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ABSTRACT

Comparing Cournot and Bertrand Equilibria Revisited

by Jim Y. Jin*

This paper compares Cournot and Bertrand equilibria with mixed products, linear demand and cost functions. It is found that a firm's price (output) need not be higher (lower) in Cournot equilibrium. However, given any number of firms and a mixture of complement and substitute products, every firm's price margin/output ratio is always higher in Cournot equilibrium, and the weighted squared outputs (price margins) are higher (lower) in Bertrand equilibrium. When price (quantity) competition is a supermodular game, consumer surplus (social welfare) is higher in price competition. Nevertheless, price competition results in more market concentration measured by Herfindahl index.

ZUSAMMENFASSUNG

Ein erneuter Vergleich von Cournot- und Bertrand-Gleichgewichten

In diesem Beitrag werden Cournot- und Bertrand-Gleichgewichte für den Fall des Produktemix, linearer Nachfrage- und Kostenfunktionen verglichen. Als Ergebnis kann festgestellt werden, daß der Preis (Output) eines Unternehmens im Cournot-Gleichgewicht nicht höher (niedriger) sein muß. Angenommen, es gibt eine beliebige Zahl von Unternehmen und ein Produktemix von Komplementär- und Substitutionsgütern, dann ist der Quotient aus Preisspanne/Output immer höher im Cournot-Gleichgewicht und die gewichteten Quadrate der Outputs (Preisspannen) sind höher (niedriger) im Bertrand-Gleichgewicht. Wenn Preis-(Mengen)-Wettbewerb als supermodulares Spiel betrachtet wird, dann ist der Konsumentenüberschuß (soziale Wohlfahrt) im Falle des Preiswettbewerbs höher. Trotzdem führt Preiswettbewerb zur höherer Marktkonzentration, gemessen am Herfindahl-Index.

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1. Introduction

Since Bertrand's famous criticism on Cournot's homogeneous duopoly model, there has been a widely held conjecture, if not a belief, that price competition results in lower prices and higher outputs than does quantity competition. A comparison of these two benchmark oligopoly models has been widely undertaken in the literature. Examples include Hathaway and Rickard (1979) who examined a duopoly market with general demand and cost functions. They found that at least one firm's price is higher in Cournot equilibrium than in Bertrand equilibrium. Under duopoly Singh and Vives (1984) further showed that both firms' prices are higher and outputs are lower in quantity competition than in price competition, if each firm can make a profit when the other's price or output is zero. Cheng (1985), obtained similar results with a geometric approach. Extending the analysis to a general oligopoly model, Vives (1985) and Okuguchi (1987) obtained a nice conclusion that under certain conditions, prices are lower in Bertrand equilibrium.

These papers significantly improved our understanding of the two models. However, some questions remain unanswered. First, are prices (outputs) always lower (higher) in price competition? Okuguchi (1987) gave a counter example in which prices (outputs) are higher (lower) in Bertrand equilibrium than in Cournot equilibrium. However, his demand function $x_i = 1 - p_i - 3p_j$, i, j = 1, 2, can not be generated from a concave utility function. It has not been known whether this can happen if the demand function is derived from a concave utility function. We will show that even if demand is linear and derived from a concave utility function, marginal costs are constant, a firm's price need not be higher in Cournot equilibrium when goods are complements, and output need not be lower when goods are substitutes. Then if both complement and substitute goods exist, neither a price must be higher nor an output must be lower in Cournot equilibrium. We actually can say little whether price competition is more competitive.

Secondly, as a Bertrand equilibrium does not necessarily imply lower prices or higher outputs with a mixture of complement and substitute products, is there any other criterion by which it is viewed more competitive than a Cournot equilibrium? In fact, most previous papers did not allow mixed products[1], though the opposite is often true in the reality. This paper shows that when demand and cost functions are linear, the ratio of price margin to output and the weighted sum of squared price margins or output can be used as such criterion.

The third question: what is the fundamental force which makes price competition more competitive than quantity competition? One may think substitute goods as an essential factor. This explanation seems very consistent with Bertrand's original argument and could be easily accepted by economists[2]. However, Singh and Vives (1984) found that in duopoly "*quantities are lower and prices higher in Cournot than in Bertrand competition, regardless of whether the goods are substitutes or complements*" (pp. 549). It suggests that the substitutability is not essential. This paper examines the crucial importance of the concavity of utility function. Additional to that, the supermodularity can ensure a partial ordering in prices or outputs with non-linear demand and cost functions. When quantity (price) competition is supermodular, social welfare (consumer surplus) is higher in price competition. However, price competition results in a higher market concentration measured by Herfindale index.

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[1]The exception is Okuguchi's model (1987) where products can be divided into two groups, substitutable within groups and complementary between two groups.

[2]For example, in a seemingly type error, Cheng wrote: "*Cournot equilibrium prices (quantities) are higher than Bertrand equilibrium prices (quantities) . . . if the goods are substitutes (complements)*" (1985, pp. 146)

The next section introduces the linear oligopoly model. Section 3 gives examples that some firm's price (output) can be lower (higher) in Cournot equilibrium. Section 4 presents two criterion by which price competition can be said more competitive than quantity competition. Section 5 discusses supermodular games and Herfindale index. The last section concludes and considers future research.

2. Linear model

There are n firms. Each firm produces one product. Firm i's output is denoted by x_i and its price by p_i. The product vector is denoted by $\mathbf{x}$ and the price vector by $\mathbf{p}$. Every firm i has a constant marginal cost c_i. Denote the n×1 cost vector by $\mathbf{c}$. Firm i chooses its price p_i or output x_i to maximize its profit $x_i(p_i-c_i)$.

