Federal Reserve Bank of Minneapolis Quarterly Review
Summer 2003, Vol. 27, No. 3, pp. 17–33

How Severe Is the Time-Inconsistency Problem in Monetary Policy?

Stefania Albanesi
Assistant Professor of Economics
Fuqua School of Business
Duke University
and Research Affiliate
Centre for Economic Policy Research

V. V. Chari
Paul W. Frenzel Land Grant
Professor of Liberal Arts
and Professor of Economics
University of Minnesota
and Consultant
Research Department
Federal Reserve Bank of Minneapolis

Lawrence J. Christiano
Alfred W. Chase Chair in Business Institutions
and Professor of Economics
Northwestern University
and Consultant
Research Department
Federal Reserve Bank of Minneapolis
and Visiting Scholar
Federal Reserve Bank of Chicago
and Federal Reserve Bank of Cleveland

Abstract

This study analyzes two monetary economies, a cash-credit good model and a limited-participation model. In these models, monetary policy is made by a benevolent policymaker who cannot commit to future policies. The study defines and analyzes Markov equilibrium in these economies and shows that there is no time-inconsistency problem for a wide range of parameter values. The study originally appeared in a book, *Advances in Economics and Econometrics: Theory and Applications* © 2003 by Cambridge University Press.

The history of inflation in the United States and other countries has occasionally been quite bad. Are the bad experiences the consequence of policy errors? Or does the problem lie with the nature of monetary institutions? The second possibility has been explored in a long literature, which starts at least with the work of Kydland and Prescott (1977) and Barro and Gordon (1983). This study seeks to make a contribution to that literature.

The Kydland-Prescott and Barro-Gordon literature focuses on the extent to which monetary institutions allow policymakers to commit to future policies. A key result is that if policymakers cannot commit to future policies, inflation rates are higher than if they can commit. That is, there is a *time-inconsistency problem* that introduces a systematic inflation bias. This study investigates the magnitude of the inflation bias in two standard general equilibrium models. One is the cash-credit good model of Lucas and Stokey (1983). The other is the limited-participation model of money described by Christiano, Eichenbaum, and Evans (1997). We find that in these models, for a large range of parameter values, there is no time-inconsistency problem and no inflation bias.

In the Kydland-Prescott and Barro-Gordon literature, equilibrium inflation in the absence of commitment is the outcome of an interplay between the benefits and costs of inflation. For the most part, this literature consists of reduced-form models. Our general equilibrium models incorporate the kinds of benefits and costs that seem to motivate the reduced-form specifications. (For related general equilibrium models, see Ireland 1997; Chari, Christiano, and Eichenbaum 1998; and Neiss 1999.)

To understand these benefits and costs, we must first explain why money is not neutral in our models. In both models, at the time the monetary authority sets its money growth rate, some nominal variable in the economy has already been set. In the cash-credit good model, this variable is the price of a subset of intermediate goods. Here, as in the work of Blanchard and Kiyotaki (1987), some firms must post prices in advance and are required to meet all demand at their posted price. In the limited-participation model, a portfolio choice variable is set in advance. In both models, higher than expected money growth tends—other things being the same—to raise output. The rise in output raises welfare because the presence of monopoly power in our model economies implies that output and employment are below their efficient levels. These features give incentives to the monetary authority to make money growth rates higher than expected. Thus, inflation clearly has benefits in these models.

Turning to the costs of inflation, we first discuss the cash-credit good model. We assume that cash good consumption must be financed by using money carried over from the previous period. If the money growth rate is high, the price of the cash good is high, and the quantity of cash goods consumed is low. This mechanism tends to reduce welfare as the money growth rate rises.

In the cash-credit good model, the monetary authority balances the output-increasing benefits of high money growth against the costs of the resulting fall in cash good consumption. Somewhat surprisingly, we find that there is a large subset of parameter values in which the costs of inflation dominate the benefits at all levels of inflation and money growth above the ex ante optimal rate. As a result, in this model, for these parameter values, the unique equilibrium yields the same outcome as under commitment. That is, there is no time-inconsistency problem and no inflation bias.

In our limited-participation model, at all interest rates higher than zero, increases in money growth tend to stimulate employment by reducing the interest rate. As a result, there is no equilibrium with a positive interest rate. When the interest rate is already zero, further reductions are not possible. In this model, additional money generated by the monetary authority simply accumulates as idle balances at the financial intermediary. The unique Markov equilibrium in this model has a zero interest rate. Again, there is no time-inconsistency problem and no inflation bias.

Should we conclude from our results here that lack of commitment in monetary policy cannot account for the bad inflation outcomes that have occurred? We think such a conclusion is premature. Research on the consequences of lack of commitment in dynamic general equilibrium models is still in its infancy. Elsewhere, in Albanesi, Chari, and Christiano 2002, we have displayed a class of empirically plausible models in which lack of commitment may in fact lead to high and volatile inflation. The key difference between the model in that work and the models studied here lies in the modeling of money demand. Taken together, these findings suggest that a resolution of the importance of time inconsistency in monetary policy depends on the details of money demand. As our understanding about the implications for time inconsistency in dynamic models grows, we may discover other features of the economic environment that are crucial for determining the severity of the time-inconsistency problem. It is too soon to tell whether the ultimate conclusion will be consistent with the implications of the models studied here.

The study is organized as follows. First, we analyze a cash-credit good model with arbitrary monetary policy. This section sets up the basic framework for analyzing purposeful monetary policy. Interestingly, we also obtain some new results on multiplicity of equilibria under mild deflations. Next, we analyze the same model when monetary policy is chosen by a benevolent policymaker without commitment. Then, we analyze a limited-participation model. Finally, we conclude.

The Basic Framework

Here we develop a version of the Lucas-Stokey cash-credit good model. There are three key modifications: we introduce monopolistic competition, as do Blanchard and Kiyotaki (1987); we modify the timing in the cash-in-advance constraint, as do Svensson (1985) and Nicolini (1998); and we consider nonstationary equilibria. The agents in the model are a representative household and representative intermediate and final good producing firms. A policy for the monetary authority is a sequence of growth rates for the money supply. We consider arbitrary monetary policies and define and characterize the equilibrium. We show that in the best equilibrium with commitment, monetary policy follows the Friedman rule in the sense that the nominal interest rate is zero (Friedman 1969). Following Cole and Kocherlakota (1998), we show that there is a nontrivial class of monetary policies that support the best equilibrium. We show that only one of the policies in this class is robust; the others are fragile. Specifically, we show that only the policy in which money growth deflates at the pure rate of time preference supports the best equilibrium as the unique outcome. We show that the other policies are fragile in the sense that there are many equilibria associated with them.

The Agents

The Household

A household's utility function is

$$(1)\qquad \sum_{t=0}^{\infty}\beta^t u(c_{1t},c_{2t},n_t),\ u(c_1,c_2,n) = \log c_1 + \log c_2 + \log(1-n)$$

where c_{1t}, c_{2t}, and n_t denote consumption of cash goods, consumption of credit goods, and employment, respectively, in time period t.

The sequence of events in a period is as follows. At the beginning of the period, the household trades in a securities market in which it allocates nominal assets between money and bonds. After trading in the securities market, the household supplies labor and consumes cash and credit goods.

For securities market trading, the constraint is

$$(2)\qquad A_t \geq M_t + B_t$$

where A_t denotes beginning-of-period t nominal assets, M_t denotes the household's holdings of money, or cash, B_t denotes the household's holdings of interest-bearing bonds, and A_0 is given. Cash goods must be paid for with money received from securities market trading. The cash-in-advance constraint is given by

$$(3)\qquad P_{1t}c_{1t} \leq M_t$$

where P_{1t} is the period t price of cash goods. Let P_{1t-1} and P_{2t-1} denote the period $t-1$ prices of cash and credit goods; R_{t-1}, the gross interest rate; and W_{t-1}, the wage rate in period $t-1$. The household's sources of cash during securities market trading are cash left over from consuming goods in the previous period, $M_{t-1} - P_{1t-1}c_{1t-1}$; earnings on bonds accumulated in the previous period, $R_{t-1}B_{t-1}$; transfers received from the monetary authority, T_{t-1}; labor income in the previous period, $W_{t-1}n_{t-1}$; and profits in the previous period, D_{t-1}. Finally, the household pays debts, $P_{2t-1}c_{2t-1}$, owed from its period $t-1$ purchases of credit goods during securities market trading. These considerations are summarized in the following securities market constraint:

$$(4)\qquad A_t = W_{t-1}n_{t-1} - P_{2t-1}c_{2t-1} + (M_{t-1}-P_{1t-1}c_{1t-1}) + R_{t-1}B_{t-1} + T_{t-1} + D_{t-1}.$$

We place the following restriction on the household's ability to borrow:

$$(5)\qquad A_{t+1} \geq -(1/q_{t+1})\sum_{j=1}^{\infty} q_{t+j+1}(W_{t+j}+T_{t+j}+D_{t+j})$$

for $t = 0, 1, 2, \ldots$, where $q_t = \prod_{j=0}^{t-1} 1/R_j$ and $q_0 \equiv 1$. Condition (5) says that the household can never borrow more than the maximum present value of future income.

