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Application of the American Real Flexible Switch Options Methodology

A Generalized Approach*

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Abstract

The paper deals with the inclusion of flexibility in financial decision-making under risk. It describes the application of the real options methodology with the possibility of sequential multinomial decision-making. The basic intention is to describe and apply a generalized approach and methodology of the flexibility modeling and valuation based on multiple choices and non-symmetrical switching costs under risk. The stochastic dynamic Bellman optimization principle is explained and applied. The optimization criterion of the present expected value is derived and used. Likewise, an option valuation approach based on replication strategy and risk-neutral probability is applied. An illustrative example of the application of the real multinomial flexible non-symmetrical switch options methodology is presented for three chosen modes. The option flexible values are computed. The usefulness, effectiveness, and suitability of applying the generalized flexibility model in company valuation and project evaluation is verified and confirmed. The significance of applying the generalized methodology in transition market economies is discussed and verified.

1. Introduction

Real options represent a flexible approach to financial decision-making in the strategic decisions of non-financial companies concerning real assets (assets, debt, equity, investments, commodity, electricity, temperature, land, research costs, technology, processes, and production). Flexibility is an important aspect that is often neglected in financial decision-making and valuation. This finding concerns both highly developed countries and transition market economies.

The real options methodology is based on the financial options methodology applied to real assets. In comparison with the traditional passive strategies, it takes into consideration active measures in real projects managing in the future. Examples of flexible actions include project abandonment, temporary shutdown of production, expansions, contractions, changes in technological processes, the design of production structure parameters, sales, purchases, etc.

The real options topic has been a focus of attention for the academic and managerial community for several years and represents a fundamental innovation in corporate finance. The key texts concerning the real options methodology and applications include for instance (Black, Scholes, 1973), (Brennan, Schwarz, 1985), (McDonald, Siegel, 1986), (Kulatilaka, 1993), (Kulatilaka, Trigeorgis, 1994), (Dixit, Pindyck, 1994), (Sick, 1995), (Smith, Nau, 1995), (Trigeorgis, 1998), (Brennan, Trigeorgis, 1999), (Bellalah, 2001), (Howel et al., 2001), (Ronn, 2002), (Vollert, 2002), (Smit, Trigeorgis, 2004), and (Brandao, Dyer, 2005).

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We can distinguish several types of real options models, from simplified to complex ones. These models can be categorized according to the following basic characteristics: (i) valuation approach (replication strategy, arbitrage principle, martingale approach, complete market, incomplete market); (ii) payoff function (plain vanilla, path dependent, look back, barrier); (iii) choice variability (binary, chooser, exchange, multiple switching); (iv) inclusion of switching costs (neglected, symmetrical, non-symmetrical); (v) number of underlying factors (one-factor, spread, rainbow, basket, multi-factor); (vi) random process (Brown, mean-reverting, jump diffusion, Schwartz); (vii) exercise moment possibility (European, American, Bermudian, swing); (viii) mathematical solution (continuous, discrete-binomial, discrete-trinomial, discrete--multinomial, random scenarios simulation). Furthermore, the published particular features and research findings will be introduced later in the appropriate parts of the paper.

The intention and motivation of the paper is to propose, describe, and apply a specific generalized complex real options model. The novelty and uniqueness of the model consists in the particular combination of basic characteristics introduced. The model is characterized by: (i) a replication valuation strategy under a complete market; (ii) a plain vanilla payoff function; (iii) multiple switching among modes; (iv) non-symmetrical switching costs; (v) one-factor; (vi) a geometric Brown process; (vii) an American option type; (viii) discrete-binomial approximation.

The crucial features of the proposed model are multiple choices, American options, and non-symmetrical switching costs in particular. These real option models come out of the stochastic dynamic programming based on Bellman optimization principle. This approach is for example introduced in (Kulatilaka, 1988), (Trigeorgis, 1991), (Dixit, Pindyck, 1994), (Kulatilaka, Trigeorgis, 1994), (Smith, Nau, 1995), (Trigeorgis, 1998), (Baldwin, Clark, 2000), (Duckworth, Zervos, 2001), (Chung-Li Tseng, Barz, 2002), (Weston, 2002), (Dangla, Wirl, 2004), (Fontes, 2008), (Dulluri, Raghavan, 2008), and (Erraisa, Sadowsky, 2008).

Another goal of the paper is to demonstrate the application necessity and usefulness of the proposed methodology in financial decision-making in both transition and highly developed market economies.

The paper is organized as follows. The first section is devoted to the description of the real options methodology without switching costs based on risk-neutral probability and the replication strategy. A generalized methodology of sequential multinomial American options, non-symmetrical switching options and cash-flow will be described in the second section. Then a discrete stochastic dynamic Bellman optimization model with the present expected optimization criterion will be applied. The last section gives an illustrative example of the applied generalized methodology for three modes, including an evaluation of flexibility value and a sensitivity analysis.