The representative consumer has a quadratic and strictly concave utility function $u(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} - 0.5\mathbf{x}B\mathbf{x}$, where $\mathbf{a}$ is an n×1 positive vector and B is a symmetric n×n matrix. Since $u(\mathbf{x})$ is strictly concave, B is positive definite. The consumer chooses a consumption bundle $\mathbf{x}$ to maximize her surplus $u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}$ given $\mathbf{p}$. Since $u(\mathbf{x})$ is strictly concave in $\mathbf{x}$, a vector $\mathbf{x}$ satisfying the first order condition $\mathbf{a} - B\mathbf{x} - \mathbf{p} = \mathbf{0}$ is indeed optimal. This gives us the inverse demand function in quantity competition:

$$\mathbf{p}(\mathbf{x}) = \mathbf{a} - B\mathbf{x} \qquad (1)$$

Since B is positive definite, $\frac{\partial p_i}{\partial x_i} < 0$ because it is the ith diagonal element of -B. Let Λ be an n×n diagonal matrix whose ith diagonal element is $\lambda_i \equiv -\frac{\partial p_i}{\partial x_i}$. Denote the output and price vector in Cournot equilibrium by $\mathbf{x}^C$ and $\mathbf{p}^C$.

Rearranging (1) we obtain the demand function in price competition:

$$\mathbf{x}(\mathbf{p}) = B^{-1}(\mathbf{a}-\mathbf{p}) \qquad (2)$$

Since B is positive definite, B^{-1} exists and is positive definite. $\frac{\partial x_i}{\partial p_i} < 0$ because it equals the ith diagonal element of $-B^{-1}$. Let D be the diagonal matrix whose ith diagonal element is $d_i \equiv -\frac{\partial x_i}{\partial p_i}$. Denote the output and price vector in Bertrand equilibrium by $\mathbf{x}^B$ and $\mathbf{p}^B$.

In quantity competition, firm i chooses x_i to maximize its profit $x_i(p_i-c_i)$ where p_i is given by (1). In Cournot equilibrium the first order condition holds for every x_i, i.e. $p_i - c_i - \lambda_i x_i = 0$ for all i, or $\mathbf{p}^C = \mathbf{c} + \Lambda\mathbf{x}^C$. This condition and (1) together imply $\mathbf{a} - B\mathbf{x}^C = \mathbf{c} + \Lambda\mathbf{x}^C$. Notice that $B + \Lambda$ is positive definite and its inverse $(B+\Lambda)^{-1}$ exists. We can solve the Cournot equilibrium output as

$$\mathbf{x}^C = (B+\Lambda)^{-1}(\mathbf{a}-\mathbf{c}) \qquad (3)$$

Given a vector $\mathbf{a} - \mathbf{c}$, (3) gives a unique solution. Let I be an identity matrix. Substituting (3) into $\mathbf{p}^C = \mathbf{c} + \Lambda\mathbf{x}^C$, we get the unique Cournot equilibrium price as

$$\mathbf{p}^C = (I+B\Lambda^{-1})^{-1}(\mathbf{a}-\mathbf{c}) + \mathbf{c} \qquad (4)$$

In price competition, each firm i chooses p_i to maximize its profit $x_i(p_i-c_i)$ given the demand function (2). In Bertrand equilibrium every firm i's price satisfies the first order condition $x_i - d_i(p_i-c_i) = 0$. Thus we have $\mathbf{x}^B = D(\mathbf{p}^B-\mathbf{c})$. This condition and (2) imply $B^{-1}(\mathbf{a}-\mathbf{p}^B) = D(\mathbf{p}^B-\mathbf{c})$. Notice that $B^{-1} + D$ is positive definite and its inverse $(B^{-1}+D)^{-1}$ exists. We can solve the Bertrand equilibrium price vector $\mathbf{p}^B = (B^{-1}+D)^{-1}(B^{-1}\mathbf{a}+D\mathbf{c})$. Again, it is unique. We can write it as

$$\mathbf{p}^B = (I+BD)^{-1}(\mathbf{a}-\mathbf{c}) + \mathbf{c} \qquad (5)$$

Substituting (5) into $\mathbf{x}^B = D(\mathbf{p}^B-\mathbf{c})$, we solve the unique Bertrand equilibrium output vector:

$$\mathbf{x}^B = (B+D^{-1})^{-1}(\mathbf{a}-\mathbf{c}) \qquad (6)$$

To make the equilibrium comparison meaningful, assume that every firm is active in both Cournot and Bertrand equilibria, i.e. $\mathbf{x}^C > \mathbf{0}$ and $\mathbf{x}^B > \mathbf{0}$. Given (3) and (6) we need $(B+\Lambda)^{-1}(\mathbf{a}-\mathbf{c}) > \mathbf{0}$ and $(B+D^{-1})^{-1}(\mathbf{a}-\mathbf{c}) > \mathbf{0}$. Then (4) and (5) imply $\mathbf{p}^C > \mathbf{c}$ and $\mathbf{p}^B > \mathbf{c}$. All firms make positive profits. Given these conditions next we show that $\mathbf{p}^C \geq \mathbf{p}^B$ and $\mathbf{x}^B \geq \mathbf{x}^C$ may not hold.

