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Rearrangements and Sequential Rank Order Dominance
A Result with Economic Applications*

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Réarrangements et Dominance en Quantiles Séquentielle
Un Résultat avec Applications Économiques

Résumé

L'analyse distributive implique généralement des comparaisons de distributions hétérogènes où les individus diffèrent dans plus d'un attribut. Dans le cas particulier où il y a deux attributs et où la distribution de l'un de ces deux attributs est fixée, on peut faire appel à la dominance en quantiles séquentielle pour comparer les distributions. Nous considérons le cas dégénéré où tous les individus diffèrent par rapport à l'attribut dont la distribution est fixée et nous montrons que, si une distribution domine une autre distribution au sens du critère des quantiles séquentiel, alors la première peut être obtenue à partir de la seconde au moyen d'une suite finie de permutations favorables, et réciproquement. Nous présentons trois exemples où les permutations favorables se révèlent avoir des implications intéressantes d'un point de vue normatif.

Mots-clés : Réarrangements, Permutation Favorable, Dominance en Quantiles Séquentielle, Appariement, Mobilité, Impatience.

Rearrangements and Sequential Rank Order Dominance
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Abstract

Distributive analysis typically involves comparisons of heterogeneous distributions where individuals differ in more than just one attribute. In the particular case where there are two attributes and where the distribution of one of these two attributes is fixed, one can appeal to sequential rank order dominance for comparing distributions. We consider the degenerate case where all individuals differ with respect to the attribute whose distribution is fixed and we show that sequential rank order domination of one distribution over another implies that the dominating distribution can be obtained from the dominated one by means of a finite sequence of favourable permutations, and conversely. We provide three examples where favourable permutations prove to have interesting implications from a normative point of view.

Keywords: Rearrangements, Favourable Permutations, Sequential Rank Order Dominance, Matching, Mobility, Impatience

JEL: D31, D63, I32.

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1. Introduction

It is not uncommon for distributive analysis to involve comparisons of heterogeneous distributions where individuals differ in two attributes, for instance income and health status. A related example is provided by comparisons of income profiles which indicate the income received by an individual at given dates. The assessment of intergenerational mobility is another area that involves comparisons of multidimensional distributions: in the simplest case, those of the parents and those of the children. On the one hand, Atkinson and Bourguignon (1982) have shown that the ranking of such distributions by utilitarian unanimity proves to be identical to the one implied by bidimensional first order stochastic dominance, when unanimity is sought over the class of submodular utility functions. On the other hand, it can be easily checked that favourable permutations always result in an increase in social welfare when it is evaluated by the utilitarian rule and when the utility function is submodular (see, e.g., Moyes (2011)). However, it is still an open question as to whether the converse statement holds, or equivalently whether it is possible to derive the dominating distribution from the dominated one – where domination is understood in the sense of bidimensional first order stochastic dominance – by means of appropriate combinations of such permutations.

In the particular case where the distribution of one of the two attributes is fixed, bidimensional first order stochastic dominance reduces to sequential rank order dominance (see, e.g., Atkinson and Bourguignon (1987)). To illustrate things, consider the case where every individual in the population is identified by her health status and her income. Suppose further that an individual's health status falls into a finite set of ordered categories like "bad health", "average" or "good health". The sequential rank order criterion consists in comparing, first the quantile distributions of income for those individuals in bad health, then the quantile distributions of income for those individuals in the combined bad or average health categories, and finally the quantile distributions of income for the entire population. If one distribution is ranked above another at each stage of the process, then it is declared to be better according to the sequential rank order criterion. In this note, we consider the degenerate case where all individuals differ with respect to the attribute whose distribution is fixed, i.e. no two individuals have the same value of the attribute, and we show that sequential rank order domination of one distribution over another implies that the dominating distribution can be obtained from the dominated one by means of a finite sequence of favourable permutations, and conversely.

The organisation of the note is as follows. We introduce the notation, as well as the definitions of rank order and sequential rank order dominance in Section 2. We also provide some preliminary results that will be useful later on. We present and prove in Section 3 our main result according to which sequential rank order domination of one distribution by another implies that the dominating distribution can be obtained from the dominated one by means of successive favourable permutations. We examine in Section 4 three examples that involve more or less explicitly favourable permutations and where the application of the sequential rank order criterion proves to be relevant. Finally, Section 5 concludes the paper.

2. Notation, Definitions and Preliminary Results

There is a finite set of individuals $N := \{1, 2, \ldots, n\}$ with $n \geqq 2$. A distribution for population N is a list $\mathbf{u} := (u_1, \ldots, u_n)$, where $u_i \in \mathscr{D} \subset \mathbb{R}$ may be viewed as the *income* of individual i in situation $\mathbf{u}$, but other interpretations are also possible.

