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On Risk Aversion, Classical Demand Theory, and KM Preferences

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Abstract:
Building on Kihlstrom and Mirman (1974)'s formulation of risk aversion in the case of multidimensional utility functions, we study the effect of risk aversion on optimal behavior in a general consumer's maximization problem under uncertainty. We completely characterize the relationship between changes in risk aversion and classical demand theory. We show that the effect of risk aversion on optimal behavior is determined not by the riskiness of the risky good, but rather the riskiness of the utility gamble associated with each decision. We also discuss the appropriateness of an (alternative) approach to study risk aversion suggested by Selden (1978), which has been widely popularized in the field of macroeconomics through the parametric model of Epstein and Zin (1989) (henceforth, the Selden-EZ approach). We show that the Selden-EZ approach cannot disentangle risk aversion from tastes, and, thus, cannot be used to isolate the effect of risk aversion.

Keywords: Classical Demand Theory, Consumer Choice, Epstein-Zin preferences, Risk Aversion, Selden Preferences

JEL Classification: D01, D81, D91

1 Introduction

One of the central questions in the field of economics of uncertainty is the influence of attitudes toward risk (i.e., the effect of risk aversion) on optimal decisions. Arrow (1965) and Pratt (1964) first looked at this question in the early 1960s, in the context of a portfolio problem. Since then, the analysis of the behavior of risk-averse individuals facing risk has been set in the context of Arrow-Pratt theory, most notably in the fields of insurance and finance. The analysis has also been extended by Kihlstrom and Mirman (1974) (henceforth, KM) to multidimensional utility functions in situations in which the goods are not perfect substitutes (e.g., a dynamic environment). In particular, KM show that to generalize the Arrow-Pratt approach to the multidimensional case, the issue of separating tastes from attitudes toward risk must be dealt with. Specifically, the effect of risk aversion on behavior in the multidimensional case must take account of the problem of disentangling tastes and attitudes toward risk. To achieve this, KM consider utility functions that differ by a concave transformation, and, thus, preserve ordinal preferences over gambles. There is also another approach suggested by Selden (1978), which has been widely popularized in the field of macroeconomics through the parametric model of Epstein and Zin (1989) (henceforth, the Selden-EZ approach).[1] The basis for this approach is the certainty equivalence of the one-dimensional Arrow-Pratt theory of risk aversion.

In this paper, we show that the Selden-EZ approach yields choices over gambles that are inconsistent with ordinal preferences. This is due to the fact that the ordinal preferences are distorted by the Selden-EZ approach. In contrast, the KM approach using concave transformations of the utility function alters the expected marginal rate of substitution in a way that is consistent with ordinal preferences. This is a subtle point because Selden-EZ preferences do represent the same deterministic preferences. However, it does not follow that preferences over gambles remain the same. To show this, we consider the case of two identical gambles (in terms of utility outcomes). Under the KM approach, an individual is indifferent between the

[1] See also Kreps and Porteus (1978).

two identical utility gambles. However, this is not the case under the Selden-EZ approach. In particular, for some values of the certainty equivalent, an individual chooses the gamble for which the distribution of the values of the risky good first-order stochastically dominates. In other words, the effect of risk aversion cannot be isolated since the choice for the particular gamble is unrelated to the issue of riskiness. We then consider the case of two distinct gambles, with one worse than the other (in terms of utility gambles). We show that the Selden-EZ approach leads to a reversal of preference ordering for some values of the certainty equivalent, i.e., the worst gamble can be chosen. This is never the case under the KM approach.

We then proceed to study the effect of risk aversion on optimal decisions using the KM approach. Specifically, we consider a general consumer's maximization problem under uncertainty subject to a budget constraint. In the stochastic environment, there is a sure good and a risky good. The good is risky due to the presence of randomness in the budget constraint. The sure good is chosen before the realization of the random variable is observed. The risky good is a residual, i.e., the risky good depends on the outcome of the random parameter through the budget constraint. The set up is thus a generalization of Arrow-Pratt's portfolio model in which the goods are perfect substitutes. We consider three cases of randomness: random income, random price of the sure good, and random price of the risky good.[2] In each case, we study the effect of risk aversion on optimal decisions.

We show that in the random income case as well as the case of the random price of the sure good, the effect of risk aversion is to decrease the amount of the sure normal good. While if the price of the risky good is random, then the effect of risk aversion on the amount of the sure good is ambiguous. These results follow from the fact that a more risk-averse individual is not concerned by the riskiness of the risky good, but rather by the riskiness of the utility gamble associated with the consumption bundle of the sure and risky goods. This is merely an implication of the expected utility maximization problem

[2] Note that the majority of the literature on risk aversion has been set in the context of the static portfolio problem, which is equivalent to the price of the risky good being random.

faced by the consumer. In fact, when the holding of the risky good increases, it decreases the riskiness of the utility gamble faced by the individual. In other words, a concave transformation of the utility function implies that the more risk-averse individual prefers gambles whose corresponding utility gambles are less risky. This result pinpoints the rationale of the consumer's decisions not as choice on the amount of risky versus sure goods, but rather as a choice on a set of utility gambles ordered in terms of their riskiness.[3]

Because the riskiness of the utility gambles yield the incentive for the consumer to choose an optimal gamble, through the maximization of expected utility, this leads to the counterintuitive result, which was originally observed by Ross (1981). Indeed, Ross (1981) provides an example showing that the Arrow-Pratt definition of risk aversion fails to deliver the right "intuitive" results. In particular, Ross (1981) writes that "in the portfolio problem, as wealth rises individuals whose risk aversion declines in the Arrow-Pratt sense do not necessarily increase their holding of riskier assets." Convinced of the intuitive idea that more risk aversion implies a smaller amount of the riskier asset (or good), Ross (1981) introduces a stronger measure of risk aversion that is necessary to accommodate the phenomenon observed when there are several independent sources of risk, i.e., that the more risk averse individual actually chooses to consume more of the risky asset. In fact, in the example provided by Ross (1981), the individual chooses more of the risky asset (or good) because it reduces the riskiness of the utility gamble.

In our general approach to the consumer problem, we relate classical demand theory to the theory of risk aversion. In particular, we show that the influence of risk aversion can be separated into independent components, i.e., income effect and substitution effect, as in classical demand theory with the outcome depending on their interplay. The interaction between the income and substitution effects as a determinant of the effect of risk aversion on optimal choice has also been noted in KM in the context of a consumer-saving problem. KM show that the effect of changing risk aversion depends on on

[3] See Bommier et al. (2011) who extends the insight in Diamond and Stiglitz (1974) by pointing out a direct relationship between increasing risk aversion and preferences ordering over gambles, i.e., a more risk-averse individual prefers less risky utility gambles.

the effect of the rate of return on savings in the classical (certainty) case, that is, the impact of the interplay between the income and substitution effects. We generalize the KM result by showing that in a two-dimensional setting the effect of changing risk aversion depends on the source of randomness as well as the influence of ordinal effects, i.e., the income and substitution effects.

Our results on the effect of risk aversion are summarized in three Propositions, which are inserted between relevant examples and figures that illustrate the Propositions. Proposition 3.1 studies the case in which income is random. Here, the direction of the effect of risk aversion depends on the income effect for the sure good. When the sure good is normal, the optimal decision of a more risk-averse individual is always to consume more of the risky good, which decreases the riskiness of the associated utility gamble. It is precisely this case that contradicts the Ross intuition that more risk-averse individual prefers more of the sure (or riskless) good. In Proposition 3.4, we show that, if the price of the sure normal good is random, a more risk-averse individual always chooses more of the risky good, since the income and substitution effects pull in the same direction. Here, the pure substitution effect provides an incentive to consume less of the sure good. In this case, the utility gambles become less risky with more of the risky good, yielding an incentive for the consumer to move in that direction. Proposition 3.7 shows that, if the price of the risky normal good is random, as in the traditional portfolio problem, then the choice of the normal sure good becomes ambiguous. In this case, there is an incentive through the pure substitution effect to increase the amount of the sure good, i.e., increasing the sure good less a less risky utility gamble. Hence, the income and substitution effects pull in opposite directions, and depending on the relative strengths of these effects, the individual may be led to consume more or less of the normal good. Note that Proposition 3.7 encompasses the Arrow-Pratt's result, namely a decreased in holding of a risky good in the event of an increase in risk aversion, which is a special case and is due to the fact that, in the one-dimensional case (i.e., perfect substitutes), there is no income effect, which implies that the utility gamble becomes less risky as the amount of the sure good is increased. However, when there is an income effect, this result holds only when the substitution effect is stronger than the income effect.

The paper is organized as follows. In Section 2, we show that the Selden-EZ approach cannot disentangle risk aversion from tastes, and, thus, cannot be used to isolate the effect of risk aversion. In Section 3, we use the KM approach to study the effect of risk aversion on optimal decisions. Section 4 concludes.

2 Risk Aversion and Concave Transformations

To study the effect of risk aversion on behavior in the multidimensional case, tastes and attitudes toward risk must be disentangled. This issue does not arise for the class of one-dimensional strictly increasing utility functions since tastes are represented by the natural ordering on the real line, i.e., $x > y$ means that $x \succ y$. However, the relationship between the utility representation, risk aversion, and tastes is much more delicate in the multidimensional case since there is no natural order. In other words, different utility functions incorporate different tastes as well as different attitudes toward risk so that the link between risk aversion and risk averse behavior is not clearly identified. For instance, Kihlstrom and Mirman (1974) provide an example in which the preference between a sure outcome and a gamble depend solely on tastes and not on risk aversion. To see this, let $U^1(x, y)$ and $U^2(x, y)$ be two distinct utility functions yielding indifference curves of the type IC_1 and IC_2, respectively, as depicted in Figure 1. Let (x_A, y_A) and (x_B, y_B) be two distinct consumption bundles such that $U^1(x_A, y_A) > U^1(x_B, y_B)$ and $U^2(x_A, y_A) < U^2(x_B, y_B)$. Consider choosing between the sure outcome yielding (x_A, y_A) and a gamble yielding (x_A, y_A) with probability $\pi \in (0, 1]$ and (x_B, y_B) with probability $1-\pi$. Consistent with Figure 1, an individual with preferences $U^1(x, y)$ prefers the sure outcome, while an individual with preferences $U^2(x, y)$ prefers the gamble.[4] The individual with preferences $U^2(x, y)$ acts in a seemingly more risk-averse way than the individual with preferences

[4]In other words, $U^1(x_A, y_A) > \pi U^1(x_A, y_A) + (1-\pi)U^1(x_B, y_B)$, while $U^2(x_A, y_A) < \pi U^2(x_A, y_A) + (1-\pi)U^2(x_B, y_B)$.