3. Examples

Singh and Vives (1984) considered a linear duopoly model. The consumer utility function can be written as $a_1x_1 + a_2x_2 - 0.5(b_1x_1^2+2rx_1x_2+b_2x_2^2)$, where a_i and b_i are positive. Additionally, $b_1b_2 > r^2$, so the utility is strictly concave. Also, it is assumed that $a_ib_j - a_jr > 0$ for $i \neq j$. So the demand function in price competition has a positive intercept. It implies that a firm's demand is positive when both prices are zero. Under these conditions, they found the difference between a firm's Bertrand and Cournot equilibrium outputs, $x_i^B - x_i^C = (a_ib_j-a_jr)r^2/(b_1b_2-r^2)(4b_1b_2-r^2)$, and the difference between prices, $p_i^C - p_i^B = a_ir^2/(4b_1b_2-r^2)$. Hence they conclude that "*Quantities are lower and prices are higher in Cournot than in Bertrand competition*" (pp. 549).

Their proof clearly shows that $a_ib_j - a_jr > 0$ is essential for the comparison. If one allows firms to have positive and different marginal costs, equivalently a_i must be replaced by $a_i - c_i$ as pointed by Singh and Vives. Then $x_i^B > x_i^C$ requires $(a_i-c_i)b_j - (a_j-c_j)r > 0$. This means that firm i is able to sell some products when both firms charge prices equal to their marginal costs. This condition is stronger than to assume positive equilibrium outputs in both quantity and price competition. For example, let $a_i = b_i = 1$, $r = 0.8$, $c_1 = 0.3$ and $c_2 = 0$. Thus, $(a_1-c_1)b_2 = 0.7 < (a_2-c_2)r = 0.8$. In fact, in this case the demand function is $p_i = 1 - x_i - 0.8x_j$ in quantity competition and $x_i = (0.2-p_i+0.8p_j)/0.36$ in price competition. One can check that $x_1^C = 5/28 > x_1^B = (5/28)\times(19/27)$. Firm 1's output is larger in Cournot equilibrium.

When goods are substitutes, firms can make positive profits only if $a_i > c_i$ for i = 1 and 2. Given $b_1b_2 > r^2$, $(a_1-c_1)b_2 < (a_2-c_2)r$ and $(a_1-c_1)b_2 < (a_2-c_2)r$ can not hold simultaneously. So we can not have both firms' outputs higher in quantity competition.

Therefore, for a linear duopoly with a symmetric demand function but different marginal costs, outputs need not be higher in price competition unless we require that every firm has a positive demand when prices are set at marginal costs. The requirement simply means $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ in our linear oligopoly model. The result of Singh and Vives clearly showed that this requirement is sufficient in a duopoly case given other standard assumptions. Then, a natural question is whether it holds true in oligopoly. More precisely, the question is: if $\mathbf{x}(\mathbf{c}) > \mathbf{0}$, $\mathbf{x}^C > \mathbf{0}$ and $\mathbf{x}^B > \mathbf{0}$, whether we have $\mathbf{x}^B \geq \mathbf{x}^C$ with $n > 2$.

Let us consider a three firm case. Assume a strictly concave utility function $x_1 + x_2 + x_3 - 0.5(x_1^2+x_2^2+x_3^2) - 0.8(x_1x_2+x_1x_3+x_2x_3)$. This implies firm i's inverse demand function in quantity competition $p_i = 1 - x_i - 0.8\sum_{j\neq i}^{3} x_j$ and its demand function in price competition $x_i = (1-9p_i+4\sum_{j\neq i}^{3} p_j)/2.6$. Further assume $c_1 = 0.1$, and $c_2 = c_3 = 0$. Clearly the condition $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ is satisfied. One can check that $\mathbf{x}^C > \mathbf{0}$ and $\mathbf{x}^B > \mathbf{0}$ also hold. Nevertheless, we have $x_1^C = 23/108 > x_1^B = 567/2860$. Firm 1's output is lower in price competition. With more than two firms, the condition $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ is not enough to ensure $\mathbf{x}^B \geq \mathbf{x}^C$.

We need to explain why the situation changes from duopoly to oligopoly. Consider an n firm case with a symmetric B, whose diagonal elements are 1 and the off-diagonals are r > 0. Goods are substitutes. B is positive definite only if r < 1. In our notation then, $\lambda_i = 1$ and $d_i = d = \{1+(n-2)r\}/(1-r)\{1+(n-1)r\}$ for all i. In this case $\mathbf{x}^C > \mathbf{0}$ and $\mathbf{x}^B > \mathbf{0}$ are guaranteed if $B^{-1}(\mathbf{a}-\mathbf{c}) = \mathbf{x}(\mathbf{c}) > \mathbf{0}$. From (3) and (6) we solve $\mathbf{x}^B - \mathbf{x}^C = (d-1)(dB+I)^{-1}(I+B^{-1})^{-1}\mathbf{x}(\mathbf{c})$. As $d > 1$, $\mathbf{x}^B \geq \mathbf{x}^C$ requires $T^{-1}\mathbf{x}(\mathbf{c}) \geq \mathbf{0}$

where $T = dB + (d+1)I + B^{-1}$. The off-diagonal elements in dB and B^{-1} are equal to rd and $-r/(1-r)\{1+(n-1)r\}$ respectively. Thus the off-diagonal elements of T are equal to $(n-2)r^2/(1-r)\{1+(n-1)r\} \geq 0$. When n = 2, it is zero and T is a positive diagonal matrix. $\mathbf{x}^B \geq \mathbf{x}^C$ is guaranteed given $\mathbf{x}(\mathbf{c}) > \mathbf{0}$. When n > 2, however, the off-diagonal elements of T become strictly positive since $0 < r < 1$. This implies negative off-diagonal elements in T^{-1}. If $x_i(\mathbf{c})$ is significantly smaller than other $x_j(\mathbf{c})$'s, $\mathbf{x}_i^B - \mathbf{x}_i^C$ will be negative. This is why the condition $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ is not sufficient to guarantee higher outputs in price competition.

