i},z_{-i},\theta} u_i\left(\left(a_i',a_{-i}\right),\theta\right)\psi\left(\theta\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j,z_j\right)\right)\phi\left(\left(z_i,z_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right) \\ = & \sum_{a_{-i},t_{-i},z_{-i},\theta} u_i\left(\left(a_i',a_{-i}\right),\theta\right)\nu\left(\left(a_i,a_{-i}\right),\left(t_i,t_{-i}\right),\theta\right), \end{aligned}$$

and thus Nash equilibrium conditions are implied by the correlated equilibrium conditions on ν.

Conversely, suppose that β is a Bayes Nash equilibrium of $\left(G,S,\widetilde{S}\right)$. Now $\beta_i\left(a_i|\left(t_i,z_i\right)\right) > 0$ implies

$$\begin{aligned} & \sum_{a_{-i},t_{-i},z_{-i},\theta} u_i\left(\left(a_i,a_{-i}\right),\theta\right)\psi\left(\theta\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j,z_j\right)\right)\phi\left(\left(z_i,z_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right) \\ \geq & \sum_{a_{-i},t_{-i},z_{-i},\theta} u_i\left(\left(a_i',a_{-i}\right),\theta\right)\psi\left(\theta\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j,z_j\right)\right)\phi\left(\left(z_i,z_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right). \end{aligned}$$

for each $a_i' \in A_i$. Thus

$$\begin{aligned} & \sum_{z_i\in Z_i}\beta_i\left(a_i|\left(t_i,z_i\right)\right)\sum_{a_{-i},t_{-i},z_{-i},\theta} u_i\left(\left(a_i,a_{-i}\right),\theta\right)\psi\left(\theta\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j,z_j\right)\right)\phi\left(\left(z_i,z_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right) \\ \geq & \sum_{z_i\in Z_i}\beta_i\left(a_i|\left(t_i,z_i\right)\right)\sum_{a_{-i},t_{-i},z_{-i},\theta} u_i\left(\left(a_i',a_{-i}\right),\theta\right)\psi\left(\theta\right)\pi\left(\left(t_i,t_{-i}\right)|\theta\right)\left(\prod_{j\neq i}\beta_j\left(a_j|t_j,z_j\right)\right)\phi\left(\left(z_i,z_{-i}\right)|\left(t_i,t_{-i}\right),\theta\right). \end{aligned}$$

But

$$\begin{aligned} & \sum_{z_i \in Z_i} \beta_i \left(a_i | \left(t_i, z_i \right) \right) \sum_{a_{-i}, t_{-i}, z_{-i}, \theta} u_i \left(\left(a_i', a_{-i} \right), \theta \right) \psi \left(\theta \right) \pi \left(\left(t_i, t_{-i} \right) | \theta \right) \left(\prod_{j \neq i} \beta_j \left(a_j | t_j, z_j \right) \right) \phi \left(\left(z_i, z_{-i} \right) | \left(t_i, t_{-i} \right), \theta \right) \\ = & \sum_{a_{-i}, t_{-i}, \theta} u_i \left(\left(a_i', a_{-i} \right), \theta \right) \nu_\beta \left(\left(a_i, a_{-i} \right), \left(t_i, t_{-i} \right), \theta \right) \end{aligned}$$

and thus BNE conditions imply that ν_β is a BCE. ■

An alternative formulation of this result would be to say that BCE captures the implications of common certainty of rationality (and the common prior assumption) in the game (G, S), since requiring BNE in some game with augmented information is equivalent to describing a belief closed subset where the game (G, S) is being played and there is common certainty of rationality. Thus this is an incomplete information analogue of the Aumann (1987) characterization of correlated equilibrium for complete information games and thus - as described in more detail in the next section - corresponds to the "partial Bayesian approach" of Forges (1993), with the difference that she works with the reduced game - integrating out the payoff states Θ.

This result characterizes the behavior consistent with common knowledge of rationality in an incomplete information setting. Aumann and Dreze (2008) have recently given a characterization of the possible "values" of a complete information game, i.e., the set of interim expected utilities which would be consistent with common knowledge of rationality and payoffs of the game. They show that this set is equal to the set of interim expected utilities that might arise in a correlated equilibrium of the "doubled" game where each action of each player had an identical copy. We conjecture that an analogous argument will provide an analogous result in the present incomplete information setting.

For completeness, we report the corollary that arises from applying Theorem 1 in the special case where information structure S is null. We then have:

Corollary 1 *A probability distribution $\mu \in \Delta(A \times \Theta)$ is a Bayes correlated equilibrium of basic game G if and only if it is a BNE action-state distribution of (G, S) for some information structure S.*

4 Bayes Correlated Equilibrium and the Bayesian Solution

Forges (1993) is titled and identifies "five legitimate definitions of correlated equilibrium in games with incomplete information;" Forges (2006) describes a mistake in Forges (1993) that leads to a sixth definition, the Bayesian solution. Bayes correlated equilibrium is a weakening of the weakest of these solutions for an incomplete information game (G, S), the Bayesian solution. The weakening arises because we allow outcomes to depend on states that no player knows, while Forges' solutions do not. In this Section, we first (in Section 4.1) describe the Bayesian solution and explain how Bayes correlated equilibria can be seen as Bayesian solutions if we add a dummy player who knows what the true state is but does not take any actions. Then (in Section 4.2), for completeness, we report on the relation of Bayes correlated equilibrium and the Bayesian solution to four stronger solution concepts surveyed in Forges (1993), (2006). Finally (in Section 4.3), we discuss the "universal Bayesian approach" in Forges (1993), which roughly corresponds - in our language - to Bayes correlated equilibria of the basic game (with the null information structure).

Before we start, let us highlight a few differences between our formulation of games and solution concepts from that of Forges (1993), which may be helpful in understanding the relation. An important difference is that we include the distribution of payoff states Θ, which are not necessarily known to players, in our solution concept, while she integrates out payoff states. Three less important differences in the formulation that it is helpful to bear in mind are:

1. While we directly define solution concepts for (G, S) as subsets of action-type-state distributions $\Delta(A \times T \times \Theta)$, she characterizes the set of equilibrium payoffs satisfying a set of restrictions which implicitly define the solution concept in our sense.

2. While we work with a "basic game", $G = \left((A_i, u_i)_{i=1}^I, \psi\right)$, describing prior and payoffs and an "information structure" $S = \left((T_i)_{i=1}^I, \pi\right)$, she distinguishes between the "decision problem with incomplete information," $(A_i, u_i)_{i=1}^I$ and includes the prior on payoff states in her description of the "information scheme".

3. While we and Forges (2006) allow for any finite number of players, Forges (1993) focussed on the two player case for simplicity.

4.1 Bayesian Solution

Recall that the only feasibility condition we imposed in defining Bayes correlated equilibrium was the consistency requirement (Definition 2) that the action-type-state distribution implied the distribution on exogenous variables (types and states) was that of the game (G, S). If the solution concept is intended to capture the outcome of communication among the players, perhaps by allowing for an uninformed mediator, it is natural to impose the additional restriction that the distribution over states conditional on agents' types is not changed conditional on the mediator's recommendations:

Definition 10 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is join feasible for (G,S) if, for all $a \in A$ and $t \in T$ such that*

$$\sum_{\theta \in \Theta} \nu(a,t,\theta) > 0,$$

we have

$$\frac{\nu(a,t,\theta)}{\sum_{\theta' \in \Theta} \nu(a,t,\theta')} = \frac{\psi(\theta)\pi(t|\theta)}{\sum_{\theta' \in \Theta} \psi(\theta')\pi(t|\theta')} \tag{7}$$

for all $\theta \in \Theta$.

This restriction is equivalent to requiring that there exists a mediator strategy $f : T \to \Delta(A)$ such that

$$\nu(a,t,\theta) = \psi(\theta)\pi(t|\theta) f(a|t)$$

for all $a \in A, t \in T, \theta \in \Theta$. This assumption is (implicitly) maintained in all Forges' solution concepts for (G,S) and is made explicit in Lehrer, Rosenberg, and Shmaya (2011) and Lehrer, Rosenberg, and Shmaya (2010) (e.g., condition 4 on page 676 in Lehrer, Rosenberg, and Shmaya (2010)).

Definition 11 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a Bayesian solution of (G,S) if it is consistent, join feasible and obedient.*

This is the solution concept discussed in Section 4.4 of Forges (1993) and one of the two discussed in section 2.5 of Forges (2006). Lehrer, Rosenberg, and Shmaya (2011) refer to this as a "global equilibrium." It also corresponds to the set of jointly coherent outcomes in Nau (1992), justified from no arbitrage conditions. Forges and Koessler (2005) provide a justification if players are able to certify their types to the mediator.

The following is a trivial (one player) example showing that Bayes correlated equilibrium is a more permissive solution concept than the Bayesian solution for (G,S). Suppose there is one player, $I = 1$, and two states, $\Theta = \{\theta, \theta'\}$. Let the basic game $G = (A_1, u_1, \psi)$ be defined by $A_1 = \{a_1, a_1'\}$, $u_1(a_1, \theta) = 2$, $u_1(a_1, \theta') = -1$ and $u_1(a_1', \theta) = u_1(a_1', \theta') = 0$, and $\psi(\theta) = \psi(\theta') = \frac{1}{2}$. And consider the null information structure S_0. Consistency (5), obedience (6) and join feasibility (7) together imply that

$$\mu(a_1,\theta) = \mu(a_1,\theta') = \frac{1}{2} \text{ and } \mu(a_1',\theta) = \mu(a_1',\theta') = 0.$$

This is thus the unique Bayesian solution. However, consistency (5) implies only that

$$\begin{aligned} \mu(a_1,\theta) + \mu(a_1',\theta) &= \frac{1}{2}, \\ \mu(a_1,\theta') + \mu(a_1',\theta') &= \frac{1}{2}, \end{aligned}$$

and obedience (6) implies only that

$$\begin{aligned} 2\mu\left(a_1,\theta\right)-\mu\left(a_1,\theta'\right) &\geq 0, \\ 2\mu\left(a_1',\theta\right)-\mu\left(a_1',\theta'\right) &\leq 0. \end{aligned}$$

There are many Bayesian correlated equilibria satisfying the above constraints. The one maximizing the player's utility has

$$\mu\left(a_1,\theta\right)=\mu\left(a_1',\theta'\right)=\frac{1}{2} \quad \text{and} \quad \mu\left(a_1,\theta'\right)=\mu\left(a_1',\theta\right)=0.$$

In Section 4.5, Forges (1993) discusses how more solution concepts are conceivable, including by dropping join feasibility, and gives an example like the above illustrating this point.

We can link Bayes correlated equilibria to the Bayesian solution in two ways. First, say that an information structure has no distributed uncertainty if combining the agents' information would allow them to deduce the state θ. Thus:

Definition 12 *Information structure S satisfies no distributed uncertainty if there exists $g:T\rightarrow\Theta$ such that $\pi\left(t|\theta\right)>0\Rightarrow\theta=g\left(t\right)$*

An important class of environments where this condition will always be satisfied is private value environments. This would be modelled in our language by setting $\Theta=\Theta_1\times\cdots\times\Theta_I$, each $T_i=\Theta_i$ and let

$$\pi\left(t|\theta\right)=\begin{cases} 1, & \text{if} \quad t=\theta; \\ 0, & \text{if} \quad t\neq\theta. \end{cases}$$

As an example, in Bergemann, Brooks, and Morris (2011) we study first price auctions where bidders know their own values of a signal object. This is a private value environment and thus has no distributed uncertainty. By the next observation, the Bayesian solutions and Bayes correlated equilibria coincide in this setting.

Now we have:

Lemma 3 *If S satisfies no distributed uncertainty then any consistent $\nu\in\Delta\left(A\times T\times\Theta\right)$ is join feasible and thus, for any basic game G, any Bayes correlated equilibrium of $\left(G,S\right)$ is a Bayesian solution of $\left(G,S\right)$.*

Secondly, given any incomplete information game (G,S), we can add an extra "dummy player" 0 who has only a trivial action choice but who observes the state θ.[1] The Bayesian solutions of the game with the dummy player added will correspond to the Bayes correlated equilibria. Formally, fix a basic game

[1] We are grateful to Atsushi Kajii for suggesting that we pursue this dummy player interpretation of Bayes correlated equilibrium.

$G = \left((A_i, u_i)_{i=1}^I, \psi\right)$ and information structure $S = \left((T_i)_{i=1}^I, \pi\right)$. Consider the modified basic game $\widetilde{G} = \left(\left(\widetilde{A}_i, \widetilde{u}_i\right)_{i=0}^I, \psi\right)$ with $\widetilde{A}_0 = \{a_0\}$, $\widetilde{A}_i = A_i$ for $i = 1, ..., I$ and $\widetilde{u}_i\left(\left(a_0, (a_j)_{j=1}^I\right), \theta\right) = u_i\left((a_j)_{j=1}^I, \theta\right)$ for $i = 1, ..., I$ (the form of $\widetilde{u}_0$ does not matter since the dummy player 0 does not have any action choice); and modified information structure $\widetilde{S} = \left(\left(\widetilde{T}_i\right)_{i=0}^I, \widetilde{\pi}\right)$, with $\widetilde{T}_0 = \Theta$, $\widetilde{T}_i = T_i$ for $i = 1, ..., I$ and

$$\widetilde{\pi}\left(\left(t_0, (t_i)_{i=1}^I\right)|\theta\right) = \begin{cases} \pi\left((t_i)_{i=1}^I|\theta\right), & \text{if} \quad t_0 = \theta; \\ 0, & \text{if} \quad t_0 \neq \theta. \end{cases}$$

Say that distribution $\widetilde{\nu} \in \Delta\left(\widetilde{A} \times \widetilde{T} \times \Theta\right)$ corresponds to distribution $\nu \in \Delta(A \times T \times \Theta)$ if

$$\widetilde{\nu}\left(\left(a_0, (a_i)_{i=1}^I\right), \left(t_0, (t_i)_{i=1}^I\right), \theta\right) = \begin{cases} \nu\left((a_i)_{i=1}^I, (t_i)_{i=1}^I, \theta\right), & \text{if} \quad t_0 = \theta; \\ 0, & \text{if} \quad t_0 \neq \theta. \end{cases}$$

Now we have:

Lemma 4 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is a Bayes correlated equilibrium of (G, S) if and only if the corresponding $\widetilde{\nu} \in \Delta\left(\widetilde{A} \times \widetilde{T} \times \Theta\right)$ is a Bayesian solution of the game $\left(\widetilde{G}, \widetilde{S}\right)$ with added dummy player.*

4.2 Four More Solution Concepts

We now describe four more definitions of correlated equilibrium for an incomplete information game (G, S) from Forges (1993), (2006) which strengthen the Bayesian solution. It is useful to divide restrictions into two classes: *feasibility* constraints on the distribution of action-type-state profiles, which are required to hold independent of the payoff functions, and *incentive* constraints which are rationality constraints on players' action choices. The closest solutions to our notion of Bayes correlated equilibrium rely only on additional feasibility constraints, maintaining obedience as the only incentive compatibility constraint.

The Bayesian solution concept allowed players to learn about other players' types from the mediator's recommendation. The following condition removes this possibility:

Definition 13 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is belief invariant for (G, S) if, for all $t_i \in T_i$ and $a_i \in A_i$ such that*

$$\sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) > 0,$$

we have

$$\frac{\sum_{a_{-i} \in A_{-i}, \theta \in \Theta} \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta)}{\sum_{a_{-i} \in A_{-i}, t'_{-i} \in T_{-i}, \theta \in \Theta} \nu\left((a_i, a_{-i}), \left(t_i, t'_{-i}\right), \theta\right)} = \frac{\sum_{\theta \in \Theta} \psi(\theta)\, \pi((t_i, t_{-i})|\theta)}{\sum_{t'_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\, \pi\left(\left(t_i, t'_{-i}\right)|\theta\right)} \tag{8}$$

for each $t_{-i} \in T_{-i}$.

This is condition 3 on page 676 in Lehrer, Rosenberg, and Shmaya (2010). As Forges (2006) puts it, "the omniscient mediator can use his knowledge of the types to make his recommendations but the players should not be able to infer anything on the others' types from their recommendations." This restriction is added to give the first strengthening of the Bayesian solution:

Definition 14 *A probability distribution* $\nu \in \Delta(A \times T \times \Theta)$ *is a belief invariant Bayesian solution of* (G, S) *if it is consistent, join feasible, belief invariant and obedient.*

This is the second solution concept discussed in Section 2.5 of Forges (2006); it was discussed informally in Section 4.4 of Forges (1993) but it was then mistakenly claimed that it was equivalent to agent normal form correlated equilibrium. This solution concept is also used in Lehrer, Rosenberg, and Shmaya (2010), (2011). Because they do not work with the reduced game, i.e., they explicitly discuss payoff states like Θ, it follows that they must explicitly impose a join feasibility restriction.

The belief invariant Bayesian solution allows the mediator to use information about players' types to make a recommendation to players. Suppose that the mediator has no information about the players' types when deciding what strategy to recommend as a function of the players' types. This is reflected in the next feasibility restriction. A pure strategy in the incomplete information game is function $b_i : T_i \to A_i$. Write B_i for the set of pure strategies of agent i and B for the set of pure strategy profiles, $B = B_1 \times \cdots \times B_I$.

Definition 15 *Distribution* $\nu \in \Delta(A \times T \times \Theta)$ *is agent normal form feasible for* (G, S) *if there exists* $q \in \Delta(B)$ *such that*

$$\nu(a, t, \theta) = \psi(\theta)\,\pi(t|\theta) \sum_{\{b \in B | b(t) = a\}} q(b) \tag{9}$$

for each $a \in A$, $t \in T$ *and* $\theta \in \Theta$.