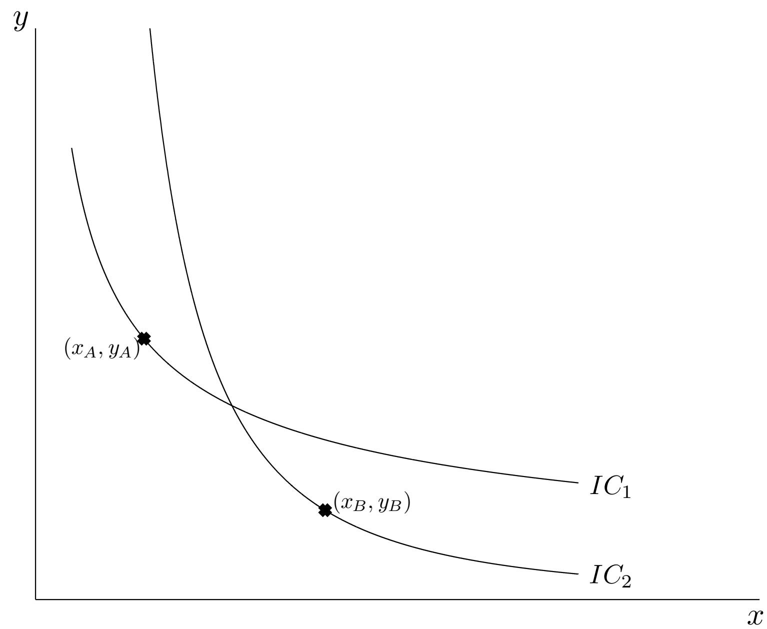

Figure 1: KM Example

$U^1(x, y)$, but is not more risk-averse. Rather, it is the composition of goods in the gamble that is preferred.

Two approaches have been suggested to disentangle tastes from risk aversion, and, thus, to analyze the effect of risk aversion on behavior. The first established in Kihlstrom and Mirman (1974) (henceforth, KM) considers the class of utilities that are concave transformations. Formally, let

$$U(x, y) = u_1(x) + u_2(y), \tag{1}$$

$u_1', u_2' > 0, u_1'', u_2'' \leq 0$, be the utility associated with the consumption profile $(x, y) \in \mathbb{R}^2_+$.[5] Given (1), for any gamble g on $(x, \tilde{y})$ in which x is the sure

[5]We consider two-dimensional utility functions with additive functions only for clarity. The discussion applies to any dimension as well as more general utility functions.

good and $\tilde{y}$ is the risky good, the KM utility function is

$$W_{KM}(x, \tilde{y}) = \mathbb{E}_{\tilde{y}} v_{KM}\left(u_1(x) + u_2(\tilde{y})\right), \tag{2}$$

where $\mathbb{E}_{\tilde{y}}$ is the expectation operator with respect to $\tilde{y}$, and v_{KM} is a strictly increasing and concave function, $v'_{KM} > 0, v''_{KM} \leq 0$.[6]

In the KM approach, a more concave v_{KM} (and, thus, a more concave W_{KM}) means that the agent is more risk-averse. Hence, KM defines risk aversion by using the concave transformation of the utility function. In doing so, KM restricts attention to utility representations that differ by a concave transformation. Note that, with the KM approach, the measure of risk aversion (i.e., the concavity of v_{KM}) is independent of any gamble.

The KM approach can be used to study the effect of risk aversion on behavior because concave transformations of the utility function alter the expected marginal rate of substitution in a way that is consistent with ordinal preferences. To see this, consider the two gambles,

$$g_A \equiv \left(\pi \circ (x_A, \underline{y}_A), (1-\pi) \circ (x_A, \overline{y}_A)\right), \tag{3}$$

$$g_B \equiv \left(\pi \circ (x_B, \underline{y}_B), (1-\pi) \circ (x_B, \overline{y}_B)\right), \tag{4}$$

where, for $i = A, B$, $\underline{y}_i < \overline{y}_i$ and $\pi \in [0,1]$ is the probability of receiving $(x_i, \underline{y}_i)$ in gamble i. We make two further restrictions. First, the gambles are not on the same vertical lines, i.e., $x_A < x_B$. Second, $\underline{y}_A > \underline{y}_B$ and $\overline{y}_A > \overline{y}_B$, i.e., $\tilde{y}_A$ first-order stochastically dominates $\tilde{y}_B$.

Suppose that ordinal preferences over the bundles are as depicted in Figure 2, i.e.,

$$u_1(x_A) + u_2(\underline{y}_A) = u_1(x_B) + u_2(\underline{y}_B), \tag{5}$$

$$u_1(x_A) + u_2(\overline{y}_A) = u_1(x_B) + u_2(\overline{y}_B). \tag{6}$$

Proposition 2.1 states that a KM concave transformation does not alter the ordering of these two gambles. Indeed, from (3), (4), (5), and (6), the KM

[6]Note that, in this formulation, W_{KM} cannot be additive.

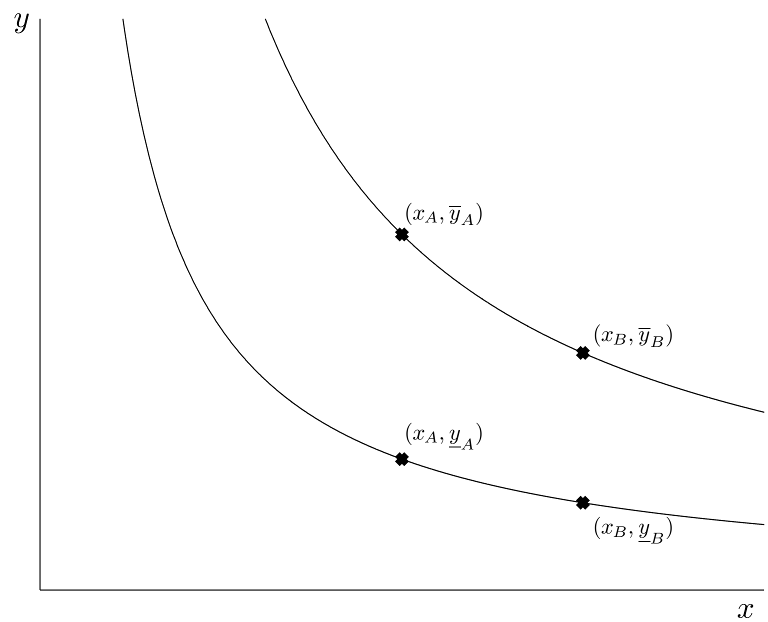

Figure 2: Case 1

utilities for the two gambles are identical, i.e.,

$$\pi v_{KM}\left(u_1(x_A) + u_2(\underline{y}_A)\right) + (1-\pi)v_{KM}\left(u_1(x_A) + u_2(\overline{y}_A)\right)$$
$$= \pi v_{KM}\left(u_1(x_B) + u_2(\underline{y}_B)\right) + (1-\pi)v_{KM}\left(u_1(x_B) + u_2(\overline{y}_B)\right). \quad (7)$$

Formally,

Proposition 2.1. *Suppose* (5) *and* (6) *hold. Under KM preferences, for any concave transformation* v_{KM}*, an individual is indifferent between gamble* A *and gamble* B*.*

Suppose next that ordinal preferences over the bundles are as depicted in Figure 3, i.e.,

$$u_1(x_A) + u_2(\underline{y}_A) < u_1(x_B) + u_2(\underline{y}_B), \quad (8)$$
$$u_1(x_A) + u_2(\overline{y}_A) = u_1(x_B) + u_2(\overline{y}_B). \quad (9)$$

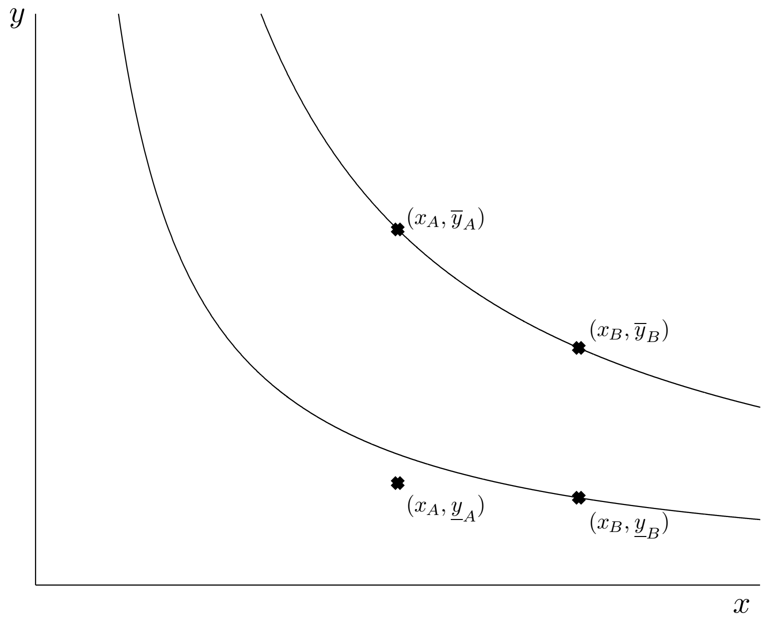

Figure 3: Case 2

That is, in terms of utility levels, gamble A is strictly worse than gamble B. Proposition 2.2 states that, regardless of the concave transformation v_{KM}, gamble B is always strictly preferred to gamble A. Indeed, for $\pi \in [0, 1)$, $W_{KM}(x_A, \tilde{y}_A) < W_{KM}(x_B, \tilde{y}_B)$.[7] Formally,

Proposition 2.2. *Suppose* (8) *and* (9) *hold, and* $\pi \in [0, 1)$. *Under KM preferences, for any concave transformation* v_{KM}, $g_B \succ g_A$.