The household's problem is to maximize its utility (1) subject to its restrictions (2)–(5) and the nonnegativity constraints: $n_t, c_{1t}, c_{2t}, 1-n_t \geq 0$. If $R_t < 1$ for any t, this problem does not have a solution. We assume throughout that $R_t \geq 1$.

Firms

We adopt a variant of the production framework of Blanchard and Kiyotaki (1987). (In developing firm problems, we delete the time subscript.) The firms in our model produce either final goods or intermediate goods.

In each period, there are two types of perfectly competitive, final good firms: those that produce cash goods and those that produce credit goods. Their production functions are that

$$(6)\qquad y_1 = \left[\int_0^1 y_1(\omega)^{\lambda} d\omega\right]^{1/\lambda}$$

$$(7)\qquad y_2 = \left[\int_0^1 y_2(\omega)^{\lambda} d\omega\right]^{1/\lambda}$$

where y_1 denotes output of the cash good, y_2 denotes output of the credit good, and $y_i(\omega)$ is the quantity of the intermediate good of type ω used to produce good i and $0 < \lambda < 1$. The final good firms solve this problem:

$$(8)\qquad \max_{y_i,\{y_i(\omega)\}} P_i y_i - \int_0^1 P_i(\omega)y_i(\omega)\, d\omega$$

for $i = 1, 2$. Solving this problem leads to the following demand curves for each intermediate good:

$$(9)\qquad y_i(\omega) = y_i[P_i/P_i(\omega)]^{1/(1-\lambda)}$$

for $i = 1, 2$.

Intermediate good firms are monopolists in the product market and competitors in the market for labor. They set prices for their goods and are then required to supply whatever final good producers demand at those prices. The intermediate good firms solve this problem:

$$(10)\qquad \max_{y_i(\omega)} P_i(\omega)y_i(\omega) - Wn_i(\omega)$$

for $i = 1, 2$, where W is the wage rate, subject to a production technology, $y_i(\omega) = n_i(\omega)$, and the demand curve in (9). Profit maximization leads the intermediate good firms to set prices according to a markup over marginal costs:

$$(11)\qquad P_1(\omega) = W/\lambda$$

$$(12)\qquad P_2(\omega) = W/\lambda.$$

The Monetary Authority

In period t, the monetary authority transfers T_t units of cash to the representative household. These transfers are financed by printing money. Let g_t denote the growth rate of the money supply. Then $T_t = (g_t-1)M_t$, where M_0 is given and $M_{t+1} = g_tM_t$. A monetary policy is an infinite sequence, g_t, $t = 0, 1, 2, \ldots$.

Equilibrium

Now we define an equilibrium, given an arbitrary specification of monetary policy, and discuss the best equilibrium achievable by some monetary policy. This equilibrium is one in which the nominal interest rate is zero. Thus, the Friedman rule is optimal in this model. We then discuss the set of policies that support the best equilibrium.

□ *Definitions*

DEFINITION 1. *A* private sector equilibrium *is a set of sequences,* $\{P_{1t},P_{2t},W_t,R_t,c_{1t},c_{2t},n_t,B_t,M_t,g_t\}$*, with the following properties:*

- *Given the prices and the government policies, the quantities solve the household problem.*
- *The firm optimality conditions in* (11) *hold.*
- *The various market-clearing conditions hold:*

$$c_{1t} + c_{2t} = n_t \tag{13}$$

$$B_t = 0 \tag{14}$$

$$M_{t+1} = M_t g_t. \tag{15}$$

DEFINITION 2. *A* Ramsey equilibrium *is a private sector equilibrium with the highest level of utility.*

We now develop a set of statements that, together with (9)–(15), allow us to characterize a private sector equilibrium. From (11)–(12) it follows that $P_{1t} = P_{2t}$. Let $P_t = P_{1t} = P_{2t}$. Combining the household and the firm first-order conditions, we get, for all t, that

$$c_{2t}/(1-c_{1t}-c_{2t}) = \lambda \tag{16}$$

$$P_{t+1}c_{1t+1} = \beta R_t P_t c_{1t} \tag{17}$$

$$R_t = c_{2t}/c_{1t} \geq 1 \tag{18}$$

and

$$P_t c_{1t} - M_t \leq 0 \tag{19}$$

$$(R_t-1)(P_t c_{1t}-M_t) = 0. \tag{20}$$

In equilibrium, with $B_t = 0$, the household's transversality condition is that

$$\lim_{t\to\infty}\beta^t M_t/P_t c_{1t} = 0. \tag{21}$$

The nonnegativity constraint on leisure implies that

$$c_{1t} + c_{2t} \leq 1. \tag{22}$$

We summarize these statements in the form of a proposition:

PROPOSITION 1. Characterization of Equilibrium. *A sequence,* $\{P_{1t},P_{2t},W_t,R_t,c_{1t},c_{2t},n_t,B_t,M_t,g_t\}$*, is an equilibrium if and only if* (11)–(22) *and* $P_{1t} = P_{2t} = P_t$ *are satisfied. Furthermore, for any* $R_t \geq 1$*, there exists a private sector equilibrium with employment and consumption allocations uniquely determined, for all t, by*

$$n_{1t} = c_{1t} = \lambda/[\lambda + (1+\lambda)R_t] \tag{23}$$

$$n_{2t} = c_{2t} = R_t c_{1t}. \tag{24}$$

Proof. Statements (16)–(22) are the resource constraints and the necessary and sufficient conditions for household and firm optimization. Necessity and sufficiency in the case of the firms are obvious, and in the case of the households, the results are derived formally in Appendix A.

We now turn to the second part of the proposition. We need to verify that prices and a monetary policy can be found such that, together with the given sequence of interest rates and (23)–(24), they constitute a private sector equilibrium. First, by construction of (23)–(24), it can be verified that (13)–(16) and (18) are satisfied. It can also be verified that (22) is satisfied. Second, let $P_0 = M_0/c_{1,0}$, and use this and (17) to compute P_t, for $t = 1, 2, 3, \ldots$. This construction ensures that (17) for all t and (19)–(20) for $t = 0$ are satisfied. Next, compute $M_t = P_t c_{1t}$ for $t = 1, 2, \ldots$, so that (19)–(20) are satisfied for all t. Finally, (21) is satisfied because $0 < \beta < 1$ and $M_t/(P_t c_{1t}) = 1$.
Q.E.D.

We use this proposition to characterize the Ramsey equilibrium:

PROPOSITION 2. Ramsey Equilibrium Yields Friedman Rule. *Any Ramsey equilibrium has the property* $R_t = 1$ *for all t and employment and consumption allocations given in* (23)–(24).

Proof. The Ramsey equilibrium solves this problem:

$$\max_{\{R_t\geq 1\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t\{2\log(c_{1t}) + \log R_t + \log[1 - (1+R_t)c_{1t}]\} \tag{25}$$

where c_{1t} is given by (23)–(24). This problem is equivalent to the static problem $\max_{R\geq 1} f(R)$, where $f(R) = 2\log(c_1) + \log R + \log[1 - (1+R)c_1]$, $c_1 = \lambda/[\lambda + (1+\lambda)R]$. This function is concave in R and is maximized at the corner solution $R = 1$.
Q.E.D.

□ *Policies*

We now turn to the set of policies that are associated with a Ramsey equilibrium. The next proposition shows that there is a continuum of such policies. It is the analog of Proposition 2 of Cole and Kocherlakota (1998, p. 7).

PROPOSITION 3. Policies Associated With Ramsey Equilibrium. *There exists a private sector equilibrium with* $R_t = 1$ *for all t if and only if*

$$M_t/\beta^t \geq \kappa \tag{26}$$

with $\kappa > 0$ *for all t, and*

$$\lim_{T\to\infty} M_T \to 0. \tag{27}$$

Proof. Consider the necessity of (26) and (27). Suppose we have an equilibrium satisfying $R_t = 1$ and (11)–(24). From (18) and (16), letting $c_t \equiv c_{1t} = c_{2t}$, we obtain that

$$c_t = c = \lambda/(1+2\lambda) \tag{28}$$

for all t. From (17) we obtain that

$$P_t c_t = \beta^t P_0 c > 0. \tag{29}$$

Substituting (29) into (21), we get that

$$\lim_{t\to\infty}\beta^t(M_t/P_t c_t) = \lim_{t\to\infty}\beta^t(M_t/\beta^t P_0 c) = 0 \tag{30}$$

so that (27) is satisfied. From the cash-in-advance constraint in (19)–(20), we know that

$$(31)\quad \beta^t P_0 c \le M_t$$

for each t, which implies (26).

Consider sufficiency. Suppose (26) and (27) are satisfied and $R_t = 1$. We must verify that the other nonzero prices and quantities can be found which satisfy (11)–(22). Let $c_{1t} = c_{2t} = c$ in (28) for all t. Let $P_t = \beta^t P_0$, where $P_0 > 0$ will be specified in the paragraphs that follow. These two specifications guarantee (16)–(18). Condition (27), together with the given specification of prices and consumption, guarantees (21). Finally, it is easily verified that when $0 < P_0 \le \kappa/c$ is set, the cash-in-advance constraint in (19)–(20) holds for each t. Q.E.D.