2. Description of Real Option Valuation without Switching Costs

The generalized valuation principle is called the martingale principle; see e.g. (Musiela, Rutkowski, 1997) and (Luenberger, 1998). The principle is defined such that a value has to be equal to its expected future value, implying that the random process shows no trend. In case of a risk-neutral approach, the martingale methodology is defined as the ratio of the random value and a risk-free asset. So, after rearranging

$$V_t = e^{-r\cdot dt} \cdot \widehat{E}\left(V_{t+dt}\right) \tag{1}$$

where V_t is the value, r is the risk-free rate, dt is the time interval, and $\widehat{E}\left(V_{t+dt}\right)$ is the risk-neutral expected value. We can gain the same result for a complete market under the replication valuation strategy or the arbitrage principle.

2.1 Derivation and Description of Replication Strategy

One of the basic approaches to derivatives valuation under a complete market is the replication strategy; see e.g. (Sick, 1995) and (Smith, Nau, 1995). Having derived the replication strategy, we assume a compact (effective) market and an asset that pays income (dividends, coupons, etc.) proportional to the asset price. The replication strategy will be applied to a discrete binomial model and one risk (random) factor. The model is a discrete one and for the sake of simplicity intra-interval continuous compounding is applied.

The replication strategy is based on creating a portfolio from an underlying asset S and a risk-free asset B such that for every situation the derivative value is replicated; this means that the derivative value equals the portfolio value.

The portfolio value at appraising time t is $\Pi_t \equiv a \cdot S_t + B_t = f_t$;

the portfolio value at time $t + dt$ given a rising price is

$\Pi_{t+dt} \equiv a \cdot S^u_{t+dt} + B_t \cdot e^{r\cdot dt} = f^u_{t+dt}$

and the portfolio value at time $t + dt$ given a declining price is

$\Pi_{t+dt} \equiv a \cdot S^d_{t+dt} + B_t \cdot e^{r\cdot dt} = f^d_{t+td}$

where Π_t is the portfolio value, S is the underlying asset value, a is the amount of underlying asset, B is the risk-free asset value, f is the derivative value, r is the risk--free rate, u (d) are the indexes of growth (decline) of the underlying asset, and S^u_{t+dt} $\left(S^d_{t+dt}\right)$ are their prices in up-movements (down-movements).

By solving the three equations for variables a, B, and f_t, we can get a general formula for the derivative price,

$$f_t = e^{-r\cdot dt} \cdot \left\{ f^u_{t+dt} \cdot \left[\frac{e^{r\cdot dt} \cdot S_t - S^d_{t+dt}}{S^u_{t+dt} - S^d_{t+dt}} \right] + f^d_{t+dt} \cdot \left[\frac{S^u_{t+dt} - e^{r\cdot dt} \cdot S_t}{S^u_{t+dt} - S^d_{t+dt}} \right] \right\} \tag{2}$$

This is the general formula for derivative price valuation by the replication strategy, which can be written as follows,

$$f_t = e^{-r\cdot dt} \cdot \left[f^u_{t+dt} \cdot \left(\widehat{p}\right) + f^d_{t+dt} \cdot \left(1 - \widehat{p}\right)\right], \text{ or } f_t = e^{-r\cdot dt} \cdot \widehat{E}\left(f_{t+dt}\right) \tag{3}$$

Here $$\widehat{p} = \frac{e^{r\cdot dt} \cdot S_t - S^d_{t+dt}}{S^u_{t+dt} - S^d_{t+dt}} \tag{4}$$

implies the risk-neutral probability of an up-movement and $\widehat{E}\left(f_{t+dt}\right)$ is the risk-neutral expected value.

The derivative price is determined as the present value of the expected value in the following period. This probability can be considered neither as market growth nor as a subjective probability. Due to (3) the derivative price is equal to the present value of the risk-neutral expected value of the subsequent period, which conforms to the generalized martingale principle, see (1).

There are several ways of calibrating the generalized model, see e.g. (Cox, Ross, Rubinstein, 1979), (Jarrow, Rudd, 1983), (Boyle, 1988), (Boyle, Evnine, Gibbs, 1989), (Madan, Milne, Shefrin, 1989), (Kamrad, Ritchken, 1991), (Trigeorgis, 1991), (Kulatilaka, 1993), (Smith, Nau, 1995), and (Luenberger, 1998).