The second approach suggested by Selden (1978), which has been widely popularized in the field of macroeconomics through the parametric model of Epstein and Zin (1989) (henceforth, the Selden-EZ approach) uses the

[7] From (3), (4), (8), and (9), for $\pi \in [0, 1)$,

$$\pi v_{KM}\left(u_1(x_A) + u_2(\underline{y}_A)\right) + (1-\pi) v_{KM}\left(u_1(x_A) + u_2(\overline{y}_A)\right)$$
$$< \pi v_{KM}\left(u_1(x_B) + u_2(\underline{y}_B)\right) + (1-\pi) v_{KM}\left(u_1(x_B) + u_2(\overline{y}_B)\right). \tag{10}$$

certainty equivalent as a measure of risk aversion. Formally, given (1), for any gamble g on $(x, \tilde{y})$, the Selden-EZ utility function is

$$W_S(x, \tilde{y}) = u_1(x) + u_2(\mu(\tilde{y}, v_S)), \tag{11}$$

where

$$\mu(\tilde{y}, v_S) = v_S^{-1}(\mathbb{E}_{\tilde{y}} v_S(\tilde{y})) \tag{12}$$

is the certainty equivalent. Here, $\mathbb{E}_{\tilde{y}}$ is the expectation operator with respect to $\tilde{y}$ and v_S is a strictly increasing and concave function, $v_S' > 0, v_S'' \leq 0$. In the Selden-EZ approach, a decrease in $\mu(\tilde{y}, v_S)$ due to a more concave v_S is used to mean that the agent is more risk averse. The basis for this approach is the certainty equivalence of the one dimensional Arrow-Pratt theory of risk-aversion. However, while there is an equivalence between a positive risk premium (or a certainty equivalent) and a concave transformation of the utility function in the one-dimensional case, this is not true in the multidimensional case.

In fact, unlike KM preferences, Selden-EZ preferences distort the expected marginal rate of substitution in a way that yields choices that are inconsistent with ordinal preferences. Selden-EZ preferences do not fall into the same category as the KM preferences because Selden-EZ preferences do not follow from a concave transformation. Indeed, a change in the concavity of v_S is equivalent to a concave transformation on the second utility function u_2. This partial concave transformation in Selden-EZ preferences is the reason that the Selden-EZ utility representation conflates tastes with risk aversion. Moreover, unlike the KM measure of risk aversion, the Selden-EZ measure of risk aversion can only be studied when there is a specific gamble. Indeed, without a gamble, preferences revert to the original deterministic preferences, so that v_S is only relevant with respect to a specific gamble.

The problems with the choice of gambles in the Selden-EZ approach is subtler than in the KM example of Figure 1. The KM example does not apply to Selden-EZ preferences because Selden-EZ preferences represent the same deterministic preferences, i.e., the same indifference curves. However, Selden-EZ preferences do not represent consistent preferences over gambles since changes in the concavity of v_S also changes tastes for gambles. In order to show this inconsistency, we need a more subtle example using the fact that deterministic preferences are the same. In fact, we can use the gambles defined by (3) and (4) to show that an inconsistency arises. Suppose that the ordinal preferences over the bundles are as depicted in Figure 2. In contrast to Proposition 2.1, Proposition 2.3 states that the Selden-EZ approach alters the ordering of these two gambles. In fact, gamble A can be preferred to gamble B because the expected return on the risky good $\tilde{y}_A$ is strictly greater than the expected return on the risky good $\tilde{y}_B$. This is important because it shows that Selden-EZ preferences disregard tastes in favor of first-order stochastic dominance on the value of outcomes in the risky good. Moreover, the fact that Selden-EZ preferences chooses gamble A is unrelated to the riskiness of the values of the risky good. In fact, from Figure 2, even though gamble A is preferred, $\overline{y}_A - \underline{y}_A > \overline{y}_B - \underline{y}_B$.

Proposition 2.3. *Suppose* (5) *and* (6) *hold. Under Selden-EZ preferences, gamble A can be strictly preferred to gamble B.*

Proof. Let

$$f_A(\pi) = u_1(x_A) + u_2\left(v_S^{-1}\left(\pi v_S(\underline{y}_A) + (1-\pi)v_S(\overline{y}_A)\right)\right), \tag{13}$$

$$f_B(\pi) = u_1(x_B) + u_2\left(v_S^{-1}\left(\pi v_S(\underline{y}_B) + (1-\pi)v_S(\overline{y}_B)\right)\right), \tag{14}$$

be the Selden-EZ utilities as a function of π. From (5), (6), (13), and (14), $f_A(0) = f_B(0)$ and $f_A(1) = f_B(1)$. Moreover,

$$f_A'(\pi) = \frac{u_2'\left(v_S^{-1}\left(\pi v_S(\underline{y}_A) + (1-\pi)v_S(\overline{y}_A)\right)\right)\left(v_S(\underline{y}_A) - v_S(\overline{y}_A)\right)}{v_S'\left(v_S^{-1}\left(\pi v_S(\underline{y}_A) + (1-\pi)v_S(\overline{y}_A)\right)\right)} < 0, \tag{15}$$

$$f_B'(\pi) = \frac{u_2'\left(v_S^{-1}\left(\pi v_S(\underline{y}_B) + (1-\pi)v_S(\overline{y}_B)\right)\right)\left(v_S(\underline{y}_B) - v_S(\overline{y}_B)\right)}{v_S'\left(v_S^{-1}\left(\pi v_S(\underline{y}_B) + (1-\pi)v_S(\overline{y}_B)\right)\right)} < 0. \tag{16}$$

Evaluating (15) and (16) at $\pi = 1$ yields

$$f'_A(\pi)|_{\pi=1} = \frac{u'_2(\underline{y}_A)\Big(v_S(\underline{y}_A) - v_S(\overline{y}_A)\Big)}{v'_S(\underline{y}_A)} < 0, \tag{17}$$

$$f'_B(\pi)|_{\pi=1} = \frac{u'_2(\underline{y}_B)\Big(v_S(\underline{y}_B) - v_S(\overline{y}_B)\Big)}{v'_S(\underline{y}_B)} < 0. \tag{18}$$

When $\underline{y}_A$ is close to $\underline{y}_B$,

$$f'_A(\pi)|_{\pi=1} < f'_B(\pi)|_{\pi=1} < 0, \tag{19}$$

so that for some $\pi \in (0,1)$ close to $\pi = 1$, $f_A(\pi)|_{\pi\approx 1} > f_B(\pi)|_{\pi\approx 1}$, i.e., gamble A is strictly preferred to gamble B. □

Suppose next that the ordinal preferences over the bundles are as depicted in Figure 3. In contrast to Proposition 2.2, Proposition 2.4 states that the ordering over the two gambles can be inconsistent with ordinal preferences. That is, gamble A which is strictly worse (in terms of utility outcomes) than gamble B can be chosen under the Selden-EZ approach. Moreover, the fact that Selden-EZ preferences chooses gamble A is unrelated to the riskiness of the utilities corresponding to the values of the risky good. In fact, from Figure 3, even though $u_2(\overline{y}_A) - u_2(\underline{y}_A) > u_2(\overline{y}_B) - u_2(\underline{y}_B)$, gamble A is preferred. It should also be noted that, for given $\pi \in (0,1)$ for which gamble A is strictly preferred to gamble B, increasing the concavity of v_S eventually leads to a reversal of the ordering of the gambles, i.e., for very concave v_S, gamble B is preferred to gamble A. Indeed, as v_S becomes more concave, the certainty equivalent tends toward the lowest utility, and, from (8), the individual no longer neglects the issue of tastes and jumps back to gamble B.

Proposition 2.4. *Suppose* (8) *and* (9) *hold. Under Selden-EZ preferences, gamble A can be preferred to gamble B.*

Proof. From (8), (9), (13), and (14), $f_A(0) = f_B(0)$ and $f_A(1) < f_B(1)$.

Moreover, evaluating (15) and (16) at $\pi = 0$ yields

$$f'_A(\pi)|_{\pi=0} = \frac{u'_2(\overline{y}_A)\Big(v_S(\underline{y}_A) - v_S(\overline{y}_A)\Big)}{v'_S(\overline{y}_A)} < 0, \tag{20}$$

$$f'_B(\pi)|_{\pi=0} = \frac{u'_2(\overline{y}_B)\Big(v_S(\underline{y}_B) - v_S(\overline{y}_B)\Big)}{v'_S(\overline{y}_B)} < 0. \tag{21}$$

When u and v_S are such that both[8]

$$\frac{u'_2(\overline{y}_A)}{v'_S(\overline{y}_A)} < \frac{u'_2(\overline{y}_B)}{v'_S(\overline{y}_B)} \tag{23}$$

and

$$v_S(\underline{y}_A) - v_S(\overline{y}_A) < v_S(\underline{y}_B) - v_S(\overline{y}_B), \tag{24}$$

then

$$0 > f'_A(\pi)|_{\pi=0} > f'_B(\pi)|_{\pi=0}, \tag{25}$$

so that for some $\pi \in (0,1)$ close to $\pi = 0$, $f_A(\pi)|_{\pi\approx 0} > f_B(\pi)|_{\pi\approx 0}$, i.e., gamble A is strictly preferred to gamble B. □

Propositions 2.3 and 2.4 show that the certainty equivalent in the multi-dimensional case cannot be compared in a meaningful way when considering gambles that are on different vertical lines, i.e., $g_i \equiv (\pi \circ (x_i, y_i), (1-\pi) \circ (x'_i, y'_i))$, $x_i \neq x'_i, y_i \neq y'_i$).[9] In fact, implicit in the comparison across different vertical lines are the tastes or preferences corresponding to the points on these two different vertical lines. Changing the concavity of v_S in Selden-EZ preferences thus conflate risk aversion and tastes.

As noted, the inconsistency regarding ordinal preferences occurs because

[8] This occurs when, for all z,

$$\frac{u''_2(z)}{u'_2(z)} < \frac{v''_S(z)}{v'_S(z)}. \tag{22}$$

[9] Only gambles that have their same first argument (i.e., gambles on the same vertical line) can be compared, e.g., $g_i \equiv (\pi \circ (x, y_i), (1-\pi) \circ (x, y'_i))$, $y_i \neq y'_i$ using the certainty equivalent approach. That is, it is only when restricting attention to gambles on a vertical line that an increase in the concavity of v_S (yielding a decrease in the certainty equivalent) is related to risk aversion.

the expected marginal rate of substitution is distorted by the Selden-EZ approach. To see this, we now present the expected marginal rate of substitution under both KM and Selden-EZ preferences. Consider the gamble

$$g \equiv (\pi \circ (x, y + \varepsilon), (1 - \pi) \circ (x, y - \varepsilon)) \tag{26}$$

for $\pi \in (0, 1)$ and $y > \varepsilon \geq 0$.