Proposition 3 shows that there are many policies that implement the Ramsey outcome. However, many of these policies are fragile in the sense that they can yield worse outcomes than the Ramsey outcome. The next proposition characterizes the set of equilibria associated with mild monetary deflations in which the (stationary) growth rate of the money supply satisfies $\beta < g < 1$.

PROPOSITION 4. Fragility of Mild Monetary Deflations. *If* $\beta < g < 1$, *then the following are equilibrium outcomes:*

- $R_t = 1$, $c_{1t} = c_{2t} = \lambda/[1+2\lambda]$ *for all t*, $P_{t+1}/P_t = \beta$, *and* $M_t/P_t \to \infty$.
- $R_t = g/\beta$, $c_{1t} = \lambda/[\lambda + (1+\lambda)g/\beta]$, $c_{2t} = (g/\beta)c_{1t}$, $P_{t+1}/P_t = g$ *for all t, and* M_t/P_t *is independent of t.*
- $R_t = g/\beta$ *for* $t \le t^*$, $R_t = 1$ *for* $t > t^*$ *for* $t^* = 0, 1, 2, \ldots$, $c_{1t} = \lambda/[\lambda + (1+\lambda)R_t]$, $c_{2t} = R_t c_{1t}$, *and*

$$(32)\quad P_{t+1}/P_t = \begin{cases} g, \text{ for } t = 0, 1, \ldots, t^* - 1, \text{ for } t^* > 0; \\ (1+2\lambda)g/[\lambda + (g/\beta)(1+\lambda)], \text{ for } t = t^*; \\ \beta, \text{ for } t = t^* + 1, t^* + 1, \ldots . \end{cases}$$

Proof. That these are all equilibria may be confirmed by verifying that (11)–(22) are satisfied. Q.E.D.

This proposition does not characterize the entire set of equilibria that can occur with $\beta < g < 1$. It gives a flavor of the possibilities, however. For example, the last outcome above indicates that there is a countable set of equilibria (one for each possible t^*) in which the consumption and employment allocations are not constant and the interest rate switches down to unity after some period. Although there do exist equilibria in which consumption and employment are not constant, these equilibria appear to be limited. For example, it can be shown that there is no equilibrium in which the interest rate switches up from unity in some period; that is, there does not exist an equilibrium in which $R_{t^*} = 1$ and $R_{t^*+1} > 1$ for some t^*. To see this, suppose the contrary. Then, from (17), $\beta P_{t^*}c_{1t^*} = P_{t^*+1}c_{1t^*+1} = M_{t^*+1}$, since the cash-in-advance constraint must be binding in period $t^* + 1$. However, $M_{t^*} \ge P_{t^*}c_{1t^*}$ implies that $\beta \ge g$, a contradiction. Furthermore, we can also show that there do not exist equilibria in which the interest rate changes and is always greater than unity, that is, in which $R_t \ne R_{t+1} \ne 1$. So, although the set of equilibria with nonconstant interest rates (and, hence, nonconstant consumption) is limited, Proposition 4 indicates that it does exist.

Proposition 4 indicates that mild monetary deflations are fragile. It turns out, however, that a deflationary policy of the kind advocated by Friedman (1969) is robust in the sense that it always yields the Ramsey outcome.

PROPOSITION 5. Robustness of Friedman Deflation. *Suppose that* $g_t = \beta$. *Then all equilibria are Ramsey equilibria.*

Proof. To show that if $g_t = \beta$, then $R_t = 1$, suppose the contrary. That is, suppose that $R_t > 1$ for some t. Therefore, $P_t c_{1t} = M_t$. Also, $P_{t+1}c_{1t+1} \le M_{t+1}$. By (17) we find that $1/(P_t c_{1t}) = \beta R_t/(P_{t+1}c_{1t+1})$, so that $(1/M_t) \ge \beta R_t(1/M_{t+1})$, or $g_t \ge \beta R_t$, which is a contradiction. Q.E.D.

It is worth pointing out that since the interest rate is constant, so are real allocations. There is, however, a continuum of equilibria in which the price level is different. In all of these equilibria, $P_{t+1}/P_t = \beta$. These equilibria are indexed by the initial price level, P_0, which satisfies $P_0 \le M_0(1+2\lambda)/\lambda$ and $P_{t+1}/P_t = \beta$.

Markov Equilibrium in a Cash-Credit Good Model

In this section, we analyze a version of the cash-credit model in which a benevolent government chooses monetary policy optimally but without commitment. We find that time inconsistency is not a problem in this model.

The Economy

We consider a more general utility function of the constant elasticity of substitution form:

$$(33)\quad u(c_1,c_2,n) = [1/(1-\sigma)] \times \{[\alpha c_1^\rho + (1-\alpha)c_2^\rho]^{1/\rho}(1-n)^\gamma\}^{1-\sigma}.$$

Note that this utility function is a generalization of the one used in the preceding section. Here, we focus on the Markov equilibrium of this model.

The timing in the model is as follows. A fraction, μ_1, of intermediate good producers in the cash good sector and a fraction, μ_2, of intermediate good producers in the credit good sector set prices at the beginning of the period. These firms are referred to as *sticky price firms*. We show in what follows that all sticky price firms set the same price. Denote this price by P^e. This price, all other prices, and all nominal assets in this version of the model are scaled by the aggregate, beginning-of-period money stock. Then the monetary authority chooses the growth rate of the money supply. Finally, all other decisions are made.

The state of the economy at the time the monetary authority makes its decision is P^e.[1] The monetary authority makes its money growth decision conditional on P^e. We denote the gross money growth rate by G and the monetary policy rule by $X(P^e)$. The state of the economy after the monetary authority makes its decision is $S = (P^e,G)$.

Definitions

With these definitions of the economy's state variables, we proceed now to discuss the decisions of firms, households, and the monetary authority.

□ *Firms*

Recall that profit maximization leads intermediate good firms to set prices as a markup over the wage rate; see equations (11)–(12). Denote by $\hat{P}(S)$ the price set by the $1 - \mu_1$ intermediate good firms in the cash good sector and the $1 - \mu_2$ intermediate good firms in the credit good sector that set their prices after the monetary authority makes its decision; these are the *flexible price firms*. For the μ_1 and μ_2 sticky price cash and credit good firms, respectively, and the $1 - \mu_1$ and $1 - \mu_2$ flexible price cash and credit good firms, respectively, the markup rule implies that

$$(34) \quad P^e = W\big(P^e, X(P^e)\big)/\lambda$$

$$(35) \quad \hat{P}(S) = W(S)/\lambda$$

for $0 < \lambda < 1$, where $W(S)$ denotes the nominal wage rate. In this model of monopolistic competition, output and employment are demand determined. That is, output and employment are given by (11)–(12). Let $P_i(S)$ denote the price of the cash and credit good for $i = 1, 2$, respectively. Let $y_{ij}(S)$, $i, j = 1, 2$, denote the output of the intermediate good firms, where the first subscript denotes whether the good is a cash good ($i = 1$) or a credit good ($i = 2$), and the second subscript indicates whether the good is produced by a sticky price ($j = 1$) or a flexible price ($j = 2$) firm.

□ *The Household*

In terms of the household's problem, it is convenient to write the constraints in recursive form. The analog of (2) is

$$(36) \quad M + B \leq A$$

where, recall, A denotes beginning-of-period nominal assets, M denotes the household's holdings of cash, and B denotes the household's holdings of interest-bearing bonds. Here, nominal assets, money, and bonds are all scaled by the aggregate stock of money. We impose a no-Ponzi constraint of the form $B \leq \bar{B}$, where $\bar{B}$ is a large, finite upper bound. The household's cash-in-advance constraint is

$$(37) \quad M - P_1(S)c_1 \geq 0$$

where c_1 denotes the quantity of the cash good. Nominal assets evolve over time as follows:

$$(38) \quad 0 \leq W(S)n + [1 - R(S)]M - P_1(S)c_1 - P_2(S)c_2 + R(S)A + (G-1) + D(S) - GA'$$

where c_2 denotes the quantity of credit goods purchased. In (38), $R(S)$ denotes the gross nominal rate of return on bonds, $D(S)$ denotes household profits after lump-sum taxes, and A' denotes the next period asset holdings. Finally, B has been substituted out of the asset equation using (36). Notice that A' is multiplied by G. This modification is necessary because of the way we have scaled the stock of nominal assets.

Consider the household's asset, good, and labor market decisions. Given that the household expects the monetary authority to choose policy according to X in the future, the household solves the following problem:

$$(39) \quad v(A,S) = \max_{n,M,A',c_i;i=1,2} u(c_1,c_2,n) + \beta v\big(A', P^e, X(P^e)\big)$$

subject to (36), (37), (38), and nonnegativity on allocations. In (39), v is the household's value function. The solution to (39) yields decision rules of the form $n(A,S)$, $M(A,S)$, $A'(A,S)$, and $c_i(A,S)$, for $i = 1, 2$. We refer to these decision rules, together with the production decisions of firms, $y_{ij}(S)$, for $i, j = 1, 2$, as *private sector allocation rules*. We refer to the collection of prices, P^e, $\hat{P}(S)$, $W(S)$, $R(S)$, and $P_i(S)$, for $i = 1, 2$, as *pricing rules*.