Similarly, are prices always higher in Cournot equilibrium than in Bertrand equilibrium? Singh and Vives (1984) showed that this is true in duopoly if and only if $a_i > 0$. In presence of positive marginal cost, a_i should be replaced by $a_i - c_i$ and the condition becomes $a_i > c_i$. This means that firm i can sell it at a price no less than its marginal cost when the other firm produces nothing. When goods are substitutes, this is plausible. However, it may not be so with complement goods. For instance, a car without gasoline is useless. A car may not be sold at its marginal cost when gasoline is not available. Without the condition $a_i > c_i$, both firms' Cournot and Bertrand equilibrium outputs can be still positive.

For instance, assume a demand function $p_i = 1 - x_i + 0.8x_j$ and marginal costs $c_1 = 1.2$ and $c_2 = 0$. Here firm 1's price is lower than its cost when firm 2's output is zero. The demand function in price competition is $x_i = (1.8-p_i-0.8p_j)/0.36$. We can calculate equilibrium prices, $p_1^C = 55.4/42 < p_1^B = 57/42$. Firm 1's price is lower in Cournot equilibrium. Even if we consider a linear duopoly with symmetric demand, prices need not be lower in Bertrand equilibrium without the condition $a_i > c_i$ for i = 1 and 2.

Given $a_1 < c_1$, a_2 must be larger than c_2, otherwise firms can not sell any products even with marginal cost pricing. Thus the price of the other firm must be higher in Cournot equilibrium. This was shown by Hathaway and Rickard (1979).

The next question is whether the condition $a_i > c_i$ for all i guarantees higher prices in Cournot equilibrium in oligopoly with n > 2. We consider a three firm oligopoly with a symmetric and strictly concave utility function $x_1 + x_2 + x_3 - 0.5(x_1^2+x_2^2+x_3^2) + 0.4(x_1x_2+x_1x_3+x_2x_3)$. The products are complements. The corresponding demand function in quantity competition is $p_i = 1 - x_i + 0.4\sum_{j\neq i}^{3} x_j$ and in price competition $x_i = (7-3p_i-2\sum_{j\neq i}^{3} p_j)/1.4$. Further assume $c_1 = 0.9$, $c_2 = c_3 = 0$. Clearly the conditions $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ and $\mathbf{a} > \mathbf{c}$ are satisfied. One can check that $\mathbf{x}^C > \mathbf{0}$ and $\mathbf{x}^B > \mathbf{0}$ hold. Nevertheless, we have $p_1^C = 37/30 < p_1^B = 1.24$. Firm 1 sets a higher price in price competition than in quantity competition, even though $a_i > c_i$ holds for all i.

Now we look at why the situation changes from 2 firms to 3 firms. We again assume the diagonal elements of B are 1, but let its off-diagonal elements to be equal to r < 0. As B is positive definite, we have r > -1/(n-1). Now $\mathbf{a} > \mathbf{c}$ ensures $\mathbf{x}(\mathbf{c}) > \mathbf{0}$, $\mathbf{x}^C > \mathbf{0}$ and $\mathbf{x}^B > \mathbf{0}$. Given (4) and (5) we know $\mathbf{p}^C > \mathbf{p}^B$ requires $T^{-1}(\mathbf{a}-\mathbf{c}) > \mathbf{0}$, where $T = dB + (d+1)I + B^{-1}$. As shown before, the off-diagonal element of T is $(n-2)r^2/(1-r)\{1+(n-1)r\}$. If n > 2, it is strictly positive, so the off-diagonal elements of T^{-1} are negative. We will have $p_1^C < p_1^B$ if $a_i - c_i$ is relatively small. Therefore $\mathbf{a} > \mathbf{c}$ is nor enough to ensure $\mathbf{p}^C \geq \mathbf{p}^B$ with more than two firms.

We have shown that a firm's output can be lower in price competition when goods are substitutes, and a price can be higher when goods are complements. Given these results, one has to say that in general, when both complement and substitute goods are allowed, neither prices are necessarily higher nor outputs necessarily lower in Cournot equilibrium. The strong result, namely, $\mathbf{p}^C \geq \mathbf{p}^B$ or $\mathbf{x}^B \geq \mathbf{x}^C$ does not hold even if demand is linear, derived from a concave utility function, and all the conditions $\mathbf{x}^C > \mathbf{0}$, $\mathbf{x}^B > \mathbf{0}$, $\mathbf{a} > \mathbf{c}$ and $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ are satisfied. A question then arises: in oligopoly with mixed products can we retain any notion that a Bertrand equilibrium is more competitive than a Cournot equilibrium?