Using (2), the KM utility function is

$$W_{KM}(x, \tilde{y}) = \pi v_{KM}(u_1(x) + u_2(y + \varepsilon)) + (1 - \pi) v_{KM}(u_1(x) + u_2(y - \varepsilon)), \tag{27}$$

where $v'_{KM} > 0, v''_{KM} \leq 0$. Here, the expected marginal rate of substitution is

$$\frac{\partial y}{\partial x} = -\frac{u'_1(x)}{\rho(v_{KM}) u'_2(y + \varepsilon) + (1 - \rho(v_{KM})) u'_2(y - \varepsilon)}, \tag{28}$$

where

$$\rho(v_{KM}) \equiv \frac{\pi v'_{KM}(u_1(x) + u_2(y + \varepsilon))}{\pi v'_{KM}(u_1(x) + u_2(y + \varepsilon)) + (1 - \pi) v'_{KM}(u_1(x) + u_2(y - \varepsilon))}. \tag{29}$$

Note that, for a given gamble, since the two values of $\tilde{y}$ occur on separate indifference curves, the expected marginal rate of substitution is a convex combination of the marginal rates of substitution under certainty. Using (11), the Selden-EZ utility function is rewritten as

$$W_S(x, y + \tilde{\varepsilon}) = u_1(x) + u_2(\mu(y + \tilde{\varepsilon}, v_S)), \tag{30}$$

where

$$\mu(y + \tilde{\varepsilon}, v_S) = v_S^{-1}(\pi v_S(y + \varepsilon) + (1 - \pi) v_S(y - \varepsilon)) \tag{31}$$

is the certainty equivalent. Here, the expected marginal rate of substitution is

$$\frac{\partial y}{\partial x} = -\frac{u'_1(x)}{u'_2(\mu(y + \tilde{\varepsilon}, v_S)) \frac{\partial \mu(y + \tilde{\varepsilon}, v_S)}{\partial y}} < 0. \tag{32}$$

On the one hand, from (28), the KM approach affects the weights on the marginal utilities of the second argument, without affecting the values on the marginal utilities themselves. On the other hand, On the other hand, from (32), with the Selden-EZ approach, the marginal utility of the second argument is evaluated at the certainty equivalent and is distorted by the derivative of the certainty equivalent with respect to the outcome of y. This distortion has the effect of changing the ordering preferences over the gambles.

3 The Effect of Risk Aversion

In this section, we study the effect of risk aversion on the optimal choice of the consumption profile $(x, \tilde{y}) \in \mathbb{R}_+^2$ with utility function $U(x, \tilde{y})$, $U_1, U_2 > 0, U_{11}, U_{22} < 0$. In the stochastic environment, x is the *sure* good, while $\tilde{y}$ is the *risky* good due to the presence of randomness in the budget constraint. Using the KM utility representation, the consumer's maximization problem under uncertainty is

$$\max_x W_{KM}(x, \tilde{y}(x)) = \max_x \mathbb{E}_{\tilde{y}(x)} v_{KM} \left(U(x, \tilde{y}(x)) \right), \tag{33}$$

where $\mathbb{E}_{\tilde{y}(x)}$ is the expectation operator over $\tilde{y}(x)$, and v_{KM} is a strictly increasing and concave function, $v'_{KM} > 0, v''_{KM} \leq 0$. Note that the risky good depends on x through the budget constraint, i.e., $y(x) = (I - P_x x)/P_y$, where I is income, and P_x and P_y are the prices of goods x and y, respectively. The effect of risk aversion is studied in three different cases: random income, random price for the sure good, and random price for the risky good.[10]

[10] As noted, the Selden-EZ approach cannot be used to study the effect of risk aversion because it does not disentangle risk aversion and tastes. Specifically, by choosing the optimal gamble $(x, \tilde{y}(x))$ on a particular vertical line, the consumer must compare gambles on different vertical lines, thereby conflating tastes and attitudes toward risk, in which case Selden-EZ preferences does not measure the pure effect of risk aversion on decisions.

3.1 Random Income

When income is random, (33) is rewritten as

$$\max_x \mathbb{E}_{\tilde{I}} v_{KM}\left(U\left(x, \frac{\tilde{I} - P_x x}{P_y}\right)\right), \tag{34}$$

where $\mathbb{E}_{\tilde{I}}$ is the expectation operator for $\tilde{I}$. Proposition 3.1 states that the effect of risk aversion depends on the income effect when only income is random. The change in consumption due to a change in risk aversion does not result from a change in income as in the usual income effect. Instead Proposition 3.1 deals with the distribution of utilities associated with random income and the effect of that distribution of utilities on the choice of the consumption bundle as the consumer becomes more risk averse. In particular, when the sure good is normal, a more risk-averse individual always consumes more of the risky good.

Proposition 3.1. *Given* (34)*, a more risk-averse individual*

1. *decreases the amount of a normal good x,*
2. *increases the amount of an inferior good x, and*
3. *does not change the amount of good x if there is no income effect.*

Proof. See Appendix A. □

This counter-intuitive result is explained by the fact that the individual faces a *utility* gamble with each possible choice of the sure good x. The riskiness of the utility gamble is implicit in the optimal trade-off between the sure good and the risky good and is crucial to the choice of the individual, overshadowing the relevance of the riskiness of the good $\tilde{y}$. In fact, a more risk-averse individual chooses a level of consumption that reduces the riskiness of the utility levels associated with random income. To see this, we proceed in two steps. We first establish a relationship between the income effect and the types of utility gambles an individual faces. We then explain how optimal behavior is changed when risk-aversion increases. To that end, it is convenient to adopt a simple distribution for income, i.e., $\tilde{I} \sim \left(\pi \circ \underline{I}, (1-\pi) \circ \overline{I}\right)$, $\pi \in [0,1]$.

Income Effect and Utility Gambles. The income effect is key in explaining how changes in x affect the riskiness of the utility gambles. To see this, let $x_{\underline{I}}$ and $x_{\overline{I}}$ be the optimal consumption for the sure good when $\pi = 1$ and $\pi = 0$, respectively. For nondegenerate distributions of income, $x \in [\min\{x_{\underline{I}}, x_{\overline{I}}\}, \max\{x_{\underline{I}}, x_{\overline{I}}\}]$ is the range of possible choices. We consider two cases.

Suppose that the sure good is normal, i.e., $x_{\underline{I}} < x_{\overline{I}}$, and let $MU(x, I) \equiv U_1\left(x, \frac{I - P_x x}{P_y}\right) - U_2\left(x, \frac{I - P_x x}{P_y}\right)\frac{P_x}{P_y}$ be the marginal utility of consumption under income $I \in \{\underline{I}, \overline{I}\}$. Then, for any choice of x, the marginal utility under low income at the corresponding point of the lower budget constraint is smaller than the marginal utility under high income at the corresponding point of the upper budget constraint. Moreover, when the marginal utility under low income is tangent to the corresponding budget constraint, (i.e., $x = x_{\underline{I}}$), then the marginal utility under high income is strictly positive. Hence, for $x \in [x_{\underline{I}}, x_{\overline{I}}]$, the difference between utility levels $U\left(x, \frac{\overline{I} - P_x x}{P_y}\right) - U\left(x, \frac{\underline{I} - P_x x}{P_y}\right)$ is positive and strictly increasing in $x \in [x_{\underline{I}}, x_{\overline{I}}]$. In other words, a decrease in x brings the two utility levels closer together. In terms of gambles, this means that a decrease in x results in a less risky utility gamble.

The relationship between x and the riskiness of the utility gamble is clearly shown in Figure 4 when the sure good is normal, i.e., $x_{\underline{I}} < x_{\overline{I}}$. The straight lines represent the budget constraints under low income and high income, while the convex lines are indifference curves. Note that the bundles $(x_{\underline{I}}, y_{\underline{I}}(x_{\underline{I}}))$ and $(x_{\overline{I}}, y_{\overline{I}}(x_{\overline{I}}))$ are the optimal bundles under certain low income and certain high income, respectively.[11] When income is random, choosing x implies choosing the utility gamble

$$g(x) \equiv \left(\pi \circ U\left(x, \frac{\underline{I} - P_x x}{P_y}\right), (1-\pi) \circ U\left(x, \frac{\overline{I} - P_x x}{P_y}\right)\right) \tag{35}$$

for $x \in [x_{\underline{I}}, x_{\overline{I}}]$. From Figure 4, the choice $x_{\overline{I}}$ has a utility gamble corre-

[11] For $I, I' \in \{\underline{I}, \overline{I}\}$, let $y_I(x_{I'}) \equiv (I - P_x x_{I'})/P_y$.

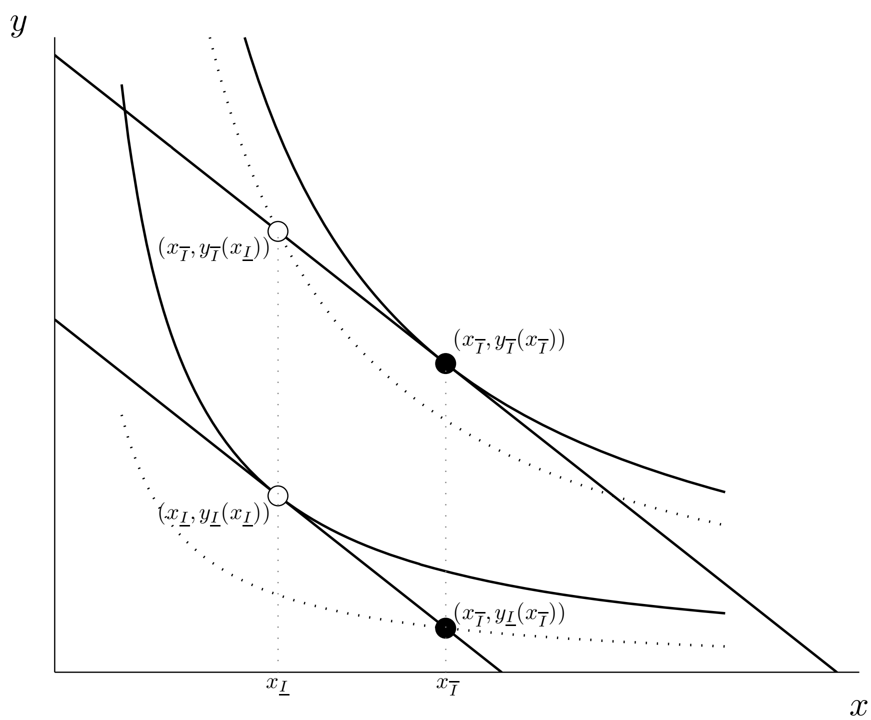

Figure 4: Utility Gambles with Normal Good x

sponding to the solid circles, while the choice $x_{\underline{I}}$ has a utility gamble corresponding to the empty circles. Hence, the gamble $g(x_{\underline{I}})$ is less risky than the gamble $g(x_{\overline{I}})$. In general, as shown in Figure 4 this implies that, for $x, x' \in [x_{\underline{I}}, x_{\overline{I}}], x < x'$,

$$U\left(x', \frac{\underline{I} - P_x x'}{P_y}\right) < U\left(x, \frac{\underline{I} - P_x x}{P_y}\right) < U\left(x, \frac{\overline{I} - P_x x}{P_y}\right) < U\left(x', \frac{\overline{I} - P_x x'}{P_y}\right). \tag{36}$$