□ *The Monetary Authority*

The monetary authority chooses the current money growth rate, G, to solve the problem

$$(40) \quad \max_G v(1,S)$$

where, recall, the state of the economy $S = (P^e, G)$. Let $X(P^e)$ denote the solution to this problem. We refer to this solution as the *monetary policy rule*.

Markov Equilibrium

□ *Definitions*

We now define a Markov equilibrium. This equilibrium requires that the household and firms optimize and markets clear.

DEFINITION 3. *A* Markov equilibrium *is a set of private sector allocation rules, pricing rules, a monetary policy rule, and a value function for the household such that*

- *The value function, v, and the private sector rules solve* (39).
- *Intermediate good firms optimize; that is,* (34) *is satisfied, final good prices satisfy*

$$(41) \quad P_i(S) = [\mu_i (P^e)^{\lambda/(\lambda-1)} + (1-\mu_i)\hat{P}(S)^{\lambda/(\lambda-1)}]^{(\lambda-1)/\lambda}$$

  for $i = 1, 2$, *and the output of intermediate good firms,* $y_{ij}(S)$, *is given by the analog of* (9).
- *Asset markets clear; that is,* $A'(1,S) = 1$ *and* $M(1,S) = 1$.
- *The labor market clears; that is,*

$$(42) \quad n(1,S) = \mu_1 y_{11}(S) + (1-\mu_1)y_{12}(S) + \mu_2 y_{21}(S) + (1-\mu_2)y_{22}(S).$$

- *The monetary authority optimizes; that is,* $X(P^e)$ *solves* (40).

Notice that our notion of Markov equilibrium has built into it the idea of sequential optimality captured in game-theoretic models by subgame perfection. In particular, we require that for any deviation by the monetary authority from $X(P^e)$, the resulting allocations be the ones that would actually occur, that is, the ones that would be in the best interests of the household and firms and would clear markets.

We now define a *Markov equilibrium outcome:*

DEFINITION 4. *A* Markov equilibrium outcome *is a set of numbers, n,* c_1, c_2, y_{ij} $(i, j = 1, 2)$, P^e, *W, R,* P_1, P_2, *and g, satisfying* $n = n(1,P^e,g)$, $c_1 = c_1(1,P^e,g), \ldots,$ *and* $g = X(P^e)$.

□ *Analysis*

Here we characterize the Markov equilibrium. In particular, we provide sufficient conditions for the Ramsey outcomes to be Markov equilibrium outcomes. We also provide sufficient conditions for the Markov equilibrium to be unique. Combining these conditions, we obtain sufficient conditions for the unique Markov equilibrium to yield the Ramsey outcomes.

In developing these results, we find it convenient to recast the monetary authority's problem as choosing $\hat{P}$ rather than G. First, we analyze the private sector allocation and pricing rules. Then, we analyze the monetary authority's problem.

We use the necessary and sufficient conditions of private sector maximization and market-clearing to generate the private sector allocation and pricing rules. The conditions are given by the following:

$$(43) \quad -u_3/u_2 = \lambda(\hat{P}/P_2)$$

$$(44) \quad [(1/P_1)-c_1](R-1) = 0$$

$$(45) \quad R = (u_1/u_2)(P_2/P_1)$$

$$(46) \quad n_i = c_i[\mu_i(P_i/P^e)^{1/(1-\lambda)} + (1-\mu_i)(P_i/\hat{P})^{1/(1-\lambda)}]$$

for $i = 1, 2$;

$$(47) \quad n = n_1 + n_2$$

$$(48) \quad P_i = [\mu_i(P^e)^{\lambda/(\lambda-1)} + (1-\mu_i)\hat{P}^{\lambda/(\lambda-1)}]^{(\lambda-1)/\lambda}$$

for $i = 1, 2$; and

$$(49) \quad Gu_1/P_1 = \beta R v_1\big(1,P^e,X(P^e)\big).$$

Notice that the growth rate of the money supply, G, appears only in (49). Equations (43)–(48) constitute eight equations in the eight unknowns, c_1, c_2, n_1, n_2, n, P_1, P_2, and R. Given values for P^e and $\hat{P}$, we see that these equations can be solved to yield functions of the following form:

$$(50) \quad c_1(P^e,\hat{P}),\ c_2(P^e,\hat{P}),\ \ldots,\ R(P^e,\hat{P}).$$

Replacing $\hat{P}$ in (50) by a pricing function, $\hat{P}(P^e,G)$, we obtain the allocation and pricing rules in a Markov equilibrium.

The pricing function, $\hat{P}(P^e,G)$, is obtained from equation (49). This equation can be thought of as yielding a function, $G(P^e,\hat{P})$. The pricing function, $\hat{P}(P^e,G)$, is obtained by inverting $G(P^e,\hat{P})$. It is possible that the inverse of $G(P^e,\hat{P})$ is a correspondence. In this case, $\hat{P}(P^e,G)$ is a selection from the correspondence. Any such selection implies a range of equilibrium prices, $\hat{P}$.

Given the function, $\hat{P}(P^e,G)$, the monetary authority's problem can be thought of in either of two equivalent ways: either it chooses G or it chooses $\hat{P}$. The monetary authority's decision problem is simplified in our setting because its choice of $\hat{P}$ has no impact on future allocations. As a result, the authority faces a static problem.

The allocation functions in (50) can be substituted into

$$(51) \quad U(P^e,\hat{P}) = u\big(c_1(P^e,\hat{P}),\ c_2(P^e,\hat{P}),\ n(P^e,\hat{P})\big).$$

Then define

$$(52) \quad P(P^e) = \arg\max_{\hat{P}\in D} U(P^e,\hat{P}).$$

The function, $P(P^e)$, is the monetary authority's best response, given P^e. Equilibrium requires that $P(P^e) = P^e$. This procedure determines the expected price P^e, the actual price $\hat{P}$, and the eight allocations and other prices just described. Given these values, we can determine the equilibrium growth rate of the money supply by evaluating $G(P^e,P^e)$.

In what follows, we assume that the first-order conditions of the monetary authority's problem characterize a maximum. In quantitative exercises we have done using these models, we have found that the first-order conditions in the neighborhood of a Ramsey outcome do in fact characterize the global maximum of the monetary authority's problem.

Next, we show that for a class of economies, the Ramsey outcomes are Markov equilibrium outcomes. Recall that a Ramsey equilibrium is a private sector equilibrium with $R = 1$. In Appendix B, we prove the following result:

PROPOSITION 6. Markov Is Ramsey. *Suppose that*

$$(53) \quad (1-\rho)(1-\mu_1) \geq \mu_2[(1-\alpha)/\alpha]^{1/(1-\rho)}.$$

Then there exists a Markov equilibrium with $R = 1$.

The intuition for this proposition is as follows. A benefit of expansionary monetary policy is that it leads to an increase in the demand for goods which have fixed prices. This increase in demand tends to raise employment. Other things being the same, welfare rises because employment has been inefficiently low. A principal cost of expansionary monetary policy is that it tends to reduce employment in the cash good sector. The reason for this reduction in employment is that nominal consumption of the cash good is predetermined, while its price rises as a result of the increase in flexible intermediate good prices. It is possible that the reduction in employment in the cash good sector is so large that overall employment and welfare fall. Indeed, it can be shown that if the sufficient condition of the proposition is met, employment falls with an increase in the money growth rate in the neighborhood of the Ramsey equilibrium. The monetary authority has an incentive to contract the money supply. This incentive disappears only if the nominal interest rate is zero.

In what follows, we assume that $\hat{P}(P^e,G)$ is a continuous function of G. This restriction is not innocuous. We have constructed examples in which, for a given value of G, there is more than one set of values of private sector allocations and prices that satisfy the conditions for private sector optimization and market-clearing.[2] Thus, it is possible to construct private sector allocation and pricing rules that are discontinuous functions of G. The assumption of continuity plays an important role in the proof of uniqueness given in Appendix C. In the next proposition, we provide sufficient conditions for uniqueness of the Markov equilibrium:

PROPOSITION 7. Uniqueness of Markov Equilibrium. *Suppose that*

- $\rho = 0$ *and* $\sigma = 1$.
- $1 - \mu_1 > \mu_2[(1-\alpha)/\alpha]$
- $\{\lambda + [\gamma\alpha/(1-\alpha)]\}(1-\mu_1) \geq [(1-\lambda)\gamma\mu_2]/\alpha$.

Then, in the class of Markov equilibria in which $\hat{P}(P^e,G)$ *is a continuous function of* G*, there exists an equilibrium with* $R = 1$*, and there is no equilibrium with outcome* $R >$ 1.

We conjecture that if we allow a discontinuous pricing function, $\hat{P}(P^e,G)$, then there exist Markov equilibria with $R > 1$, even under the conditions of this proposition.

Markov Equilibrium in a Limited-Participation Model

Here we analyze the set of Markov equilibria in a limited-participation model. Our model is adapted from that of Lucas (1990). (See Christiano 1991 and Fuerst 1992 for similar formulations.) In limited-participation models, asset market frictions make money not neutral. In our model, we assume two kinds of frictions: the ability of households to participate in asset markets is limited, and the source of funds for firms to pay for labor is restricted. Specifically, at the beginning of each period, households deposit a portion of their nominal assets with a competitive financial intermediary. The financial intermediary uses these funds together with transfers from the monetary authority to make loans to firms at a competitively determined nominal interest rate. The firms use these loans to pay for their labor input, and they pay off their loans to the financial intermediary at the end of the period with proceeds from sales. The key frictions are, then, that households cannot change the amount of their deposits with the financial intermediary after the monetary authority chooses its transfers and firms cannot use proceeds from current sales to pay workers.