4. Mixed products

Although we have found possible situations where prices could be lower and outputs could be higher in price competition, the reverse probably remains true in the most cases, including mixed products. However, the difficulty is: if one still claims that price competition is more competitive, in which sense does he mean that? In this section we will show that price competition is more competitive in two restricted senses: in the terms of the equilibrium price margin/output ratio and weighted sum of squared outputs and price margins. Although these criterion are weaker than the individual comparison for every firm's price and output, it gives a general idea about the overall competitiveness in price and quantity competition. In certain extent we can still claim that overall price competition is more competitive. Since we do not impose restrictions on product relations, we need to constrain ourselves in a linear model for the tractability.

(3) - (6) reveal that the differences between two equilibria are due to two positive diagonal matrices D^{-1} and Λ. Recall that λ_i is the ith diagonal element of B, and d_i is the ith diagonal element of B^{-1}. We can prove (see Appendix A) that $d_i\lambda_i \geq 1$ and the equality holds only if good i is independent. The first order conditions in firm i's output and price imply $(p_i^C-c_i)/x_i^C = \lambda_i$ and $(p_i^B-c_i)/x_i^B = 1/d_i$. As $\lambda_i > 1/d_i$ for every i, we have $(p_i^C-c_i)/x_i^C > (p_i^B-c_i)/x_i^B$ for all firms. This gives us the following

Proposition 1: Every firm's price margin/output ratio is higher in Cournot equilibrium than in Bertrand.

In general, a lower price margin/output ratio in Bertrand equilibrium does not imply a higher output and a lower price in the absolute sense. Nevertheless, if $p_i^C \leq$

p^B_i and $x^C_i \geq x^B_i$, we would have $(p^C_i-c_i)/x^C_i < (p^B_i-c_i)/x^B_i$. So Proposition 1 immediately implies

Corollary 1: No firm has a higher output and a lower price in Cournot equilibrium than in Bertrand equilibrium.

The second criterion by which we can say that price competition is more competitive is to compare weighted squared outputs and price margins. In particular regarding the equilibrium outputs, we choose the fixed weights λ_i - $1/d_i > 0$. It can be shown that $\sum_{i=1}^{n}(\lambda_i-1/d_i)\{(x^B_i)^2-(x^C_i)^2\} > 0$. Regarding the price margins, we choose d_i - $1/\lambda_i > 0$ to be the weights. We get $\sum_{i=1}^{n}(d_i-1/\lambda_i)\{(p^C_i-c_i)^2-(p^B_i-c_i)^2\} > 0$ (see Appendix B).

Proposition 2: The sum of squared outputs (price margins) in Bertrand equilibrium weighted by λ_i - $1/d_i$ (d_i-1/λ_i) is always higher (lower).

Obviously this result is not very strong because the specific weights. However, there are certain advantages of these particular weights. First, in our linear model, these weights are fixed and do not depend on a particular equilibrium outcome. Second, the weighted sums are independent of the units of products. Since all weights are positive, the inequality will be violated if $p^C_i \leq p^B_i$ or $x^C_i \geq x^B_i$ for all i. Thus Proposition 2 immediately implies

Corollary 2: At least one firm's price is higher and one firm's output is lower in Cournot equilibrium than in Bertrand equilibrium.

As we can not claim a Bertrand equilibrium more competitive in terms of prices and outputs, the two criterion above can be used as alternative measurements. These criterion have some advantages as well as disadvantages comparing to prices and outputs. Lower prices and higher outputs both indicate stronger competition. Yet as we have shown, a firm's lower price does not necessarily mean a larger output, and vise versa. The ratio of price margin to output reflects both prices and outputs. The weighted sum of squared outputs can be interpreted as a weighted "average output", and the sum of squared prices as a weighted "average price". This "average output" is always lower and the "average price" higher in Cournot equilibrium.

We derived our result in a linear oligopoly model. Actually, linear demand is needed only to guarantee that $\frac{\partial p_i(\mathbf{x}^B)}{\partial x_i}\frac{\partial x_i(\mathbf{p}^C)}{\partial p_i} \geq 1$. When the utility function is strictly concave but not necessarily quadratic, demand may be non-linear. We still have $\frac{\partial p_i(\mathbf{x}^B)}{\partial x_i}\frac{\partial x_i(\mathbf{p}^B)}{\partial p_i} \geq 1$ and $\frac{\partial p_i(\mathbf{x}^C)}{\partial x_i}\frac{\partial x_i(\mathbf{p}^C)}{\partial p_i} \geq 1$, but not necessarily $\frac{\partial p_i(\mathbf{x}^B)}{\partial x_i}\frac{\partial x_i(\mathbf{p}^C)}{\partial p_i} \geq 1$. However, if $\frac{\partial x_i(\mathbf{p}^B)}{\partial p_i} \geq \frac{\partial x_i(\mathbf{p}^C)}{\partial p_i}$ or $\frac{\partial p_i(\mathbf{x}^C)}{\partial x_i} \geq \frac{\partial p_i(\mathbf{x}^B)}{\partial x_i}$ holds, $\frac{\partial p_i(\mathbf{x}^B)}{\partial x_i}\frac{\partial x_i(\mathbf{p}^C)}{\partial p_i} \geq 1$ is guaranteed. Hence if every firm's sales is not more sensitive to its price, or its price is not less sensitive to its output in Bertrand equilibrium than in Cournot equilibrium, the previous results are valid for non-linear demand cases.