One can show that agent normal form feasibility implies belief invariance. This restriction is added to give the second stronger solution concept:

Definition 16 *A probability distribution* $\nu \in \Delta(A \times T \times \Theta)$ *is an agent normal form correlated equilibrium of* (G, S) *if it is consistent, join feasible, agent normal form feasible (and thus belief invariant) and obedient.*

This is the solution concept discussed in Section 4.2 of Forges (1993) and Section 2.3 of Forges (2006). It corresponds to applying the complete information definition of correlated equilibrium to the agent normal form of the reduced incomplete information game. It was also studied by Samuelson and Zhang (1989) and Cotter (1994). The solution concept only makes sense on the understanding that the players receive a recommendation for each type but do not learn what recommendation they would have received if they had been different types. If they did learn the whole strategy that the mediator choose for them in the strategic form game, then an extra incentive compatibility condition would be required:

Definition 17 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is strategic form incentive compatible for (G, S) if there exists such that*

$$\nu(a, t, \theta) = \psi(\theta)\,\pi(t|\theta) \sum_{\{b \in B | b(t) = a\}} q(b) \tag{10}$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$; and, for each $i = 1, ..., I$, $t_i \in T_i$, $a_i \in A_i$ and $b_i \in B_i$ such that $b_i(t_i) = a_i$, we have

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\,\pi(t|\theta) \left(\sum_{\{b_{-i} \in B_{-i} | b_{-i}(t_{-i}) = a_{-i}\}} q(b_i, b_{-i}) \right) u_i((a_i, a_{-i}), \theta) \\ \geq & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\,\pi(t|\theta) \left(\sum_{\{b_{-i} \in B_{-i} | b_{-i}(t_{-i}) = a_{-i}\}} q(b_i, b_{-i}) \right) u_i((a_i', a_{-i}), \theta) \end{aligned} \tag{11}$$

for all $a_i' \in A_i$.

Note that this condition implies both agent normal form feasibility and obedience. This restriction gives the third stronger solution concept:

Definition 18 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a strategic form correlated equilibrium of (G, S) if it is consistent, join feasible and strategic form incentive compatible (and thus agent normal form feasible, belief invariant and obedient).*

This is the solution concept discussed in Section 4.1 of Forges (1993) and Section 2.2 of Forges (2006). This solution concept was studied by Cotter (1991).

Thus far we have simply been adding restrictions, so that the solution concept have become stronger as we go from Bayesian solution, to belief invariant Bayesian solution, to agent normal form correlated equilibrium, to strategic form correlated equilibrium. For the Bayesian solution, an omniscient mediator who observes players' types for free is assumed. For agent normal form and strategic form correlated equilibrium, the players' types cannot play a role in the selection of recommendations to the players. An intermediate assumption is that the players can report their types to the mediator, but will do so truthfully only if it is incentive compatible to do so. Write $\xi_\nu : T \times \Theta \to A$ for the mediator's recommendation strategy implied by $\nu \in \Delta(A \times T \times \Theta)$, so that, for each $t \in T$ and $\theta \in \Theta$ with $\sum_{a' \in A} \nu(a', t, \theta) > 0$,

$$\xi_\nu(a|t, \theta) = \frac{\nu(a, t, \theta)}{\sum_{a' \in A} \nu(a', t, \theta)}, \quad \text{for each } a \in A.$$

Definition 19 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is truth telling for (G,S) if, for each $i = 1,...,I$ and $t_i \in T_i$, we have*

$$\begin{aligned} & \sum_{a \in A, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\, \pi((t_i, t_{-i})|\theta)\, \xi_\nu((a_i, a_{-i})\,|\,(t_i, t_{-i}), \theta)\, u_i((a_i, a_{-i}), \theta) \\ \geq & \sum_{a \in A, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta)\, \pi((t_i, t_{-i})|\theta)\, \xi_\nu\left((a_i, a_{-i})\,|\,\left(t_i', t_{-i}\right), \theta\right) u_i((\delta_i(a_i), a_{-i}), \theta)\,; \end{aligned} \quad (12)$$

for all $t_i' \in T_i$ and $\delta_i : A_i \to A_i$.

Note that this condition implies obedience (Definition 3). One can show that this condition is implied by strategic form incentive compatibility. Now we have the fifth solution concept:

Definition 20 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a communication equilibrium of (G,S) if it is consistent, join feasible and truth-telling (and thus obedient).*

This is the solution concept discussed in Section 4.3 of Forges (1993) and Section 2.4 of Forges (2006), and developed earlier in the work of Myerson (1982) and Forges (1986).

Thus we have Forges' five solution concepts for the incomplete information game (G,S):

1. Bayesian solution (Definition 11);

2. Belief invariant Bayesian solution (Definition 14);

3. Agent normal form correlated equilibrium (Definition 16);

4. Strategic form correlated equilibrium (Definition 18); and

5. Communication equilibrium (Definition 20).

As documented by Forges (1993), (2006) and implied by the above definitions, we have that the Bayesian solution [1] is weaker than the belief invariant Bayesian equilibrium solution [2], which is weaker than the agent normal form correlated equilibrium [3], which is weaker than the strategic form correlated equilibrium [4]; and also the Bayesian solution [1] is weaker than communication equilibrium [5] which is weaker than strategic form correlated equilibrium [4]. Examples reported in Forges (1993), (2006) show that each weak inclusion is strict and that the belief invariant Bayesian solution [2] and agent normal form correlated equilibrium [3] cannot be ranked relative to communication equilibrium [5]. Our definition of Bayes correlated equilibrium is weaker than the Bayesian solution, the weakest of Forges' five, because we do not maintain join feasibility.

4.3 The Universal Bayesian Approach

In Section 6, Forges (1993) considers a "universal Bayesian approach" in which a prior "information scheme" (in our language, prior on Θ and information structure) is not taken as given. Thus her "universal Bayesian solution" is defined for $(A_i, u_i)_{i=1}^{I}$. Expressing her ideas in the language of action-state distributions, she studies the following solution concept.

Definition 21 *A probability distribution $\mu \in \Delta(A \times \Theta)$ is a universal Bayesian solution of $(A_i, u_i)_{i=1}^{I}$ if it satisfies (6).*

Thus a probability distribution $\mu \in \Delta(A \times \Theta)$ is Bayes correlated equilibrium of $G = \left((A_i, u_i)_{i=1}^{I}, \psi\right)$ if and only if it is a universal Bayesian solution and satisfies (5). Recall that Corollary 1 showed that $\mu \in \Delta(A \times \Theta)$ is a Bayes correlated equilibrium of the basic game G if and only if there exists an information structure S and a Bayes Nash equilibrium action-type-state distribution $\nu \in \Delta(A \times T \times \Theta)$ of (G, S) which induces $\mu \in \Delta(A \times \Theta)$. This then corresponds to Forges' Proposition 4 when applied to the solution concept of Nash equilibrium (although she states the results in terms of equilibrium payoffs rather than distributions). As she notes, her Proposition 4 is a natural incomplete information generalization of Aumann (1987) and our Theorem 1 and Corollary 1 are also incomplete information generalizations of Aumann (1987) stated in different terms.

5 Comparing Information Structures

An important result for our robust predictions agenda is that as players become more informed, the set of possible predictions must be reduced, since obedience constraints will become tighter. Put like this, it sounds like a tautology. The subtle part of presenting a formal version of this claim is identifying the right notion of "more informed than" under which it is true. In this Section, we first introduce our notion of more informed than (Section 5.1) and show that it is necessary and sufficient for reducing the set of predictions (Section 5.2). Our notion of "more informed than" is a variant of the notion of "non-communicating garbling" of Lehrer, Rosenberg, and Shmaya (2010), (2011). We study the relation in detail (in Section 5.3) and describe existing results of Gossner (2000), Lehrer, Rosenberg, and Shmaya (2010), (2011) (in Section 5.4).

5.1 "More Informed Than"

Our formal definition of "more informed" works as follows. We require that there exists a joint distribution over states and signals in both information structures such that (i) the marginals on each information structure are correct; and (ii) from the point of view of each individual player, information in the less informed information structure is a garbling of information in the more informed one. Thus we have the following formal definition:

Definition 22 *Information structure S is more informed than S' if there exist $\sigma : T \times \Theta \to \Delta(T')$ and, for each i, $\xi_i : T_i \to \Delta(T_i')$, such that*

$$\pi'(t'|\theta) = \sum_{t \in T} \sigma(t'|t,\theta)\, \pi(t|\theta) \tag{13}$$

for each $t' \in T'$ and $\theta \in \Theta$, satisfying also that for each $i = 1, ..., I$, $t_i \in T_i$, $t_i' \in T_i'$,

$$\sum_{t_{-i}' \in T_{-i}'} \sigma\left(\left(t_i', t_{-i}'\right) | \left(t_i, t_{-i}\right), \theta\right) = \xi_i\left(t_i'|t_i\right) \tag{14}$$

for all $t_{-i} \in T_{-i}$ and $\theta \in \Theta$.

If S is more informed than S', we refer to σ as the mapping transforming S into S'. Information structure S is informationally equivalent to S' if S is more informed than S' and S' is more informed than S.

It is easy to check that if θ is a singleton, then any information structure is informationally equivalent to any other.

Informational equivalence has an characterization in terms of higher order belief equivalence.

Definition 23 *Two information structures $S^1 = \left(\left(T_i^1\right)_{i=1}^I, \pi^1\right)$ and $S^2 = \left(\left(T_i^2\right)_{i=1}^I, \pi^2\right)$ are higher order belief equivalent if there exists a third type space $S^* = \left(\left(T_i^*\right)_{i=1}^I, \pi^*\right)$ and, for each i, $f_i^1 : T_i^1 \to T_i^*$ and $f_i^2 : T_i^2 \to T_i^*$ such that (1) for all k, t^* and θ,*

$$\pi^k\left(\left\{t^k | f^k\left(t^k\right) = t^*\right\} | \theta\right) = \pi^*\left(t^*|\theta\right); \tag{15}$$

(2) for all k, i, $t_i \in T_i^k$, θ, θ'

$$\frac{\pi^k\left(t_i|\theta\right)}{\pi^k\left(t_i|\theta'\right)} = \frac{\pi^*\left(f_i^k\left(t_i\right)|\theta\right)}{\pi^*\left(f_i^k\left(t_i\right)|\theta'\right)}. \tag{16}$$

It is easy (but notationally burdensome) to show that two information structures are higher order belief equivalent in this sense if and only if, for any prior over states, they generate the same probability distribution over beliefs and higher order beliefs (i.e., Mertens-Zamir types). We present a formal statement of this equivalence in the appendix where we also give the proof of the following lemma:

Lemma 5 *Two information structures are informationally equivalent if and only if they are higher order belief equivalent.*

We can also interpret our definition of the "more informed than" relation directly by observing that the "augmentation" of information we discussed in Section 3 is a canonical way of making an information structure more informed in the sense of the definition. To see this, recall that we considered an information structure $S = \left((T_i)_{i=1}^I, \pi\right)$ and an "augmentation" of that information structure, $\widetilde{S} = \left((Z_i)_{i=1}^I, \phi\right)$, where $\phi : \Theta \times T \to Z$. Taken together, S and $\widetilde{S}$ describe a new information structure $\widetilde{S} \circ S = \left((Z_i \times T_i)_{i=1}^I, \phi \circ \pi\right)$, where $\phi \circ \pi : \Theta \to \Delta\left((Z_i \times T_i)_{i=1}^I\right)$ is defined by

$$\phi \circ \pi \left((z_i, t_i)_{i=1}^I \middle| \theta \right) = \pi \left((t_i)_{i=1}^I \middle| \theta \right) \phi \left((z_i)_{i=1}^I \middle| (t_i)_{i=1}^I, \theta \right).$$

Now we have:

Lemma 6 *Information structure $\widetilde{S} \circ S$ is more informed that S, for any information structure S and augmentation $\widetilde{S}$.*

Proof. Fix information structure $\left((T_i)_{i=1}^I, \pi\right)$ and augmentation $\widetilde{S} = \left((Z_i)_{i=1}^I, \phi\right)$.
Define $\sigma : (Z_i \times T_i)_{i=1}^I \times \Theta \to (T_i)_{i=1}^I$ and $\xi_i : Z_i \times T_i \to \Delta(T_i)$ by:

$$\sigma \left((t_i)_{i=1}^I \middle| \left(z_i, \widetilde{t}_i\right)_{i=1}^I, \theta \right) = \begin{cases} 1, & \text{if } \left(\widetilde{t}_i\right)_{i=1}^I = (t_i)_{i=1}^I; \\ 0, & \text{if otherwise.} \end{cases}$$

and

$$\xi_i \left(t_i \middle| \left(z_i, \widetilde{t}_i\right) \right) = \begin{cases} 1, & \text{if } \widetilde{t}_i = t_i; \\ 0, & \text{if otherwise.} \end{cases}$$

Now observe that

$$\begin{aligned} & \sum_{\widetilde{t} \in T, z \in Z} \sigma \left((t_i)_{i=1}^I \middle| \left(z_i, \widetilde{t}_i\right)_{i=1}^I, \theta \right) \phi \circ \pi \left(\left(z_i, \widetilde{t}_i\right)_{i=1}^I \middle| \theta \right) \\ = & \sum_{z \in Z} \phi \circ \pi \left((z_i, t_i)_{i=1}^I \middle| \theta \right) \\ = & \pi \left((t_i)_{i=1}^I \middle| \theta \right), \end{aligned}$$

and

$$\begin{aligned} \sum_{t_{-i} \in T_{-i}} \sigma \left((t_i, t_{-i}) \middle| (z_j)_{j=1}^I, \left(\widetilde{t}_i, \widetilde{t}_{-i}\right), \theta \right) &= \sigma \left(\left(t_i, \widetilde{t}_{-i}\right) \middle| (z_j)_{j=1}^I, \left(\widetilde{t}_i, \widetilde{t}_{-i}\right), \theta \right) = \begin{cases} 1, & \text{if } \widetilde{t}_i = t_i; \\ 0, & \text{if otherwise;} \end{cases} \\ &= \xi_i \left(t_i \middle| \left(z_i, \widetilde{t}_i\right) \right). \end{aligned}$$

■

We omit the straightforward proof of transitivity of the "more informed than" relation:

Lemma 7 *If information structure S is more informed than S' and S' is more informed than S'' then S is more informed than S''.*

Now we can use the previous two results to give a tighter characterization of the relation between augmented information structures and the "more informed than" relation:

Lemma 8 *Information structure S is more informed than S' if and only if there exists an augmentation $\widetilde{S}$ such that S is informationally equivalent to $\widetilde{S} \circ S'$.*

Proof. Suppose that S is more informed than S'. Consider the augmentation $\widetilde{S} = \left((Z_i)_{i=1}^I , \phi \right)$ where each $Z_i = T_i$ and $\phi : T' \times \Theta \to T$ is defined by

$$\phi\left(t|t',\theta\right) = \frac{\pi\left(t|\theta\right)\sigma\left(t'|t,\theta\right)}{\sum\limits_{\widetilde{t}\in T}\pi\left(\widetilde{t}|\theta\right)\sigma\left(t'|\widetilde{t},\theta\right)}.$$

To see that S is more informed than $\widetilde{S} \circ S'$, let $\widehat{\sigma} : T \times \Theta \to \Delta\left((T_i \times T_i')_{i=1}^I \right)$ and $\widehat{\xi}_i : T_i \to \Delta\left(T_i \times T_i'\right)$ be defined by

$$\widehat{\sigma}\left(\left(\widetilde{t}_i, t_i'\right)_{i=1}^I \middle| (t_i)_{i=1}^I , \theta \right) = \begin{cases} \sigma\left((t_i')_{i=1}^I \middle| (t_i)_{i=1}^I , \theta \right), & \text{if} \quad \left(\widetilde{t}_i\right)_{i=1}^I = (t_i)_{i=1}^I ; \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

and

$$\widehat{\xi}_i\left(\left(\widetilde{t}_i, t_i'\right) \middle| t_i \right) = \begin{cases} \xi_i\left(t_i'|t_i\right), & \text{if} \quad \widetilde{t}_i = t_i; \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Now observe that

$$\begin{aligned} \phi \circ \pi'\left(\left(t_i, t_i'\right)_{i=1}^I \middle| \theta \right) &= \pi'\left(\left(t_i'\right)_{i=1}^I \middle| \theta \right) \phi\left((t_i)_{i=1}^I \middle| \left(t_i'\right)_{i=1}^I , \theta \right) \\ &= \frac{\pi'\left((t_i')_{i=1}^I \middle| \theta \right) \pi\left((t_i)_{i=1}^I \middle| \theta \right) \sigma\left((t_i')_{i=1}^I \middle| (t_i)_{i=1}^I , \theta \right)}{\sum\limits_{\widetilde{t}\in T} \pi\left(\left(\widetilde{t}_i\right)_{i=1}^I \middle| \theta \right) \sigma\left((t_i')_{i=1}^I \middle| \left(\widetilde{t}_i\right)_{i=1}^I , \theta \right)} \\ &= \pi\left((t_i)_{i=1}^I \middle| \theta \right) \sigma\left((t_i')_{i=1}^I \middle| (t_i)_{i=1}^I , \theta \right), \end{aligned}$$

and

$$\begin{aligned} \sum_{\widetilde{t}_{-i}\in T_{-i}, t'_{-i}\in T'_{-i}} \widehat{\sigma}\left(\left(\widetilde{t}_j, t_j'\right)_{j=1}^I \middle| (t_j)_{j=1}^I , \theta \right) &= \begin{cases} \sum\limits_{t'_{-i}\in T'_{-i}} \sigma\left(\left(t_j'\right)_{j=1}^I \middle| (t_j)_{j=1}^I , \theta \right), & \text{if} \quad \widetilde{t}_i = t_i; \\ 0, & \text{if} \quad \text{otherwise.} \end{cases} \\ &= \begin{cases} \xi_i\left(t_i'|t_i\right), & \text{if} \quad \widetilde{t}_i = t_i; \\ 0, & \text{if} \quad \text{otherwise.} \end{cases} \\ &= \widehat{\xi}_i\left(\left(\widetilde{t}_i, t_i'\right) \middle| t_i \right). \end{aligned}$$