Definition 2.1. Given two income distributions $\mathbf{u} := (u_1, \ldots, u_n), \mathbf{v} := (v_1, \ldots, v_n) \in \mathscr{D}^n$, we say that $\mathbf{u}$ *component-wise dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq \mathbf{v}$, if and only if:

$$u_h \geqq v_h, \ \forall \ h = 1, 2, \ldots, n. \tag{2.1}$$

We indicate respectively by $\sim$ and $>$ the symmetric and asymmetric components of $\geq$ defined in the usual way, and we note that $\mathbf{u} \sim \mathbf{v}$ if and only if $u_h = v_h$, for all $h = 1, 2, \ldots, n$. The non-decreasing rearrangement of an income distribution $\mathbf{u} := (u_1, \ldots, u_n) \in \mathscr{D}^n$ is indicated by $\mathbf{u}_{(\ )} := (u_{(1)}, u_{(2)}, \ldots, u_{(n)})$, where $u_{(1)} \leqq u_{(2)} \leqq \cdots \leqq u_{(n)}$.

Definition 2.2. Given two income distributions $\mathbf{u} := (u_1, \ldots, u_n), \mathbf{v} := (v_1, \ldots, v_n) \in \mathscr{D}^n$, we say that $\mathbf{u}$ *rank order dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_{RO} \mathbf{v}$, if and only if:

$$u_{(h)} \geqq v_{(h)}, \ \forall \ h = 1, 2, \ldots, n. \tag{2.2}$$

We indicate respectively by $\sim_{RO}$ and $>_{RO}$ the symmetric and asymmetric components of $\geq_{RO}$, and we note that $\mathbf{u} \sim_{RO} \mathbf{v}$ if and only if $u_{(h)} = v_{(h)}$, for all $h = 1, 2, \ldots, n$, in which case $\mathbf{u}$ is a permutation of $\mathbf{v}$. Component-wise dominance implies rank-order dominance, but the converse implication is false as the next result demonstrates.

Lemma 2.1. *Let* $\mathbf{u}, \mathbf{v} \in \mathscr{D}^n$ *and consider the following two statements:*

(a) $\mathbf{u} \geq \mathbf{v}$.

(b) $\mathbf{u} \geq_{RO} \mathbf{v}$.

Then, statement (a) *implies statement* (b)*, but the converse implication does not hold.*

Proof.
(a) $\Longrightarrow$ (b). By definition, we have $u_h \geqq v_h$, for all $h \in \{1, 2, \ldots, n\}$. Consider the indices i and j defined by $u_i \geqq u_h$, for all $h \in \{1, 2, \ldots, n\}$, and $v_j \geqq v_h$, for all $h \in \{1, 2, \ldots, n\}$.

Case 1: $i = j$. Then $u_{(n)} = u_i \geqq v_i = v_{(n)}$ and $u_h \geqq v_h$, for all $h \in \{1, 2, \ldots, n\} \setminus \{i\}$. Let

$$\tilde{\mathbf{u}} := (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n); \tag{2.3a}$$

$$\tilde{\mathbf{v}} := (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n); \tag{2.3b}$$

$$\hat{\mathbf{u}} := (u_{(n)}) = (u_i); \tag{2.3c}$$

$$\hat{\mathbf{v}} := (v_{(n)}) = (v_i). \tag{2.3d}$$

Then, we have $\tilde{\mathbf{u}} \geq \tilde{\mathbf{v}}$ and $\hat{\mathbf{u}} \geq \hat{\mathbf{v}}$.

Case 2: $i \neq j$. We have $u_i \geqq u_j \geqq v_j \geqq v_i$, from which we deduce that $u_{(n)} = u_i \geqq v_j = v_{(n)}$ and $u_j \geqq v_i$. In addition, $u_h \geqq v_h$, for all $h \in \{1, 2, \ldots, n\} \setminus \{i, j\}$. Denote as $\mathbf{v}^*$ the permutation of distribution $\mathbf{v}$ defined by

$$\mathbf{v}^* := (v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n)\,, \tag{2.4}$$

assuming that $i < j$. From what precedes, we deduce that $\mathbf{u} \geq \mathbf{v}^*$. Define next

$$\tilde{\mathbf{u}} := (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_j \ldots, u_n); \tag{2.5a}$$

$$\tilde{\mathbf{v}} := (v_1^*, \ldots, v_{i-1}^*, v_{i+1}^*, \ldots, v_j^* \ldots, v_n^*) = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_i, \ldots, v_n); \tag{2.5b}$$

$$\hat{\mathbf{u}} := (u_{(n)}) = (u_i); \tag{2.5c}$$

$$\hat{\mathbf{v}} := (v_{(n)}) = (v_j). \tag{2.5d}$$

Then, we obtain $\tilde{\mathbf{u}} \geq \tilde{\mathbf{v}}$ and $\hat{\mathbf{u}} \geq \hat{\mathbf{v}}$. Repeating this argument n times, we finally obtain two distributions $\hat{\mathbf{u}} = \mathbf{u}_{(\,)}$ and $\hat{\mathbf{v}} = \mathbf{v}_{(\,)}$ such that $\hat{\mathbf{u}} \geq \hat{\mathbf{v}}$.