5. Supermodular game

In our linear model, when prices are strategic complements, i.e. $\frac{\partial^2 x_i}{\partial p_i \partial p_j} \geq 0$ for all j ≠ i, price competition is a supermodular game because $\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \geq 0$ for j ≠ i. If goods are complements, i.e. $\frac{\partial^2 u_i}{\partial x_i \partial x_j} \geq 0$ for all j ≠ i, quantity competition is supermodular as $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} \geq 0$ for j ≠ i. When a game is supermodular, it is easier to compare the equilibrium outcomes in price and quantity competition. In fact, when Vives (1985) and Okuguchi (1987) showed that prices are higher in Cournot equilibrium they both assume that price competition is supermodular. In this section we make use of the results of supermodular games by Vives (1990) and Milgrom and Roberts (1990) to compare Cournot and Bertrand equilibrium outputs. For the proof of their results we recommend reading the original papers, especially Vives (1990).

As we did before, we assume that demand is derived from a strictly concave utility function, but it need not be linear. Marginal costs are non-decreasing, but not necessarily constant. To make our comparison meaningful, we assume both quantity and price competition yield unique interior equilibrium solutions. In addition to these, we assume $\frac{\partial^2\pi_i}{\partial x_i \partial x_j} \geq 0$ for all $j \neq i$, and π_i is quasi-concave in x_i.

In Bertrand equilibrium, the first order condition for p_i implies $x_i^B + \{p_i^B - c_i'(x_i^B)\}\frac{\partial x_i}{\partial p_i} = 0$ where $c_i'(x_i^B)$ is the marginal cost. The derivative of profit with respect to x_i, $\frac{\partial \pi_i}{\partial x_i} = p_i^B - c_i'(x_i^B) + x_i^B\frac{\partial p_i}{\partial x_i} = \{x_i^B + [p_i^B - c_i'(x_i^B)]\frac{\partial x_i}{\partial p_i}\}/\frac{\partial x_i}{\partial p_i} + x_i^B(\frac{\partial p_i \partial x_i}{\partial x_i \partial p_i} - 1)/\frac{\partial x_i}{\partial p_i}$. The first term is zero. As $\frac{\partial p_i \partial x_i}{\partial x_i \partial p_i} > 1$, the second term is negative. So $\frac{\partial \pi_i}{\partial x_i} < 0$ in Bertrand equilibrium. Every firm can raise its profit by lowering its output. As $\pi_i(\mathbf{x})$ is quasi-concave in x_i, given other firms' Bertrand equilibrium outputs $\mathbf{x}_{-i}^B$, every firm i's optimal output is lower than the current level in Bertrand equilibrium, x_i^B. According to Vives' Theorem 5.1 (1990), there exists an equilibrium in quantities with all outputs lower than those in Bertrand equilibrium. It is the Cournot equilibrium. Thus, the equilibrium outputs are lower in Cournot equilibrium when quantity competition is a supermodular game. In a linear duopoly model, this happens if goods are complements. In our earlier examples where one firm's output is lower in Bertrand equilibrium, products must be substitutes.

Similar argument goes through when price competition is supermodular and profits are quasi-concave in prices. The conclusion of Vives (1985) and Okuguchi (1987) holds even if one drops the assumption $\left|\frac{\partial^2\pi_i}{\partial p_i^2}\right| > \sum_{j\neq i}^{n} \left|\frac{\partial^2\pi_i}{\partial p_i \partial p_j}\right|$ for all i by Okuguchi and the assumption $\mathbf{x}(\mathbf{c}) > \mathbf{0}$ by Vives.

When prices are lower, consumer surplus is obviously higher. When outputs are higher, one can draw a definite conclusion in terms of social welfare even with non-linear demand and cost functions. We can write the utility in Cournot equilibrium as $u(\mathbf{x}^C) = u(\mathbf{x}^B) + u'(\mathbf{x}^B)(\mathbf{x}^C-\mathbf{x}^B) + 0.5(\mathbf{x}^C-\mathbf{x}^B)u''(\mathbf{z})(\mathbf{x}^C-\mathbf{x}^B)$, and firm i's cost $c_i(x_i^C) = c_i(x_i^B) + c_i'(x_i^B)(x_i^C-x_i^B) + 0.5c_i''(w_i)(x_i^C-x_i^B)^2$. Assume that marginal costs are non-decreasing, so $c_i''(w_i) \geq 0$. Then the difference between the social welfare in Cournot and Bertrand equilibria is $u(\mathbf{x}^C) - \sum_{i=1}^{n}c_i(x_i^C) - u(\mathbf{x}^B) + \sum_{i=1}^{n}c_i(x_i^B)$ is equal to $\{u'(\mathbf{x}^B)-\mathbf{c}'(\mathbf{x}^B)\}\cdot(\mathbf{x}^B-\mathbf{x}^C) + 0.5(\mathbf{x}^C-\mathbf{x}^B)u''(\mathbf{z})(\mathbf{x}^C-\mathbf{x}^B) - 0.5\sum_{i=1}^{n}c_i''(w_i)(x_i^C-x_i^B)^2$, where $\mathbf{c}'(\mathbf{x}^B)$ is the vector of marginal costs in Bertrand equilibrium. Since the consumer maximizes her surplus, $u'(\mathbf{x}^B) = \mathbf{p}^B > \mathbf{c}'(\mathbf{x}^B)$, the first term is negative if $\mathbf{x}^B \geq \mathbf{x}^C$. The second is also negative due to the concavity of the utility function. The last term is negative as marginal costs are non-decreasing. Thus the social welfare is higher if outputs are higher.

Proposition 3: If price (quantity) competition is a supermodular game and every firm's profit is quasi-concave in its price (output), prices (outputs) are higher (lower) in Cournot equilibrium, and consumer surplus (social welfare) is lower.