To see that $\widetilde{S} \circ S'$ is more informed than S, let $\overline{\sigma} : (T_i \times T'_i)_{i=1}^I \times \Theta \to \Delta(T)$ and $\overline{\xi}_i : T_i \times T'_i \to \Delta(T_i)$ be defined by

$$\overline{\sigma}\left((t_i)_{i=1}^I \middle| \left(\widetilde{t}_i, t'_i\right)_{i=1}^I, \theta \right) = \begin{cases} 1, & \text{if} \quad \left(\widetilde{t}_i\right)_{i=1}^I = (t_i)_{i=1}^I ; \\ 0, & \text{if} \quad \text{otherwise;} \end{cases}$$

and

$$\overline{\xi}_i\left(t_i | \left(\widetilde{t}_i, t'_i\right)\right) = \begin{cases} 1, & \text{if} \quad \widetilde{t}_i = t_i; \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Now observe that

$$\begin{aligned} & \sum_{\widetilde{t} \in T, t' \in T'} \overline{\sigma}\left((t_i)_{i=1}^I \middle| \left(\widetilde{t}_i, t'_i\right)_{i=1}^I, \theta \right) \phi \circ \pi' \left(\left(\widetilde{t}_i, t'_i\right)_{i=1}^I \middle| \theta \right) \\ = & \sum_{t' \in T'} \phi \circ \pi' \left(\left(t_i, t'_i\right)_{i=1}^I \middle| \theta \right) \\ = & \ \pi \left((t_i)_{i=1}^I \middle| \theta \right) \end{aligned}$$

and

$$\begin{aligned} \sum_{t_{-i} \in T_{-i}} \overline{\sigma}\left((t_i, t_{-i}) | \left(\widetilde{t}_j, t'_j\right)_{j=1}^I, \theta \right) &= \begin{cases} 1, & \text{if} \quad \widetilde{t}_i = t_i; \\ 0, & \text{if} \quad \text{otherwise;} \end{cases} \\ &= \overline{\xi}_i\left(t_i | \left(\widetilde{t}_i, t'_i\right)\right). \end{aligned}$$

Now suppose that S is informationally equivalent to $\widetilde{S} \circ S'$. This in particular means that S is more informed than $\widetilde{S} \circ S'$. By Lemma 6, $\widetilde{S} \circ S'$ is more informed than S'. So by transitivity (Lemma 7), S is more informed than S'. ■

5.2 Comparative Statics of Information

We present the main result of this paper showing that more information reduces the set of Bayes correlated equilibria. Let us write $BCE(G, S)$ for the subset of $\Delta(A \times \Theta)$ consisting of all BCE action-state distributions of (G, S). We say that information structure S is "BCE-larger" than S' if it supports more outcomes in Bayes correlated equilibrium. Thus:

Definition 24 *Information structure S' is BCE-larger than information structure S if $BCE(G, S) \subseteq BCE(G, S')$ for all games G. Information structure S' is BCE-equivalent to information structure S if S' is BCE-larger than S and S is BCE-larger than S'.*

Now we have:

Theorem 2 *S' is BCE-larger than S if and only if S is more informed than S'.*

We will prove below that if S is more informed than S', then S' is BCE-larger than S. The method of proof is to show that if S is more informed than S', we can take any game G and BCE action-state distribution of (G, S) and show by construction that it is also a BCE action-state distribution of (G, S'). We use the fact that S is more informed than S' (and thus give rise to stronger obedience constraints) in constructing the BCE for (G, S').

We prove the converse in the appendix (Section **??**). The method of proof is as follows. We fix an information structure S and consider a class of "higher order beliefs" games $G_{S,\varepsilon}$ indexed by $\varepsilon > 0$ with the property that players can report ε-approximations of their higher order beliefs about Θ. We show that $(G_{S,\varepsilon}, S)$ will have a BCE where all types in S report their types truthfully, giving rise to an action-state distribution μ^*. Now consider an information structure S' which is BCE-larger than S. We must have that for every $\varepsilon > 0$, $(G_{S,\varepsilon}, S')$ has a BCE inducing action-state distribution μ^*. We show that this property implies that S is more informed than S'.

Proof. We will prove here that if S is more informed than S', then S' is BCE-larger than S. In particular, let $\nu \in \Delta(A \times T \times \Theta)$ be any BCE of (G, S). We will construct $\nu' \in \Delta(A \times T' \times \Theta)$ which is a BCE of (G, S') which gives rise to the same action-state distribution as ν.

Write $V_i(a_i, a_i', t_i)$ for the expected utility for agent i under distribution ν if he is type t_i, receives recommendation a_i but chooses action a_i', so that

$$V_i\left(a_i, a_i', t_i\right) \triangleq \sum_{a_{-i}\in A_{-i}, t_{-i}\in T_{-i}, \theta\in\Theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right) \nu\left(\left(a_i, a_{-i}\right), \left(t_i, t_{-i}\right), \theta\right).$$

Now - by Definition 3 - for each $i = 1, ..., I$, $t_i \in T_i$ and $a_i \in A_i$, we have

$$V_i\left(a_i, a_i, t_i\right) \geq V_i\left(a_i, a_i', t_i\right) \tag{17}$$

for each $a_i' \in A_i$; and, by Definition 2, for all $t \in T$ and $\theta \in \Theta$, we have

$$\sum_{a\in A} \nu\left(a, t, \theta\right) = \psi\left(\theta\right) \pi\left(t|\theta\right). \tag{18}$$

Now suppose that S is more informed than S' and that σ is the mapping that transforms S to S' in that definition. Define $\nu' \in \Delta(A \times T' \times \Theta)$ by

$$\nu'\left(a, t', \theta\right) = \sum_{t\in T} \nu\left(a, t, \theta\right) \sigma\left(t'|t, \theta\right). \tag{19}$$

By construction, for all $t' \in T'$ and $\theta \in \Theta$,

$$\begin{aligned}
\sum_{a\in A} \nu'\left(a,t',\theta\right) &= \sum_{a\in A, t\in T} \nu\left(a,t,\theta\right)\sigma\left(t'|t,\theta\right), \text{ by (19)} \\
&= \sum_{t\in T} \psi\left(\theta\right)\pi\left(t|\theta\right)\sigma\left(t'|t,\theta\right), \text{ by (18)} \\
&= \psi\left(\theta\right)\pi'\left(t'|\theta\right), \text{ because } \sigma \text{ transforms } S \text{ to } S'.
\end{aligned}$$

Thus ν' satisfies the consistency condition (Definition 2) to be a BCE of (G, S'). Symmetrically, write $V_i'\left(a_i, a_i', t_i'\right)$ for the expected utility for agent i under distribution ν' if he is type t_i', receives recommendation a_i but chooses action a_i', so that

$$V_i'\left(a_i, a_i', t_i'\right) \triangleq \sum_{a_{-i}\in A_{-i}, t'_{-i}\in T'_{-i}, \theta\in\Theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right)\nu'\left(\left(a_i, a_{-i}\right), \left(t_i', t_{-i}'\right), \theta\right).$$

Now ν' satisfies the obedience condition (Definition 3) to be a correlated equilibrium of (G, S') if for each $i = 1, ..., I$, $t_i' \in T_i'$ and $a_i \in A_i$,

$$V_i'\left(a_i, a_i, t_i'\right) \geq V_i'\left(a_i, a_i', t_i'\right)$$

for all $a_i' \in A_i$. But

$$\begin{aligned}
V_i'\left(a_i, a_i', t_i'\right) &= \sum_{a_{-i}\in A_{-i}, t'_{-i}\in T'_{-i}, \theta\in\Theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right)\nu'\left(\left(a_i, a_{-i}\right), \left(t_i', t_{-i}'\right), \theta\right) \\
&= \sum_{a_{-i}\in A_{-i}, t'_{-i}\in T'_{-i}, \theta\in\Theta, t\in T} u_i\left(\left(a_i', a_{-i}\right), \theta\right)\nu\left(\left(a_i, a_{-i}\right), \left(t_i, t_{-i}\right), \theta\right)\sigma\left(t'|t,\theta\right), \\
&\quad \text{by the definition of } \nu', \text{ see (19)} \\
&= \sum_{a_{-i}\in A_{-i},, t\in T, \theta\in\Theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right)\nu\left(\left(a_i, a_{-i}\right), \left(t_i, t_{-i}\right), \theta\right) \sum_{t'_{-i}\in T'_{-i}} \sigma\left(\left(t_i', t_{-i}'\right) | \left(t_i, t_{-i}\right), \theta\right) \\
&= \sum_{a_{-i}\in A_{-i}, t\in T, \theta\in\Theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right)\nu\left(\left(a_i, a_{-i}\right), \left(t_i, t_{-i}\right), \theta\right)\xi_i\left(t_i'|t_i\right), \\
&= \sum_{t_i\in T_i} \xi_i\left(t_i'|t_i\right)\left[\sum_{a_{-i}\in A_{-i}, t_{-i}\in T_{-i}, \theta\in\Theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right)\nu\left(\left(a_i, a_{-i}\right), \left(t_i, t_{-i}\right), \theta\right)\right] \\
&= \sum_{t_i\in T_i} \xi_i\left(t_i'|t_i\right) V_i\left(a_i, a_i', t_i\right). \qquad (20)
\end{aligned}$$

Now for each $i = 1, ..., I$, $t_i' \in T_i'$ and $a_i \in A_i$,

$$\begin{aligned}
V_i'\left(a_i, a_i, t_i'\right) &= \sum_{t_i\in T_i} \xi_i\left(t_i'|t_i\right) V_i\left(a_i, a_i, t_i\right), \text{ by (20)} \\
&\geq \sum_{t_i\in T_i} \xi_i\left(t_i'|t_i\right) V_i\left(a_i, a_i', t_i\right), \text{ by (17) for each } t_i \in T_i \\
&= V_i'\left(a_i, a_i', t_i'\right), \text{ by (20)}
\end{aligned}$$

for each $a_i' \in A_i$. Thus ν' is a BCE of (G, S'). By construction ν' and ν induce the same distribution in $\Delta(A \times \Theta)$. Since this argument started with an arbitrary BCE ν of (G, S) and an arbitrary G, we have $BCE(G, S) \subseteq BCE(G, S')$ for all games G. ■

An immediate corollary of our result is:

Corollary 2 *Information structure S' is BCE-equivalent to S if and only if S' is informationally equivalent to S'.*

This result can be seen as the common prior / incomplete information correlated equilibrium analogue of an non-common prior / incomplete rationalizability result of Dekel, Fudenberg, and Morris (2007). Dekel, Fudenberg, and Morris (2007) suggested a weak definition of incomplete information rationalizability ("interim correlated rationalizability") that captured the implications of common certainty of rationality. It was weaker than some alternative definitions in not imposing restrictions on how a player might believe that opponents' behavior was correlated with payoff states; in particular, the correlations could convey information not known to any player. Dekel, Fudenberg, and Morris (2007) showed that two types in any type space were equivalent in terms of interim correlated rationalizable actions in all games if and only if they mapped to the same Mertens-Zamir hierarchy of higher order beliefs about Θ.

The definition of Bayes correlated equilibrium in this paper is weaker than alternative definitions of incomplete information correlated equilibrium in allowing arbitrary correlation with payoff states. It is the natural equilibrium / common prior analogue to the solution concept of interim correlated rationalizability. It captures the implications of common certainty of rationality and the common prior assumption (assuming no further feasibility restrictions are imposed). Corollary 2 shows that the set of BCE action-state distributions of (G, S) are the same as the set of BCE action-state distributions of (G, S') for all games G if and only if S is informationally equivalent to S'; and, as we noted above, S is informationally equivalent to S' if and only if they generate the same probability distribution over Mertens-Zamir hierarchies.

We can add a further connection. Ely and Peski (2006) consider a finer definition of incomplete information rationalizability that does not allow unexplained correlation between an opponent's behavior and the payoff state: Dekel, Fudenberg, and Morris (2007) call this solution concept "interim independent rationalizability." Ely and Peski (2006) show that two types in any type space are equivalent in terms of interim independent rationalizable actions in all games if and only if they mapped to the same "hierarchies of conditional beliefs" that they describe. The Bayesian solution can be seen as an equilibrium / common prior analogue of interim independent rationalizability. And Tang (2010) has shown that two information structures are "Bayesian solution equivalent" to each other if and only if they give rise to the same probability distributions over "hierarchies of conditional beliefs," i.e., the hierarchies introduced by Ely and Peski (2006).

5.3 Garblings

Lehrer, Rosenberg, and Shmaya (2010), (2011) introduced an elegant language for comparing information structures. Our "more informed than" relation is a variation on one of their conditions. In this Section, we review their language and discuss the connection with our "more informed than" relation. In Section 5.4, we discuss the results on the impact of changing information structures on the set of equilibria that they and Gossner (2000) have derived for other solution concepts.

Definition 25 *Information structure S' is a garbling of S if there exists $\xi : T \rightarrow \Delta(T')$ and satisfying*

$$\pi'\left(t'|\theta\right) = \sum_{t \in T} \pi\left(t|\theta\right) \xi\left(t'|t\right)$$

for each $t' \in T'$ and $\theta \in \Theta$. The map ξ is called a garbling that transforms S to S'.

This says that the join of the information in S' is a garbling in the sense of Blackwell (1951) of the join of the information in S. Garbling ξ is *non-communicating* if, for each $i = 1, ..., I$, $t_i \in T_i$, $t'_i \in T_i$,

$$\sum_{t'_{-i} \in T'_{-i}} \xi\left(\left(t'_i, t'_{-i}\right) \mid (t_i, t_{-i})\right) = \sum_{t'_{-i} \in T'_{-i}} \xi\left(\left(t'_i, t'_{-i}\right) \mid \left(t_i, \widetilde{t}_{-i}\right)\right)$$

for all $t_{-i}, \widetilde{t}_{-i} \in T_{-i}$.

Definition 26 *Information structure S' is a non-communicating garbling of S if there exists a non-communicating garbling ξ that transforms S into S'.*

This condition requires that each player's information in S' is a Blackwell garbling of his information in S. If garbling ξ is a non-communicating garbling, we write $\xi_i\left(t'_i|t_i\right)$ for the (t_{-i} independent) probability of t'_i conditional on t_i, i.e.,

$$\xi_i\left(t'_i|t_i\right) \equiv \sum_{t'_{-i} \in T_i} \xi\left(\left(t'_i, t'_{-i}\right) \mid (t_i, t_{-i})\right).$$

Garbling ξ is *coordinated* if there exist $\lambda \in \Delta(\{1, ..., K\})$ and, for each i, $\xi_i : T_i \times \{1, ..., K\} \rightarrow \Delta(T_i)$ such that

$$\xi\left(t'|t\right) = \sum_{k=1}^{K} \lambda(k) \prod_{i=1}^{I} \xi_i\left(t'_i|t_i, k\right)$$

for each $t \in T$ and $t' \in T'$.

Definition 27 *Information structure S' is a coordinated garbling of S if there exists a coordinated garbling ξ that transforms S into S'.*

A garbling is independent if it is coordinated with $K = 1$, so that there exists, for each i, $\xi_i : T_i \rightarrow \Delta(T_i)$ such that

$$\xi(t'|t) = \prod_{i=1}^{I} \xi_i(t'_i|t_i)$$

for each $t \in T$ and $t' \in T'$.

Definition 28 *Information structure S' is an independent garbling of S if there exists a independent garbling ξ that transforms S into S'*

Lehrer, Rosenberg, and Shmaya (2010), (2011) note that, by definition, an independent garbling is a coordinated garbling, a coordinated garbling is a non-communicating garbling and a non-communicating garbling is a garbling, and present elegant examples showing that none of the reverse implications is true.

Our Definition 22 says that an information structure S is more informed than information structure S' if S' is non-communicating garbling of S in the sense of Definition 25, with the twist that we allow ξ to be a function of Θ as well as T. Thus if S' is a non-communicating garbling of S, then S is more informed than S'. But a robust example in the Appendix (Section 9.4) shows that the converse is not true.

One way to further understand the connection is to introduce a $0th$ "dummy player" (as we did in Section 4.1) into both information structures S and S' who observes θ perfectly under both information structures. Write (as we did in Section 4.1), $\widetilde{S}$ and $\widetilde{S}'$ for the information structures that arise if we add the dummy player. Now we have:

Lemma 9 *Information structure S is more informed than information structure S' if and only if $\widetilde{S}'$ is a non-communicating garbling of information structure $\widetilde{S}$.*

5.4 The Existing Literature

Say that an information structure S is larger than S' under a given equilibrium concept if, for every game G, every action-state distribution induced by an equilibrium of (G, S') is also induced by an equilibrium of (G, S). Information structure S is equivalent to S' under a given equilibrium concept if S is larger than S' and S' is larger than S under that equilibrium.

Theorem 2.8 in Lehrer, Rosenberg, and Shmaya (2011) shows that

1. Two information structures are equivalent under Bayes Nash equilibrium if and only if they are independent garblings of each other.

2. Two information structures are equivalent under Agent Normal Form correlated equilibrium (Definition 16) if and only if they are coordinated garblings of each other.

3. Two information structures are equivalent under the Belief Invariant Bayesian Solution (Definition 14) if and only if they are non-communicating garblings of each other.

Lehrer, Rosenberg, and Shmaya (2011) note that it is a Corollary 2 - that two information structures are equivalent under Bayes correlated equilibrium if and only if they are informationally equivalent - has the same format as the above results in Lehrer, Rosenberg, and Shmaya (2011), and could surely be shown elegantly and more directly using their methods.