$\neg$ [(a) $\Longrightarrow$ (b)]: Choose $\mathbf{u} := (4, 2, 2)$ and $\mathbf{v} := (1, 1, 3)$, which are not comparable according to the component-wise ordering, though $\mathbf{u}_{(\,)} := (2, 2, 4) \geq (1, 1, 3) =: \mathbf{v}_{(\,)}$. □

The next lemma is well-known (see, e.g., Saposnik (1981)) and it simply states that the ranking of income distributions by utilitarian unanimity over the class of non-decreasing utility functions is equivalent to the ranking implied by the rank order criterion. More precisely, letting

$$\mathbf{\Phi} := \{\phi : \mathscr{D} \to \mathbb{R} \mid \phi \text{ is non-decreasing}\}, \tag{2.6}$$

we have the following result:

Lemma 2.2. *Let* $\mathbf{u}, \mathbf{v} \in \mathscr{D}^n$. *Then, statements* (a) *and* (b) *below are equivalent:*

(a) $\sum_{h=1}^n \phi(u_h) \geqq \sum_{h=1}^n \phi(v_h),\ \forall\ \phi \in \mathbf{\Phi}$.

(b) $\mathbf{u} \geq_{RO} \mathbf{v}$.

This result mirrors the standard equivalence between first order stochastic dominance and utilitarian unanimity (see, e.g., Fishburn and Vickson (1978)).

From now on, we assume (i) that individuals can be distinguished on the basis of a variable which we call for convenience *ability* and (ii) that no two individuals have the same ability. What is important here is that individuals are ranked according to ability which means that individual i is strictly less able than individual j whenever $i < j$. In other words, we can interpret the set N as an ordered list of n distinct ability levels and $\mathbf{u} := (u_1, \ldots, u_n) \in \mathscr{D}^n$ as a distribution of incomes over the set of (ordered) abilities. We indicate by $\mathbf{u}^h := (u_1, u_2, \ldots, u_h)$ the distribution of the incomes received by the individuals with abilities equal to h or less and we note that $\mathbf{u}^n = \mathbf{u}$. We let

$$\mathbf{u}^h_{(\,)} := (u^h_{(1)}, u^h_{(2)}, \ldots, u^h_{(h-1)}, u^h_{(h)}) \tag{2.7}$$

stand for the rearrangement of $\mathbf{u}^h$ such that

$$u^h_{(1)} \leqq u^h_{(2)} \leqq \cdots \leqq u^h_{(h-1)} \leqq u^h_{(h)}. \tag{2.8}$$

Definition 2.3. Given two distributions $\mathbf{u} := (u_1, \ldots, u_n), \mathbf{v} := (v_1, \ldots, v_n) \in \mathscr{D}^n$, we say that $\mathbf{u}$ *sequential rank order dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_{SRO} \mathbf{v}$, if and only if:

$$\mathbf{u}^h \geq_{RO} \mathbf{v}^h, \; \forall\, h = 1, 2, \ldots, n-1, \text{ and } \mathbf{u}^n \sim_{RO} \mathbf{v}^n. \tag{2.9}$$

If in addition $\mathbf{u}^k >_{RO} \mathbf{v}^k$, for some $k < n$, then we say that $\mathbf{u}$ *sequential rank order strictly dominates* $\mathbf{v}$, which we write $\mathbf{u} >_{SRO} \mathbf{v}$.

Important for subsequent developments is the fact that, if $\mathbf{u} \geq_{SRO} \mathbf{v}$, then $\mathbf{u}$ is a permutation of $\mathbf{v}$. Making use of (2.2), (2.7) and (2.8), we note that $\mathbf{u} \geq_{SRO} \mathbf{v}$ amounts to requiring that

$$u^h_{(g)} \geqq v^h_{(g)}, \forall\, g = 1, 2, \ldots, h, \forall\, h = 1, 2, \ldots, n-1, \text{ and} \tag{2.10a}$$

$$u^n_{(g)} = v^n_{(g)}, \forall\, g = 1, 2, \ldots, n. \tag{2.10b}$$

Among the different possible rearrangements of the elements of a distribution, the following one will play a crucial role in the paper.

Definition 2.4. Given two distributions $\mathbf{u} := (u_1, \ldots, u_n), \mathbf{v} := (v_1, \ldots, v_n) \in \mathscr{D}^n$, we say that $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by means of a favourable permutation* if there exists two individuals $i, j \in N$ with $i < j$ such that

$$v_i = u_j < u_i = v_j; \text{ and} \tag{2.11a}$$

$$v_g = u_g, \forall\, g \neq i, j. \tag{2.11b}$$

The third lemma constitutes the analogue of Lemma 2.2 in the case of heterogeneous income distributions by appealing to the sequential rank order criterion. Before we state the result, we need to introduce the following class of n-tuples of functions:

$$\boldsymbol{\Psi} := \left\{ \boldsymbol{\psi} := (\psi_1, \ldots, \psi_n) \mid \psi'_h(s) \geqq \psi'_{h+1}(s), \forall\, s \in \mathscr{D}, \forall\, h = 1, 2, \ldots, n-1 \right\}. \tag{2.12}$$

Then, we have:

Lemma 2.3. *Consider two heterogeneous distributions* $\mathbf{u}, \mathbf{v} \in \mathscr{D}^n$*. Then, statements* (a) *and* (b) *below are equivalent:*

(a) $\sum_{h=1}^n \psi_h(u_h) \geqq \sum_{h=1}^n \psi_h(v_h), \; \forall\; \boldsymbol{\psi} := (\psi_1, \ldots, \psi_n) \in \boldsymbol{\Psi}$.