Suppose next that the good is inferior, i.e., $x_{\underline{I}} > x_{\overline{I}}$, so that the marginal utility under high income is smaller than the marginal utility under low income at the corresponding point on the budget constraint. Hence, for $x \in [x_{\underline{I}}, x_{\overline{I}}]$, the difference between utility levels $U\left(x, \frac{\underline{I}-P_x x}{P_y}\right) - U\left(x, \frac{\overline{I}-P_x x}{P_y}\right)$ is positive and strictly decreasing in $x \in [x_{\underline{I}}, x_{\overline{I}}]$. In other words, an increase in x brings the two utility levels closer together. In terms of gambles, this means that an increase in x results in a less risky utility gamble, as depicted in Figure 5, where the utility gamble associated with x^* is less risky than the utility gamble corresponding to x. In general, this implies that, for $x, x' \in [x_{\overline{I}}, x_{\underline{I}}], x < x'$,

$$U\left(x, \frac{\underline{I} - P_x x}{P_y}\right) < U\left(x', \frac{\underline{I} - P_x x'}{P_y}\right) < U\left(x', \frac{\overline{I} - P_x x'}{P_y}\right) < U\left(x, \frac{\overline{I} - P_x x}{P_y}\right). \tag{37}$$

Optimal Utility Gamble. Having shown that the income effect determines the direction of a reduction in the riskiness of a gamble, we next turn to the optimal behavior. Without loss of generality, we define two different KM utility representations, $W^1_{KM}(x, \tilde{y}(x)) = U(x, \tilde{y}(x))$ and $W^2_{KM}(x, \tilde{y}(x)) = \varphi(U(x, \tilde{y}(x)))$, $\varphi' > 0, \varphi'' < 0$, so that W^2_{KM} is strictly more risk-averse than W^1_{KM}.

Recall that $MU(x, I) \equiv U_1\left(x, \frac{I-P_x x}{P_y}\right) - U_2\left(x, \frac{I-P_x x}{P_y}\right)\frac{P_x}{P_y}$ is the marginal utility of consumption for $I \in \{\underline{I}, \overline{I}\}$. Then, the first-order conditions corresponding to preferences W^1_{KM} and W^2_{KM} are

$$\pi MU(x, \underline{I}) + (1 - \pi) MU(x, \overline{I}) = 0, \tag{38}$$

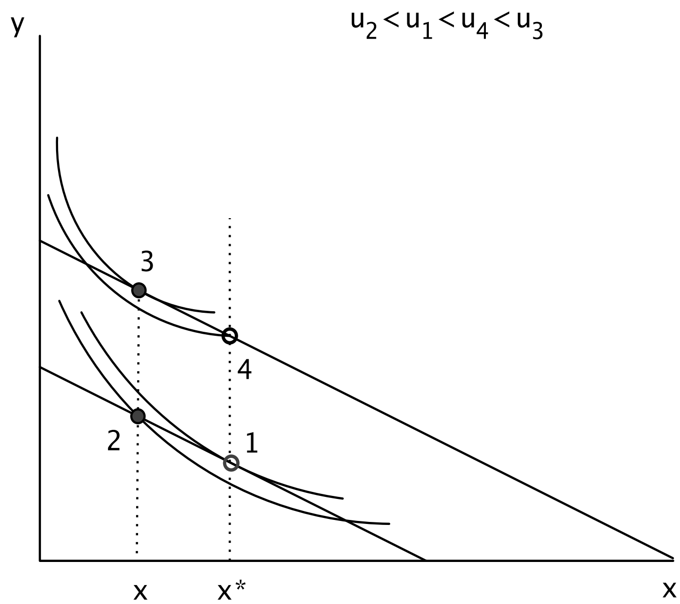

Figure 5: Utility Gambles with Inferior Good x

and

$$\pi\rho(x,\underline{I},\varphi')MU(x,\underline{I}) + (1-\pi)\rho(x,\overline{I},\varphi')MU(x,\overline{I}) = 0, \tag{39}$$

respectively. Here,

$$\rho\left(x,\underline{I},\varphi'\right) = \frac{\varphi'\left(U\left(x,\frac{\underline{I}-P_x x}{P_y}\right)\right)}{\varphi'\left(U\left(x,\frac{\underline{I}-P_x x}{P_y}\right)\right) + \varphi'\left(U\left(x,\frac{\overline{I}-P_x x}{P_y}\right)\right)}, \tag{40}$$

is a weighting function that depends on the risk aversion of the individual, $\rho(x,\underline{I},\varphi') = 1 - \rho(x,\overline{I},\varphi') \in [0,1]$. Note that risk-aversion measured by the function φ enters the first-order condition only through the weighting function ρ. Remark 3.2 states the effect of risk-aversion on the weighting function.

Remark 3.2. *When income is random, the more risk-averse individual adds more weight to the low value of income, i.e.,* $\rho(x,\underline{I},\varphi') > 1/2$.

Given Remark 3.2, the effect of risk aversion is determined by the income effect, which orders the marginal utilities.

Remark 3.3. *When the good is normal,* $MU(x,\underline{I}) < MU(x,\overline{I})$*, while, an inferior good yields* $MU(x,\underline{I}) > MU(x,\overline{I})$.

Combining Remarks 3.2 and 3.3 implies that a more risk-averse individual puts more weight on the lower marginal utility, which corresponds to the low income when the good is normal and the high income when the good is inferior. Hence, a more risk-averse agent decreases the amount of the sure good if and only if it is normal.

It is worth noting that before imposing the more risk averse transformation φ, expected utility maximization yields a trade-off between the sure good and the risky good. However, the introduction of φ changes that trade-off by giving the more risk-averse individual an incentive to choose a less risky utility gamble. In the random income case, this is done by reducing the amount of the sure good x. From this vantage point, it appears that the cardinality of the utility function determines the consumer's choice. However, it is clear from Figure 4, that, for any normal good x, it is the ordinal preference that

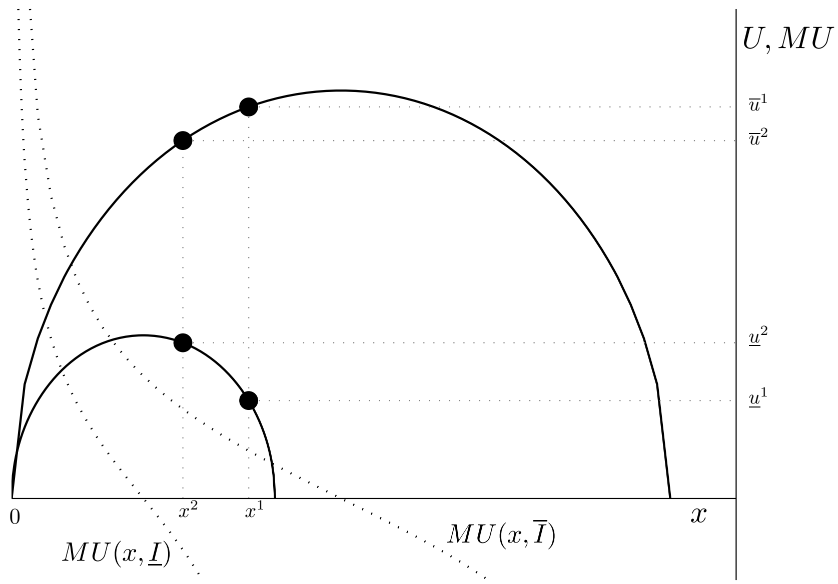

Figure 6: Optimal Utility Gamble with Normal Good x

dictates a decrease in the amount of the sure good x, which results in a less risky utility gamble.

Figure 6 further illustrates the effect of risk aversion on optimal behavior when preferences are Cobb-Douglas, i.e., $U(x, y) = x^\alpha y^{1-\alpha}$, $\alpha \in (0, 1)$.[12] Here, the sure good x is normal. The solid lines represent the utility functions, while the dotted decreasing lines represent the marginal utility functions. The points x^1 and x^2 are the optimal bundles corresponding to preferences W^1_{KM} and W^2_{KM}, respectively. From Figure 6, an increase in risk-aversion adds more weight to the marginal utility under low income, which decreases the amount of the sure good, i.e., $x^1 > x^2$, so as to reduce the riskiness of the utility gamble, i.e., $\underline{u}^1 < \underline{u}^2 < \overline{u}^2 < \overline{u}^1$.

[12] To generate the graph, we set $P_x = P_y = 1$ and $I \in \{2, 5\}$.

3.2 Random Prices

Having shown that the effect of risk aversion depends on the income effect when income is random, we next study the cases of random prices. Here, the relative strength of the income and substitution effects determine the effect of risk aversion on the optimal choice of x.

Random Price of the Sure Good. When the price of the sure good is random, (33) is rewritten as

$$\max_x \mathbb{E}_{\tilde{P}_x} v_{KM}\left(U\left(x, \frac{I - \tilde{P}_x x}{P_y}\right)\right), \tag{41}$$

where $\mathbb{E}_{\tilde{P}_x}$ is the expectation operator for $\tilde{P}_x$. Proposition 3.4 states that the effect of risk aversion is determined by the interplay of the income and substitution effects.