Lehrer, Rosenberg, and Shmaya (2011) do not report results for the "larger than" relation, like our Theorem 2. The intuitive explanation why we would not expect such results to exist is that "more information" or "less garbling" will generally (under solution concepts stronger than Bayes correlated equilibrium) add incentive constraints but also remove feasibility constraints. We were able to prove a "larger than" characterization because the BCE solution concept ensures that feasibility constraints do not change as the informativeness of the information structure increases.

To further understand the connection, we can re-interpret our result as a "larger than" result about the Bayesian solution if we impose constraints on the information structures being compared to make sure that feasibility constraints do not change.

Corollary 3 *Consider two information structures S and S' with the property that there is a player who perfectly observes the state θ under both S and S'. Then S' is Bayesian solution larger than S if and only if S' is a non-communicating garbling of S.*

This follows easily from Theorem 2 and our observations about adding dummy players in Sections 4.1 and 5.3.

Lehrer, Rosenberg, and Shmaya (2010) consider common interest games. Say that information structure S is better than S' under a given solution concept if, for every common interest game G, the maximum (common) equilibrium payoff is higher in (G, S) than (G, S'). They show:

1. (Theorem 3.5) Information structure S is better than S' under Bayes Nash equilibrium if and only if S' is a coordinated garbling of S.

2. (Theorem 4.2) Information structure S is better than S' under Agent Normal Form correlated equilibrium (Definition 16) if and only if S' is a coordinated garbling of S.

3. (Theorem 4.2) Information structure S is better than S' under Strategic Form correlated equilibrium (Definition 18) if and only if S' is a coordinated garbling of S.

4. (Theorem 4.5) Information structure S is better than S' under the Belief Invariant Bayesian Solution (Definition 14) if and only if S' is a non-communicating garbling of S.

5. (Theorem 4.6) Information structure S is better than S' under Communication equilibrium (Definition 20) if and only if S' is a garbling of S.

Gossner (2000) studies Bayes Nash equilibrium only as a solution concept. His focus is on complete information games but also reports results for incomplete information games. The idea of his results is that more correlation possibilities are better for the set of BNE that can be supported. To state Gossner's result, write $BNE(G,S)$ for the set of BNE action-state distributions of (G,S) (see Definition 6), i.e., the set of distributions on $A \times \Theta$ that can be induced by a BNE of (G,S).

Definition 29 *Information structure S is BNE-larger than information structure S' if $BNE(G,S') \subseteq BNE(G,S)$ for all basic games G.*

An independent garbling ξ is *faithful* if whenever for each i, $t_i \in T_i$ and $t_i' \in T_i'$ with $\xi_i(t_i'|t_i) > 0$, we have

$$\frac{\psi(\theta)\,\pi'\left(\left(t_i', t_{-i}'\right)|\theta\right)}{\sum\limits_{\widetilde{t}_{-i}' \in T_{-i}', \widetilde{\theta} \in \Theta} \psi\left(\widetilde{\theta}\right)\pi'\left(\left(t_i', \widetilde{t}_{-i}'\right)|\widetilde{\theta}\right)} = \frac{\psi(\theta) \sum\limits_{t_{-i} \in T_{-i}} \pi\left((t_i, t_{-i})|\theta\right)\left(\prod\limits_{j \neq i} \xi_j\left(t_j'|t_j\right)\right)}{\sum\limits_{t_{-i} \in T_{-i}, \widetilde{\theta} \in \Theta} \psi\left(\widetilde{\theta}\right)\pi\left((t_i, t_{-i})|\widetilde{\theta}\right)}$$

for all $t_{-i}' \in T_{-i}'$ and $\theta \in \Theta$.

Definition 30 *Information structure S' is a faithful independent garbling of S if there exists a faithful independent garbling ξ that transforms S into S'.*

Intuitively, this states that information structure S allows more correlation possibilities than S' but does not give more information about beliefs and higher order beliefs about payoff states. Now we have:

Proposition 1 *Information structure S is BNE-larger than S' if and only if S' is a faithful independent garbling of S.*

This is Theorem 19 in Gossner (2000). [In the briefly described (Section 6) statement of Gossner's result, his definition of BNE-larger ("richer" in his language) refers only to distributions over action profiles, and not over action profiles and Θ; however his arguments would apply the above result.] An interesting special case is when S' is uninformative, i.e., contains neither information about Θ nor correlation opportunities, so that there exist, for each i, $\lambda_i \in \Delta(T_i')$ such that

$$\pi'(t'|\theta) = \prod_{i=1}^{I} \lambda_i(t_i')$$

for all $t' \in T'$ and $\theta \in \Theta$. In this case, $BNE(G, S')$ is just equal to the independent distributions over actions generated by Nash equilibria in the basic game G. This S' is a faithful independent garbling of S for *any* S which is not informative about Θ: simply set

$$\xi(t'|t) = \prod_{i=1}^{I} \lambda_i(t'_i)$$

for all $t' \in T'$ and $\theta \in \Theta$. Now $BNE(G, S)$ contains $BNE(G, S')$ because there are weakly more correlation possibilities in S.

6 The Robust Predictions Agenda

An important motivation for the analysis of the Bayes correlated equilibria is that they represent a robust prediction for an observer who knows that the game G is being played, but only knows that players have at observed information structure S but does not know if they have observed more. In Bergemann and Morris (2011b), we examine BCE in a class of continuum player, continuum action, symmetric, linear best response games. In Section 7 below, we discuss how the (finite player, finite action) results of this paper can be adapted to that setting. In Bergemann, Brooks, and Morris (2011), we study Bayes correlated equilibria of a first price auction with finitely many valuations and a continuum of bids, and then characterize which information structures increase and decrease the seller's revenue in that setting. In this Section, we will briefly illustrate the logic of the approach by considering Bayes correlated equilibria in the degenerate case of single player games.

The idea of characterizing what might happen across a range of information structures naturally arises in a variety of contexts. Kamenica and Gentzkow (forthcoming) consider a classic sender-receiver problem where the "sender" knows the state, a "receiver" (with perhaps different preferences) will take an action and the sender must send a message to the receiver. In a twist from the standard "cheap talk" literature, Kamenica and Gentzkow (forthcoming) assume that the sender can commit ex ante (before observing the state) to any communication strategy. Thus the problem reduces to a sender choosing the optimal information structure for the receiver in a one player game (decision problem). The set of outcomes that could be induced (ignoring the sender's preferences) is the set of BCE of the one player game. Caplin and Martin (2011) consider an "ideal observer" who sees a "decision maker" making many choices from action sets.[2] The ideal observer knows a true stochastic mapping from actions to outcomes (and thus utilities) but does not know what the decision maker's perception of each choice situation, i.e., his belief about the true stochastic map. Even without knowing those beliefs, one can impose constraints on observable choice

[2] We thank Jonathan Weinstein for bringing this work, and its relation to Bayes correlated equilibrium, to our attention.

behavior. These too will correspond to BCE of a single player game. Caplin and Martin (2011) introduce this framework to analyze limited perception and run experiments to test rationality within the model and the nature of information-based framing effects that are revealed. In Section 8.2, we discuss Chwe (2006) which characterizes the observable implications of incentive constraints; here too the problem is one of making predictions without observing the underlying signals.

In this Section, we briefly illustrate the idea of Bayes correlated equilibria, and their implications for robust predictions, in a one player, two state, two action example. Chwe (2006), Kamenica and Gentzkow (forthcoming) and Caplin and Martin (2011) all illustrate their results with such examples, and thus we are replicating some of their formal analysis.

There is one player, and we will thus drop the player subscripts. There are two states, $\Theta = \{\theta_0, \theta_1\}$. Consider the game G with $A = \{a_0, a_1\}$; $u(a_0, \theta_0) = \kappa$, $u(a_1, \theta_1) = 1 - \kappa$ and $u(a_0, \theta_1) = u(a_1, \theta_0) = 0$; and $\psi(\theta_0) = \xi$ and $\psi(\theta_1) = 1 - \xi$, with $\kappa, \xi \in [0, 1]$. Thus the payoff matrix is

	θ_0	θ_1
a_0	κ	0
a_1	0	$1-\kappa$

Note that this parameterization of payoffs and beliefs is without loss of generality (if we are interested in predictions not utilities) up to the assumption that action a_1 is not dominated. Consider an arbitrary information structure $S = (T, \pi)$, where T is a finite set and write $\pi_k(t)$ for the probability of signal t in state θ_k.

For a motivation like that in Kamenica and Gentzkow (forthcoming), let θ_0 and θ_1 represent "innocence" and "guilt" and let a_0 and a_1 represent "acquittal" and "conviction".

We are interested in Bayes correlated equilibria of the game (G, S). Suppose that the mediator recommends action a_1 if the player observes signal t in state θ_k with probability $\beta_k(t)$ (and thus a_0 with probability $1 - \beta_k(t)$). Thus the mediator's behavior is given by (β_1, β_2) with each $\beta_k : T \to [0, 1]$. Now if the player observes signal t and is advised to take action a_1, he attaches probability

$$\frac{\xi \pi_0(t) \beta_0(t)}{\xi \pi_0(t) \beta_0(t) + (1-\xi) \pi_1(t) \beta_1(t)}$$

to state θ_0 and thus follows the recommendation if

$$(1-\xi) \pi_1(t) \beta_1(t) (1-\kappa) \geq \xi \pi_0(t) \beta_0(t) \kappa, \tag{21}$$

or

$$\beta_1(t) \geq \left(\frac{\kappa}{1-\kappa}\right) \left(\frac{\xi}{1-\xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) \beta_0(t). \tag{22}$$

If the player observes signal t and is advised to take action a_0, he attaches probability

$$\frac{\xi\pi_0(t)(1-\beta_0(t))}{\xi\pi_0(t)(1-\beta_0(t))+(1-\xi)\pi_1(t)(1-\beta_1(t))}$$

to state θ_0 and thus follows the recommendation if

$$(1-\xi)\pi_1(t)(1-\beta_1(t))(1-\kappa)\leq\xi\pi_0(t)(1-\beta_0(t))\kappa,$$

or

$$(1-\xi)\pi_1(t)\beta_1(t)(1-\kappa)\geq\xi\pi_0(t)\beta_0(t)\kappa+(1-\xi)\pi_1(t)(1-\kappa)-\xi\pi_0(t)\kappa, \tag{23}$$

or

$$\beta_1(t)\geq\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right)\beta_0(t)+\left(1-\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right)\right). \tag{24}$$

Now the two obedience constraints (22) and (24) can be combined in the constraint that

$$\beta_1(t)\geq\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right)\beta_0(t)+\max\left(0,1-\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right)\right). \tag{25}$$

Now distribution $\nu\in\Delta(A\times T\times\Theta)$ is a Bayes correlated equilibrium if and only if

$$\nu(a,t,\theta)=\begin{cases}(1-\xi)\pi_1(t)\beta_1(t), & \text{if } (a,\theta)=(a_1,\theta_1);\\ (1-\xi)\pi_1(t)(1-\beta_1(t)), & \text{if } (a,\theta)=(a_0,\theta_1);\\ \xi\pi_0(t)\beta_0(t), & \text{if } (a,\theta)=(a_1,\theta_0);\\ \xi\pi_0(t)(1-\beta_0(t)), & \text{if } (a,\theta)=(a_0,\theta_0);\end{cases}$$

for some (β_1,β_2) satisfying (25).

To understand how the set of BCE vary with different information structures, we can consider some extreme points. Consider the player's ex ante utility:

$$\sum_{t\in T}(\xi\kappa\pi_0(t)(1-\beta_0(t))+(1-\xi)(1-\kappa)\pi_1(t)\beta_1(t)). \tag{26}$$

(note that for comparison of utility, the parameterization of payoffs is not longer without loss of generality). This is maximized by setting $\beta_0(t)=0$ and $\beta_1(t)=1$ for all $t\in T$, giving maximum ex ante utility

$$\overline{U}(S)=\xi\kappa+(1-\xi)(1-\kappa).$$

We write this as a function of the information structure S, although it turns out to be independent of the information structure. Now we find the BCE minimizing the player's ex ante utility. An alternative writing of the obedience condition (re-writing (21) and (23)), we have that

$$(1-\xi)\pi_1(t)\beta_1(t)(1-\kappa)-\xi\pi_0(t)\beta_0(t)\kappa\geq\max\{0,(1-\xi)\pi_1(t)(1-\kappa)-\xi\pi_0(t)\kappa\}. \tag{27}$$

This condition immediately gives a lower bound of

$$\sum_{t\in T}\max\left\{\xi\kappa\pi_0\left(t\right),\left(1-\xi\right)\pi_1\left(t\right)\left(1-\kappa\right)\right\}.$$

This bound can be obtained by setting $\beta_0\left(t\right)=\beta_1\left(t\right)=1$ if

$$\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0\left(t\right)}{\pi_1\left(t\right)}\right)\leq 1, \tag{28}$$

and $\beta_0\left(t\right)=\beta_1\left(t\right)=0$ if

$$\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0\left(t\right)}{\pi_1\left(t\right)}\right)>1. \tag{29}$$

Thus in the utility minimizing BCE, each type will take his most preferred action if he had no additional information beyond his type. This gives minimum ex ante utility

$$\underline{U}\left(S\right)=\sum_{t\in T}\max\left\{\xi\kappa\pi_0\left(t\right),\left(1-\xi\right)\pi_1\left(t\right)\left(1-\kappa\right)\right\}.$$

Thus we have a robust prediction that with information structure S ex ante utility will be in the interval $\left[\underline{U}\left(S\right),\overline{U}\left(S\right)\right]$. The perfect information system S^* has $T=\{t_0,t_1\}$, $\pi_0\left(t_0\right)=1$ and $\pi_1\left(t_1\right)=1$, this minimum utility will equal the maximum utility

$$\underline{U}\left(S^*\right)=\overline{U}\left(S^*\right)=\xi\kappa+\left(1-\xi\right)\left(1-\kappa\right).$$

With null information system S_0 has $T=\{t^*\}$, $\pi_0\left(t^*\right)=\pi_1\left(t^*\right)=1$ and thus minimum utility

$$\underline{U}\left(S_0\right)=\max\left\{\xi\kappa,\left(1-\xi\right)\left(1-\kappa\right)\right\}.$$

Intuitively more information will increase the minimum ex ante utility and not change the maximum ex ante utility. Changes in information structure that will most increase ex ante utility are those that have signals with likelihood ratio $\frac{\pi_0(t)}{\pi_1(t)}$ a long way from $\frac{(1-\kappa)(1-\xi)}{\kappa\xi}$.

For a more interesting question, we might be interested in bounding the probability of conviction, independently of guilt or innocence (following Caplin and Martin (2011), we can think of an aggressive district attorney). To answer this, consider the probability that action a_1 is chosen,

$$\sum_{t\in T}\left(\xi\pi_0\left(t\right)\beta_0\left(t\right)+\left(1-\xi\right)\pi_1\left(t\right)\beta_1\left(t\right)\right). \tag{30}$$

This is maximized for each t subject to the obedience constraint by setting $\beta_0\left(t\right)=\beta_1\left(t\right)=1$ if

$$\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0\left(t\right)}{\pi_1\left(t\right)}\right)\leq 1,$$

and $\beta_1(t) = 1$ and $\beta_0(t)$ solves

$$\beta_0(t) = \left(\frac{1-\kappa}{\kappa}\right)\left(\frac{1-\xi}{\xi}\right)\left(\frac{\pi_1(t)}{\pi_0(t)}\right),$$

otherwise. Write

$$h(t) = \xi\pi_0(t) + (1-\xi)\pi_1(t),$$

for the unconditional probability that signal t is realized and

$$g(t) = \frac{(1-\xi)\pi_1(t)}{\xi\pi_0(t) + (1-\xi)\pi_1(t)},$$

for the posterior probability of guilt. Now the maximum probability that action 1 is taken is

$$\overline{\Pi}(S) = \sum_{t\in T} h(t)\min\left\{1, \frac{g(t)}{\kappa}\right\}.$$

while by a symmetric argument the minimum probability will be

$$\underline{\Pi}(S) = \sum_{t\in T} h(t)\max\left\{0, 1-\frac{1-g(t)}{1-\kappa}\right\}$$

Thus we have a robust prediction that with information structure S the probability of action a_1 will be in the interval $\left[\underline{\Pi}(S), \overline{\Pi}(S)\right]$. The perfect information system S^* has

$$\underline{\Pi}(S^*) = \overline{\Pi}(S^*) = 1-\xi.$$

With null information system S_0 we have

$$\underline{\Pi}(S_0) = \max\left(0, 1-\frac{\xi}{1-\kappa}\right) \text{ and } \overline{\Pi}(S_0) = \min\left(1, \frac{1-\xi}{\kappa}\right).$$

Intuitively more information will increase the minimum probability and decrease the maximum.

7 Symmetric and Anonymous Games

In this section, we specialize our analysis to the case of symmetric games, where there is symmetry across players in payoffs in the basic game G and symmetry across signals in the information structure S. Thus players' labels are assumed to not matter for the description of the game. We will be focusing attention on "exchangeable equilibria," where players' behavior in equilibrium does not break the underlying symmetry in the game. In a recent MIT Ph.D., Stein (2011) introduces and analyzes "exchangeable correlated equilibria" in a complete information setting. Our definitions are presumably incomplete information generalizations of his definitions, although we have not examined the relation in detail. The restriction to exchangeable equilibria will be implicit within our definitions.

Once we have an exchangeable finite player, finite action, finite state version of Bayes correlated equilibrium, it is then possible to present analogue results for continuum player, continuum action, continuum state games. This will then provide a foundation for our modelling in Bergemann and Morris (2011b).