(b) $\mathbf{u} \geq_{SRO} \mathbf{v}$.

Proof.

(a) $\Longrightarrow$ (b). Let $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)$ and consider the n-tuple $\boldsymbol{\psi} := (\psi_1, \ldots, \psi_n)$ defined by $\psi_h(s) := \lambda_h \, \phi(s)$, for all $h = 1, 2, \ldots, n$. Clearly, $\boldsymbol{\psi} := (\psi_1, \ldots, \psi_n) \in \boldsymbol{\Psi}$ provided that $\phi'(s) \geqq 0$ and $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$, which we assume. Choosing successively $\boldsymbol{\lambda} := (1, 0, 0, \ldots, 0, 0)$, $\boldsymbol{\lambda} := (1, 1, 0, \ldots, 0, 0), \ldots, \boldsymbol{\lambda} := (1, 1, 1, \ldots, 1, 0)$, $\boldsymbol{\lambda} := (1, 1, 1, \ldots, 1, 1)$, and $\boldsymbol{\lambda} := (-1, -1, -1, \ldots, -1, -1)$, condition (a) reduces to

$$\sum_{g=1}^{h} \phi(u_g) \geqq \sum_{g=1}^{h} \phi(v_g), \forall\, h = 1, 2, \ldots, n-1, \text{ and} \tag{2.13a}$$

$$\sum_{g=1}^{n} \phi(u_g) = \sum_{g=1}^{n} \phi(v_g), \tag{2.13b}$$

which holds for all functions $\phi \in \mathbf{\Phi}$. Appealing to Lemma 2.2, we conclude that $\mathbf{u}^h \geq_{RO} \mathbf{v}^h$, for all $h = 1, 2, \ldots, n-1$, and $\mathbf{u}^n \sim_{RO} \mathbf{v}^n$, hence $\mathbf{u} \geq_{SRO} \mathbf{v}$.

(b) $\Longrightarrow$ (a). Given any $\mathbf{u} := (u_1, \ldots, u_n) \in \mathscr{D}^n$, we have the following equality:

$$\sum_{h=1}^{n} \psi_h(u_h) = \sum_{h=1}^{n} \left[\psi_h(u_h) \pm \sum_{k=1}^{h-1} \psi_h(u_k) \right] = \sum_{h=1}^{n-1} \left[\sum_{k=1}^{h} (\psi_h(u_k) - \psi_{h+1}(u_k)) \right] + \sum_{k=1}^{n} \psi_n(u_k). \tag{2.14}$$

Letting $f_h(s) := \psi_h(s) - \psi_{h+1}(s)$, for $h = 1, 2, \ldots, n-1$, and $f_n(s) := \psi_n(s)$, and upon substituting into (2.14), we obtain

$$\sum_{h=1}^{n} \psi_h(u_h) = \sum_{h=1}^{n-1} \left[\sum_{k=1}^{h} f_h(u_k) \right] + \sum_{k=1}^{n} f_n(u_k) = \sum_{h=1}^{n} \sum_{i=1}^{h} f_h(u^h_{(i)}). \tag{2.15}$$

Because $f'_h(s) := \psi'_h(s) - \psi'_{h+1}(s)$, in the light of (2.15), condition (a) can be equivalently rewritten as

$$\sum_{h=1}^{n} (\psi_h(u_h) - \psi_h(v_h)) = \sum_{h=1}^{n} \sum_{i=1}^{h} \left(f_h(u^h_{(i)}) - f_h(v^h_{(i)}) \right) \geqq 0, \tag{2.16}$$

for all $f_h(s)$ that are non-decreasing in s. Making use of the *Mean Value Theorem*, condition (2.16) is equivalent to

$$\sum_{h=1}^{n-1} \sum_{i=1}^{h} f'_h(\xi^h_i) \left[u^h_{(i)} - v^h_{(i)} \right] + \sum_{i=1}^{n} f'_n(\xi^n_i) \left[u^n_{(i)} - v^n_{(i)} \right] \geqq 0, \tag{2.17}$$

for some $\xi^h_i \in (u^h_{(i)}, v^h_{(i)})$, for all $i \in \{1, 2, \ldots, h\}$ and all $h \in \{1, 2, \ldots, n\}$. Given (2.10) and since $f'_h(s) = \psi'_h(s) - \psi'_{h+1}(s) \geqq 0$, for all s and all $h = 1, 2, \ldots, n-1$, we conclude that it is sufficient for (2.17) to hold that $\mathbf{u} \geq_{SRO} \mathbf{v}$. □

The connection between the class of n-tuples $\mathbf{\Psi}$ and the class of submodular functions is easily recognised if we let $g(s, h) := \psi_h(s)$, for all $s \in \mathscr{D}$ and all $h \in \{1, 2, \ldots, n\}$. Indeed, a function $g : \mathscr{D} \times \mathscr{D} \to \mathbb{R}$ is called *submodular* if $g(u+\delta, v+\varepsilon) - g(u, v+\varepsilon) \leqq g(u+\delta, v) - g(u, v)$, for all $(u, v) \in \mathscr{D} \times \mathscr{D}$ and all $\delta, \varepsilon > 0$. When the function g is differentiable in its first argument, this reduces to the condition that $g_{(1)}(s, h) = \psi'_h(s) \geqq \psi'_{h+1}(s) = g_{(1)}(s, h+1)$, for all $s \in \mathscr{D}$ and all $h \in \{1, 2, \ldots, n-1\}$. Sometimes, one also says that g is *L-subadditive* (see, e.g., Marshall and Olkin (1979, Chapter 6, Section D).