Proposition 3.4. *Given* (41)*, a more risk-averse individual*

1. *decreases the amount of a normal good x, and*
2. *increases the amount of an inferior good x if and only if the income effect is stronger than the substitution effect.*

Proof. See Appendix A. □

To explain the results in Proposition 3.4, it is convenient to adopt a simple distribution for the price of the sure good, i.e., $\tilde{P}_x \sim \left(\pi \circ \underline{P}_x, (1-\pi) \circ \overline{P}_x\right)$, $\pi \in [0, 1]$. Without loss of generality, we define two different KM utility representations, $W^1_{KM}(x, \tilde{y}(x)) = U(x, \tilde{y}(x))$ and $W^2_{KM}(x, \tilde{y}(x)) = \varphi(U(x, \tilde{y}(x)))$, $\varphi' > 0, \varphi'' < 0$, so that W^2_{KM} is strictly more risk-averse than W^1_{KM}.

Letting $MU(x, P_x) \equiv U_1\left(x, \frac{I-P_x x}{P_y}\right) - U_2\left(x, \frac{I-P_x x}{P_y}\right)\frac{P_x}{P_y}$ be the marginal utility of consumption for $P_x \in \{\underline{P}_x, \overline{P}_x\}$, the first-order conditions corresponding to preferences W^1_{KM} and W^2_{KM} are

$$\pi MU(x, \underline{P}_x) + (1-\pi) MU(x, \overline{P}_x) = 0, \tag{42}$$

and

$$\pi\rho(x,\underline{P}_x,\varphi')MU(x,\underline{P}_x)+(1-\pi)\rho(x,\overline{P}_x,\varphi')MU(x,\overline{P}_x)=0, \tag{43}$$

respectively. Here,

$$\rho\left(x,\underline{P}_x,\varphi'\right)=\frac{\varphi'\left(U\left(x,\frac{I-\underline{P}_x x}{P_y}\right)\right)}{\varphi'\left(U\left(x,\frac{I-\underline{P}_x x}{P_y}\right)\right)+\varphi'\left(U\left(x,\frac{I-\overline{P}_x x}{P_y}\right)\right)}, \tag{44}$$

is a weighting function that depends on the risk aversion of the individual, $\rho(x,\underline{P}_x,\varphi')=1-\rho(x,\overline{P}_x,\varphi')\in[0,1]$. Note that risk-aversion measured by the function φ enters the first-order condition only through the weighting function ρ, as in the case of random income. Remark 3.5 states the effect of risk-aversion on the weighting function when P_x is random.

Remark 3.5. *When the price of the sure good is random, the more risk-averse individual adds less weight to the low value of P_x, i.e., $\rho(x,\underline{P}_x,\varphi')<1/2$.*

The effect of risk aversion is determined by the income and substitution effects, which orders the marginal utilities. Abstracting for a moment that we are dealing with only two values for P_x, the sign of the derivative of $MU(x,P_x)$ with respect to P_x is useful in ordering the marginal utilities. Formally,

$$\frac{\partial MU(x,P_x)}{\partial P_x}=\underbrace{-\left(U_{12}-U_{22}\frac{P_x}{P_y}\right)\frac{x}{P_y}}_{=IE_{P_x}}\underbrace{-\frac{U_2}{P_y}}_{=SE_{P_x}}, \tag{45}$$

where IE_{P_x} and $SE_{P_x}<0$ are proportional to and of the same sign as the income effect and the substitution effect, respectively, related to a change in P_x.

Remark 3.6. *When the good is normal, both the income and substitution effects are negative, so that $MU(x,\underline{P}_x)>MU(x,\overline{P}_x)$. When the good is inferior, i.e., $IE_{P_x}>0$, the relative strengths of the income and substitution effects determine the ordering of the marginal utilities. For instance, if the*

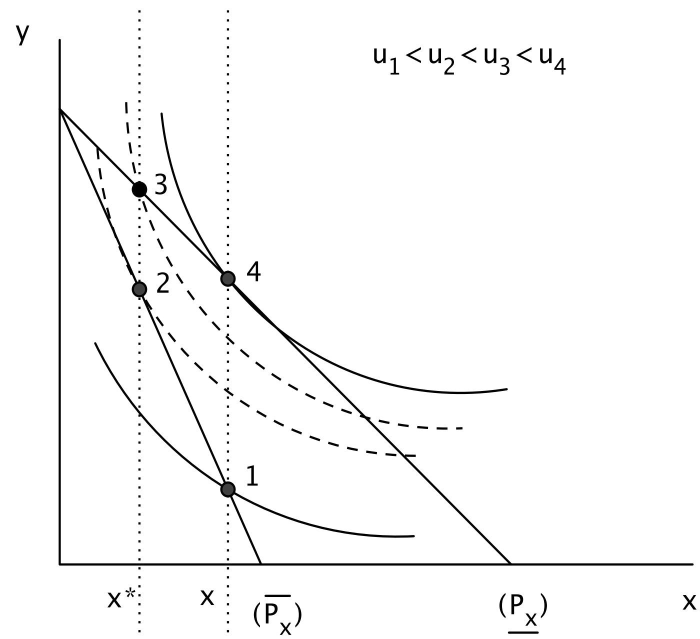

Figure 7: Utility Gambles with Normal Good x and Random P_x

(positive) income effect is stronger than the (negative) substitution effect, then, from (45)*,* $MU(x, \underline{P}_x) < MU(x, \overline{P}_x)$*.*

Remarks 3.5 and 3.6 explain the result in Proposition 3.4. In particular, when the good is normal, a more risk-averse individual individual puts less weight to the marginal utility corresponding to the high price of x. Proposition 3.4 is illustrated in Figure 7 for the case of a normal good x. Due to the randomness of P_x, the slope of the budget constraint makes the utility gamble less risky as consumption decreases. Specifically, when P_x is random, the pure substitution effect induces a squeeze in the utility gamble in the direction of less quantity of the sure good, from x to x^*, which increases the amount of the risky good y.

Random Price of the Risky Good. When the price of the risky good

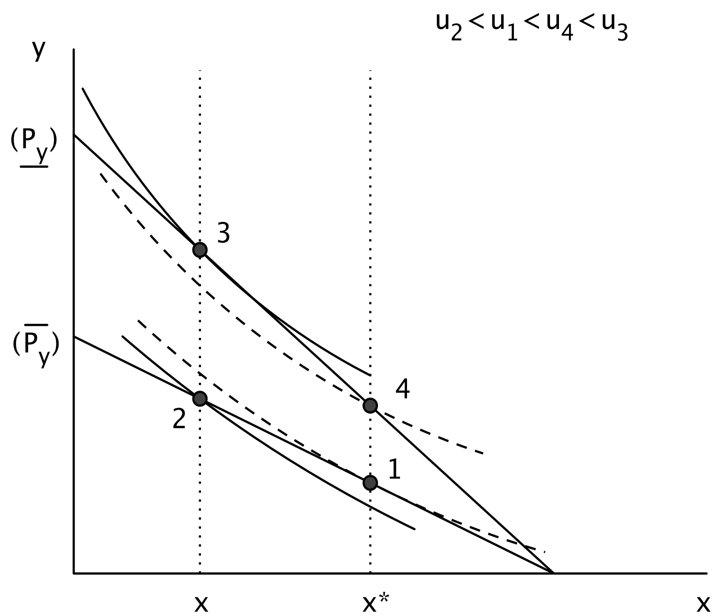

Figure 8: Utility Gambles with Normal Good x and Random P_y

is random, (33) is rewritten as

$$\max_x \mathbb{E}_{\tilde{P}_y} v_{KM}\left(U\left(x, \frac{I - P_x x}{\tilde{P}_{\tilde{y}}}\right)\right), \tag{46}$$

where $\mathbb{E}_{\tilde{P}_y}$ is the expectation operator for $\tilde{P}_y$. Proposition 3.7 states that the effect of risk aversion is again determined by the interplay of the income and substitution effects.

Proposition 3.7. *Given* (46)*, a more risk-averse individual*

1. *decreases the amount of a normal good x if and only if the income effect is stronger than the substitution effect, and*

2. *increases the amount of an inferior good x.*

Proof. See Appendix A. □

Proposition 3.7 is illustrated in Figure 8, which shows that, when the price of the risky good is random, the substitution effect induces a squeeze in the utility gamble by increasing the amount of the sure good from x to x^*.

3.3 Examples

Propositions 3.1, 3.4, and 3.7 establish the connection between risk aversion and classical demand theory implicit in KM, and explain, in that context, the limits of the intuition of the Ross (1981) critique on risk aversion. In particular, increasing the amount of the risky good when risk aversion increases, is natural and not counterintuitive, as thought by Ross (1981) and his followers. We now illustrate our results by considering specific classes of preferences: Cobb-Douglas, Leontief, and quasi-linear utility functions.

Suppose that preferences are Cobb-Douglas, i.e., $U(x, y) = x^\alpha y^\beta, \alpha, \beta > 0$, so that x is a normal good. Note that the results in Propositions 3.1, 3.4, and 3.7 continue to hold, even when the utility function is concave or convex, as long as there is an interior solution to the constrained optimization problem. For random income, more risk aversion has the effect of decreasing the amount of x. This result is even stronger when the price of x is random, since the income and the substitution effects go in the same direction. However, when the price of y is random, the income and the substitution effects not only go in opposite directions, but cancel each other out with Cobb-Douglas preferences. This is exactly the consumption-saving problem discussed in KM, in which the rate of return (the price of y) is random.