7.1 The Finite Case

As before, there are I players and finite state space Θ. A "basic game" G now consists of (1) a common action set A; (2) a common utility function $u : A \times \Delta_I(A) \times \Theta \rightarrow \mathbb{R}$; , where $u(a, h, \theta)$ is a player's payoff if he chooses action a, the distribution of actions among the I players is $h \in \Delta_I(A)$ and the state is θ. (For any finite set X, we write $\Delta_I(X)$ for the set of probability distributions on X with the property that for any random variable ξ, $\mathbb{P}(\xi \in X) \in \left\{0, \frac{1}{I}, \frac{2}{I}, ..., 1\right\}$; and (3) a full support prior $\psi \in \Delta(\Theta)$. Thus a basic symmetric game $G = (A, u, \psi)$. A symmetric "information structure" S now consists of (1) a common set of types or "signals" T; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(\Delta_I(T))$. Now $\pi(\theta) \in \Delta(\Delta_I(T))$ is a probability distribution over the realized distribution of signals in the population. Thus $S = (T, \pi)$. Now (G, S) describes a standard (symmetric) Bayesian game.

If $\xi \in \Delta_I(A \times T)$ is a distribution over action-signal pairs, write $\operatorname{marg}_T \xi \in \Delta_I(T)$ for the marginal distribution over signals, so

$$\operatorname*{marg}_T \xi(t) = \sum_{a \in A} \xi(a, t)$$

for each $t \in T$; write $\operatorname{marg}_A \xi \in \Delta_I(A)$ for the marginal distribution over actions, so

$$\operatorname*{marg}_A \xi(a) = \sum_{t \in T} \xi(a, t)$$

for each $a \in A$. If $\nu \in \Delta(\Delta_I(A \times T) \times \Theta)$ is a distribution over action-signal pair distributions and states, write $\operatorname{marg}_{\Delta_I(T) \times \Theta} \nu \in \Delta(\Delta_I(T) \times \Theta)$ for the marginal distribution over realized distributions of signals and states, so

$$\operatorname*{marg}_{\Delta_I(T) \times \Theta} \nu(g, \theta) = \sum_{\{\xi \in \Delta_I(A \times T) : \operatorname{marg}_T \xi = g\}} \nu(\xi, \theta)$$

for each $g \in \Delta_I(T)$ and $\theta \in \Theta$. Finally, write $\pi \circ \psi$ for the probability distribution on $\Delta_I(T) \times \Theta$ induced by $\psi \in \Delta(\Theta)$ and $\pi : \Theta \rightarrow \Delta(\Delta_I(T))$, so

$$\pi \circ \psi(g, \theta) = \psi(\theta) \pi(g | \theta)$$

for each $g \in \Delta_I(T)$ and $\theta \in \Theta$.

Definition 31 (Bayes Correlated Equilibrium)
A probability distribution $\nu \in \Delta(\Delta_I(A \times T) \times \Theta)$ is a Bayes correlated equilibrium (BCE) of (G, S) if

$$\sum_{\xi \in \Delta_I(A \times T), \theta \in \Theta} u(a, marg_A \xi, \theta) \xi(a, t) \nu(\xi, \theta) \geq \sum_{\xi \in \Delta_I(A \times T), \theta \in \Theta} u(a', marg_A \xi, \theta) \xi(a, t) \nu(\xi, \theta); \tag{31}$$

for each $t \in T$, $a \in A$ and $a' \in A$; and

$$marg_{\Delta_I(T) \times \Theta} \nu = \pi \circ \psi. \tag{32}$$

In the special case of a null information structure (so there are no signals), then the obedience condition (31) for $\mu \in \Delta(\Delta_I(A) \times \Theta)$ will be

$$\sum_{g \in \Delta_I(A), \theta \in \Theta} u(a, g, \theta)\, g(a)\, \mu(g, \theta) \geq \sum_{g \in \Delta_I(A), \theta \in \Theta} u\left(a', g, \theta\right) g(a)\, \mu(g, \theta);$$

for each $a \in A$ and $a' \in A$ while the consistency condition (32) will be

$$\mathrm{marg}_{\Theta}\mu = \psi.$$

7.2 The Continuum Case

There is a continuum $[0,1]$ of players and state space Θ. A "basic game" G now consists of (1) a common action set $A \subseteq \mathbb{R}$; (2) a common utility function $u : A \times \Delta(A) \times \Theta \rightarrow \mathbb{R}$; where $u(a, h, \theta)$ is a player's payoff if he chooses action a, the distribution of actions among the continuum players is $h \in \Delta(A)$ and the state is θ; and (3) a full support prior $\psi \in \Delta(\Theta)$. Thus $G = (A, u, \psi)$. An "information structure" S now consists of (1) a common set of types or "signals" T; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(\Delta(T))$. Now $\pi(\theta) \in \Delta(\Delta(T))$ is a probability distribution over realized distributions of signals in the population. Thus $S = (T, \pi)$. Now (G, S) describes a standard continuum (symmetric and anonymous) Bayesian game.

Now the definitions for the continuum case are as before, except that distributions are over a continuum population and summations are replaced with integrals. We omit the measurability conditions that will be required in general (they are not an issue for applications we are considering with well defined densities).

As before:

- if $\xi \in \Delta(A \times T)$ is a distribution over action-signal pairs, write $marg_T\xi(t) \in \Delta(T)$ and $marg_A\xi \in \Delta(A)$ for the marginal distributions over signals and actions respectively;
- if $\nu \in \Delta(\Delta(A \times T) \times \Theta)$, write $marg_{\Delta(T)\times\Theta}\nu \in \Delta(\Delta(T) \times \Theta)$ for the marginal distribution over realized distributions of signals and states;
- write $\pi \circ \psi$ for the probability distribution on $\Delta(T) \times \Theta$ induced by $\psi \in \Delta(\Theta)$ and $\pi : \Theta \rightarrow \Delta(\Delta(T))$.

Definition 32 (Bayes Correlated Equilibrium)
A probability distribution $\nu \in \Delta(\Delta(A \times T) \times \Theta)$ is a Bayes correlated equilibrium (BCE) of (G, S) if

$$\int_{\xi \in \Delta(A \times T), \theta \in \Theta} u(a, marg_A\xi, \theta)\, \xi(a, t)\, d\nu \geq \int_{\xi \in \Delta(A \times T), \theta \in \Theta} u\left(a', marg_A\xi, \theta\right) \xi(a, t)\, d\nu; \tag{33}$$

for each $t \in T$, $a \in A$ *and* $a' \in A$; *and*

$$marg_{\Delta(T)\times\Theta}\nu = \pi \circ \psi. \tag{34}$$

In the special case of a null information structure (so there are no signals), then the obedience condition (33) for $\mu \in \Delta(\Delta(A) \times \Theta)$ will be

$$\int\limits_{g \in \Delta(A), \theta \in \Theta} u(a, g, \theta)\, g(a)\, d\mu \geq \int\limits_{g \in \Delta(A), \theta \in \Theta} u\left(a', g, \theta\right) g(a)\, d\mu;$$

for each $a \in A$ and $a' \in A$ while the consistency condition (34) will be

$$marg_{\Theta}\mu = \psi.$$

8 Discussion

8.1 Payoff Type Spaces

In a body of work collected in Bergemann and Morris (2012), we studied a robust mechanism environments in a setting where agents knew their own "payoff types", there was common knowledge of how utilities depended on the profile of payoff types, but agents were allowed to have any beliefs and higher order beliefs about others' payoff types. In Bergemann and Morris (2007), we discussed a game theoretic framework underlying this work. Here we briefly how this environment maps into the setting of this paper.

Suppose that Θ is a product space with $\Theta = \Theta_1 \times \cdots \times \Theta_I$. Consider the special information structure where agent i's set of possible signals is Θ_i, and each agent i observes the realization $\theta_i \in \Theta_i$, so $S^{**} = \left((\Theta_i)_{i=1}^{I}, id\right)$, where id is the identity map $id : \Theta \to \Theta$ with $id(\theta) = \theta$ for all θ. Now the set of Bayes correlated equilibria of a game (G, S) describe all the distributions over payoff type profiles and actions consistent with the common prior and common knowledge of rationality. Bergemann and Morris (2007) - in the language of this paper - is an analysis of the structure of Bayes correlated equilibria with the special information structure S^{**}.

8.2 Signed Covariance

Chwe (2006) analyzes statistical implications of incentive compatibility in general, and in particular statistical implications of correlated equilibrium play. We can state his main observation in the language of our paper. Fix any basic game G. Fix any Bayes correlated equilibrium $\mu \in \Delta(A \times \Theta)$ of the basic game (i.e., the game with the null information structure). Fix a player i and action $a_i^* \in A_i$. Consider the random variable $\mathbb{I}_{a_i^*}$ on $A \times \Theta$ that as indicator function takes value 1 if a_i^* is played and 0 otherwise:

$$\mathbb{I}_{a_i^*}(a,\theta) \triangleq \begin{cases} 1, & \text{if} \quad a_i = a_i^*; \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Fix any other action $a_i' \in A_i$. Let $\Pi_{a_i^*,a_i'}$ be the random variable on $A\times\Theta$ equal to the payoff gain to player i of choosing action a_i^* rather than a_i':

$$\Pi_{a_i^*,a_i'}(a,\theta) \triangleq u_i\left(\left(a_i^*,a_{-i}\right),\theta\right) - u_i\left(\left(a_i',a_{-i}\right),\theta\right).$$

Then, conditional on a_i^* or a_i' being played, the random variables $\mathbb{I}_{a_i^*}$ and $\Pi_{a_i^*,a_i'}$ have positive covariance. This is the content of the main result in Chwe (2006). As he notes, this is not merely a re-writing of the incentive compatibility constraints, since these are linear in probabilities while the covariance is quadratic in probabilities. Thus his signed conditional covariance result is a necessary property of second order statistics of a Bayes correlated equilibrium.

We sketch a formal statement and proof. The expectations of $\mathbb{I}_{a_i^*}$, $\Pi_{a_i^*,a_i'}$ and their product, under μ, conditional on the event $\{a_i^*,a_i'\}$ occurring, are:

$$\begin{aligned}
\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}\left|\left\{a_i^*,a_i'\right\}\right.\right) &= \frac{\sum\limits_{a_{-i},\theta}\mu\left(\left(a_i^*,a_{-i}\right),\theta\right)}{\sum\limits_{a_{-i},\theta}\mu\left(\left(a_i^*,a_{-i}\right),\theta\right)+\sum\limits_{a_{-i},\theta}\mu\left(\left(a_i',a_{-i}\right),\theta\right)} \\
\mathbb{E}_\mu\left(\Pi_{a_i^*,a_i'}\left|\left\{a_i^*,a_i'\right\}\right.\right) &= \frac{\sum\limits_{a_{-i},\theta}\left(\mu\left(\left(a_i^*,a_{-i}\right),\theta\right)+\mu\left(\left(a_i',a_{-i}\right),\theta\right)\right)\left(u_i\left(\left(a_i^*,a_{-i}\right),\theta\right)-u_i\left(\left(a_i',a_{-i}\right),\theta\right)\right)}{\sum\limits_{a_{-i},\theta}\left(\mu\left(\left(a_i^*,a_{-i}\right),\theta\right)+\mu\left(\left(a_i',a_{-i}\right),\theta\right)\right)} \\
\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}\Pi_{a_i^*,a_i'}\left|\left\{a_i^*,a_i'\right\}\right.\right) &= \frac{\sum\limits_{a_{-i},\theta}\mu\left(\left(a_i^*,a_{-i}\right),\theta\right)\left(u_i\left(\left(a_i^*,a_{-i}\right),\theta\right)-u_i\left(\left(a_i',a_{-i}\right),\theta\right)\right)}{\sum\limits_{a_{-i},\theta}\left(\mu\left(\left(a_i^*,a_{-i}\right),\theta\right)+\mu\left(\left(a_i',a_{-i}\right),\theta\right)\right)}.
\end{aligned}$$

Now the incentive compatibility condition that that player i prefers a_i^* to a_i' when advised to play a_i^* can be written as

$$\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}\Pi_{a_i^*,a_i'}\left|\left\{a_i^*,a_i'\right\}\right.\right) \geq 0 \tag{35}$$

while that incentive compatibility condition that player i prefers a_i' to a_i^* when advised to play a_i' is

$$\mathbb{E}_\mu\left(\left(1-\mathbb{I}_{a_i^*}\right)\Pi_{a_i^*,a_i'}\left|\left\{a_i^*,a_i'\right\}\right.\right) \leq 0$$

which can be re-written as

$$\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}\Pi_{a_i^*,a_i'}\left|\left\{a_i^*,a_i'\right\}\right.\right) \geq \mathbb{E}_\mu\left(\Pi_{a_i^*,a_i'}\left|\left\{a_i^*,a_i'\right\}\right.\right). \tag{36}$$

Now the covariance of $\mathbb{I}_{a_i^*}$ and $\Pi_{a_i^*,a_i'}$, conditional on $\{a_i^*, a_i'\}$, is

$$\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}\Pi_{a_i^*,a_i'}|\left\{a_i^*, a_i'\right\}\right) - \mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}|\left\{a_i^*, a_i'\right\}\right)\mathbb{E}_\mu\left(\Pi_{a_i^*,a_i'}|\left\{a_i^*, a_i'\right\}\right)$$

If $\mathbb{E}_\mu\left(\Pi_{a_i^*,a_i'}|\{a_i^*, a_i'\}\right) \leq 0$, (35) and $\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}|\{a_i^*, a_i'\}\right) \geq 0$ imply that this is non-negative. If $\mathbb{E}_\mu\left(\Pi_{a_i^*,a_i'}|\{a_i^*, a_i'\}\right) \geq 0$, (36) and $\mathbb{E}_\mu\left(\mathbb{I}_{a_i^*}|\{a_i^*, a_i'\}\right) \leq 1$ imply that this is non-negative.

9 Appendix

The appendix collects the remaining proofs and additional material

9.1 Higher Order Belief Equivalence

We present a formal argument that the notion of higher order belief equivalence presented earlier in Definition 23 indeed captures all the information contained in the hierarchical belief types of Mertens and Zamir (1985). Fix Θ. Let $X^0 = \Theta$, and define $X^k = X^{k-1} \times \left[\Delta\left(X^{k-1}\right)\right]^{I-1}$. An element of $\left(\Delta\left(X^k\right)\right)_{k=0}^{\infty} \triangleq H$ is called a hierarchy (of beliefs).

Now a prior $\psi \in \Delta(\Theta)$ and an information system $S = \left((T_i)_{i=1}^{I}, \pi\right)$ together define a finite common prior type space (ψ, S). We can associate such a common prior type space with a probability distribution over H^I as follows. For each i and $t_i \in T_i$, write $\widehat{\pi}_i^1[t_i] \in \Delta(\Theta) = \Delta\left(X^0\right)$ for his posterior under a uniform prior on Θ, so

$$\widehat{\pi}_i^1[t_i](\theta) = \frac{\sum\limits_{t_{-i}\in T_{-i}} \pi\left((t_i, t_{-i})|\theta\right)\psi(\theta)}{\sum\limits_{\theta'\in\Theta, t_{-i}\in T_{-i}} \pi\left((t_i, t_{-i})|\theta'\right)\psi\left(\theta'\right)}.$$

Write $\widehat{\pi}_i^2(t_i) \in \Delta\left(\Theta \times (\Delta(\Theta))^{I-1}\right) = \Delta\left(X^1\right)$ for his belief over Θ and the first order beliefs of other players, so

$$\widehat{\pi}_i^2[t_i]\left(\theta, \pi_{-i}^1\right) = \frac{\sum\limits_{\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^1(t_j)=\pi_j^1 \text{ for each } j\neq i\right\}} \pi\left((t_i, t_{-i})|\theta\right)\psi(\theta)}{\sum\limits_{\theta'\in\Theta,\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^1(t_j)=\pi_j^1 \text{ for each } j\neq i\right\}} \pi\left((t_i, t_{-i})|\theta'\right)\psi\left(\theta'\right)}.$$

Proceeding inductively for $k \geq 2$, write $\widehat{\pi}_i^k(t_i) \in \Delta\left(X^{k-1}\right)$ for his belief over Θ and the $(k-1)$th order beliefs of other players, so

$$\widehat{\pi}_i^k[t_i]\left(\theta, \pi_{-i}^{k-1}\right) = \frac{\sum\limits_{\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^{k-1}(t_j)=\pi_j^{k-1} \text{ for each } j\neq i\right\}} \pi\left((t_i, t_{-i})|\theta\right)\psi(\theta)}{\sum\limits_{\theta'\in\Theta,\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^{k-1}(t_j)=\pi_j^{k-1} \text{ for each } j\neq i\right\}} \pi\left((t_i, t_{-i})|\theta'\right)\psi(\theta)}.$$

Now each $\widehat{\pi}_i^k : T_i \to \Delta\left(X^{k-1}\right)$, we can define $\widehat{\pi}_i : T_i \to H$ by

$$\widehat{\pi}_i[t_i] = \left(\widehat{\pi}_i^1[t_i], \widehat{\pi}_i^2[t_i], ...\right)$$

and $\widehat{\pi} : T \to H^I$ by

$$\widehat{\pi}[t] = \left(\widehat{\pi}_i[t_i]\right)_{i=1}^{I}$$

Now we can identify (ψ, S) with a probability distribution $\chi_{\psi,S} \in \Delta\left(H^I\right)$ defined by

$$\chi_{\psi,S}\left((\pi_i)_{i=1}^I\right) = \sum_{\theta,\left\{t:\widehat{\pi}[t]=(\pi_i)_{i=1}^I\right\}} \pi\left(t|\theta\right)\psi\left(\theta\right).$$

Lemma 10 *The following statements are equivalent:*

1. *Information structures S^1 and S^2 are higher order belief equivalent;*
2. $\chi_{\psi,S^1} = \chi_{\psi,S^2}$ *for all* $\psi \in \Delta\left(\Theta\right)$*;*
3. $\chi_{\psi,S^1} = \chi_{\psi,S^2}$ *for some* $\psi \in \Delta_{++}\left(\Theta\right)$*.*