Applying the approach of Cox, Ross, and Rubinstein (1979) we can express the underlying asset price, under proportional continuous income *c*, according to the geometric Brown process as follows: $S_{t+dt}^{u} = S_t \cdot e^{u+c\cdot dt}$; $S_{t+dt}^{d} = S_t \cdot e^{d+c\cdot dt}$. Because $e^u = e^{\sigma\cdot\sqrt{dt}}$, $e^d = e^{-\sigma\cdot\sqrt{dt}}$, then after substitution into (4) and re-arranging we can get the particular risk-neutral probability formula

$$\widehat{p} = \frac{e^{(r-c)\cdot dt} - e^d}{e^u - e^d} \tag{5}$$

This formula can be generalized after substituting for the risk-neutral probability growth parameter $\widehat{g} = (r-c)\cdot dt$, as follows $\widehat{p} = \dfrac{e^{\widehat{g}} - e^d}{e^u - e^d}$.

2.2 Valuation Procedure for American Options without Switching Costs

Applying the replication strategy described above, the option pricing procedure using the discrete binomial model respecting a stochastic dynamic programming model and risk-neutral probability can be divided into the following steps.

(i) Determination of the risk-neutral growth parameter $\widehat{g}$.

(ii) Underlying asset modeling

(a) A subjective approach based on expert estimation and forecasting.

(b) An objective approach based on statistical estimation of a random underlying asset on a market data basis (e.g. an arithmetic, geometric Brown process, a mean-reversion process, the Vasicek, Schwartz, CIR or Ito process, etc.).

In the case of the geometric Brown process according to the Cox, Ross, and Rubinstein (1979) calibration we first compute the up-movement and down--movement indexes, which characterize the volatility coinciding with market volatility, then $e^u = e^{\sigma\cdot\sqrt{dt}}$, $e^d = e^{-\sigma\cdot\sqrt{dt}}$ and $S_{t+dt}^{u} = S_t \cdot e^u$, $S_{t+dt}^{d} = S_t \cdot e^d$.

(iii) At the maturity day, *T*, the option price is equal to the intrinsic value $f_T^u = g_T^u$, respective $f_T^d = g_T^d$. The computation of the intrinsic value (the pay-off function), *g*, depends on the option type. For example, for a call option, $g_t^u = \max\left(S_t^u - X;0\right)$, and for a put option, $g_t^u = \max\left(X - S_t^u;0\right)$, *X* being the exercise price.

(iv) Working backwards from the end to the beginning of the binomial tree, the price of the option is calculated at every node and also at the initial node according to the formulas.

The price for a European option is $f_t = e^{-r \cdot dt} \cdot \left[f_{t+dt}^{u} \cdot (\widehat{p}) + f_{t+dt}^{d} \cdot (1-\widehat{p}) \right]$.

The price for an American option, which can be exercised whenever during a pre-specified period, is $f_t = \max\limits_{q \in S} \left(g_t^q \right)$, which means

$$f_t = \max_{q \in S \text{ or } q = S+1} \left\{ g_t^q ; g_t^{S+1} = e^{-r \cdot dt} \cdot \left[f_{t+dt}^{u} \cdot (\widehat{p}) + f_{t+dt}^{d} \cdot (1-\widehat{p}) \right] \right\}$$

Function g_t^q represents exercise of the option, while $g_t^{\acute{S}+1}$ represents non--exercise of the option. Parameter q represents the choice (option) of process, generally called a mode, $\widehat{p}$ depicts the risk-neutral probability defined previously. At the beginning of the period, f_0 is then the sought price of the option.

(v) Determination of the decision-making variables, Q_t,

$$Q_t = \arg\max_{q \in S \text{ or } q = S+1} \left\{ g_t^q ; g_t^{S+1} = e^{-r \cdot dt} \cdot \left[f_{t+dt}^{u} \cdot (\widehat{p}) + f_{t+dt}^{d} \cdot (1-\widehat{p}) \right] \right\}$$

The function argmax represents the argument of the maximum of the function, so the decision parameter q corresponds to the maximum value of the objective function.

(vi) A sensitivity analysis concerning the input data.

3. Description of Real Option Valuation with Switching Costs

Dynamic programming represents an optimal management problem for finding the optimal decision-making trajectory. It is a multi-period optimization method based on Bellman optimization principle. By contrast with deterministic programming, stochastic dynamic programming means that the whole process is of a random type.

The optimization of the whole process under this approach means that it is possible to optimize particular periods separately. The final system state depends on all previous states and also on the initial state. The optimal decision is made with respect to the future possible states and also to future forward-looking decision-making.

Bellman optimality principle, which is considered to be an axiom, means that whatever the initial decision is, the following decisions have to be of optimal strategy with respect to the previous decision.

The application assumptions of the principle are that the process must be divisible into periods and the objective function must be separable. Thus, the optimization objective function must be expressible as the aggregation of the optimization functions for particular periods. The calculation procedure is performed recurrently from the final period to the initial period, i.e., in the opposite direction to the process flow.