Suppose next that preferences are Leontief, i.e., $U(x, y) = u(\min\{x, y\})$, $u' > 0, u'' < 0$, so that x is a normal good. Then, there is no substitution effect and the income effect determines the direction of the change along with an increase in risk aversion. In particular, regardless of the source of risk, an increase in risk aversion always decreases the amount of the sure good in favor of the risky good. To see this, consider the income distribution $\tilde{I} \sim (1 - \pi \circ \underline{I}, \pi \circ \overline{I})$, $\underline{I} < \overline{I}$. For simplicity assume $P_x = P_y = 1$, so that the individual faces

$$\max_x \mathbb{E}u\left(\min\left\{x, \tilde{I}-x\right\}\right). \tag{47}$$

The optimal solution lies in $\frac{\underline{I}}{2} \leq x \leq \frac{\overline{I}}{2}$.[13] Hence, the problem faced by the individual is $\mathbb{E}u\left(\min\left\{x, \widetilde{I}-x\right\}\right) = \pi u\left(x\right) + \left(1-\pi\right)u\left(\underline{I}-x\right)$, so that the first-order condition is

$$\pi u'(x) - (1-\pi)u'(\underline{I}-x) = 0 \qquad \Longrightarrow \frac{u'(x)}{u'(\underline{I}-x)} = \frac{1-\pi}{\pi}. \tag{48}$$

Consider next a more risk-averse individual, i.e., the maximization problem is

$$\max_x \pi\varphi(u(x)) + (1-\pi)\varphi(u(\underline{I}-x)), \tag{49}$$

$\varphi' > 0, \varphi'' < 0$. The first-order condition is

$$\pi\varphi'(u(x^*))u'(x^*) - (1-\pi)\varphi'(u(\underline{I}-x^*))u'(\underline{I}-x^*) = 0, \tag{50}$$

so that

$$\frac{u'(x^*)}{u'(\underline{I}-x^*)} = \frac{1-\pi}{\pi}\frac{\varphi'(u(\underline{I}-x^*))}{\varphi'(u(x^*))} \tag{51}$$

since φ is strictly concave and $\frac{\underline{I}}{2} \leq x \leq \frac{\overline{I}}{2}$. Then,

$$\frac{\varphi'(u(\underline{I}-x^*))}{\varphi'(u(x^*))} > 1$$

therefore,

$$\frac{u'(x^*)}{u'(\underline{I}-x^*)} = \frac{1-\pi}{\pi}\frac{\varphi'(u(\underline{I}-x^*))}{\varphi'(u(x^*))} > \frac{u'(x)}{u'(\underline{I}-x)}$$

$$u'(x^*) > u'(x) \Rightarrow x^* < x,$$

as shown in Figure 9.

[13] Note that if $x < \frac{\underline{I}}{2}$, then the outcome is strictly worse than choosing $x = \frac{\underline{I}}{2}$, while, if $x > \frac{\overline{I}}{2}$, then the outcome is strictly worse than choosing $x = \frac{\overline{I}}{2}$. Note also that in this case, the optimal solution has both x and y positive, i.e., there is no corner solution, in which either $x = 0$ or $y = 0$. However, on the interval $\frac{\underline{I}}{2} \leq x \leq \frac{\overline{I}}{2}$, there can be "corner solutions", when $x = \frac{\underline{I}}{2}$ or $x = \frac{\overline{I}}{2}$, which correspond to the most risk-averse and the most risk-loving choices respectively.

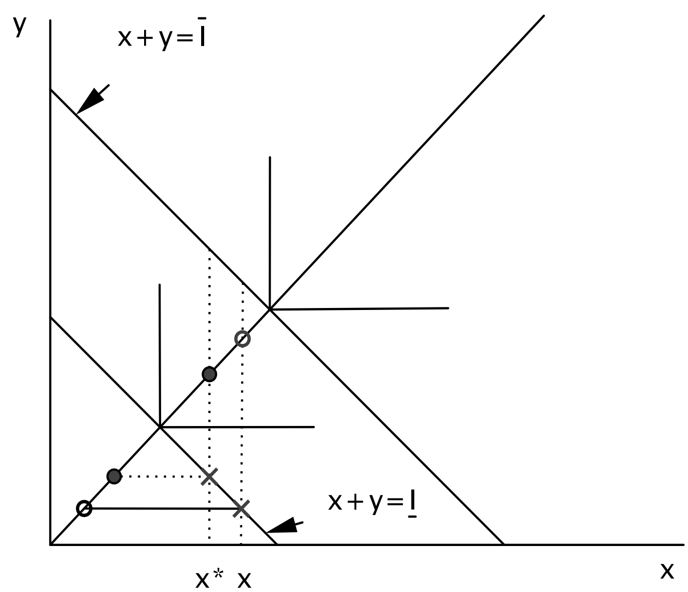

Figure 9: Leontief Preferences

In other words, the more risk-averse individual consumes even more of the risky good due to the presence of only income effects.[14] This decrease in the amount of the sure good x yields a less risky utility gamble as depicted in Figure 9. Note that, with Leontief preferences, the amount of x always decreases because there is only a pure income effect, i.e., without substitution. That is why Leontief preferences yield results opposite to the Arrow-Pratt's portfolio problem, in which there is no income effect and only substitution effect.

Suppose finally that preferences are quasi-linear, where x is a normal good. These preferences shed light on the Arrow-Pratt result, i.e., an increase in risk aversion increases the amount of income invested in the safe asset (i.e., allocated to the sure good in our context). We now demonstrate that Arrow-Pratt's result holds due to the absence of the income effect, and that the source of uncertainty lies in the rate of return of the risky asset. Specifically, we consider two cases and show stark difference in results between the two. First, consider the case in which there is no income effect for the sure good x, i.e., $U(x, y) = u_1(x) + y, u_1' > 0, u_1'' < 0$. Hence, when income is random, since there is no income effect, risk aversion has no effect on the amount of x. When the price of x is random, increased risk aversion cause the amount of x to decrease solely due to the substitution effect. However, for random price of the risky good, the substitution effect dominates (since there is no income effect), which implies that the amount of x increases along with an increase in risk aversion. This result generalizes the result in the Arrow-Pratt portfolio problem. In fact, it is only in this case that increasing risk aversion increases the amount of the sure good without reference to income and substitution effects. However, the result is not robust to a slight modification in the utility function. To see this, consider the quasi-linear utility function, i.e., $U(x, y) = x + u_2(y), u_2' > 0, u_2'' < 0$. In this case, the good x is normal, so that if either income or the price of x is random, risk aversion decreases the amount of the sure good x. On the other hand, for random price of the

[14] Note that if the initial choice is $x = \frac{I}{2}$, then the consumer is making the most risk-averse choice. Therefore a more risk-averse transformation cannot reduce the level of x.

risky good, the income and substitution effects pull in opposite directions. If the income effect is dominant, then an increase in risk aversion leads to a decrease in the amount of the sure good x. This last result illustrates that the Arrow-Pratt result is solely due to the absence of an income effect on x. Finally, note that another version of the Arrow-Pratt theorem is that, if income increases, then an individual with decreasing risk aversion reduces the amount of the sure good. This result is not general and is due only to the fact that, in the portfolio problem, there is no income effect for the sure good.

4 Final Remarks

In this paper, we completely characterize the relationship between changes in risk aversion and classical demand theory in the case of a single source of uncertainty. We show that a more risk averse consumer generally decreases the amount placed in the sure good. In addition, we show that it is the utility gambles that determine the choice of a more risk-averse agent between the sure good and the risky good. This provides an explanation for certain paradoxical behaviors of an individual who becomes more risk-averse. The paper also paves a path for some immediate interesting questions. In particular, one could ask what the relationship between risk aversion and classical demand theory implies for changes in income in which the consumer is decreasingly risk-averse. This is especially interesting in light of Arrow-Pratt's result that in the portfolio case increasing income results in an increase in the risky assert if and only if the consumer is decreasingly risk averse with income.

A Proofs

We recall Theorem 236 in Hardy et al. (1964), which we appeal in the proofs of the propositions.

Lemma A.1. *If* $\lambda_1'(t), \lambda_2'(t) > 0$ *or* $\lambda_1'(t), \lambda_2'(t) < 0$*, then* $\mathbb{E}_{\tilde{t}}\lambda_1(\tilde{t})\lambda_2(\tilde{t}) > \mathbb{E}_{\tilde{t}}\lambda_1(\tilde{t}) \cdot \mathbb{E}_{\tilde{t}}\lambda_2(\tilde{t})$*. If* $\lambda_1'(t) > 0, \lambda_2'(t) < 0$ *or* $\lambda_1'(t) < 0, \lambda_2'(t) > 0$*, then* $\mathbb{E}_{\tilde{t}}\lambda_1(\tilde{t})\lambda_2(\tilde{t}) < \mathbb{E}_{\tilde{t}}\lambda_1(\tilde{t}) \cdot \mathbb{E}_{\tilde{t}}\lambda_2(\tilde{t})$.

Proof of Proposition 3.1. Without loss of generality, consider two different KM utility representations, $W_{KM}^1(x, \tilde{y}(x)) = \mathbb{E}_{\tilde{I}} U\left(x, (\tilde{I} - P_x x)/P_y\right)$ and $W_{KM}^2(x, \tilde{y}(x)) = \varphi\left(U\left(x, (\tilde{I} - P_x x)/P_y\right)\right)$, $\varphi' > 0, \varphi'' < 0$. Here, W_{KM}^2 is strictly more risk-averse than W_{KM}^1.

From (34), the first-order condition corresponding to preferences W_{KM}^1 is

$$\mathbb{E}_{\tilde{I}} \underbrace{\left[U_1\left(x, \frac{\tilde{I} - P_x x}{P_y}\right) - U_2\left(x, \frac{\tilde{I} - P_x x}{P_y}\right)\frac{P_x}{P_y}\right]}_{\equiv h(x, \tilde{I}, P_x, P_y)} = 0, \tag{52}$$

while the first-order condition corresponding to preferences W_{KM}^2 is

$$\mathbb{E}_{\tilde{I}} \underbrace{\varphi'\left(U\left(x, (\tilde{I} - P_x x)/P_y\right)\right)}_{\equiv f(x, \tilde{I}, P_x, P_y)} \cdot \underbrace{\left[U_1\left(x, \frac{\tilde{I} - P_x x}{P_y}\right) - U_2\left(x, \frac{\tilde{I} - P_x x}{P_y}\right)\frac{P_x}{P_y}\right]}_{\equiv h(x, \tilde{I}, P_x, P_y)} = 0. \tag{53}$$

Let x^1 and x^2 be the optimal choice of the sure good satisfying (52) and (53), respectively.

Given the definition of $f(x, I, P_x, P_y)$ in (53), $\varphi'' < 0, U_2 > 0$ imply that $\partial f / \partial I < 0$. We now consider three cases.

1. Suppose first that x is a normal good. Then, given the definition of $h(x, I, P_x, P_y)$ in (52) or (53),

$$\frac{\partial h}{\partial I} = \underbrace{U_{12} \cdot \frac{x}{P_y} - U_{22} \cdot \frac{P_x x}{P_y^2}}_{= IE_I} > 0, \tag{54}$$

where IE_I is proportional and of the same sign as the income effect related to a change in income.

Since $\partial f/\partial I < 0, \partial h/\partial I > 0$, Lemma A.1 and (52) imply that

$$\mathbb{E}_{\tilde{I}} f(x, \tilde{I}, P_x, P_y) h(x, \tilde{I}, P_x, P_y) < \mathbb{E}_{\tilde{I}} f(x, \tilde{I}, P_x, P_y) \cdot \mathbb{E}_{\tilde{I}} h(x, \tilde{I}, P_x, P_y) = 0. \tag{55}$$

Since x^1, x^2 are unique interior solutions, $x^1 > x^2$, i.e., a more risk-averse individual decreases the consumption of a normal sure good.