Proof. We argue that (1) implies (2) by induction. By (16),

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,1}\left[t_i\right] = \widehat{\pi}_i^{k,1}\left[t_i'\right].$$

Now suppose that

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,l}\left[t_i\right] = \widehat{\pi}_i^{k,l}\left[t_i'\right].$$

By (15), we have

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,l+1}\left[t_i\right] = \widehat{\pi}_i^{k,l+1}\left[t_i'\right].$$

But since the premise of the inductive step holds for $l = 1$, we have that for all l

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k,l}\left[t_i\right] = \widehat{\pi}_i^{k,l}\left[t_i'\right].$$

and thus

$$f_i^k\left(t_i\right) = f_i^k\left(t_i'\right) \Rightarrow \widehat{\pi}_i^{k}\left[t_i\right] = \widehat{\pi}_i^{k}\left[t_i'\right].$$

Clearly (2) implies (3). Now suppose that (3) holds. Let $T_i^* = range\left(\widehat{\pi}_i^1\right) = range\left(\widehat{\pi}_i^2\right)$. Let $f_i^k\left(t_i\right) = \widehat{\pi}_i^k\left(t_i\right)$. By construction, properties (15) and (16) hold with respect to type space $S^* = \left(\left(T_i^*\right)_{i=1}^I, \pi^*\right)$. ■

9.2 Proof of Lemma 5

For a pair of mappings $\sigma^1 : T^1 \times \Theta \to \Delta\left(T^2\right)$ and $\sigma^2 : T^2 \times \Theta \to \Delta\left(T^1\right)$, we write $\sigma^2 \circ \sigma^1 : T^1 \times \Theta \to \Delta\left(T^1 \times \Theta\right)$ for the induced Markov process obtained by starting with $\left(t^1, \theta\right)$, picking t^2 according to $\sigma^2\left(\cdot|t^1, \theta\right)$, picking $\widetilde{t}^1$ according to $\sigma^1\left(\cdot|t^2, \theta\right)$, and setting $\widetilde{\theta} = \theta$. Thus

$$\sigma^2 \circ \sigma^1\left(\widetilde{t}^1, \widetilde{\theta}|t^1, \theta\right) = \begin{cases} \sum\limits_{t^2} \sigma^1\left(t^2|t^1, \theta\right)\sigma^2\left(\widetilde{t}^1|t^2, \theta\right) & \text{if} \quad \widetilde{\theta} = \theta; \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

A Markov process $\xi : Z \to \Delta(Z)$ is *idempotent* if applying it twice gives the same Markov matrix, i.e., defining $\xi \circ \xi : Z \to \Delta(Z)$ by

$$\xi \circ \xi\left(z'|z\right) = \sum_{z''} \xi\left(z'|z''\right) \xi\left(z''|z\right).$$

$\xi \circ \xi$ is idempotent if $\xi \circ \xi = \xi$ or

$$\xi\left(z'|z\right) = \sum_{z''} \xi\left(z'|z''\right) \xi\left(z''|z\right).$$

Lemma 11 *If information structure S^1 is informationally equivalent to S^2, then there exist mappings $\widetilde{\sigma}^1$ and $\widetilde{\sigma}^2$ such that $\widetilde{\sigma}^1 \circ \widetilde{\sigma}^2$ is idempotent and that preserve the more informed than conditions, (13) and (14) in each direction.*

Proof. Suppose that S^1 is informationally equivalent to S^2. Then there exist mappings σ^1 and σ^2 showing that S^1 is more informed than S^2 and S^2 is more informed than S^1 respectively. Consider the finite Markov process $\xi \triangleq \sigma^2 \circ \sigma^1$. By the theory of Markov chains, there exists a limit if we keep applying this Markov process, $\xi^\infty \triangleq \left(\sigma^2 \circ \sigma^1\right)^\infty$. By construction, ξ^∞ is idempotent. Now let

$$\widetilde{\sigma}^1 \triangleq \sigma^1, \tag{37}$$

and

$$\widetilde{\sigma}^{2,\infty} \triangleq \xi^\infty \circ \sigma^2. \tag{38}$$

By hypothesis and construction, $\widetilde{\sigma}^1$ satisfies satisfies (13) and (14). Now, we show that

$$\widetilde{\sigma}^{2,1} \triangleq \xi \circ \sigma^2$$

also satisfies (13) and (14), and the repeated application of the ξ composition, then establishes that the limit $\widetilde{\sigma}^{2,\infty}$ satisfies (13) and (14). First we show that (13) holds for $\widetilde{\sigma}^{2,1}$, or:

$$\pi^1\left(t^1|\theta\right) = \sum_{t^2 \in T^2} \widetilde{\sigma}^{2,1}\left(t^1|t^2,\theta\right) \pi^2\left(t^2|\theta\right),$$

now

$$\begin{aligned}\sum_{t^2\in T^2}\widetilde{\sigma}^{2,1}\left(t^1|t^2,\theta\right)\pi^2\left(t^2|\theta\right) &= \sum_{t^2\in T^2}\left(\sum_{\widetilde{t}^1\in T^1}\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\right)\sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\right)\pi^2\left(t^2|\theta\right)\\ &= \sum_{\widetilde{t}^1\in T^1}\left(\sum_{\widetilde{t}^2\in T^2}\left(\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\sum_{t^2\in T^2}\sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\pi^2\left(t^2|\theta\right)\right)\right)\\ &= \sum_{\widetilde{t}^1\in T^1}\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\right)\pi^1\left(\widetilde{t}^1|\theta\right) \quad \text{by (13)}\\ &= \sum_{\widetilde{t}^2\in T^2}\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\left(\sum_{\widetilde{t}^1\in T^1}\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\pi^1\left(\widetilde{t}^1|\theta\right)\right)\right)\\ &= \sum_{\widetilde{t}^2\in T^2}\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\pi^2\left(\widetilde{t}^2|\theta\right) \quad \text{by (13)}\\ &= \pi^1\left(t^1|\theta\right),\end{aligned}$$

which is precisely the claim that we wanted to establish.

Second, we show that (14) holds for $\widetilde{\sigma}^{2,1}$, i.e. for:

$$\widetilde{\sigma}^2\left(t^1|t^2,\theta\right) = \sum_{\widetilde{t}^1\in T^1}\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\right)\sigma^2\left(\widetilde{t}^1|t^2,\theta\right),$$

we have that

$$\sum_{t^1_{-i}}\widetilde{\sigma}^2\left(t^1|t^2,\theta\right) = \sum_{t^1_{-i}}\left(\sum_{\widetilde{t}^1\in T^1}\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\right)\sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\right)$$

is independent of t^2_{-i} and θ. Now, we can rewrite

$$\begin{aligned}&\sum_{t^1_{-i}}\left(\sum_{\widetilde{t}^1\in T^1}\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\right)\sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\right)\\ =& \sum_{\widetilde{t}^1\in T^1}\sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\sum_{t^1_{-i}}\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\right),\end{aligned}$$

and by hypothesis

$$\sum_{t^1_{-i}}\sigma^2\left(t^1|\widetilde{t}^2,\theta\right) \triangleq \xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right),$$

is independent of $\widetilde{t}^2_{-i}$ and θ, and thus

$$\sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\sum_{t^1_{-i}}\sigma^2\left(t^1|\widetilde{t}^2,\theta\right)\right) = \sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2\in T^2}\left(\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\right)$$

$$= \sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2_i\in T^2_i}\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\sum_{\widetilde{t}^2_{-i}\in T^2_{-i}}\left(\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right.\right.\right.$$

and, again by hypothesis,

$$\sum_{\widetilde{t}^2_{-i}\in T^2_{-i}}\left(\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right) \triangleq \xi^1_i\left(\widetilde{t}^2_i|\widetilde{t}^1_i\right),$$

is independent of $\widetilde{t}^1_{-i}$ and θ, and thus

$$\begin{aligned}
&\sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2_i\in T^2_i}\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\sum_{\widetilde{t}^2_{-i}\in T^2_{-i}}\left(\sigma^1\left(\widetilde{t}^2|\widetilde{t}^1,\theta\right)\right)\right)\\
=\ &\sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2_i\in T^2_i}\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\xi^1_i\left(\widetilde{t}^2_i|\widetilde{t}^1_i\right)\right)\\
=\ &\sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2_i\in T^2_i}\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\xi^1_i\left(\widetilde{t}^2_i|\widetilde{t}^1_i\right)\right)
\end{aligned}$$

and hence

$$\sum_{\widetilde{t}^2_i\in T^2_i}\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\xi^1_i\left(\widetilde{t}^2_i|\widetilde{t}^1_i\right) \triangleq \xi_i\left(t^1_i|\widetilde{t}^1_i\right),$$

is independent of θ, and thus

$$\begin{aligned}
\sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\left(\sum_{\widetilde{t}^2_i\in T^2_i}\xi^2_i\left(t^1_i\left|\widetilde{t}^2_i\right.\right)\xi^1_i\left(\widetilde{t}^2_i|\widetilde{t}^1_i\right)\right) &= \sum_{\widetilde{t}^1 \in T^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\xi_i\left(t^1_i|\widetilde{t}^1_i\right)\\
&= \sum_{\widetilde{t}^1_i\in T^1_i}\left(\xi_i\left(t^1_i|\widetilde{t}^1_i\right)\sum_{\widetilde{t}^1_{-i}\in T^1_{-i}}\sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\right),
\end{aligned}$$

and again by hypothesis,

$$\sum_{\widetilde{t}^1_{-i}\in T^1_{-i}}\sigma^2\left(\widetilde{t}^1|t^2,\theta\right),$$

is independent of t^2_{-i} and θ, thus establishing the second claim. Now, we can repeat the operation by recursion to obtain the same result for $\widetilde{\sigma}^{2,\infty} \triangleq \xi^\infty \circ \sigma^2$. ■

We restate Lemma 3.4 of Lehrer, Rosenberg, and Shmaya (2011) in the pointwise version, i.e. conditional on θ, needed for our result.

Lemma 12 (Lemma 3.4, Lehrer, Rosenberg, and Shmaya (2011)) *Let T^1 and T^2 be two finite sets, let $\sigma^1 : \Theta \times T^1 \to \Delta\left(T^2\right)$ and $\sigma^2 : \Theta \times T^2 \to \Delta\left(T^1\right)$ be a pair of stochastic maps, and let $\pi^1\left(t^1|\theta\right) \in \Delta\left(T^1\right)$, $\pi^2\left(t^2|\theta\right) \in \Delta\left(T^2\right)$, be such that:*

$$\pi^2\left(t^2|\theta\right) = \sum_{t^1 \in T^1} \sigma^1\left(t^2|t^1,\theta\right)\pi^1\left(t^1|\theta\right), \qquad \pi^1\left(t^1|\theta\right) = \sum_{t^2 \in T^2} \sigma^2\left(t^1|t^2,\theta\right)\pi^2\left(t^2|\theta\right).$$

If $\xi = \sigma^2 \circ \sigma^1$ is idempotent, then:

$$\pi^1\left(t^1\,|\theta\right)\left(\sum_{\widetilde{t}^1} \xi\left(\widetilde{t}^1\,|t^1\right)\sigma^1\left(t^2|\widetilde{t}^1,\theta\right)\right) = \pi^2\left(t^2\,|\theta\right)\left(\sum_{\widetilde{t}^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\xi\left(t^1\,|\widetilde{t}^1\right)\right).$$

Definition 33 *Two information structures $S^1 = \left(\left(T_i^1\right)_{i=1}^I, \pi^1\right)$ and $S^2 = \left(\left(T_i^2\right)_{i=1}^I, \pi^2\right)$ are* simply informationally equivalent *if they are* informationally equivalent *and there exists a probability distribution $\nu : \Theta \to \Delta\left(T^1 \times T^2\right)$ such that, for each $k = 1, 2$, π^k is the marginal of ν on T^k:*

$$\pi^k\left(t^k|\theta\right) = \sum_{t^l} \nu\left(t^k, t^l|\theta\right); \tag{39}$$

and for each $k = 1, 2$ and $i = 1, .., I$,

$$\frac{\sum_{t_{-i}^l} \nu\left(\left(t_i^l, t_{-i}^l\right), \left(t_i^k, t_{-i}^k\right)|\theta\right)}{\sum_{t^l} \nu\left(t^k, t^l|\theta\right)}$$

is independent of t_{-i}^k and θ.

Lemma 13 *S^1 is informationally equivalent to S^2 if and only if they are simply informationally equivalent.*

Proof. Suppose $S^1 = \left(\left(T_i^1\right)_{i=1}^I, \pi^1\right)$ and $S^2 = \left(\left(T_i^2\right)_{i=1}^I, \pi^2\right)$ are simply informationally equivalent, then by Definition 33, they are informationally equivalent.

Suppose $S^1 = \left(\left(T_i^1\right)_{i=1}^I, \pi^1\right)$ and $S^2 = \left(\left(T_i^2\right)_{i=1}^I, \pi^2\right)$ are informationally equivalent, then we identify a distribution $\nu : \Theta \to \Delta\left(T^1 \times T^2\right)$ with the properties required by Definition 33. Let

$$\nu\left(t^1, t^2|\theta\right) \triangleq \pi^1\left(t^1\,|\theta\right)\left(\sum_{\widetilde{t}^1} \xi\left(\widetilde{t}^1\,|t^1\right)\sigma^1\left(t^2|\widetilde{t}^1,\theta\right)\right), \tag{40}$$

and we can also define

$$\overline{\nu}\left(t^1, t^2|\theta\right) \triangleq \pi^2\left(t^2\,|\theta\right)\left(\sum_{\widetilde{t}^1} \sigma^2\left(\widetilde{t}^1|t^2,\theta\right)\xi\left(t^1\,|\widetilde{t}^1\right)\right). \tag{41}$$

Now by the asserting of Lemma 3.4 of Lehrer, Rosenberg, and Shmaya (2011), it follows that for all θ, t_1, t_2

$$\nu\left(t^1, t^2|\theta\right) = \overline{\nu}\left(t^1, t^2|\theta\right).$$

We first establish the marginal property of ν. We first integrate out t^2, using (40):

$$\begin{aligned}
\sum_{t^2} \nu\left(t^1, t^2|\theta\right) &= \pi^1\left(t^1|\theta\right) \sum_{t^2} \left(\sum_{\widetilde{t}^1} \xi\left(\widetilde{t}^1|t^1\right) \sigma^1\left(t^2|\widetilde{t}^1, \theta\right) \right) \\
&= \pi^1\left(t^1|\theta\right) \left(\sum_{\widetilde{t}^1} \xi\left(\widetilde{t}^1|t^1\right) \sum_{t^2} \sigma^1\left(t^2|\widetilde{t}^1, \theta\right) \right) \\
&= \pi^1\left(t^1|\theta\right) \sum_{\widetilde{t}^1} \xi\left(\widetilde{t}^1|t^1\right) \\
&= \pi^1\left(t^1|\theta\right).
\end{aligned}$$

Next we integrate out t^1, using (41):

$$\begin{aligned}
\sum_{t^1} \nu\left(t^1, t^2|\theta\right) &= \sum_{t^1} \pi^2\left(t^2|\theta\right) \left(\sum_{\widetilde{t}^1} \sigma^2\left(\widetilde{t}^1|t^2, \theta\right) \xi\left(t^1|\widetilde{t}^1\right) \right) \\
&= \pi^2\left(t^2|\theta\right) \left(\sum_{\widetilde{t}^1} \sigma^2\left(\widetilde{t}^1|t^2, \theta\right) \left(\sum_{t^1} \xi\left(t^1|\widetilde{t}^1\right) \right) \right) \\
&= \pi^2\left(t^2|\theta\right) \left(\sum_{\widetilde{t}^1} \sigma^2\left(\widetilde{t}^1|t^2, \theta\right) \right) \\
&= \pi^2\left(t^2|\theta\right).
\end{aligned}$$

Now, we establish the independence property, namely that:

$$\frac{\sum_{t^l_{-i}} \nu\left(\left(t^k_i, t^k_{-i}\right), \left(t^l_i, t^l_{-i}\right)|\theta\right)}{\sum_{t^l} \nu\left(t^k, t^l|\theta\right)}, \tag{42}$$

is independent of t^k_{-i} and θ. We start with $l = 2$:

$$\nu\left(t^l, t^k|\theta\right) = \pi^k\left(t^k|\theta\right) \left(\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k \middle| t^k\right) \sigma^k\left(t^l|\widetilde{t}^k, \theta\right) \right)$$

and hence

$$\begin{aligned}
\frac{\sum_{t^l_{-i}} \nu\left(t^l, t^k|\theta\right)}{\sum_{t^l} \nu\left(t^k, t^l|\theta\right)} &= \frac{\sum_{t^l_{-i}} \pi^k\left(t^k\,|\theta\right)\left(\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,|t^k\right)\sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right)\right)}{\sum_{t^l} \pi^k\left(t^k\,|\theta\right)\left(\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,|t^k\right)\sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right)\right)} \\
&= \frac{\left(\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,|t^k\right)\sum_{t^l_{-i}} \sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right)\right)}{\left(\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,|t^k\right)\sum_{t^l} \sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right)\right)} \\
&= \frac{\left(\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,|t^k\right)\sum_{t^l_{-i}} \sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right)\right)}{\sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,|t^k\right)} \\
&= \sum_{\widetilde{t}^k} \xi\left(\widetilde{t}^k\,\middle|t^k\right)\sum_{t^l_{-i}} \sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right). \qquad (43)
\end{aligned}$$

But by the hypothesis of informational equivalence, each interior sum

$$\sum_{t^l_{-i}} \sigma^k\left(\left(t^l_i, t^l_{-i}\right)|\widetilde{t}^k, \theta\right)$$

is independent of θ and $\widetilde{t}^k_{-i}$, and hence so is any weighted sum over these individual sums.