The assumptions of stochastic dynamic programming are: the process is divisible into particular periods; the periods are characterized by possible random states;

the particular decisions are depicted by the mode (e.g. technology, equipment, process, stage of development); and the total objective function must be separable such that it is expressed as the aggregation of particular objective functions.

The problem solved by stochastic dynamic programming is formulated in such a way that regardless of the initial state it is necessary to determine a decision trajectory that coincides with the optimal total objective function. The basic point is the division of the whole process (an N-period extreme process) into particular periods, and for every period the optimal solution is found. So, at the beginning of every period the system is in a certain mode and, according to the period optimization criteria result, a decision follows on whether to transit to a new mode or to keep the existing mode. The solution procedure consists in transforming the whole process into the successive finding of particular optimal solutions. The backward recurrent procedure is then applied.

3.1 Derivation of the Recurrent Formula for the Present Expected Value Criteria

Present value is one of the basic principles and criteria of financial decision-making. This optimization criterion fulfils the condition of separation, so dynamic programming according to Bellman optimization principle can be employed. For decision-making under risk, maximization of the present expected value is applied for the optimal choice of mode and trajectory. Subsequently, the recurrent present expected value formula is derived and explained.

Under the assumption that the cash flow of a given period is paid at the beginning of the period, the present value of the cash flow is formulated as follows:

$$V_N = \widehat{E}\left[\sum_{t=0}^{N-1} \beta_t \cdot x_t\right] = \sum_{t=0}^{N-1} \beta_t \cdot \widehat{E}(x_t)$$

where V_N is the value with N periods to the final period and x_t is the cash flow at the beginning of the particular period. The discount factor is generally depicted as $\beta_t = (1+R)^{-t}$; for the sake of simplicity $\beta = \beta_1 = (1+R)^{-1}$. The equation can be rewritten as follows:

$$V_N = x_0 + \sum_{t=1}^{N-1} \beta_t \cdot \widehat{E}(x_t) = x_0 + \beta \cdot \sum_{t=1}^{N-1} \beta_{t-1} \cdot \widehat{E}(x_t) = x_0 + \beta \cdot \left[x_1 + \sum_{t=2}^{N-1} \beta_{t-1} \cdot \widehat{E}(x_t)\right]$$

The value of the particular period can therefore be expressed recurrently depending on the subsequent period as follows

$$V_N = x_0 + \beta \cdot \widehat{E}[V_{N-1}], \text{ where } \widehat{E}[V_{N-1}] = x_1 + \sum_{t=2}^{N-1} \beta_{t-1} \cdot \widehat{E}(x_t)$$

Analogously for the following period

$$V_{N-1} = x_1 + \sum_{t=2}^{N-1} \beta_{t-1} \cdot \widehat{E}(x_t) = x_1 + \beta \cdot \sum_{t=2}^{N-1} \beta_{t-2} \cdot \widehat{E}(x_t) = x_1 + \beta \cdot \left[x_2 + \sum_{t=3}^{N-1} \beta_{t-2} \cdot \widehat{E}(x_t)\right]$$

It is apparent that the value of a particular period can again be expressed in terms of the subsequent period

$$V_{N-1} = x_1 + \beta \cdot \widehat{E}\left(V_{N-2}\right), \text{ where } V_{N-2} = x_2 + \sum_{t=3}^{N-1} \beta_{t-2} \cdot \widehat{E}\left(x_t\right)$$

Generally, the recurrent formula for every period is

$$V_{N-k} = x_k + \sum_{t=k+1}^{N-1} \beta_{t-k} \cdot \widehat{E}\left(x_t\right) = x_k + \beta \cdot \sum_{t=k+1}^{N-1} \beta_{t-(k+1)} \cdot \widehat{E}\left(x_t\right) =$$

$$= x_k + \beta \cdot \left[x_{k+1} + \sum_{t=k+2}^{N-1} \beta_{t-(k+1)} \cdot \widehat{E}\left(x_t\right) \right]$$

and so any period value can be determined in terms of the subsequent period in this way

$$V_{N-k} = x_k + \beta \cdot \widehat{E}\left[V_{N-k-1}\right]$$

Here *N-k* depicts the number of periods until the end of the first phase.

The value of the last period can be written as follows:

$$V_1 = x_{N-1} + \beta \cdot \widehat{E}\left(V_0\right)$$

In the preceding paragraphs the recurrent formulae of the present expected value was shown. Now, the possibility of deciding about the mode choice will be carried out under the present expected value optimization criteria.