The assumption that individuals, whatever their abilities, have different incomes in the two situations under comparison simplifies things. In conjunction with Lemma 2.3, our last technical result confirms that there is no loss of generality when comparing heterogeneous income distributions by means of utilitarian unanimity in order to restrict attention to the subpopulation of individuals whose incomes differ in the two situations. Given two heterogeneous distributions $\mathbf{u}, \mathbf{v} \in \mathscr{D}^n$, we define:

$$S := \{h \in N \mid u_h = v_h\}; \tag{2.18a}$$

$$T := \{h \in N \mid u_h \neq v_h\}. \tag{2.18b}$$

Given the n-tuple $\boldsymbol{\psi} := (\psi_1, \ldots, \psi_n) \in \mathbf{\Psi}$, we denote by $\boldsymbol{\psi}(T) := ((\psi_h)_{h \in T})$ its restriction to T and by $\mathbf{\Psi}(T)$ the set of such profiles. Then, we have the following obvious result:

Lemma 2.4. *Let* $\mathbf{u}, \mathbf{v} \in \mathscr{D}^n$*. Then, statements* (a) *and* (b) *below are equivalent:*

(a) $\sum_{h=1}^{n} \psi_h(u_h) \geqq \sum_{h=1}^{n} \psi_h(v_h),\ \forall\ \boldsymbol{\psi} := (\psi_1, \ldots, \psi_n) \in \boldsymbol{\Psi}$.

(b) $\sum_{h \in T} \psi_h(u_h) \geqq \sum_{h \in T} \psi_h(v_h),\ \forall\ \boldsymbol{\psi}(T) \in \boldsymbol{\Psi}(T)$.

3. Main Result

We are now in a position to state our main result which establishes the connection between sequential rank order dominance and favourable permutations.

Theorem 3.1. *Let* $\mathbf{u}, \mathbf{v} \in \mathscr{D}^n$*. Then, statements* (a) *and* (b) *below are equivalent:*

(a) $\mathbf{u}$ *is obtained from* $\mathbf{v}$ *by means of a finite sequence of favourable permutations.*

(b) $\mathbf{u} \geq_{SRO} \mathbf{v}$.

Proof.

(a) $\Longrightarrow$ (b). Suppose that $\mathbf{u}$ is obtained from $\mathbf{v}$ by means of a single favourable permutation so that

$$\mathbf{u} = (v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n), \tag{3.1}$$

where $v_i < v_j$. Then, we have:

$$1 \leqq h \leqq i-1 : \mathbf{u}^h \sim_{RO} \mathbf{v}^h \qquad \text{because } \mathbf{u}^h = \mathbf{v}^h, \tag{3.2a}$$

$$i \leqq h \leqq j-1 : \mathbf{u}^h >_{RO} \mathbf{v}^h \qquad \text{by Lemma 2.1}, \tag{3.2b}$$

$$j \leqq h \leqq n : \mathbf{u}^h \sim_{RO} \mathbf{v}^h \qquad \text{because } \mathbf{u}^h \text{ is a permutation of } \mathbf{v}^h, \tag{3.2c}$$

and we conclude that $\mathbf{u} \geq_{SRO} \mathbf{v}$. When more than one favourable permutation is needed to convert $\mathbf{u}$ into $\mathbf{v}$, the result follows by invoking the transitivity of the sequential rank order criterion.

(b) $\Longrightarrow$ (a). Thanks to Lemma 2.4, we assume without loss of generality that

$$u_h \neq v_h,\ \forall\, h \in \{1, 2, \ldots, n\}. \tag{3.3}$$

Now consider the indices i and j defined as follows:

$$i := \min\left\{h \in \{1, 2, \ldots, n\} \mid u_h \geqq u_g,\ \forall\, g \in \{1, 2, \ldots, n\}\right\}; \tag{3.4a}$$

$$j := \min\left\{h \in \{1, 2, \ldots, n\} \mid v_h \geqq v_g,\ \forall\, g \in \{1, 2, \ldots, n\}\right\}. \tag{3.4b}$$

From the definition of the indices i and j, we have

$$u_h < u_i,\ \forall\, h = 1, 2, \ldots, i-1; \tag{3.5a}$$

$$u_i \geqq u_h,\ \forall\, h = i+1, i+2, \ldots, n; \tag{3.5b}$$

$$v_h < v_j,\ \forall\, h = 1, 2, \ldots, j-1; \tag{3.5c}$$

(3.5d) $$v_j \geqq v_h, \forall\, h = j+1, i+2, \ldots, n.$$

Furthermore, $u_i := \max\{u_h\} = \max\{v_h\} =: v_j$, because $\mathbf{u}$ is a permutation of $\mathbf{v}$. We note that by construction $i < j$. Indeed, we cannot have $i = j$, for, if it were the case, then $u_i = v_i$, which is ruled out by (3.3). Suppose next that $j < i$ and consider the distributions $\mathbf{u}^j := (u_1, u_2, \ldots, u_j)$ and $\mathbf{v}^j := (v_1, v_2, \ldots, v_j)$. From definition of v_j, we have

(3.6) $$u^j_{(j)} < u_i = v_j = v^j_{(j)}.$$

Hence $\neg\, [\mathbf{u}^j \geq_{RO} \mathbf{v}^j]$, which contradicts the fact that $\mathbf{u} \geq_{SRO} \mathbf{v}$ by assumption.