The limited-participation model lets us study the set of Markov equilibria in a model in which the source of monetary nonneutrality is quite different from that in the cash-credit goods model with sticky prices. Interestingly, we find that in this model also, there is no time-inconsistency problem.

Description of Model

We first briefly describe the model. The sequence of events is as follows. The household starts each period with nominal assets, and it must choose how much to deposit in a financial intermediary. The monetary authority then chooses how much to transfer to the financial intermediary. The financial intermediary makes loans to firms, which must borrow the wage bill before they produce. The household makes its consumption and labor supply decisions, and firms make production decisions. Money is not neutral because a household cannot change its deposit decision after the monetary authority chooses its transfer.

Let Q denote the aggregate deposits made by the representative household, and let G denote the growth rate of the money supply chosen by the monetary authority. In this section, as in the preceding section, all prices and quantities of nominal assets are scaled by the aggregate stock of money. Let $S = (Q,G)$ denote the state of the economy after these decisions are made.

The household's utility function is

$$(54) \qquad \sum_{t=0}^{\infty} \beta^t u(c_t,n_t),\ u(c,n) = \log(c) + \gamma \log(1-n)$$

where c_t and n_t denote period t consumption and labor, respectively. We write the household's problem recursively. We start with the problem solved by the household after the monetary authority has made its transfer. Let A denote the household's beginning-of-period nominal assets. Let q denote its deposits. Both variables have been scaled by the aggregate beginning-of-period stock of money. The consumption, employment, and asset accumulation decisions solve

$$(55) \qquad w(A,q,S) = \max_{c,n,M'} u(c,n) + \beta v(A')$$

subject to

$$(56) \qquad P(S)c \le W(S)n + A - q$$

and

$$(57) \qquad GA' = R(S)[q + (G-1)] + D(S) + W(S)n + A - q - P(S)c.$$

In (55), v is the household's value function at the beginning of the next period, before the household makes next period's deposit decision. Also, $R(S)$ is the gross interest rate, $P(S)$ is the price of the consumption good, $W(S)$ is the wage rate, and $D(S)$ is the profit from firms. The choice of q solves the following dynamic programming program:

$$(58) \qquad v(A) = \max_q w(A,q,S^e)$$

where S^e is the state if the monetary authority does not deviate from its policy decision; that is, $S^e = \big(Q,X(Q)\big)$, where $X(Q)$ is the monetary authority's policy rule.

The production sector in this model is exactly as in the cash-credit good model, with one exception. To pay for the labor that they hire during the period, intermediate good producing firms must borrow in advance from the financial intermediary at a gross interest rate $R(S)$. Thus, the marginal dollar cost of hiring a worker is $R(S)W(S)$, so that, by the type of reasoning in the cash-credit good model, we find that $R(S)W(S)/P(S) = \lambda$.

The financial intermediary behaves competitively. It receives Q from the household and $G - 1$ on the household's behalf from the monetary authority. When $R(S) > 1$, the financial intermediary lends all these funds in the loan market. When $R(S) = 1$, it supplies whatever is demanded, up to the funds it has available. We shall say that when $R(S) = 1$ and demand is less than available funds, there is a *liquidity trap*. At the end of the period, the financial intermediary returns its earnings, $R(S)(Q+G-1)$, to the household. Finally, if $R(S) < 1$, the financial intermediary lends no funds, and it returns $Q + G - 1$ to the household. Loan demand by firms is given by $W(S)n(S)$. Therefore, loan market-clearing requires that

$$(59) \qquad W(S)n(S) \le Q + G - 1$$

which holds with equality if $R(S) > 1$.

The monetary authority's policy function, $X(Q)$, solves this:

$$(60) \qquad X(Q) \in \arg\max_G w(1,Q,Q,G).$$

A recursive private sector equilibrium and a Markov equilibrium are defined analogously to those in the last section.

Analysis of Equilibrium

It is useful to begin our analysis of equilibrium in the limited-participation model with an analysis of outcomes under commitment. It is easy to show, as we did earlier, that the Ramsey equilibrium has $R = 1$ and can be supported by a policy that sets the growth rate of the money supply equal to β. Let c^*, n^*, W^*, R^*, P^*, and Q^* denote this Ramsey equilibrium. These variables solve the following system of equations:

(61) $\gamma c^*/(1-n^*) = W^*/P^*$

(62) $W^*/P^* = \lambda/R^*$

(63) $R^* = 1$

(64) $W^*n^* = Q^* + \beta - 1$

(65) $P^*c^* = W^*n^* + 1 - Q^*$

(66) $c^* = n^*$.

It is straightforward to verify that the usual nonnegativity constraints are satisfied. Notice that equation (61) is the household's first-order condition for labor, (62) results from firm optimization, (63) corresponds to the intertemporal Euler equation, (64) corresponds to money market-clearing, (65) is the household's cash-in-advance constraint, and (66) corresponds to good market-clearing.

Next, we analyze the Markov equilibria of our model. The necessary and sufficient conditions for allocation and pricing rules to constitute a recursive private sector equilibrium are as follows:

(67) $\gamma n(S)/[1-n(S)] = W(S)/P(S)$

(68) $W(S)/P(S) = \lambda/R(S)$

(69) $W(S)n(S) \leq Q + G - 1$ if $R(S) \geq 1$;
$\qquad 0$ if $R(S) < 1$

(70) $P(S)n(S) - W(S)n(S) \leq 1 - Q$

where (69) holds with equality if $R(S) > 1$. As already noted, if $R(S) < 1$, then the supply of funds in the loan market is zero. Also, (70) holds with equality if $R\big(Q,X(Q)\big) > 1$ and $S = \big(Q,X(Q)\big)$. That is, if along the Markov equilibrium path the net interest rate is strictly positive, then the household's cash-in-advance constraint is satisfied as a strict equality. In a deviation from the Markov equilibrium path, the cash-in-advance constraint must hold as a weak inequality, regardless of the realized interest rate.

We now establish the following proposition:

PROPOSITION 8. All Markov Equilibria Are Ramsey. *In any Markov equilibrium,* $R\big(Q,X(Q)\big) = 1$, *and the allocations and prices on the equilibrium path are the Ramsey outcomes given in* (61)–(66).

Proof. We prove this proposition in two parts. First, we construct a Markov equilibrium in which $R\big(Q,X(Q)\big) = 1$. Then we show that there is no equilibrium with $R\big(Q,X(Q)\big) > 1$.

Our constructed Markov equilibrium is as follows. Let $Q = Q^*$, where Q^* solves (61)–(66). On the equilibrium path, the monetary authority's decision rule is $X(Q^*) = \beta$. The allocation and pricing rules, $c(S)$, $n(S)$, $W(S)$, $P(S)$, and $R(S)$, in a recursive private sector equilibrium are defined as follows. For all S, $c(S) = n(S)$. For $G \leq \beta$, $R(S) = 1$, and $n(S) = n^*$, $W(S)$ is obtained from (69) with equality, and $P(S) = W(S)/\lambda$. It is then easy to show that (70) holds with inequality. For $G > \beta$ the functions are defined as follows: $n(S) = n^*$, $W(S) = w^*$, $R(S) = R^* = 1$, and $P(S) = P^*$, where the variables with the asterisk are those associated with the Ramsey equilibrium, (61)–(66). Notice that these allocation and pricing rules satisfy (67), (68), and (70) with equality and (69) with inequality.

Next we show by contradiction that there does not exist a Markov equilibrium with $R\big(Q,X(Q)\big) > 1$. Suppose, to the contrary, that there does exist such an equilibrium. Notice that it is always possible to construct a private sector equilibrium for arbitrary $G \geq \beta$ by simply setting (67)–(70) to equality. Therefore, the domain of deviation that has to be considered includes all $G > X(Q)$. Consider such a deviation. We will show that, in the private sector equilibrium associated with this deviation, $R(Q,G) < R\big(Q,X(Q)\big)$. This argument is also by contradiction. Thus, suppose that $R(Q,G) \geq R\big(Q,X(Q)\big)$. Then, because $R\big(Q,X(Q)\big) > 1$, (69) must hold as an equality at the deviation. Substituting for $P(S)$ from (68) and $W(S)n(S)$ from (69), we see that the left side of (70) becomes

(71) $[R(S)/\lambda-1](Q+G-1)$

which is larger than $[R(S)/\lambda-1][Q+X(Q)-1]$. On the equilibrium path, (70) must hold as an equality. Therefore, at the deviation, (70) must be violated. We have established that, in any deviation of the form $G > X(Q)$, $R(Q,G) < R\big(Q,X(Q)\big)$. However, from (67) and (68), this raises employment toward the efficient level, contradicting monetary authority optimization. We have established the desired contradiction. Q.E.D.