Firstly, an example for two modes A and B will be described. The initial mode will be A, and the cost of switching from A to B is depicted as $C_{A,B}$. The cost incurred with no change in mode A is depicted as $C_{A,A}$. Usually, $C_{A,A} = 0$, because no costs are incurred. A positive switching cost means a cash outlay and a negative one represents a cash inflow. Applying the previous results, the recurrent formulae for solving the problem are

$$V_N^A = \max_{A,B}\left[x_0^A - C_{A,A} + \beta \cdot \widehat{E}\left(V_{N-1}^A\right);\ x_0^B - C_{A,B} + \beta \cdot \widehat{E}\left(V_{N-1}^B\right) \right] \tag{6}$$

$$V_{N-k}^A = \max_{A,B}\left[x_k^A - C_{A,A} + \beta \cdot \widehat{E}\left(V_{N-k-1}^A\right);\ x_k^B - C_{A,B} + \beta \cdot \widehat{E}\left(V_{N-k-1}^B\right) \right] \tag{7}$$

$$V_1^A = \max_{A,B}\left[x_{N-1}^A - C_{A,A} + \beta \cdot V_0^A;\ x_{N-1}^B - C_{A,B} + \beta \cdot V_0^B \right] \tag{8}$$

Secondly, we can derive by induction the formulas for the three modes A, B, and C. The initial mode will again be A. Here, the costs of switching from A to A, A to B, and A to C are depicted as $C_{A,A}$, $C_{A,B}$, and $C_{A,C}$, respectively.

In this case, the recurrent formulas are the following:

$$V_N^A = \max_{A,B,C}\left[x_0^A - C_{A,A} + \beta \cdot \widehat{E}\left(V_{N-1}^A\right);\ x_0^B - C_{A,B} + \beta \cdot \widehat{E}\left(V_{N-1}^B\right);\ x_0^C - C_{A,C} + \beta \cdot \widehat{E}\left(V_{N-1}^C\right) \right] \tag{9}$$

$$V_{N-k}^{A}=\max_{A,B,C}\begin{bmatrix} x_k^A - C_{A,A} + \beta \cdot \widehat{E}\left(V_{N-k-1}^{A}\right);\ x_k^B - C_{A,B} + \beta \cdot \widehat{E}\left(V_{N-k-1}^{B}\right); \\ x_k^C - C_{A,C} + \beta \cdot \widehat{E}\left(V_{N-k-1}^{C}\right) \end{bmatrix} \tag{10}$$

$$V_1^A = \max_{A,B,C}\left[x_{N-1}^A - C_{A,A} + \beta \cdot V_0^A; x_{N-1}^B - C_{A,B} + \beta \cdot V_0^B; x_{N-1}^C - C_{A,C} + \beta \cdot V_0^C\right] \tag{11}$$

We can generalize the previous results for switching between a larger number of modes under the assumption that the initial mode is m and the subsequent mode is q, which is chosen from mode set S. The procedure can be generalized according to the following recurrent formulas:

$$V_N^m = \max_{q \in S}\left[x_0^q - C_{m,q} + \beta \cdot \widehat{E}\left(V_{N-1}^q\right)\right] \tag{12}$$

$$V_{N-k}^m = \max_{q \in S}\left[x_k^q - C_{m,q} + \beta \cdot \widehat{E}\left(V_{N-1-k}^q\right)\right] \tag{13}$$

$$V_1^m = \max_{q \in S}\left[x_{N-1}^q - C_{m,q} + \beta \cdot V_0^q\right] \tag{14}$$

These generalized formulas derived are similar to those in, for example, (Kulatilaka, 1988), (Kulatilaka, Trigeorgis, 1994), (Weston, 2002), and (Erraisa, Sadowsky, 2008).

3.2 Valuation Procedure for American Options with Switching Costs

In contrast to section 2.2, the valuation procedure in this section is founded on a cash-flow basis. The valuation procedure for multinomial options with non-symmetrical switch options in respect of stochastic dynamic programming on Bellman principle expressed by recurrent equations, under the discrete binomial model and risk-neutral probability, is performed in the following steps.

(i) Determination of the risk-neutral growth parameter $\widehat{g}$.

(ii) Modeling of cash flow as the underlying asset.

(a) A subjective approach based on expert estimation and forecasting.

(b) An objective approach based on statistical estimation and forecasting of a random process. In the case of a geometric Brown process according to the Cox, Ross, and Rubinstein (1979) calibration:

$$x_{t+1,s+u}^u = x_{t,s} \cdot U; \quad x_{t+1,s+d}^d = x_t \cdot D$$

where $U = e^u = e^{\sigma \cdot \sqrt{dt}}$ and $D = e^d = e^{-\sigma \cdot \sqrt{dt}}$.

(iii) At the beginning of the second phase the value for the second phase is $V_{0,s}^q$, where s is the state and q is the mode.

(iv) The value is calculated by a backward recurrent procedure from the end to the beginning of the binomial tree for the states s and modes q of a particular period in accordance with the generalized recurrent Bellman stochastic equations (12), (13), and (14). Here $\widehat{p}$ is the risk-neutral probability of an up-movement and $\widehat{q} = 1 - \widehat{p}$ is the risk-neutral probability of a down-movement.