The idea of the proof is to operate favourable permutations within the subpopulation $\{i, i+1, \ldots, j-1, j\}$ so that, at the end of this process, one obtains a new distribution $\mathbf{z}$ with the properties that (i) $u_i = z_i = v_j$, and (ii) $\mathbf{u} \geq_{SRO} \mathbf{z} >_{SRO} \mathbf{v}$. Let

(3.7) $$k := \min \{h \in \{i, i+1, \ldots, j-1\} \mid v_h \geqq v_g, \forall\, g \in \{i, i+1, \ldots, j-1\}\}.$$

By definition of the index k, we have:

(3.8a) $$v_h < v_k, \forall\, h \in \{i, i+1, \ldots, k-1\};$$

(3.8b) $$v_k \geqq v_h, \forall\, h \in \{k+1, k+2, \ldots, j-1\};$$

(3.8c) $$v_k < v_j.$$

Then, we construct a new distribution $\mathbf{w} := (w_1, \ldots, w_n)$ starting from $\mathbf{v} := (v_1, \ldots, v_n)$ by means of a favourable permutation as it is indicated below

(3.9a) $$w_h = v_h, \forall\, h \neq k, j; \text{ and}$$

(3.9b) $$w_k = v_j;\ w_j = v_k;$$

and illustrated in Table 3.1.

We now examine the distributions $\mathbf{u}^h$, $\mathbf{w}^h$ and $\mathbf{v}^h$, for $h = 1, 2, \ldots, H$. We consider successively four cases.

Case 1: $1 \leqq h \leqq i-1$. By assumption $\mathbf{u}^h \geq_{RO} \mathbf{v}^h$ and by construction $\mathbf{w}^h = \mathbf{v}^h$. Hence, $u^h_{(g)} \geqq w^h_{(g)} = v^h_{(g)}$, for all $g \in \{1, 2, \ldots, h\}$.

Table 3.1: Construction of $\mathbf{w}$ starting from $\mathbf{v}$ by means of a favourable permutation

	$1 \cdots i-1$	$\boxed{i}$	$i+1 \cdots k-1$	$\boxed{k}$	$k+1 \cdots j-1$	$\boxed{j}$	$j+1 \cdots n$
$\mathbf{u}$:	$u_1 \cdots u_{i-1}$	$\boxed{u_i}$	$u_{i+1} \cdots u_{k-1}$	u_k	$u_{k+1} \cdots u_{j-1}$	u_j	$u_{j+1} \cdots u_n$
$\mathbf{w}$:	$v_1 \cdots v_{i-1}$	v_i	$v_{i+1} \cdots v_{k-1}$	$\boxed{v_j}$	$v_{k+1} \cdots v_{j-1}$	$\boxed{v_k}$	$v_{j+1} \cdots v_n$
$\mathbf{v}$:	$v_1 \cdots v_{i-1}$	v_i	$v_{i+1} \cdots v_{k-1}$	$\boxed{v_k}$	$v_{k+1} \cdots v_{j-1}$	$\boxed{v_j}$	$v_{j+1} \cdots v_n$

Case 2: $i \leqq h \leqq j-1$. If $k > h$, the argument is the same as in Case 1, so we can restrict attention to the case in which $k \in \{i, i+1, \ldots, h-1, h\}$. We indicate by $g^* = \chi(h,k)$ the largest rank with the income v_k in the ordered distribution $\mathbf{v}^h_{(\,)}$ as shown in

$$v^h_{(1)} \leqq \cdots \leqq v^h_{(g^*-1)} \leqq \boxed{v^h_{(g^*)} \equiv v_k} < v^h_{(g^*+1)} \leqq \cdots \leqq v^h_{(h)}, \tag{3.10}$$

and we denote by

$$\rho^* := \#\left\{g \in \{1,2,\ldots,h\} \;\middle|\; v^h_{(g)} > v^h_{(g^*)} = v_k\right\} = h - g^* \tag{3.11}$$

the number of individuals who, in distribution $\mathbf{v}^h := (v_1, v_2, \ldots, v_h)$, have incomes greater than v_k. We note that $h-i+1 \leqq g^* = \chi(h,k) \leqq h$, for all $h \in \{i, i+1, \ldots, j-1\}$, or equivalently that $\rho^* \leqq i-1$. This is because by definition $v_k \geqq v_g$, for all $g \in \{i, i+1, \ldots, h-1, h\}$ and all $h \in \{i, i+1, \ldots, j-1\}$. Suppose first that $g^* = h$. Then, it follows from the definition of $\mathbf{w}^h_{(\,)}$ and the fact that $\mathbf{u}^h_{(\,)} \geq_{RO} \mathbf{v}^h_{(\,)}$ by assumption that