2. Suppose next that x is an inferior good. Then, from (54), $\partial h/\partial I < 0$. Since $\partial f/\partial I < 0, \partial h/\partial I < 0$, Lemma A.1 and (52) imply that

$$\mathbb{E}_{\tilde{I}} f(x, \tilde{I}, P_x, P_y) h(x, \tilde{I}, P_x, P_y) > \mathbb{E}_{\tilde{I}} f(x, \tilde{I}, P_x, P_y) \cdot \mathbb{E}_{\tilde{I}} h(x, \tilde{I}, P_x, P_y) = 0. \tag{56}$$

Since x^1, x^2 are unique interior solutions, $x^1 < x^2$, i.e., a more risk-averse individual increases the consumption of an inferior sure good.

3. Suppose finally that there is no income effect. Then, $x^1 = x^2$.

Proof of Proposition 3.4. Without loss of generality, consider two different KM utility representations, $W^1_{KM}(x, \tilde{y}(x)) = \mathbb{E}_{\tilde{P}_x} U\left(x, (I - \tilde{P}_x x)/P_y\right)$ and $W^2_{KM}(x, \tilde{y}(x)) = \mathbb{E}_{\tilde{P}_x} \varphi\left(U\left(x, (I - \tilde{P}_x x)/P_y\right)\right)$, $\varphi' > 0, \varphi'' < 0$. Here, W^2_{KM} is strictly more risk-averse than W^1_{KM}.

From (41), the first-order condition corresponding to preferences W^1_{KM} is

$$\mathbb{E}_{\tilde{P}_x} \underbrace{\left[U_1\left(x, \frac{I - \tilde{P}_x x}{P_y}\right) - U_2\left(x, \frac{I - \tilde{P}_x x}{P_y}\right) \frac{\tilde{P}_x}{P_y} \right]}_{\equiv h(x, I, \tilde{P}_x, P_y)} = 0, \tag{57}$$

while the first-order condition corresponding to preferences W^2_{KM} is

$$\mathbb{E}_{\tilde{P}_x} \underbrace{\varphi'\left(U\left(x, (I - \tilde{P}_x x)/P_y\right)\right)}_{\equiv f(x, I, \tilde{P}_x, P_y)} \cdot \underbrace{\left[U_1\left(x, \frac{I - \tilde{P}_x x}{P_y}\right) - U_2\left(x, \frac{I - \tilde{P}_x x}{P_y}\right) \frac{\tilde{P}_x}{P_y} \right]}_{\equiv h(x, I, \tilde{P}_x, P_y)} = 0. \tag{58}$$

Let x^1 and x^2 be the optimal choice of the sure good satisfying (57) and (58), respectively.

Given the definition of $f(x, I, P_x, P_y)$ in (53), $\varphi'' < 0, U_2 > 0$ imply that $\partial f / \partial P_x > 0$. We now consider two cases.

1. Suppose first that x is a normal good. Then, given the definition of $h(x, I, P_x, P_y)$ in (57) or (58),

$$\frac{\partial h}{\partial P_x} = \underbrace{-\left[U_{12} \cdot \frac{x}{P_y} - U_{22} \cdot \frac{P_x x}{P_y^2}\right]}_{=IE_{P_x}} \underbrace{-\frac{U_2}{P_y}}_{=SE_{P_x}} < 0, \tag{59}$$

   as both the income and the substitution effects pull in the same direction, $IE_{P_x} < 0, SE_{P_x} < 0$. Here, IE_{P_x} and SE_{P_x} are proportional and of the same sign as the income and substitution effects, respectively, related to a change in P_x.

   Since $\partial f / \partial P_x > 0, \partial h / \partial P_x < 0$, Lemma A.1 and (57) imply that

$$\mathbb{E}_{\tilde{P}_x} f(x, I, \tilde{P}_x, P_y) h(x, I, P_x, P_y) < \mathbb{E}_{\tilde{P}_x} f(x, I, \tilde{P}_x, P_y) \cdot \mathbb{E}_{\tilde{P}_x} h(x, I, P_x, P_y) = 0. \tag{60}$$

   Since x^1, x^2 are unique interior solutions, $x^1 > x^2$, i.e., a more risk-averse individual decreases the consumption of a normal sure good.

2. Suppose next that x is an inferior good, i.e., $IE_{P_x} > 0$. Then, from (59), if the income effect is stronger than the substitution effect, $\partial h / \partial P_x > 0$. Since $\partial f / \partial P_x, \partial h / \partial P_x > 0$, Lemma A.1 and (52) imply that

$$\mathbb{E}_{\tilde{P}_x} f(x, I, \tilde{P}_x, P_y) h(x, I, \tilde{P}_x, P_y) > \mathbb{E}_{\tilde{P}_x} f(x, I, \tilde{P}_x, P_y) \cdot \mathbb{E}_{\tilde{P}_x} h(x, I, \tilde{P}_x, P_y) = 0. \tag{61}$$

   Since x^1, x^2 are unique interior solutions, $x^1 < x^2$, i.e., a more risk-averse individual increases the consumption of an inferior sure good. If the income effect is weaker than the substitution effect, then, by the same argument, $x^1 > x^2$. Finally, if the income and substitution effects cancel each other, then $x^1 = x^2$.

Proof of Proposition 3.7. Without loss of generality, consider two different KM utility representations, $W^1_{KM}(x, \tilde{y}(x)) = \mathbb{E}_{\tilde{P}_{\tilde{y}}} U\left(x, (I - P_x x)/\tilde{P}_{\tilde{y}}\right)$ and $W^2_{KM}(x, \tilde{y}(x)) = \mathbb{E}_{\tilde{P}_{\tilde{y}}} \varphi\left(U\left(x, (I - P_x x)/\tilde{P}_{\tilde{y}}\right)\right)$, $\varphi' > 0, \varphi'' < 0$. Here, W^2_{KM} is strictly more risk-averse than W^1_{KM}.

From (46), the first-order condition corresponding to preferences W^1_{KM} is

$$\mathbb{E}_{\tilde{P}_y} \underbrace{\left[U_1\left(x, \frac{I - P_x x}{\tilde{P}_{\tilde{y}}}\right) - U_2\left(x, \frac{I - P_x x}{\tilde{P}_{\tilde{y}}}\right)\frac{P_x}{\tilde{P}_{\tilde{y}}}\right]}_{\equiv h(x, I, P_x, \tilde{P}_{\tilde{y}})} = 0, \tag{62}$$

while the first-order condition corresponding to preferences W^2_{KM} is

$$\mathbb{E}_{\tilde{P}_y} \underbrace{\varphi'\left(U\left(x, (I - P_x x)/\tilde{P}_{\tilde{y}}\right)\right)}_{\equiv f(x, I, P_x, \tilde{P}_{\tilde{y}})} \cdot \underbrace{\left[U_1\left(x, \frac{I - P_x x}{\tilde{P}_{\tilde{y}}}\right) - U_2\left(x, \frac{I - P_x x}{\tilde{P}_{\tilde{y}}}\right)\frac{P_x}{\tilde{P}_{\tilde{y}}}\right]}_{\equiv h(x, I, P_x, \tilde{P}_{\tilde{y}})} = 0. \tag{63}$$

Let x^1 and x^2 be the optimal choice of the sure good satisfying (62) and (63), respectively.

Given the definition of $f(x, I, P_x, P_y)$ in (53), $\varphi'' < 0, U_2 > 0$ imply that $\partial f / \partial P_y > 0$. We now consider two cases.

1. Suppose first that x is an inferior good. Then, given the definition of $h(x, I, P_x, P_y)$ in (62) or (63),

$$\frac{\partial h}{\partial P_y} = \underbrace{-\left[U_{12} \cdot \frac{x}{P_y} - U_{22} \cdot \frac{P_x}{P_y}\right] \frac{I - P_x x}{P_y^2}}_{IE_{P_y}} + \underbrace{\frac{U_2 P_x}{P_y^2}}_{SE_{P_y}} < 0, \tag{64}$$

as both the income and the substitution effects pull in the same direction, $IE_{P_y} > 0, SE_{P_y} > 0$. Here, IE_{P_y} and SE_{P_y} are proportional and of the same sign as the income and substitution effects, respectively, related to a change in the price of $\tilde{y}$.

Since $\partial f/\partial P_x, \partial h/\partial P_x > 0$, Lemma A.1 and (52) imply that

$$\mathbb{E}_{\tilde{P}_{\tilde{y}}} f(x, I, P_x, \tilde{P}_{\tilde{y}}) h(x, I, P_x, \tilde{P}_{\tilde{y}}) > \mathbb{E}_{\tilde{P}_{\tilde{y}}} f(x, I, P_x, \tilde{P}_{\tilde{y}}) \cdot \mathbb{E}_{\tilde{P}_{\tilde{y}}} h(x, I, P_x, \tilde{P}_{\tilde{y}}) = 0. \tag{65}$$

Since x^1, x^2 are unique interior solutions, $x^1 < x^2$, i.e., a more risk-averse individual increases the consumption of an inferior sure good.

2. Suppose next that x is a normal good, i.e., $IE_{P_y} < 0$. Then, from (64), if the income effect is stronger than the substitution effect, then $\partial h/\partial P_y < 0$. Since $\partial f/\partial P_y > 0, \partial h/\partial P_y < 0$, Lemma A.1 and (62) imply that

$$\mathbb{E}_{\tilde{P}_{\tilde{y}}} f(x, I, P_x, \tilde{P}_{\tilde{y}}) h(x, I, P_x, \tilde{P}_{\tilde{y}}) < \mathbb{E}_{\tilde{P}_{\tilde{y}}} f(x, I, P_x, \tilde{P}_{\tilde{y}}) \cdot \mathbb{E}_{\tilde{P}_{\tilde{y}}} h(x, I, P_x, \tilde{P}_{\tilde{y}}) = 0. \tag{66}$$

Since x^1, x^2 are unique interior solutions, $x^1 > x^2$, i.e., a more risk-averse individual decreases the consumption of a normal sure good. If the substitution effect is stronger than the substitution effect, then, by the same argument, $x^1 > x^2$. Finally, if the income and substitution effects cancel each other, then $x^1 = x^2$.

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