The valuation formula for period one to the end of the first phase is

$$V_{1,s}^{m} = \max_{q \in S} \left[x_{N-1,s}^{q} - C_{m,q} + \beta \cdot V_{0,s}^{q} \right] \tag{15}$$

The valuation formula for other periods based on the recurrent procedure is

$$V_{N-k,s}^{m} = \max_{q \in S} \left\{ x_{k,s}^{q} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V_{N-1-k,s+u}^{q} + \widehat{q} \cdot V_{N-1-k,s-d}^{q} \right] \right\} \tag{16}$$

The valuation formula at the beginning of the whole first phase (the first period) is

$$V_{N,s}^{m} = \max_{q \in S} \left\{ x_{0,s}^{q} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V_{N-1,s+u}^{q} + \widehat{q} \cdot V_{N-1,s-d}^{q} \right] \right\} \tag{17}$$

(v) Identification of the decision variant for state *s* and time *t*, $Q_{t,s}$ is

$$Q_{t,s} = \arg\max_{q \in S} \left\{ x_{k,s}^{q} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V_{N-1-k,s+u}^{q} + \widehat{q} \cdot V_{N-1-k,s-d}^{q} \right] \right\}$$

(vi) A sensitivity analysis concerning the input data.

4. Example of Company Valuation with Dynamic Flexibility Based on American Switch Options with Non-symmetrical Switching Costs

This section applies the generalized flexible approach with the possibility of a dynamic choice of particular modes (switch option) to find the optimal trajectory on an expected present value basis. Three variants will be investigated depending on the initial mode (situation) of the company: Variant 1 – Mode A (normal production), Variant 2 – Mode B (expanded production), Variant 3 – Mode C (contracted production). Switching in these three variants is assumed.

A stochastic dynamic programming model based on the binomial model (American options; non-symmetrical switching costs; replication value strategy; risk--neutral approach; expected present value objective function) will be employed. The applied model is of two-phase type. The first phase, with random cash flow, takes 4 years, and the second non-random phase is the perpetuity version. We assume that the random cash flow (the underlying asset) follows a geometric Brown process.

4.1 Computation Procedure and Results

The input data of the applied model are the following: risk-free rate r = 10 % (discount factor 0.9091); up-movement index U = 1.2; value for beginning of second phase $V_{0,s}^{q}$ for states *s* and modes *q*, see *Figure 1*. The price of the underlying asset *S* is 10, and the rate of contraction or expansion is 15 %. The risk-neutral probability of an up-movement is $\widehat{p}$ = 72.73 % and that of a down-movement is $\widehat{q} = 1 - \widehat{p} =$ = 27.27 %. *Table 1* shows the switching costs C_{ij} connected with switching between particular modes. Keeping the same mode is linked with no switching costs, of course.

FIGURE1 Valuation Procedure of Dynamic Flexible Multinomial Switch Options Methodology

I. Cash flow

	Mode A - Normal						Mode B - Expansion						Mode C - Contraction					
n/t	0	1	2	3	4	V_0	0	1	2	3	4	V_0	0	1	2	3	4	V_0
0	10.0	12.0	14.4	17.3	20.7	41.5	11.5	13.8	16.6	19.9	23.8	47.7	8.5	10.2	12.2	14.7	17.6	35.3
-2		8.3	10.0	12.0	14.4	28.8		9.6	11.5	13.8	16.6	33.1		7.1	8.5	10.2	12.2	24.5
-4			6.9	8.3	10.0	20.0			8.0	9.6	11.5	23.0			5.9	7.1	8.5	17.0
-6				5.8	6.9	13.9				6.7	8.0	16.0				4.9	5.9	11.8
-8					4.8	9.6					5.5	11.1					4.1	8.2

$$x_{t,n} = x_0 \cdot u^{(t+n)}$$

II. Pre-calculation

	Mode A - Normal					Mode B - Expansion					Mode C - Contraction				
n/t	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	62.4	61.9	58.7	51.5	37.7	66.9	66.5	63.2	56.0	43.4	57.7	57.0	53.3	45.1	32.0
-2		41.7	39.5	34.5	26.2		46.2	43.9	38.9	30.1		37.5	35.1	30.2	22.3
-4			27.0	23.7	18.2			30.7	27.0	20.9			23.3	20.1	15.5
-6				17.5	12.6				19.0	14.5				13.9	10.7
-8					8.8					10.1					7.5

$$\widehat{E}\left(V^m_{N-k,s}\right) = \beta \cdot \left[\widehat{p} \cdot V^q_{N-1-k,s+u} + \widehat{q} \cdot V^q_{N-1-k,s-d}\right]$$