Since $\mathbf{x}^B \geq \mathbf{x}^C$ and $\mathbf{p}^C \geq \mathbf{p}^B$ do not imply each other, two criterion for a more competitive outcome in price competition may not be consistent to each other. Now we compare the Herfindale indices in price and quantity competition. Let $s_i = x_i/\sum_{i=1}^{n}x_i$, the Herfindale index is $100\times\sum_{i=1}^{n}s_i^2$. To calculate this index products must be additive. Our model does not allow perfect substitutes, so we assume strong substitute goods and let matrix B be symmetric. Without loss of generality, let the diagonal elements of B be 1 and off-diagonal elements be $r > 0$.

Let $\delta > 0$ and consider an output vector $\mathbf{x} = (B+\delta I)^{-1}(\mathbf{a}-\mathbf{c})$. The diagonal elements of $(B+\delta I)^{-1}$ are $\{1+\delta+(n-2)r\}/(1+\delta-r)\{1+\delta+(n-1)\}$ and off-diagonal elements equal

$-r/(1+\delta-r)(1+\delta+(n-1)\}$. This implies $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n}(a_i-c_i)/\{1+\delta+(n-1)\}$. Let us denote $r/(1+\delta-r)$ by t. Then, we can write s_i as $\{(1+nt)(a_i-c_i)-t\sum_{j=1}^{n}(a_j-c_j)\}/\sum_{j=1}^{n}(a_j-c_j)$. Let $L \equiv \sum_{i=1}^{n}\{(1+nt)(a_i-c_i)-t\sum_{j=1}^{n}(a_j-c_j)\}^2$. It implies that $\partial L/\partial t = 2\sum_{i=1}^{n}\{[(1+nt)(a_i-c_i)-t\sum_{j=1}^{n}(a_j-c_j)][n(a_i-c_i)-\sum_{j=1}^{n}(a_j-c_j)]\} = 2(1+nt)\sum_{i=1}^{n}\{n(a_i-c_i)-\sum_{j=1}^{n}(a_j-c_j)\}^2 \geq 0$. The equality only holds when $n(a_i-c_i) = \sum_{j=1}^{n}(a_j-c_j)$ for all i. This happens when all firms are symmetric, namely $a_i - c_i$ are the same. In this case Herfindale index equals 1/n in both price and quantity competition. Otherwise L strictly increases in t. By its definition, t decreases in δ. Given (3) and (6), δ is equal to $1/d_i$ in price competition and λ_i in quantity competition. Since $d_i\lambda_i > 1$, we know that δ is smaller and t is bigger in price competition. So L is larger in price competition. As $L/\{\sum_{j=1}^{n}(a_j-c_j)\}^2$ is Herfindale index, price competition gives a higher index value than quantity competition except for the case with symmetric firms.

Proposition 4: Price competition leads to higher market concentration measured by the Herfindale index than does quantity competition.

Even when price competition leads to lower prices and larger outputs for all firms, it results in a higher market concentration than quantity competition measured by Herfindale index. This is due to the fact that given cost disparity price competition results in more asymmetric outputs and market shares. Since Herfindale index measures the concentration of market rather than the absolute prices or outputs, it gives a lower mark for quantity competition. However, as we showed here, a lower concentration does not always mean more competition. This result warns us not mixing them two.

6. Concluding remarks

It had been speculated that price competition leads to lower equilibrium prices and higher outputs than does quantity competition. This paper showed that this is not always true. When both complement and substitute goods exist, a firm could produce less or set a higher price in Bertrand equilibrium than in Cournot even if demand is linear, derived from a concave utility function and marginal costs are constant. With more than three firms, this could happen even though every firm can sell something at its marginal cost while other firms' prices equal to marginal costs or quantities equal to zero. Hence, price competition is not always more competitive in the terms of every firm's price and output.

Nevertheless, one can retain the original speculation in a restricted sense. If demand and cost functions are linear, every firm's price margin/output ratio is higher in Cournot equilibrium and a weighted sum of price margins (outputs) is always higher (lower). This holds true regardless of whether goods are substitutes or complements, or any mixture of them. The main driving force behind is the concavity of utility function. In addition to that, if the price (quantity) game is supermodular and profits are quasi-concave in firms' own decisions, consumer surplus (social welfare) is higher in Bertrand equilibrium.

This paper probably raises more questions than it answers. For instance, it is yet unknown whether one can apply the criterion of the price margin/output ratio and the weighted sum of outputs/price margins to non-linear cases. Or, probably better, one may find different criterion by which price competition is viewed more competitive under general and intuitive conditions.

Secondly, except in the cases of supermodular games, comparison between price and quantity competition was made in the terms of social welfare or consumer surplus. It would be a meaningful measurement if an unambiguous result exists. Besides our results in supermodular games, Vives (1985) has shown that in a symmetric case both social welfare and consumer surplus are higher in price competition. So far we have not known any counter example. Thus this criterion may preserve a general conclusion that price competition is more competitive.

Thirdly, as we have shown, Herfindale index is higher even though all prices may be lower and outputs higher in price competition. It suggests a trade off between efficiency and market concentration. If one takes into account of potential exit, price competition would generally allow few firms than does quantity competition. The a question arises: which situation is more competitive? It way be worthwhile to find out a robust and meaningful criterion for that.