Similarly, for $l = 1$, using (41):

$$\nu\left(t^l, t^k|\theta\right) = \pi^k\left(t^k\,|\theta\right)\left(\sum_{\widetilde{t}^l} \sigma^k\left(\widetilde{t}^l|t^k, \theta\right)\xi\left(t^l\,\middle|\widetilde{t}^l\right)\right)$$

and so

$$\begin{aligned}
\frac{\sum_{t^l_{-i}} \nu\left(t^l, t^k|\theta\right)}{\sum_{t^l} \nu\left(t^k, t^l|\theta\right)} &= \frac{\sum_{t^l_{-i}} \pi^k\left(t^k\,|\theta\right)\left(\sum_{\widetilde{t}^l} \sigma^k\left(\widetilde{t}^l|t^k, \theta\right)\xi\left(t^l\,|\widetilde{t}^l\right)\right)}{\sum_{t^l} \pi^k\left(t^k\,|\theta\right)\left(\sum_{\widetilde{t}^l} \sigma^k\left(\widetilde{t}^l|t^k, \theta\right)\xi\left(t^l\,|\widetilde{t}^l\right)\right)} \\
&= \frac{\sum_{t^l_{-i}} \left(\sum_{\widetilde{t}^l} \sigma^k\left(\widetilde{t}^l|t^k, \theta\right)\xi\left(t^l\,|\widetilde{t}^l\right)\right)}{\sum_{t^l} \left(\sum_{\widetilde{t}^l} \sigma^k\left(\widetilde{t}^l|t^k, \theta\right)\xi\left(t^l\,|\widetilde{t}^l\right)\right)} \\
&= \sum_{\widetilde{t}^l}\left(\sigma^k\left(\widetilde{t}^l|t^k, \theta\right)\sum_{t^l_{-i}} \xi\left(\left(t^l_i, t^l_{-i}\right)\middle|\widetilde{t}^l\right)\right), \qquad (44)
\end{aligned}$$

Now, we would like to show that the interior sum

$$\sum_{t^l_{-i}} \xi\left(\left(t^l_i, t^l_{-i}\right)\middle|\widetilde{t}^l\right) \qquad (45)$$

is independent of $\widetilde{t}^{l}_{-i}$, once we sum up over t^{l}_{-i}, or

$$\sum_{t^{l}_{-i}} \xi\left(\left(t^{l}_{i}, t^{l}_{-i}\right)\left|\widetilde{t}^{l}\right.\right) \triangleq \xi_i\left(t^{l}_{i}\left|\widetilde{t}^{l}_{i}\right.\right).$$

Now suppose that ξ is indeed the direct composition of $\sigma^2 \circ \sigma^1$. Then we can write for $l = 1:$

$$\xi\left(t^1\left|\widetilde{t}^1, \theta\right.\right) = \sum_{t^2} \sigma^2\left(t^1\left|t^2, \theta\right.\right)\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)$$

and picking up the sum of (45):

$$\begin{aligned}\sum_{t^1_{-i}} \xi\left(\left(t^1_i, t^1_{-i}\right)\left|\widetilde{t}^1\right.\right) &= \sum_{t^1_{-i}}\sum_{t^2} \sigma^2\left(t^1\left|t^2, \theta\right.\right)\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right) \\ &= \sum_{t^2}\left(\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)\sum_{t^1_{-i}}\sigma^2\left(t^1\left|t^2, \theta\right.\right)\right).\end{aligned}$$

Now, by the hypothesis of informational equivalence, we can write the interior sum:

$$\sum_{t^1_{-i}} \sigma^2\left(t^1\left|t^2, \theta\right.\right) \triangleq \xi_i\left(t^1_i\left|t^2_i\right.\right),$$

and hence

$$\begin{aligned}\sum_{t^2}\left(\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)\sum_{t^1_{-i}}\sigma^2\left(t^1\left|t^2, \theta\right.\right)\right) &= \sum_{t^2}\left(\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)\xi_i\left(t^1_i\left|t^2_i\right.\right)\right) \\ &= \sum_{t^2_i}\left(\xi_i\left(t^1_i\left|t^2_i\right.\right)\sum_{t^2_{-i}}\left(\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)\right)\right).\end{aligned}$$

Now, again by the hypothesis of informational equivalence:

$$\sum_{t^2_{-i}}\left(\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)\right) \triangleq \xi_i\left(t^2_i\left|\widetilde{t}^1_i\right.\right),$$

and hence

$$\sum_{t^2_i}\left(\xi_i\left(t^1_i\left|t^2_i\right.\right)\sum_{t^2_{-i}}\left(\sigma^1\left(t^2\left|\widetilde{t}^1, \theta\right.\right)\right)\right) = \sum_{t^2_i}\left(\xi_i\left(t^1_i\left|t^2_i\right.\right)\xi_i\left(t^2_i\left|\widetilde{t}^1_i\right.\right)\right)$$

and hence it is independent of $\widetilde{t}^{l}_{-i}$and θ. ■

With the equivalence result of Lemma 13, we can now restate Lemma 5 and then establish the corresponding result as follows.

Lemma 14 *Two information structures are simply informationally equivalent if and only if they are higher order belief equivalent.*

Proof. "if" Define $\nu : \Theta \to \Delta\left(T^1 \times T^2\right)$ by setting

$$\nu\left(t^1, t^2|\theta\right) = \begin{cases} \frac{\pi^1\left(t^1|\theta\right)\pi^2\left(t^2|\theta\right)}{\pi^*\left(t^*|\theta\right)}, \text{ if } f^1\left(t^1\right) = f^2\left(t^2\right) = t^* \\ 0, \text{ otherwise} \end{cases}$$

Now by (15), π^k is the marginal of ν on T^k for each k. Now

$$\sum_{t^j_{-i}} \nu\left(\left(t^1_i, t^1_{-i}\right), \left(t^2_i, t^2_{-i}\right)|\theta\right) = \begin{cases} \pi^k\left(t^k|\theta\right)\left(\dfrac{\sum\limits_{t^j_{-i}} \pi^j\left(t^j_i, t^j_{-i}|\theta\right)}{\sum\limits_{\left\{\widetilde{t}^j_i | f^j_i\left(\widetilde{t}^j_i\right) = f^j_i\left(t^j_i\right)\right\}} \sum\limits_{t^j_{-i}} \pi^j\left(\widetilde{t}^j_i, t^j_{-i}|\theta\right)}\right), \text{ if } f^j_i\left(t^j_i\right) = f^2_i\left(t^2_i\right) \\ 0, \text{ if } f^1_i\left(t^1_i\right) \neq f^2_i\left(t^2_i\right) \end{cases}$$

and so Δ

$$\frac{\sum\limits_{t^j_{-i}} \nu\left(\left(t^1_i, t^1_{-i}\right), \left(t^2_i, t^2_{-i}\right)|\theta\right)}{\pi^k\left(\left(t^k_i, t^k_{-i}\right)|\theta\right)} = \begin{cases} \left(\dfrac{\sum\limits_{t^j_{-i}} \pi^j\left(t^j_i, t^j_{-i}|\theta\right)}{\sum\limits_{\left\{\widetilde{t}^j_i | f^j_i\left(\widetilde{t}^j_i\right) = f^j_i\left(t^j_i\right)\right\}} \sum\limits_{t^j_{-i}} \pi^j\left(\widetilde{t}^j_i, t^j_{-i}|\theta\right)}\right), \text{ if } f^1_i\left(t^1_i\right) = f^2_i\left(t^2_i\right) \\ 0, \text{ if } f^1_i\left(t^1_i\right) \neq f^2_i\left(t^2_i\right) \end{cases}$$

But (16) implies that

$$\frac{\sum\limits_{t^j_{-i}} \pi^j\left(t^j_i, t^j_{-i}|\theta\right)}{\sum\limits_{\left\{\widetilde{t}^j_i | f^j_i\left(\widetilde{t}^j_i\right) = f^j_i\left(t^j_i\right)\right\}} \sum\limits_{t^j_{-i}} \pi^j\left(t^j_i, t^j_{-i}|\theta\right)}$$

is independent of t^j_{-i} and θ.

"only if" for each i and k, there exists $\xi^k_i : T^k_i \to \Delta\left(T^l_i\right)$ such that

$$\frac{\sum\limits_{t^j_{-i}} \nu\left(\left(t^1_i, t^1_{-i}\right), \left(t^2_i, t^2_{-i}\right)|\theta\right)}{\pi^k\left(\left(t^k_i, t^k_{-i}\right)|\theta\right)} = \xi^k_i\left(t^l_i | t^k_i\right)$$

Describe a equivalence class on $T^1_i \cup T^2_i$ by setting $t^k_i \sim_i t^l_i$ if either $\xi^k_i\left(t^l_i|t^k_i\right) > 0$ or $\xi^l_i\left(t^k_i|t^l_i\right) > 0$ and let $\sim^*_i$be the transitive closure of $\sim_i$. Let T^*_i be the set of equivalence classes of $\sim^*_i$and define $f^k_i : T^k_i \to T^*_i$ by letting $f^k_i\left(t^k_i\right)$ be the equivalence class containing t^k_i. Observe that $\nu\left(t^1, t^2|\theta\right) > 0$ implies that for each i, $t^1_i \sim^*_i t^2_i$. Thus we can define

$$\pi^*\left(t^*|\theta\right) = \sum_{\left\{t^1|f^1\left(t^1\right)=t^*\right\}} \nu\left(\left(t^1_i, t^1_{-i}\right), \left(t^2_i, t^2_{-i}\right)|\theta\right) = \sum_{\left\{t^2|f^2\left(t^2\right)=t^*\right\}} \nu\left(\left(t^1_i, t^1_{-i}\right), \left(t^2_i, t^2_{-i}\right)|\theta\right).$$

Now fix t_i^*. Observe that for each $t_i^1 \in T_i^1$ with $f_i^1\left(t_i^1\right) = t_i^*$,

$$\pi^1\left(t_i^1|\theta\right) = \sum_{\left\{t_i^2|f_i^2(t_i)=t_i^*\right\}} \xi_i^1\left(t_i^2|t_i^1\right)\pi^2\left(t_i^2|\theta\right),$$

and for each $t_i^2 \in T_i^2$ with $f_i^2\left(t_i^2\right) = t_i^*$,

$$\pi^2\left(t_i^2|\theta\right) = \sum_{\left\{t_i^1|f_i^1(t_i)=t_i^*\right\}} \xi_i^2\left(t_i^1|t_i^2\right)\pi^1\left(t_i^1|\theta\right).$$

This requires that (16) holds. ■

9.3 Proof of Theorem 2

Proof of Theorem 2. For a fixed finite information structure $S = \left((T_i)_{i=1}^I, \pi\right)$ and assuming a uniform prior on Θ, we define for each player i the set of possible higher order beliefs Π_i^*. For a type $t_i \in T_i$, write $\widehat{\pi}_i^1[t_i] \in \Delta(\Theta)$ for his posterior under a uniform prior on Θ, so

$$\widehat{\pi}_i^1[t_i](\theta) = \frac{\sum\limits_{t_{-i}\in T_{-i}} \pi\left((t_i, t_{-i})|\theta\right)}{\sum\limits_{\theta'\in\Theta, t_{-i}\in T_{-i}} \pi\left((t_i, t_{-i})|\theta'\right)}.$$

Write $\Pi_i^1 \subseteq \Delta(\Theta)$ for the range of $\widehat{\pi}_i^1$ and π_i^1 for a typical element of Π_i^1.

Write $\widehat{\pi}_i^2(t_i) \in \Delta\left(\Theta \times \left(\times_{j\neq i} \Pi_j^1\right)\right)$ for his belief over Θ and the first order beliefs of other players, so

$$\widehat{\pi}_i^2[t_i]\left(\theta, \pi_{-i}^1\right) = \frac{\sum\limits_{\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^1(t_j)=\pi_j^1 \text{ for each } j\neq i\right\}} \pi\left((t_i, t_{-i})|\theta\right)}{\sum\limits_{\theta'\in\Theta, t_{-i}\in T_{-i}} \pi\left((t_i, t_{-i})|\theta'\right)}.$$

Write $\Pi_i^2 \subseteq \Delta\left(\Theta \times \left(\times_{j\neq i} \Pi_j^1\right)\right)$ for the range of $\widehat{\pi}_i^2$ and π_i^2 for a typical element of Π_i^2.

Proceeding inductively for $k \geq 2$, write $\widehat{\pi}_i^k(t_i) \in \Delta\left(\Theta \times \left(\times_{j\neq i} \Pi_j^{k-1}\right)\right)$ for his belief over Θ and the $(k-1)$th order beliefs of other players, so

$$\widehat{\pi}_i^k[t_i]\left(\theta, \pi_{-i}^{k-1}\right) = \frac{\sum\limits_{\left\{t_{-i}\in T_{-i}|\widehat{\pi}_j^{k-1}(t_j)=\pi_j^{k-1} \text{ for each } j\neq i\right\}} \pi\left((t_i, t_{-i})|\theta\right)}{\sum\limits_{\theta'\in\Theta, t_{-i}\in T_{-i}} \pi\left((t_i, t_{-i})|\theta'\right)}.$$

Write $\Pi_i^k \subseteq \Delta\left(\Theta \times \left(\times_{j\neq i} \Pi_j^{k-1}\right)\right)$ for the range of $\widehat{\pi}_i^k$ and π_i^k for a typical element of Π_i^k.

Each $\widehat{\pi}_i^k$ generates a partition T_i which becomes more refined as k increases. Since each T_i is finite, the information structure has a *depth* K, so that the depth of the information structure S is smallest integer K such that

$$\widehat{\pi}_i^k(t_i) = \widehat{\pi}_i^k(t_i') \Leftrightarrow \widehat{\pi}_i^K(t_i) = \widehat{\pi}_i^K(t_i')$$

for all i and $k \geq K$. Let $\pi_i^*[t_i]$ be a list of the first Kth level beliefs of player i, so

$$\pi_i^*[t_i] = \left(\widehat{\pi}_i^k[t_i]\right)_{k=1}^K.$$

Let $\Pi_i^* \subseteq \underset{k=1,\ldots,K}{\times} \Pi_i^k$ be the range of π_i^*.

For the fixed information structure $S = \left((T_i)_{i=1}^I, \pi\right)$ and $\varepsilon > 0$, we will construct a finite "higher order beliefs game" $G_{S,\varepsilon}$. This is a variation and simplification of such a game used in Dekel, Fudenberg, and Morris (2006). For any finite set X, the Euclidean distance between two points $\zeta, \zeta' \in \Delta(X)$ is defined as

$$\|\zeta - \zeta'\| = \sqrt{\sum_{x \in X} \left(\zeta(x) - \zeta'(x)\right)^2}$$

A set of probability distributions $\Xi \subseteq \Delta(X)$ is said to be an ε-grid of $\Delta(X)$ if every point in $\Delta(X)$ is within ε of a point in Ξ. Now let A_i^1 be any ε-grid of $\Delta(\Theta)$ includes Π_i^1, i.e., all possible first order beliefs of agent i in information structure S. Let A_i^2 be any ε-grid of $\Delta\left(\Theta \times \left(\underset{j \neq i}{\times} A_j^1\right)\right)$ including Π_i^2. Inductively, for each $k = 2, \ldots, K$, let A_i^k be any ε-grid of $\Delta\left(\Theta \times \left(\underset{j \neq i}{\times} A_j^{k-1}\right)\right)$ including Π_i^k. Let

$$A_i = \underset{k=1,\ldots,K}{\times} A_i^k.$$

We want to give players an incentive to truthfully announces their higher order beliefs. We write $a_i = \left(a_i^1, \ldots, a_i^K\right)$ for a typical element of A_i. Let

$$u_i\left((a_i, a_{-i}), \theta\right) = u_i^1\left(a_i^1, \theta\right) + \sum_{k=2}^K u_i^k\left(a_i^k, a_{-i}^{k-1}\right).$$

Now let

$$u_i^1\left(a_i^1, \theta\right) = 2a_i^1(\theta) - \sum_{\theta' \in \Theta}\left(a_i^1(\theta')\right)^2$$

and, for $k \geq 2$,

$$u_i^k\left(a_i^k, a_{-i}^{k-1}\right) = 2a_i^k\left(\theta, a_{-i}^{k-1}\right) - \sum_{\theta' \in \Theta, \widetilde{a}_{-i}^{k-1} \in A_{-i}^{k-1}} \left(a_i^k\left(\theta', \widetilde{a}_{-i}^{k-1}\right)\right)^2.$$

Write ψ_0 be the uniform prior on Θ. This completes the description of the game $G_{S,\varepsilon} = \left((A_i, u_i)_{i=1}^I, \psi_0\right)$.

In the special case of information structure S, the game $(G_{S,\varepsilon}, S)$ has a BCE where each type truthfully reports his true first K levels of higher order beliefs. Observe that for every $\varepsilon > 0$, the game $(G_{S,\varepsilon}, S)$ has a BCE satisfying

$$\nu(a,t,\theta) = \begin{cases} \psi_0(\theta)\,\pi(t|\theta), & \text{if} \quad a = \pi^*[t]; \\ 0, & \text{if} \quad \text{otherwise}. \end{cases}.$$

This is a BCE because if all other players truthfully announce their types, a player's best response is to truthfully announce his type, as he is advised to do. Note that in this BCE, each players' actions are restricted to the set $\Pi_i^* \subset A_i$. This BCE induces the action-state distribution $\mu^* \in \Delta(A \times \Theta)$ satisfying

$$\mu^*(a,\theta) = \begin{cases} \psi_0(\theta) \displaystyle\sum_{\{t:a=\pi^*[t]\}} \pi(t|\theta), & \text{if} \quad a = \pi^*[t]; \\ 0, & \text{if} \quad \text{otherwise}. \end{cases} \tag{46}$$

Now fix any information structure S' which is BCE-larger than S. Then it must be true that, for every $\varepsilon > 0$, the incomplete information game $(G_{S,\varepsilon}, S')$ has a BCE inducing the action-state distribution μ^*, defined in equation (46). We will show that this implies that S is more informed than S'.