III. Maximal value

	Mode A - Normal					Mode B - Expansion					Mode C - Contraction				
n/t	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	73.4	75.3	74.8	70.9	62.2	78.4	80.3	79.8	75.9	67.2	68.4	69.9	69.1	64.7	54.4
-2		50.8	50.4	47.7	41.7		55.8	55.4	52.7	46.7		46.1	45.5	42.5	36.6
-4			33.9	32.1	28.2			38.7	36.6	32.4			29.9	28.1	24.2
-6				23.3	20.6				26.3	22.6				19.3	16.6
-8					15.6					16.6					11.6

$$V^m_{N-k,s} = \max_{q \in S}\left\{x^q_{k,s} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V^q_{N-1-k,s+u} + \widehat{q} \cdot V^q_{N-1-k,s-d}\right]\right\}$$

IV. Assigned mode

	Mode A - Normal					Mode B - Expansion					Mode C - Contraction				
n/t	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	B	B	B	B	B	B	B	B	B	B	A	A	A	A	A
-2		B	B	B	B		B	B	B	B		A	A	A	A
-4			A	A	A			B	B	B			A	A	A
-6				A	C				A	A				A	C
-8					C					A					C

$$Q_{t,s} = \arg\max_{q \in S}\left\{x^q_{k,s} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V^q_{N-1-k,s+u} + \widehat{q} \cdot V^q_{N-1-k,s-d}\right]\right\}$$

TABLE 1 Switching Costs between Modes

Switching cost $C_{i,j}$		**Subsequent mode *j***		
		A	**B**	**C**
Initial mode *i*	A	0	5	-4
	B	-3	0	∞
	C	4	∞	0

Positive values mean a cash outflow and negative ones a cash inflow. The impossibility of switching between modes is expressed by a penalty value, ∞. It is apparent that the switching costs are non-symmetrical.

The computation procedure, conforming to the methodology of stochastic dynamic programming valuation of multinomial options (see section 3.2 and equations (12), (13), and (14)), for the three initial modes is presented in *Figure 1*

The three columns show the procedure for the three initial states: Mode A (normal production), Mode B (expansion in production) and Mode C (contraction in production). The rows show the four steps of the procedure: (I) the cash flow tree, (II) the pre-calculation (the present expected value), (III) the maximum value, and (IV) the assigned mode.

The computation procedure is shown in *Figure 1* and is decomposed into the following steps.

Step 1: Cash flow calculations according to a geometric Brown process for modes A, B, and C in part I, $x^u_{t+1,s+u} = x_{t,s} \cdot U; \quad x^d_{t+1,s+d} = x_t \cdot D$.

Step 2: For time 4 the present expected value calculations $E\left(V^m_{1,s}\right) = \beta \cdot V^q_{0,s}$ are performed. So, for mode A, $37.7 = 0.9091 \cdot 41.5$. The values in part II for modes A, B, and C at time 4 are calculated similarly.

Step 3: Calculation of the maximum value for time 4 in part III according to equation $V^m_{N-k,s} = \max\limits_{q \in S}\left\{x^q_{k,s} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V^q_{N-1-k,s+u} + \widehat{q} \cdot V^q_{N-1-k,s-d}\right]\right\}$. In particular, for mode A $62.2 = \max\left\{20.7 - 0 + 37.7;\ 23.8 - 5 + 43.4;\ 17.6 - (-4) + 32\right\}$. The values in part III for modes A, B, and C at time 4 are calculated similarly.

Step 4: Due to $Q_{t,s} = \arg\max\limits_{q \in S}\left\{x^q_{k,s} - C_{m,q} + \beta \cdot \left[\widehat{p} \cdot V^q_{N-1-k,s+u} + \widehat{q} \cdot V^q_{N-1-k,s-d}\right]\right\}$ the particular modes are assigned in part III for original modes A, B, and C.

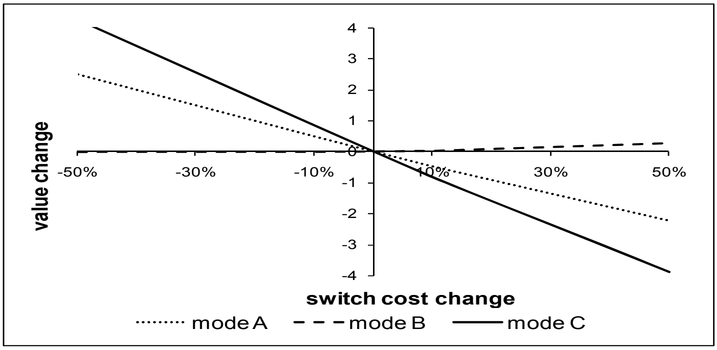
Step 5: For time 3 the present expected value due to $V^m_{N-k,s} = V^m_{N-k,s} = \beta \cdot \left[\widehat{p} \cdot V^q_{N-1-k,s+u} + \widehat{q} \cdot V^q_{N-1-k,s-d}\right]$ is calculated. So, for mode A, $51.5 = 0.9091 \cdot \left[0.7273 \cdot 62.2 + 0.2727 \cdot 41.7\right]$. The values in part II for modes A, B, and C at time 3 are calculated similarly.