$$u^h_{(g)} \geqq w^h_{(g)} = v^h_{(g)},\ \forall\, g = 1,2,\ldots,h-1, \text{ and} \tag{3.12a}$$

$$u_i = u^h_{(h)} = w^h_{(h)} = v_j > v_k = v^h_{(h)}. \tag{3.12b}$$

Consider next the case where $g^* \in \{1,2,\ldots,h-1\}$. Invoking again the definition of $\mathbf{w}^h_{(\,)}$ and the fact that $\mathbf{u}^h_{(\,)} \geq_{RO} \mathbf{v}^h_{(\,)}$ by assumption, we have

$$u^h_{(g)} \geqq w^h_{(g)} = v^h_{(g)},\ \forall\, g = 1,2,\ldots,g^*-1, \text{ and} \tag{3.13a}$$

$$u_i = u^h_{(h)} = w^h_{(h)} = v_j > v_k = v^h_{(h)}. \tag{3.13b}$$

It remains to be examined what happens when $g = g^*, g^*+1, \ldots, h-1$. By definition of the index i, we have

$$u^h_{(1)} \leqq u^h_{(2)} \leqq \cdots \leqq u^h_{(h-p)} < \boxed{u^h_{(h-p+1)} = \cdots = \boxed{u^h_{(h)} = u_i}}, \tag{3.14}$$

where $p := \#\{g \in \{i, i+1, \ldots, h\} \mid u_g = u_i\}$. Given a non-empty and finite set $A := \{a_1, a_2, \ldots, a_m\}$, where $a_i \in \mathbb{R}$, for all $i = 1,2,\ldots,m$ ($m \geqq 2$), we denote by $\max_\rho A$ the ρ^{th}-greatest element in A with $1 \leqq \rho \leqq m$. Since by definition $u^h_{(h)} = u_i > u_g$, for all $g \in \{1,2,\ldots,i-1\}$, we deduce that

$$\left\{\{u^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1}\right\} \subseteq \left\{\{u^h_{(g)}\}_{g=1,2,\ldots,h-1}\right\}, \tag{3.15}$$

and we have:

$$u^h_{(h-1)} = \max_1 \left\{\{u^h_{(g)}\}_{g=1,2,\ldots,h-1}\right\} \geqq \max_1 \left\{\{u^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1}\right\} = u^{i-1}_{(i-1)}; \qquad \rho = 1$$

$$u^h_{(h-2)} = \max_2 \left\{\{u^h_{(g)}\}_{g=1,2,\ldots,h-1}\right\} \geqq \max_2 \left\{\{u^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1}\right\} = u^{i-1}_{(i-2)}; \qquad \rho = 2$$

$$\vdots \qquad\qquad \vdots$$

$$u^h_{(h-\rho^*+1)} = \max_{\rho^*-1} \left\{\{u^h_{(g)}\}_{g=1,2,\ldots,h-1}\right\} \geqq \max_{\rho^*-1} \left\{\{u^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1}\right\} = u^{i-1}_{(i-\rho^*+1)}; \qquad \rho = \rho^*-1$$

$$u^h_{(h-\rho^*)} = \max_{\rho^*} \left\{ \{u^h_{(g)}\}_{g=1,2,\ldots,h-1} \right\} \geqq \max_{\rho^*} \left\{ \{u^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = u^{i-1}_{(i-\rho^*)}; \qquad \rho = \rho^*.$$

More compactly:

$$u^h_{(h-\rho)} = \max_{\rho} \left\{ \{u^h_{(g)}\}_{g=1,2,\ldots,h-1} \right\} \geqq \max_{\rho} \left\{ \{u^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = u^{i-1}_{(i-\rho)}, \tag{3.16}$$

for all $\rho = 1, 2, \ldots, \rho^* - 1, \rho^*$. Similarly, by definition of the index k, we have

$$v^h_{(1)} \leqq \cdots \leqq v^h_{(h-q)} \leqq \boxed{v^h_{(h-q+1)} = \cdots = \boxed{v^h_{(g^*)} \equiv v_k}} < v^h_{(g^*+1)} \leqq \cdots \leqq v^h_{(h)}, \tag{3.17}$$

where $q := \# \{g \in \{1, 2, \ldots, h\} \mid v_g = v_k\}$. Since by definition $v^h_{(g^*)} = v_k \geqq v_g$, for all $g \in \{i, i+1, \ldots, h\}$, we deduce that

$$\left\{v^h_{(g^*+1)}, v^h_{(g^*+2)}, \ldots, v^h_{(h)}\right\} \subseteq \left\{v^{i-1}_{(1)}, v^{i-1}_{(2)}, \ldots, v^{i-1}_{(i-1)}\right\}. \tag{3.18}$$