Notice that, in the Markov equilibrium we have constructed, there is a liquidity trap. If the monetary authority deviates and chooses a growth rate for the money supply greater than β, then the resulting transfers of money are simply hoarded by the financial intermediary and not lent out to firms. All allocations and prices are unaffected by such a deviation.

Conclusion

How severe is the time-inconsistency problem in monetary policy? Not severe at all, according to this study. Here we have worked with an environment that, with one exception, is similar in spirit to the one analyzed in the Kydland-Prescott and Barro-Gordon literature. The exception is that we are explicit about the mechanisms that cause unanticipated monetary injections to generate benefits and costs. Contrary to the existing literature, we have found that, in two standard general equilibrium models, there is no inflation bias at all.

What this means generally for the severity of the time-inconsistency problem is too soon to know. The result does, however, help focus future research. A comparison of our work here and elsewhere (Albanesi, Chari, and Christiano 2002) suggests, for example, that we could learn something by investigating various ways to model the demand for money.

*This study is reprinted, with the permission of the Cambridge University Press, from the book, *Advances in Economics and Econometrics: Theory and Applications,* Eighth World Congress of the Econometric Society, Volume 3, edited by Mathias Dewatripont, Lars Peter Hansen, and Stephen J. Turnovsky, pp. 123–50 (Chapter 4), Cambridge, U.K.: Cambridge University Press, 2003. The study was edited for publication in the *Federal Reserve Bank of Minneapolis Quarterly Review.*

Chari and Christiano thank the National Science Foundation for supporting this research.

†When this study was originally published, Albanesi was at Bocconi University's Innocenzo Gasparini Institute for Economic Research.

[1]Notice that we do not include the aggregate stock of money in the state. In our economy, all equilibria are neutral in the usual sense that if the initial money stock is doubled, there is an equilibrium in which real allocations and the interest rate are unaffected and all nominal variables are doubled. This consideration leads us to focus on equilibria which are invariant with respect to the initial money stock. We are certainly mindful of the possibility that there can be equilibria that depend on the money stock. For example, if there are multiple equilibria in our sense, it is possible to construct trigger strategy–type equilibria that are functions of the initial money stock. In our analysis, we exclude such equilibria, and we normalize the aggregate stock of money at the beginning of each period to unity.

[2]Specifically, we have found numerical examples in which the function, $G(P^e,\hat{P})$, displays an inverted U shape when graphed for fixed P^e with G on the vertical axis and $\hat{P}$ on the horizontal. In these examples, each fixed $\hat{P}$ implies a unique G. However, there are intervals of values of G in which a fixed G maps into two distinct $\hat{P}$s.

Appendix A
Necessary and Sufficient Conditions for Household Optimization in the Cash-Credit Good Model

This appendix develops necessary and sufficient conditions for optimality of the household problem in the cash-credit good model of the first section of the study. These derivations are included here for completeness. Many of the results here can be found in the literature. See, for example, Woodford 1994.

In what follows, we assume that

(A1) $P_{1t},\ P_{2t},\ W_t > 0$

(A2) $R_t \geq 1$

(A3) $\lim_{t\to\infty}\sum_{j=0}^{t} q_{j+1}\left(W_j+T_j+D_j\right)$ is finite.

If these conditions do not hold, there can be no equilibrium.

We begin by proving a proposition that allows us to rewrite the household's budget set in a more convenient form. We show the following:

PROPOSITION A1. *Suppose* (2), (4), *and* (A1)–(A3) *are satisfied. Then the constraint given in* (5) *is equivalent to*

(A4) $\lim_{T\to\infty} q_T A_T \geq 0.$

Proof. It is useful to introduce some new notation. Let I_t and S_t be defined by

(A5) $I_t \equiv W_t + T_t + D_t$

(A6) $S_t \equiv (R_t-1)M_t + P_{1t}c_{1t} + P_{2t}c_{2t} + W_t(1-n_t).$

Then it is straightforward to show that household nominal assets satisfy this equation:

(A7) $A_{t+1} = I_t + R_t A_t - S_t.$

We establish that (A4) implies (5). Recursively solving for assets using (A7) and (2) from t to T yields that

(A8) $q_T A_T \leq \sum_{j=0}^{T-t-1} q_{t+j+1}I_{t+j} + q_t A_t - \sum_{j=0}^{T-t-1} q_{t+j+1}S_{t+j}.$

Taking into account $q_{t+j+1}S_{t+j} \geq 0$ and rewriting (A8), we obtain that

(A9) $q_T A_T \geq q_T A_T - \sum_{j=0}^{T-t-1} q_{t+j+1}I_{t+j}.$

Fixing t, taking the limit as $T\to\infty$, and using (A4) yields (5).

We now show that (5) implies (A4). Note first that the limit in (A1)–(A3) being finite implies that

(A10) $\lim_{t\to\infty}\sum_{j=1}^{\infty} q_{t+j+1}I_{t+j} = 0.$

From using this result and (5), (A4) follows trivially. Q.E.D.

Following is the main result of this appendix:

PROPOSITION A2. *A sequence,* $\{c_{1t},c_{2t},n_t,M_t,B_t\}$, *solves the household problem if and only if the following conditions are satisfied. The Euler equations are*

(A11) $u_{1t}/P_{1t} = R_t(u_{2t}/P_{2t})$

(A12) $-u_{3t}/u_{2t} = W_t/P_{2t}$

(A13) $u_{1t}/P_{1t} = \beta R_t(u_{1t+1}/P_{1t+1})$

(A14) $(R_t-1)(P_{1t}c_{1t}-M_t) = 0.$

The transversality condition is (A4) *with equality:*

(A15) $\lim_{t\to\infty} q_t A_t = 0.$

Proof. We begin by showing that if a sequence $\{c_{1t},c_{2t},n_t,M_t,B_t\}$ satisfies (A11)–(A15), then that sequence solves the household's problem. That is, we show that

(A16) $$D = \lim_{T\to\infty}\Big[\sum_{t=0}^{T}\beta^t u(c_{1t},c_{2t},n_t) - \sum_{t=0}^{T}\beta^t u(c'_{1t},c'_{2t},n'_t)\Big] \geq 0$$

where $\{c'_{1t},c'_{2t},n'_t,M'_t,B'_t\}_{t=0}^{\infty}$ is any other feasible plan. Note first that the Euler equations imply that

(A17) $\beta^t u_{1,t} = q_t P_{1t}(u_{1,0}/P_{1,0})$

(A18) $\beta^t u_{2,t} = q_{t+1} P_{2t}(u_{1,0}/P_{1,0})$

(A19) $\beta^t u_{3,t} = -q_{t+1} W_t(u_{1,0}/P_{1,0})$

where $u_{i,t}$ is the derivative of u with respect to its ith argument. By concavity and the fact that the candidate optimal plan satisfies (A11) and (A12), we can write (A16) as

(A20) $$\begin{aligned} D &\geq \lim_{T\to\infty}(u_{1,0}/P_{1,0}) \\ &\quad\times \sum_{t=0}^{T}[q_t P_{1t}(c_{1t}-c'_{1t}) + q_{t+1}P_{2t}(c_{2t}-c'_{2t}) \\ &\qquad - q_{t+1}W_t(n_t-n'_t)] \\ &= \lim_{T\to\infty}(u_{1,0}/P_{1,0}) \\ &\quad\times \sum_{t=0}^{T} q_t\big[(S_t/R_t) + [(1-R_t)/R_t](M_t-P_{1t}c_{1t}) \\ &\qquad - (S'_t/R_t) - [(1-R_t)/R_t](M'_t-P_{1t}c'_{1t})\big] \\ &\geq \lim_{T\to\infty}(u_{1,0}/P_{1,0})\sum_{t=0}^{T}\left(q_{t+1}S_t - q_{t+1}S'_t\right) \end{aligned}$$

where the equality is obtained by using the definition of S_t and the second inequality is obtained by using $R_t \geq 1$ and where $(1-R_t)(M'_t-P_{1t}c'_{1t}) \leq 0$; see (3). Iterating on (A7) for the two plans, we can rewrite (A20) as

(A21) $$\begin{aligned} D &\geq \lim_{T\to\infty}(u_{1,0}/P_{1,0})\Big(\sum_{t=0}^{T} q_{t+1}S_t + q_{T+1}A'_{T+1} - \sum_{t=0}^{T} q_{t+1}I_t - A_0\Big) \\ &\geq \lim_{T\to\infty}(u_{1,0}/P_{1,0})\Big(\sum_{t=0}^{T} q_{t+1}S_t - \sum_{t=0}^{T} q_{t+1}I_t - A_0\Big) \\ &= \lim_{T\to\infty}(u_{1,0}/P_{1,0})q_{T+1}A_{T+1} \geq 0 \end{aligned}$$

by (A15).