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Appendix A:

Denote the ith column of B without the ith element $-\frac{\partial p_i}{\partial x_i}$ by an (n-1)×1 vector $\mathbf{b}_i$, and denote the ith column of B^{-1} without the ith element $-\frac{\partial x_i}{\partial p_i}$ by an (n-1)×1 vector β_i. Let B_{-i} denote the (n-1)×(n-1) sub-matrix of B without the ith row and ith column. As B is positive definite, so is B_{-i}. Given our definitions, we have $\frac{\partial p_i \partial x_i}{\partial x_i \partial p_i} + \boldsymbol{\beta}_i \cdot \mathbf{b}_i = 1$ and $-\frac{\partial x_i}{\partial p_i}\mathbf{b}_i + B_{-i}\boldsymbol{\beta}_i = \mathbf{0}$. Multiplying the second equation by $\boldsymbol{\beta}_i$, we get $-\frac{\partial x_i}{\partial p_i}\boldsymbol{\beta}_i \cdot \mathbf{b}_i + \boldsymbol{\beta}_i B_{-i}\boldsymbol{\beta}_i = 0$. As B_{-i} is positive definite, $\boldsymbol{\beta}_i B_{-i}\boldsymbol{\beta}_i \geq 0$. So $\frac{\partial x_i}{\partial p_i}\boldsymbol{\beta}_i \cdot \mathbf{b}_i \geq 0$. As $\frac{\partial x_i}{\partial p_i} < 0$, $\boldsymbol{\beta}_i \cdot \mathbf{b}_i \leq 0$. Hence $\frac{\partial p_i \partial x_i}{\partial x_i \partial p_i} = 1 - \boldsymbol{\beta}_i \cdot \mathbf{b}_i \geq 1$.

Further, as $\boldsymbol{\beta}_i B_{-i}\boldsymbol{\beta}_i = 0$ only when $\boldsymbol{\beta}_i = \mathbf{0}$, which implies $\mathbf{b}_i = \mathbf{0}$. So $\frac{\partial p_i \partial x_i}{\partial x_i \partial p_i} = 1$ if and only if $\frac{\partial u^2}{\partial x_i \partial x_j}$ $(j \neq i)$ i.e. good i is independent of other products. ||

Appendix B:

Our output comparison means $\mathbf{x}^B(\Lambda\text{-}D^{-1})\mathbf{x}^B - \mathbf{x}^C(\Lambda\text{-}D^{-1})\mathbf{x}^C > 0$. (3) and (6) imply that this equals $(\mathbf{a}\text{-}\mathbf{c})\{(B+D^{-1})^{-1}(\Lambda\text{-}D^{-1})(B+D^{-1})^{-1}\text{-}(B+\Lambda)^{-1}(\Lambda\text{-}D^{-1})(B+\Lambda)^{-1}\}(\mathbf{a}\text{-}\mathbf{c})$. It is equal to $\mathbf{x}^C T \mathbf{x}^C$ where $T = (B+\Lambda)(B+D^{-1})^{-1}(\Lambda\text{-}D^{-1})(B+D^{-1})^{-1}(B+\Lambda) - \Lambda + D^{-1}$. Notice that $(B+\Lambda)(B+D^{-1})^{-1} = I + (\Lambda\text{-}D^{-1})(B+D^{-1})^{-1}$. $T = 2(\Lambda\text{-}D^{-1})(B+D^{-1})^{-1}(\Lambda\text{-}D^{-1}) + (\Lambda\text{-}D^{-1})(B+D^{-1})^{-1}(\Lambda\text{-}D^{-1})(B+D^{-1})^{-1}(\Lambda\text{-}D^{-1})$, which is positive definite. Hence $\mathbf{x}^B(\Lambda\text{-}D^{-1})\mathbf{x}^B - \mathbf{x}^C(\Lambda\text{-}D^{-1})\mathbf{x}^C > 0$.

Similarly, for price margin comparison, we need to show that $(\mathbf{p}^C\text{-}\mathbf{c})(D\text{-}\Lambda^{-1})(\mathbf{p}^C\text{-}\mathbf{c}) > (\mathbf{p}^B\text{-}\mathbf{c})(D\text{-}\Lambda^{-1})(\mathbf{p}^B\text{-}\mathbf{c})$. Given (4) and (5), it means $(\mathbf{p}^B\text{-}\mathbf{c})T(\mathbf{p}^B\text{-}\mathbf{c}) > 0$, where $T = (I+DB)(I+\Lambda^{-1}B)^{-1}(D\text{-}\Lambda^{-1})(I+B\Lambda^{-1})^{-1}(I+BD) - D + \Lambda^{-1}$. Since $(I+DB)(I+\Lambda^{-1}B)^{-1} = I + (D\text{-}\Lambda^{-1})(B^{-1}+\Lambda^{-1})^{-1}$, the matrix T can be written as $2(D\text{-}\Lambda^{-1})(B^{-1}+\Lambda^{-1})^{-1}(D\text{-}\Lambda^{-1}) + (D\text{-}\Lambda^{-1})(B^{-1}+\Lambda^{-1})^{-1}(D\text{-}\Lambda^{-1})(B^{-1}+\Lambda^{-1})^{-1}(D\text{-}\Lambda^{-1})$. It is positive definite. Therefore we get $(\mathbf{p}^C\text{-}\mathbf{c})(D\text{-}\Lambda^{-1})(\mathbf{p}^C\text{-}\mathbf{c}) > (\mathbf{p}^B\text{-}\mathbf{c})(D\text{-}\Lambda^{-1})(\mathbf{p}^B\text{-}\mathbf{c})$. ||