The values for time 3 are calculated analogously to Step 3 and Step 4. The procedure goes by backwardation analogously to Steps 5, 3, and 4 for times 2, 1, and 0.

TABLE 2 Flexibility Values According to Particular Modes

Value	Starting modes		
	A	B	C
with flexibility	73.424	78.424	68.379
without flexibility	68.182	78.409	57.955
flexibility	5.243	0.015	10.424

GRAPH 1 Sensitivity of Value to Switching Costs for Particular Modes



The calculated results show the company values with flexible modes (actions) for three modes; normal, expansion and contraction. All three variants according to initial mode are investigated and evaluated. The resulting values for the three variants at the initial node of the present value in part III are 73.424, 78.424, and 68.379 monetary units.

It is apparent that if the initial mode is A, then it is optimal to switch to B, and, under unfavorable conditions, to A or C. If the initial mode is B, then the mode is maintained; only under unfavorable circumstances is it switched to A. If the initial mode is C, then it is switched immediately to A and, under unfavorable conditions, back to C. We can conclude that the initial mode significantly influences the optimal decision (switching mode) trajectory.

The flexibility values for every initial mode are given in *Table 2*. It shows the values of the company for initial modes A, B, and C from *Figure 1*. It also presents the results of the computation without flexibility and the flexibility value (calculated by subtracting the value with flexibility from the value without flexibility).

We can see in *Table 2* resulting values of modes without flexibility, values with flexibility, and flexibility values. It is apparent that mode B is the most stable and valuable. Mode C has the greatest flexibility value.

The influence of switching costs on the final value is significant. This influence is investigated by sensitivity analysis. This involves computing the change in company value dependent on the relative (percentage) change in the switching cost. The results are illustrated in *Graph 1*. The value is most influenced by the switching cost of mode C, followed by mode A. The switching cost of mode B does not significantly influence the value.

5. Conclusions

The purpose of the paper was to derive, describe, explain, and verify the possibility of applying the generalized real multinomial switch options methodology based on non-symmetrical switching costs to company decision-making. The basic approach of the paper was to apply the stochastic dynamic programming problem under Bellman optimality principle on the present expected value criterion. The replication strategy and risk-neutral approach were applied.

Firstly, the valuation of real American options without switching costs was described. As the basic principle, the replication strategy for a complete market under the risk-neutral approach was explained.

Secondly, the generalized recurrent valuation optimization formula with multinomial decisions (options) based on real American switch options with non-symmetrical switching costs was described. Bellman principle of optimality for the present expected value was derived and applied.

Thirdly, the generalized flexible real American switch options methodology with non-symmetrical switching costs was applied using the example of company valuation with options to select and switch between three modes: normal production, expansion and contraction. The same modes as the initial ones were assumed. A sensitivity analysis was carried out as well. The results showed the significance of flexibility aspects and the influence of non-symmetrical switching costs on decision-making under risk. The stability of the results was verified by means of a switching cost sensitivity analysis.

The influence of switching costs is multi-fold. If we assume that the switching costs are null or symmetrical (equal switching costs when going into and out of a mode), the optimal decision is to choose the mode with the maximum cash flow. However, for non-symmetrical switching costs and when more than two modes are used, the optimal decisions are influenced and determined by the future options (choices). This is a consequence of considerable inertia and the hysteresis effect. For example, it can be optimal to postpone a project even if the net present value is positive, or to continue a production process even though the production cash flow is temporarily under the variable cost level.

It was shown that the stability of the solution can be suitably verified by a sensitivity analysis concerning the input data, and the switching costs in particular. A generalized approach to option value sensitivity analysis based on the fuzzy sets methodology is presented for instance in (Zmeškal, 1999, 2001).

It was explained and verified that the generalized multinomial flexible switch options approach with non-symmetrical switching costs suitably models the real decision-making and valuation conditions. This generalized approach incorporates and covers the full range of decision-making aspects and features: risk, flexibility, multiple optionality (switching), time dynamics, and non-symmetrical switching costs.

The generalized real options methodology described in this paper can be viewed as a basic and common approach to financial decision-making, including company valuation and project evaluation under flexible conditions and switch options variants. This ensues from the preceding explanations and discussions. It is also possible and appropriate to apply the generalized real options methodology to a small open economy in a transition phase. The usefulness of the model lies in the fact that it allows for more realistic decision-making. There is no doubt that this approach allows us to obtain more project portfolio and decision-making variants and better manage companies' economic efficiency.

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