Now we establish that if $\{c_{1t},c_{2t},n_t,M_t,B_t\}$ is optimal, then (A11)–(A15) are true. That (A11)–(A14) are necessary is obvious. It remains to show that (A15) is necessary. Suppose (A15) is not true. We show this contradicts the hypothesis of optimality. We need only consider the case where $\lim_{T\to\infty} q_T A_T$ is strictly positive. The strictly negative case is ruled out by the preceding proposition. So suppose that

$$\text{(A22)}\quad \lim_{T\to\infty} q_T A_T = \Delta > 0.$$

We construct a deviation from the optimal sequence that is consistent with the budget constraint and results in an increase in utility. Fix some particular period, τ. We replace $c_{1\tau}$ by $c_{1\tau} + \varepsilon/P_{1\tau}$, where $0 < \varepsilon \leq \Delta/q_\tau$. Consumption in all other periods and $c_{2\tau}$ are left unchanged, as well as employment in all periods. We finance this increase in consumption by replacing M_τ with $M_\tau + \varepsilon$ and B_τ with $B_\tau - \varepsilon$. Money holdings in all other periods are left unchanged. Debt and wealth after t, B_t, A_t, $t > \tau$ are different in the perturbed allocations. We denote the variables in the perturbed plan with a prime. From (A7), we know that

$$\text{(A23)}\quad A'_{\tau+1} - A_{\tau+1} = -R_\tau\varepsilon = -(q_\tau/q_{\tau+1})\varepsilon$$

$$\text{(A24)}\quad A'_{\tau+j} - A_{\tau+j} = -R_{\tau+j-1}\cdots R_\tau\varepsilon = -(q_\tau/q_{\tau+j})\varepsilon.$$

Multiplying this last expression by $q_{\tau+j}$ and setting $T = \tau + j$, we have that

$$\text{(A25)}\quad q_T(A'_T - A_T) = -q_\tau\varepsilon.$$

Taking the limit, as $T\to\infty$, we find that

$$\text{(A26)}\quad \lim_{T\to\infty} q_T A'_T = \Delta - q_\tau\varepsilon \geq 0.$$

We conclude that the perturbed plan satisfies (A4). However, utility is clearly higher in the perturbed plan. We have a contradiction. Q.E.D.

Appendix B
Properties of a Markov Equilibrium in the Cash-Credit Good Model

In this appendix, we prove Proposition 6, the proposition that a Markov equilibrium in the cash-credit good model is a Ramsey equilibrium. We establish the result by constructing a Markov equilibrium which supports the Ramsey outcomes.

Specifically, we construct P^e, a set of private sector allocation rules, a set of pricing rules, and a monetary policy rule, all of which satisfy the conditions for a Markov equilibrium. In our analysis of Markov equilibrium in this model, we have shown that private sector allocation rules and pricing rules can equivalently be expressed as functions of the growth rate of the money supply, G, or of $\hat{P}$, the price of the flexibly priced intermediate goods. Because these representations are equivalent and it is convenient to work with $\hat{P}$, we do so here.

Construction

The construction of the Markov equilibrium is as follows. Let c_1^*, c_2^*, W^*, R^*, P^*, P_1^*, and P_2^* solve (43)–(48) with $R = 1$ and with the cash-in-advance constraint holding with equality. That is, these variables are given by

$$\text{(B1)}\quad c_1^* = \{1 + [1+(\gamma/\lambda)][(1-\alpha)/\alpha]^{1/(1-\rho)} + (\gamma/\lambda)\}^{-1}$$

$$\text{(B2)}\quad c_2^* = c_1^*[(1-\alpha)/\alpha]^{1/(1-\rho)}$$

along with $R^* = 1$, $P_1^* = P_2^* = P^* = 1/c_1^*$, and $W^* = \lambda P^*$. Let $P^e = P^*$. For $\hat{P} > P^e$, let the allocation and pricing rules solve (43)–(48) with (44) replaced by $c_1 = 1/P_1$. For $\hat{P} < P^e$, let the allocation and pricing rules solve (43)–(48) with $R = 1$. By construction, P^e and these allocation and pricing rules satisfy private sector optimality and market-clearing. We need only check optimality of the monetary authority.

Denote the derivative of U in (51) with respect to $\hat{P}$ by L,

$$\text{(B3)}\quad L = u_1c'_1 + u_2c'_2 + u_nn'$$

where u_1, u_2, and u_n denote derivatives of the utility function with respect to the cash good, the credit good, and employment, respectively. In addition, c'_1, c'_2, and n' denote derivatives of the allocation rules defined in (50) with respect to $\hat{P}$. These derivatives and all others in this appendix are evaluated at $\hat{P} = P^e$. Let L^+ be the right derivative and L^- be the left derivative associated with L. We show that when our sufficient conditions are met, $L^+ \leq 0$ and $L^- \geq 0$.

Note that

$$\text{(B4)}\quad P'_i = (1-\mu_i)$$

for $i = 1, 2$. Using (B4) and grouping terms in (B3), we obtain that

$$\text{(B5)}\quad L = u_2[(u_1/u_2)c'_1 + c'_2 + (u_3/u_2)(c'_1+c'_2)]$$
$$= (1-\lambda)u_2c_1[(c'_1/c_1) + (c_2/c_1)(c'_2/c_2)]$$

because $u_1/u_2 = R = 1$ and $-u_3/u_2 = \lambda$, when $\hat{P} = P^e$.

The Utility Function's Right Derivative . . .

We now establish that when our sufficient conditions are met, $L^+ \leq 0$. To evaluate the derivatives in (B5), we require expressions for c'_1/c_1 and c'_2/c_2.

The first of these is obtained by differentiating the binding cash-in-advance constraint:

$$\text{(B6)}\quad c'_1/c_1 = -(1-\mu_1)/P^e.$$

To obtain c'_2/c_2, note that the static labor Euler equation is given by

$$\text{(B7)}\quad [\gamma c_2/(1-n)](c/c_2)^\rho = \lambda(\hat{P}/P_2)$$

or, after substituting for c and rearranging, we get that

$$\text{(B8)}\quad [\gamma/(1-\alpha)][\alpha(c_1/c_2)^\rho + 1 - \alpha] = \lambda(\hat{P}/P_2)[(1-n_1-n_2)/c_2].$$

Differentiating both sides of this expression with respect to $\hat{P}$ and taking into account $d(\hat{P}/P_2)/d\hat{P} = \mu_2/P_2$ when $\hat{P} = P^e$, we obtain, after some manipulations, that

$$\text{(B9)}\quad [\lambda(1-c_1)/c_1 - \gamma\rho](c'_2/c_2) = \lambda(\mu_2/P_2)[(1-c_1-c_2)/c_1]$$
$$- [\lambda+\gamma\rho](c'_1/c_1).$$

Substituting for c'_1/c_1 and c'_2/c_2 from (B6) and (B9), respectively, into (B5), we obtain that

$$\text{(B10)}\quad L^+ = \big((1-\lambda)u_2c_1/\{\lambda[(1-c_1)/c_1] - \gamma\rho\}\big)$$
$$\times \Big[\big((c_2/c_1)(\lambda+\gamma\rho) - \{\lambda[(1-c_1)/c_1] - \gamma\rho\}\big)$$
$$\times [(1-\mu_1)/P^e] + (c_2/c_1)\{\lambda(\mu_2/P_2)[(1-c_1-c_2)/c_1]\}\Big].$$

The denominator of (B10) is positive. To see this, use (B1) to show that

$$\text{(B11)} \quad \lambda[(1-c_1)/c_1] - \gamma\rho = \lambda[1+(\gamma/\lambda)][(1-\alpha)/\alpha]^{1/(1-\rho)} + \gamma(1-\rho) > 0$$

because $\rho \leq 1$. We can rewrite (B10) as

$$\text{(B12)} \quad L^+ = \big(u_2c_2(1-\lambda)/P_2\{\lambda[(1-c_1)/c_1] - \gamma\rho(g/\beta)\}\big) \times \Big(-\{\lambda[(1-c_1-c_2)/c_2] - \gamma\rho[(c_1+c_2)/c_2]\}(1-\mu_1) + \lambda\mu_2[(1-c_1-c_2)/c_1]\Big).$$

Substituting for c_1 from (B1) and c_2/c_1 from (B2), we obtain that

$$\text{(B13)} \quad (1-c_1-c_2)/c_2 = (\gamma/\lambda)\{[(1-\alpha)/\alpha]^{1/(1-\rho)} + 1\}.$$

In addition,

$$\text{(B14)} \quad (1-c_1-c_2)/c_2 = (\gamma/\lambda)\{1 + [(1-\alpha)/\alpha]^{-1/(1-\rho)}\}$$

and

$$\text{(B15)} \quad (c_1+c_2)/c_2 = [(1-\alpha)/\alpha]^{-1/(1-\rho)} + 1.$$

Substituting these results into (B12), we obtain that

$$\text{(B16)} \quad L^+ = u_2c_2(1-\lambda)/P_2\{\lambda[(1-c_1)/c_1] - \gamma\rho(g/\beta)\} \times \big\{-[\lambda(\gamma/\lambda)\{1 + [(1-\alpha)\alpha]^{-1/(1-\rho)}\} - \gamma\rho\{[(1-\alpha)/\alpha]^{-1/(1-\rho)} + 1\}(1-\mu_1) + \lambda\mu_2\big((\gamma/\lambda)\{[(1-\alpha)/\alpha]^{1/(1-\rho)} + 1\}\big)\big\}.$$

Simplifying, we have that

$$\text{(B17)} \quad L^+ = \Big(\gamma u_2c_2(1-\lambda)\big\{1 + [(1-\alpha)/\alpha]^{-1/(1-\rho)}\big\} \div P_2\big\{\lambda[(1-c_1)/c_1] - \gamma\rho(g/\beta)\big\}\Big) \times \big\{-(1-\rho)(1-\mu_1) + \mu_2[(1-\alpha)/\alpha]^{1/(1-\rho)}\big\}.$$

Because the large fraction on the right side of (B17) is positive, it follows that $L^+ \leq 0$ if and only if

$$\text{(B18)} \quad (1-\rho)(1-\mu_1) \geq \mu_2[(1-\alpha)/\alpha]^{1/(1-\rho)}.$$

. . . And Left Derivative

Next, we establish that under the sufficient conditions of the proposition, $L^- \geq 0$.