For any $\nu' \in \Delta(A \times T' \times \Theta)$, write $\nu'(\cdot|a_i, t_i')$ for the induced distribution over $A_{-i} \times \Theta$ of a type t_i' of player i advised to take action $a_i \in A_i$, so that

$$\nu'\left(a_{-i},\theta|a_i,t_i'\right) = \frac{\displaystyle\sum_{t'_{-i}} \nu'\left((a_i,a_{-i}),\left(t_i',t_{-i}'\right),\theta\right)}{\displaystyle\sum_{t'_{-i},a_{-i},\widetilde{\theta}} \nu'\left((a_i,a_{-i})),\left(t_i',t_{-i}'\right),\widetilde{\theta}\right)}.$$

Now for each $\varepsilon > 0$ and let ν_ε' be any BCE of $(G_{S,\varepsilon}, S')$ inducing the action-state distribution μ^*. Note that this is an element of $\Delta(\Pi^* \times T' \times \Theta)$, since by assumption only action profiles in Π^* are chosen under μ^*. Note that a hierarchy of beliefs $\pi_i^* = \left(\pi_i^{*1}, ..., \pi_i^{*K}\right)$ can be identified in the usual way with a probability distribution over $\Pi_{-i}^* \times \Theta$. Now it is a property of best responses in the game $G_{S,\varepsilon}$ that for each type t_i' following a recommendation to play action π_i^* in ν_ε', we must have

$$\left\|\nu_\varepsilon'\left(\cdot|\pi_i^*,t_i'\right) - \pi_i^*\right\| \leq \varepsilon. \tag{47}$$

To see why, note that a first necessary condition is that player i with type t_i' and recommendation $\pi_i^* = \left(\pi_i^{*1}, ..., \pi_i^{*K}\right)$ has an incentive to set a_i^1 equal to π_i^{*1}. A necessary condition for this is that his beliefs on Θ are within ε of π_i^{*1}. Now we can argue inductively that, for each $k = 2, .., K$, a necessary condition is that player i with type t_i has an incentive to set $a_i^k = \pi_i^{*k}$. A necessary condition for this is that his beliefs on $A_{-i}^{k-1} \times \Theta$ are within ε of π_i^{*k}. But this last condition when $k = K$ is (47).

For each $\varepsilon > 0$, ν_ε' satisfies consistency and is an element of the compact set $\Delta(\Pi^* \times T' \times \Theta)$. Thus we can find a subsequence of the ν_ε' converging to a limit ν' which induces μ^*, is a BCE of $(G_{S,\varepsilon}, S')$ for every $\varepsilon > 0$ and has the property that for each type t_i' following a recommendation to play action π_i^* in ν', we have

$$\nu'\left(\cdot|\pi_i^*, t_i'\right) = \pi_i^*. \tag{48}$$

Now

$$\begin{aligned}
\nu'\left(\pi^*, t_i', \theta\right) &= \sum_{t_{-i}' \in T_{-i}} \nu'\left(\left(\pi_i^*, \pi_{-i}^*\right), \left(t_i', t_{-i}'\right), \theta\right) \\
&= \nu'\left(\pi_i^*, t_i'\right) \nu'\left(\pi_{-i}^*, \theta|\pi_i^*, t_i'\right) \\
&= \nu'\left(\pi_i^*, t_i'\right) \nu'\left(\pi_{-i}^*, \theta|\pi_i^*\right), \text{ by (48)}
\end{aligned} \tag{49}$$

But now define $\sigma : T \times \Theta \to \Delta\left(T'\right)$ by

$$\sigma\left(t'|t, \theta\right) = \nu'\left(t'|\pi^*\left(t\right), \theta\right).$$

Now

$$\begin{aligned}
\sigma\left(t_i'|\left(t_i, t_{-i}\right), \theta\right) &= \frac{\nu'\left(\pi_i^*\left(t_i\right), t_i'\right) \nu'\left(\pi_{-i}^*\left(t_{-i}\right), \theta|\pi_i^*\left(t_i\right)\right)}{\sum_{\widetilde{t}_i'} \nu'\left(\pi_i^*\left(t_i\right), \widetilde{t}_i'\right) \nu'\left(\pi_{-i}^*\left(t_{-i}\right), \theta|\pi_i^*\left(t_i\right)\right)}, \text{ by (49)} \\
&= \frac{\nu'\left(\pi_i^*\left(t_i\right), t_i'\right)}{\sum_{\widetilde{t}_i'} \nu'\left(\pi_i^*\left(t_i\right), \widetilde{t}_i'\right)} \\
&= \nu'\left(t_i'|\pi_i^*\left(t_i\right)\right).
\end{aligned}$$

But now if we define $\xi_i : T_i \to \Delta\left(T_{-i}'\right)$ by

$$\xi_i\left(t_i'|t_i\right) = \nu'\left(t_i'|\pi_i^*\left(t_i\right)\right),$$

we have established that S is more informed than S'. ■

9.4 Example of More Informed Than Information Structure

The following is a robust example of non-redundant information structures where S' is not a non-communicating garbling of S in the sense of Lehrer, Rosenberg, and Shmaya (2010), (2011), but S is more informed than S' in the sense of Definition 22 and thus - by Theorem 2 - S' is BCE-richer than S.

Suppose that there is uniform prior on $\Theta = \{\theta_1, \theta_2\}$. Information structure S has $T_1 = \{t_{11}, t_{12}\}$, $T_2 = \{t_{21}, t_{22}\}$ and $\pi : \Theta \to \Delta\left(T\right)$ given by

$\pi\left(t\|\theta_1\right)$	t_{21}	t_{22}	$\pi\left(t\|\theta_2\right)$	t_{21}	t_{22}
t_{11}	$\frac{4}{9}$	$\frac{2}{9}$	t_{11}	$\frac{1}{9}$	$\frac{2}{9}$
t_{12}	$\frac{2}{9}$	$\frac{1}{9}$	t_{12}	$\frac{2}{9}$	$\frac{4}{9}$

This information structure simply has each agent observing a conditionally independent signal with "accuracy" $\frac{2}{3}$.

Information structure S' has $T_1' = \{t_{11}', t_{12}'\}$, $T_2' = \{t_{21}', t_{22}'\}$ and $\pi' : \Theta \to \Delta(T')$ given by

$\pi'(t'\|\theta_1)$	t_{21}'	t_{22}'
t_{11}'	$\frac{13}{27}$	$\frac{2}{27}$
t_{12}'	$\frac{2}{27}$	$\frac{10}{27}$

$\pi'(t'\|\theta_2)$	t_{21}'	t_{22}'
t_{11}'	$\frac{1}{27}$	$\frac{11}{27}$
t_{12}'	$\frac{11}{27}$	$\frac{4}{27}$

First, we identify the garblings relating S and S' from each agent's point of view. Observe that

$$\begin{aligned}
\pi(t_{11}|\theta_1) &= \sum_{t_2} \pi((t_{11}, t_2)|\theta_1) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3} \\
\pi(t_{11}|\theta_2) &= \sum_{t_2} \pi((t_{11}, t_2)|\theta_2) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3} \\
\pi'(t_{11}'|\theta_1) &= \sum_{t_2'} \pi'((t_{11}', t_2')|\theta_1) = \frac{13}{27} + \frac{2}{27} = \frac{15}{27} = \frac{5}{9} \\
\pi'(t_{11}'|\theta_2) &= \sum_{t_2'} \pi'((t_{11}', t_2')|\theta_1) = \frac{1}{27} + \frac{11}{27} = \frac{12}{27} = \frac{4}{9}
\end{aligned}$$

Thus there is a unique $\xi_1 : T_1 \to \Delta(T_1')$ satisfying

$$\pi'(t_1'|\theta) = \sum_{t_1} \xi_1(t_1'|t_1)\, \pi(t_1|\theta)$$

for all t_1' and θ, and it is given by

$\xi_1(t_1'\|t_1)$	t_{11}'	t_{12}'
t_{11}	$\frac{2}{3}$	$\frac{1}{3}$
t_{12}	$\frac{1}{3}$	$\frac{2}{3}$

(50)

Symmetrically, we have

$$\begin{aligned}
\pi(t_{21}|\theta_1) &= \sum_{t_1} \pi((t_1, t_{21})|\theta_1) = \frac{4}{9} + \frac{2}{9} = \frac{2}{3} \\
\pi(t_{21}|\theta_2) &= \sum_{t_1} \pi((t_1, t_{21})|\theta_2) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3} \\
\pi'(t_{21}'|\theta_1) &= \sum_{t_1'} \pi'((t_1', t_{21}')|\theta_1) = \frac{13}{27} + \frac{2}{27} = \frac{15}{27} = \frac{5}{9} \\
\pi'(t_{21}'|\theta_2) &= \sum_{t_1'} \pi'((t_1', t_{21}')|\theta_1) = \frac{1}{27} + \frac{11}{27} = \frac{12}{27} = \frac{4}{9}
\end{aligned}$$

Thus there is a unique $\xi_2 : T_2 \to \Delta(T_2')$ satisfying

$$\pi'(t_2'|\theta) = \sum_{t_2} \xi_2(t_2'|t_2)\, \pi(t_2|\theta)$$

for all t_2' and θ, and it is given by

$\xi_2(t_2'\|t_2)$	t_{21}'	t_{22}'
t_{21}	$\frac{2}{3}$	$\frac{1}{3}$
t_{22}	$\frac{1}{3}$	$\frac{2}{3}$

(51)

Thus from each individual's point of view, under S', he is simply observing a noisy version (with accuracy $\frac{2}{3}$) of the original signal (with accuracy $\frac{2}{3}$).

Now consider the mapping $\sigma : T \times \Theta \to \Delta(T')$.

$\sigma(t'\|t,\theta)$	(t_{11}', t_{21}')	(t_{11}', t_{22}')	(t_{12}', t_{21}')	(t_{12}', t_{22}')
$(\theta_1, t_{11}, t_{21})$	$\frac{2}{3}$	0	0	$\frac{1}{3}$
$(\theta_1, t_{11}, t_{22})$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
$(\theta_1, t_{12}, t_{21})$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
$(\theta_1, t_{12}, t_{22})$	$\frac{1}{3}$	0	0	$\frac{2}{3}$
$(\theta_2, t_{11}, t_{21})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$(\theta_2, t_{11}, t_{22})$	0	$\frac{2}{3}$	$\frac{1}{3}$	0
$(\theta_2, t_{12}, t_{21})$	0	$\frac{1}{3}$	$\frac{2}{3}$	0
$(\theta_2, t_{12}, t_{22})$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Now if t is drawn according to S and t' is drawn according to σ, we get the following joint distribution $\nu : \Theta \to \Delta(T' \times T)$:

$\nu(t', t\|\theta)$	(t_{11}', t_{21}')	(t_{11}', t_{22}')	(t_{12}', t_{21}')	(t_{12}', t_{22}')
$(\theta_1, t_{11}, t_{21})$	$\frac{8}{27}$	0	0	$\frac{4}{27}$
$(\theta_1, t_{11}, t_{22})$	$\frac{2}{27}$	$\frac{2}{27}$	0	$\frac{2}{27}$
$(\theta_1, t_{12}, t_{21})$	$\frac{2}{27}$	0	$\frac{2}{27}$	$\frac{2}{27}$
$(\theta_1, t_{12}, t_{22})$	$\frac{1}{27}$	0	0	$\frac{2}{27}$
$(\theta_2, t_{11}, t_{21})$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{27}$	0
$(\theta_2, t_{11}, t_{22})$	0	$\frac{4}{27}$	$\frac{2}{27}$	0
$(\theta_2, t_{12}, t_{21})$	0	$\frac{2}{27}$	$\frac{4}{27}$	0
$(\theta_2, t_{12}, t_{22})$	0	$\frac{4}{27}$	$\frac{4}{27}$	$\frac{4}{27}$

(52)

Observe that the marginal of ν on T' is:

$\text{marg}_\nu(t'\|\theta_1)$	t_{21}'	t_{22}'
t_{11}'	$\frac{13}{27}$	$\frac{2}{27}$
t_{12}'	$\frac{2}{27}$	$\frac{10}{27}$

$\text{marg}_\nu(t'\|\theta_2)$	t_{21}'	t_{22}'
t_{11}'	$\frac{1}{27}$	$\frac{11}{27}$
t_{12}'	$\frac{11}{27}$	$\frac{4}{27}$

This is simply π'.

Also observe that σ satisfies the property that

$$\sum_{t_2'} \sigma\left(\left(t_1', t_2'\right) \mid \left(t_1, t_2\right), \theta\right) = \xi_1\left(t_1' | t_1\right) \text{ for each } t_2 \text{ and } \theta,$$

and

$$\sum_{t_1'} \sigma\left(\left(t_1', t_2'\right) \mid \left(t_1, t_2\right), \theta\right) = \xi_2\left(t_2' | t_2\right) \text{ for each } t_1 \text{ and } \theta.$$

We have now established that S is more informed than S' and thus that, for every game G, the set of Bayes correlated equilibria of (G, S') contains the Bayes correlated equilibria of (G, S).

However, S' is not a "non-communicating garbling" of S. We will show this by contradiction. For S' to be a non-communicating garbling of S, there would have to exist $\xi : T \to \Delta(T')$ satisfying

$$\pi'\left(t' | \theta\right) = \sum_{t} \xi\left(t' | t\right) \pi\left(t | \theta\right)$$

with

$$\sum_{t_2'} \xi\left(\left(t_1', t_2'\right) \mid \left(t_1, t_2\right)\right) \text{ independent of } t_2 \tag{53}$$

and

$$\sum_{t_1'} \xi\left(\left(t_1', t_2'\right) \mid \left(t_1, t_2\right)\right) \text{ independent of } t_1 \tag{54}$$

But

$$\begin{aligned}
\sum_{t_2'} \pi'\left(\left(t_1', t_2'\right) | \theta\right) &= \sum_{t} \sum_{t_2'} \xi\left(t_1', t_2' | t_1, t_2\right) \pi\left(t | \theta\right) \\
&= \sum_{t} \sum_{t_2'} \xi\left(t_1', t_2' | t_1\right) \pi\left(t | \theta\right), \text{ by (53)} \\
&= \sum_{t_1} \sum_{t_2'} \xi\left(t_1', t_2' | t_1\right) \sum_{t_2} \pi\left(\left(t_1, t_2\right) | \theta\right)
\end{aligned}$$

But we observed earlier that (50) is the unique expression $\xi_1 : T_1 \times \Delta(T_1')$ satisfying this equation, so we have

$$\sum_{t_2'} \xi\left(\left(t_1', t_2'\right) \mid \left(t_1, t_2\right)\right) = \xi_1\left(t_1' | t_1\right) \text{ for each } t_2 \tag{55}$$

and, by a symmetric argument,

$$\sum_{t_1'} \xi\left(\left(t_1', t_2'\right) \mid \left(t_1, t_2\right)\right) = \xi_2\left(t_2' | t_2\right) \text{ for each } t_1 \tag{56}$$

where ξ_2 is given by (51).

Let us focus on the probability of a fixed profile of S' signals (t'_{11}, t'_{21}) and write α_{jk} for the probability of (t'_{11}, t'_{21}) conditional on (t_{1j}, t_{2k}) under ξ, i.e.,

$$\alpha_{jk} = \xi\left((t'_{11}, t'_{21}) \,|\, (t_{1j}, t_{2k})\right) \tag{57}$$

Now

$$\begin{aligned}
\alpha_{11} &\leq \min\left(\xi_1\left(t'_{11}|t_{11}\right), \xi_2\left(t'_{21}|t_{21}\right)\right) = \xi_1\left(t'_{11}|t_{11}\right) = \frac{2}{3} \\
\alpha_{12} &\leq \min\left(\xi_1\left(t'_{11}|t_{11}\right), \xi_2\left(t'_{21}|t_{22}\right)\right) = \xi_2\left(t'_{21}|t_{22}\right) = \frac{1}{3} \\
\alpha_{21} &\leq \min\left(\xi_1\left(t'_{11}|t_{12}\right), \xi_2\left(t'_{21}|t_{21}\right)\right) = \xi_1\left(t'_{11}|t_{12}\right) = \frac{1}{3} \\
\alpha_{22} &\leq \min\left(\xi_1\left(t'_{11}|t_{12}\right), \xi_2\left(t'_{21}|t_{22}\right)\right) = \xi_1\left(t'_{11}|t_{12}\right) = \frac{1}{3}
\end{aligned} \tag{58}$$

But

$$\pi'\left(\left(t'_{11}, t'_{21}\right)|\theta_1\right) = \sum_t \xi\left(\left(t'_{11}, t'_{21}\right)|t\right)\pi\left(t|\theta_1\right)$$

requires we must have

$$\frac{13}{27} = \frac{4}{9}\alpha_{11} + \frac{2}{9}\alpha_{12} + \frac{2}{9}\alpha_{21} + \frac{1}{9}\alpha_{22}.$$

Combined with (58), this requires $\alpha_{11} = \frac{2}{3}$, $\alpha_{12} = \frac{1}{3}$, $\alpha_{21} = \frac{1}{3}$ and $\alpha_{22} = \frac{1}{3}$. However,

$$\pi'\left(\left(t'_{11}, t'_{21}\right)|\theta_2\right) = \sum_t \xi\left(\left(t'_{11}, t'_{21}\right)|t\right)\pi\left(t|\theta_2\right)$$

requires we must have

$$\frac{1}{27} = \frac{1}{9}\alpha_{11} + \frac{2}{9}\alpha_{12} + \frac{2}{9}\alpha_{21} + \frac{4}{9}\alpha_{22}.$$

which is a contradiction.

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