The expression for c_2'/c_2 is still given by (B9). To obtain c_1'/c_1, we differentiate (45) with $R = 1$ and use (B4) to get that

$$\text{(B19)} \quad [\alpha/(1-\alpha)](1-\rho)(c_2/c_1)^{1-\rho}[(c_2'/c_2)-(c_1'/c_1)] = (\mu_2-\mu_1)/P^e$$

or, because $[\alpha/(1-\alpha)](c_2/c_1)^{1-\rho} = 1$,

$$\text{(B20)} \quad (c_2'/c_2) - (c_1'/c_1) = (\mu_2-\mu_1)/(1-\rho)P^e.$$

Substituting for c_1'/c_1 from here into (B9) and collecting terms, we obtain, after simplifying, that

$$\text{(B21)} \quad \lambda(1/c_1)(c_2'/c_2) = \lambda(u_2/P_2)[(1-c_1-c_2)/c_1] + (\lambda+\gamma\rho)[(\mu_2-\mu_1)/(1-\rho)P^e].$$

Then, using (B13), we obtain that

$$\text{(B22)} \quad \lambda(1/c_1)(c_2'/c_2) = (u_2/P_2)\gamma\big\{[(1-\alpha)/\alpha]^{1/(1-\rho)} + 1\big\} + (\lambda+\gamma\rho)[(\mu_2-\mu_1)/(1-\rho)P^e].$$

Now, substituting out for c_1'/c_1 and c_2'/c_2 into (B5), we obtain, after simplifying, that

$$\text{(B23)} \quad L^- = (1-\lambda)[u_2c_1\gamma/P^e(\lambda+\gamma)]\big\{\mu_2[(1-\alpha)/\alpha]^{1/(1-\rho)} + \mu_1\big\} > 0.$$

Appendix C
Uniqueness of a Markov Equilibrium in the Cash-Credit Good Model

We prove Proposition 7, that the Markov equilibrium in our cash-credit good model is unique, by contradiction. Suppose that there exists a Markov equilibrium outcome with $R > 1$. The contradiction is achieved in two steps. First, we establish that a deviation down in $\hat{P}$ can be accomplished by some feasible deviation in G. We then establish that such a deviation is desirable. That a Markov equilibrium exists follows from Proposition 6.

A Deviation's Feasibility . . .

Let P^e denote the expected price level in the Markov equilibrium, and let G^e denote the money growth rate in the corresponding equilibrium outcome; that is, $G^e = X(P^e)$. We establish that for any $\hat{P}$ in a neighborhood, U, of P^e, there exists a G belonging to a neighborhood, V, of G^e, such that $\hat{P} = \hat{P}(P^e,G)$. Here, $\hat{P}(P^e,G)$ is the price allocation rule in the Markov equilibrium.

Substituting from (45) into (49) and using the assumptions, $\sigma = 1$ and $\rho = 0$, we obtain that

$$\text{(C1)} \quad G(P^e,\hat{P}) = P_2(P^e,\hat{P})c_2(P^e,\hat{P})[\beta/(1-\alpha)]v_1\big(1,P^e,X(P^e)\big).$$

From the analogs of (B6) and (B9) obtained for the case $g/\beta \geq 1$ and using $\rho = 0$, we can determine that $c_2(P^e,\hat{P})$ is a strictly increasing function of $\hat{P}$ for $\hat{P}$ in a sufficiently small neighborhood, U, of P^e. It is evident from (48) that P_2 is globally increasing in $\hat{P}$. This establishes that $G(P^e,\hat{P})$ is strictly increasing for $\hat{P} \in U$. By the inversion theorem, we know that $G(P^e,\hat{P})$ has a unique, continuous inverse function mapping from $V = G(P^e,U)$ to U. By continuity of $\hat{P}(P^e,G)$, we know that this inverse is $\hat{P}(P^e,G)$ itself. This establishes the desired result.

. . . And Desirability

To show that a deviation, $\hat{P} < P^e$, is desirable, we first establish properties of the private sector allocation rules and pricing rules in Markov equilibria in which the interest rate is strictly greater than one.

Let

$$\text{(C2)} \quad x^a(P^e,\hat{P}) \equiv [c_1^a(P^e,\hat{P}),\ c_2^a(P^e,\hat{P}),\ \ldots,\ R^a(P^e,\hat{P})]$$

denote the solutions to (43)–(48) with (44) replaced by the cash-in-advance constraint holding with equality. Let

$$\text{(C3)} \quad x^b(P^e,\hat{P}) \equiv [c_1^b(P^e,\hat{P}),\ c_2^b(P^e,\hat{P}),\ \ldots,\ R^b(P^e,\hat{P})]$$

denote the solutions to (43)–(48) with (44) replaced by $R = 1$. Then, for any $\hat{P}$, P^e, private sector allocations and prices must be given by either $x^a(P^e,\hat{P})$ or $x^b(P^e,\hat{P})$.

We now show that for all $\hat{P}$ in a neighborhood of P^e, the private sector allocations and prices must be given by $x^a(P^e,\hat{P})$. Consider $\hat{P} = P^e$. Solving (43)–(48) with (44) replaced by the cash-in-advance constraint holding with equality and with $\hat{P} = P^e$, we obtain that

$$\text{(C4)} \quad c_1^a(P^e,P^e) = \big(1 + [1+(\gamma/\lambda)]\{[(1-\alpha)/\alpha]R\}^{1/(1-\rho)} + (\gamma/\lambda)\big)^{-1}.$$

Solving the analogous equations for $c_1^b(P^e,P^e)$, we obtain that

$$\text{(C5)} \quad c_1^b(P^e,P^e) = \big\{1 + [1+(\gamma/\lambda)][(1-\alpha)/\alpha]^{1/(1-\rho)} + (\gamma/\lambda)\big\}^{-1}.$$

Evidently, $P^e c_1^b(P^e,P^e) > P^e c_1^a(P^e,P^e$.) By continuity, for all $\hat{P}$ in some neighborhood of P^e, we see that $\hat{P}c_1^b(\hat{P},P^e) > \hat{P}c_1^a(\hat{P},P^e)$. Because $\hat{P}c_1^a(\hat{P},P^e) = 1$, it follows that $\hat{P}c_1^b(\hat{P},P^e) > 1$ for all $\hat{P}$ in a neighborhood of P^e. Since the cash-in-advance constraint is violated, $x^b(\hat{P},P^e)$ cannot be part of a Markov equilibrium. We have established that for $\hat{P}$ in a neighborhood of P^e, private sector allocation and pricing rules must be given by $x^a(\hat{P},P^e)$.

With these allocation and pricing rules, the derivative of the utility function with respect to $\hat{P}$, evaluated at $P^e = \hat{P}$, can be shown to be

$$\text{(C6)} \quad L = \{(u_2 c_2/P_2)/\lambda[(1-c_1)/c_1] - \gamma\rho(g/\beta)\} \times [-a(g/\beta) + b(g/\beta)]$$

where g is the growth rate of money at the supposed outcome and

$$\text{(C7)} \quad a(g/\beta) = [(g/\beta)-\lambda]\left\{\lambda + \gamma + \gamma(g/\beta)^{-\rho/(1-\rho)} \times [(1-\alpha)/\alpha]^{-1/(1-\rho)}(1-\rho)\right\}(1-\mu_1)$$

$$\text{(C8)} \quad b(g/\beta) = (1-\lambda)\gamma\mu_2\left\{(g/\beta)[(1-\alpha)/\alpha]^{1/(1-\rho)} + (g/\beta)\right\} + (1-\mu_1)(1-\lambda)[\lambda+\gamma\rho(g/\beta)].$$

The second condition of Proposition 7, that $1 - \mu_1 > \mu_2[(1-\alpha) \div \alpha]$, guarantees that $a(1) \geq b(1)$. (It is easily verified that this is equivalent to condition (B18).) In addition, under the first condition, that $\rho = 0$ and $\sigma = 1$, a and b are linear with slopes a' and b', respectively, given by

$$\text{(C9)} \quad a' = \{\lambda + [\gamma\alpha/(1-\alpha)]\}(1-\mu_1)$$

$$\text{(C10)} \quad b' = (1-\lambda)\gamma\mu_2/\alpha.$$

Given the third condition, it is trivial to verify that $L < 0$ for all $g/\beta > 1$. Thus, the supposition that there is an outcome with $R > 1$ leads to the implication that the monetary authority can raise utility by reducing $\hat{P}$. This contradicts monetary authority maximization. We conclude that there are no Markov equilibrium outcomes with $R > 1$.