This implies in turn that

$$v^h_{(h)} = \max_{1} \left\{ \{v^h_{(g)}\}_{g=g^*+1,g^*+2,\ldots,h} \right\} \leqq \max_{1} \left\{ \{v^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = v^{i-1}_{(i-1)}; \qquad \rho = 1$$

$$v^h_{(h-1)} = \max_{2} \left\{ \{v^h_{(g)}\}_{g=g^*+1,g^*+2,\ldots,h} \right\} \leqq \max_{2} \left\{ \{v^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = v^{i-1}_{(i-2)}; \qquad \rho = 2$$

$$\vdots \qquad\qquad \vdots$$

$$v^h_{(h-\rho^*+2)} = \max_{\rho^*-1} \left\{ \{v^h_{(g)}\}_{g=g^*+1,g^*+2,\ldots,h} \right\} \leqq \max_{\rho^*-1} \left\{ \{v^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = v^{i-1}_{(i-\rho^*+1)}; \quad \rho = \rho^* - 1$$

$$v^h_{(h-\rho^*+1)} = \max_{\rho^*} \left\{ \{v^h_{(g)}\}_{g=g^*+1,g^*+2,\ldots,h} \right\} \leqq \max_{\rho^*} \left\{ \{v^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = v^{i-1}_{(i-\rho^*)}; \qquad \rho = \rho^*.$$

More compactly:

$$v^h_{(h-\rho+1)} = \max_{\rho} \left\{ \{v^h_{(g)}\}_{g=g^*+1,g^*+2,\ldots,h} \right\} \leqq \max_{\rho} \left\{ \{v^{i-1}_{(g)}\}_{g=1,2,\ldots,i-1} \right\} = v^{i-1}_{(i-\rho)}, \tag{3.19}$$

for all $\rho = 1, 2, \ldots, \rho^* - 1, \rho^*$. Combining (3.16) and (3.19), making use of the definition of $\mathbf{w}$ and of the fact that $\mathbf{u}^{i-1}_{(\ )} \geq_{RO} \mathbf{v}^{i-1}_{(\ )}$, we obtain

$$u^h_{(h-1)} \geqq u^{i-1}_{(i-1)} \geqq v^{i-1}_{(i-1)} \geqq v^h_{(h)} = w^h_{(h-1)} \geqq v^h_{(h-1)}; \qquad \rho = 1$$

$$u^h_{(h-2)} \geqq u^{i-1}_{(i-2)} \geqq v^{i-1}_{(i-2)} \geqq v^h_{(h-1)} = w^h_{(h-2)} \geqq v^h_{(h-2)}; \qquad \rho = 2$$

$$\vdots \qquad\qquad \vdots$$

$$u^i_{(h-\rho^*+1)} \geqq u^{i-1}_{(i-\rho^*+1)} \geqq v^{i-1}_{(i-\rho^*+1)} \geqq v^h_{(h-\rho^*+2)} = w^h_{(h-\rho^*+1)} \geqq v^h_{(h-\rho^*+1)}; \qquad \rho = \rho^* - 1$$

$$u^i_{(h-\rho^*)} \geqq u^{(i-1)}_{(i-\rho^*)} \geqq v^{i-1}_{(i-\rho^*)} \geqq v^h_{(h-\rho^*+1)} = w^h_{(h-\rho^*)} \geqq v^h_{(h-\rho^*)}; \qquad \rho = \rho^*.$$

More compactly:

$$u^h_{(h-\rho)} \geqq u^{i-1}_{(i-\rho)} \geqq v^{i-1}_{(i-\rho)} \geqq v^h_{(h-\rho+1)} = w^h_{(h-\rho)} \geqq v^h_{(h-\rho)}, \tag{3.20}$$

for all $\rho = 1, 2, \ldots, \rho^* - 1, \rho^*$. Thus, we conclude that

$$u^h_{(h-\rho)} \geqq w^h_{(h-\rho)} \geqq v^h_{(h-\rho)}, \ \forall \rho = 1, 2, \ldots, h - g^* - 1, h - g^*, \tag{3.21}$$

or equivalently

$$u^h_{(g)} \geqq w^h_{(g)} \geqq v^h_{(g)}, \forall\, g = h-1, h-2, \ldots, g^*+1, g^*. \tag{3.22}$$

Case 3: $j \leqq h \leqq n-1$. By assumption $\mathbf{u}^h \geq_{RO} \mathbf{v}^h$ and by construction $\mathbf{w}^h$ is a permutation of $\mathbf{v}^h$. Hence, $u^h_{(g)} \geqq w^h_{(g)} = v^h_{(g)}$, for all $g \in \{1, 2, \ldots, h\}$.

Case 4: $h = n$. By assumption $\mathbf{u}^n \sim_{RO} \mathbf{v}^n$ and by construction $\mathbf{w}^n$ is a permutation of $\mathbf{v}^n$. Hence, $u^n_{(g)} = w^n_{(g)} = v^n_{(g)}$, for all $g \in \{1, 2, \ldots, n\}$.

To summarise, we have $\mathbf{u}^h \geq_{RO} \mathbf{w}^h \geq_{RO} \mathbf{v}^h$, for all $h = 1, 2, \ldots, n-1$, with $\mathbf{w}^h >_{RO} \mathbf{v}^h$, for at least one h, and $\mathbf{u}^n \sim_{RO} \mathbf{w}^n \sim_{RO} \mathbf{v}^n$. Therefore $\mathbf{u} \geq_{SRO} \mathbf{w} >_{SRO} \mathbf{v}$. If $\mathbf{u} = \mathbf{w}$, then distribution $\mathbf{u}$ results from distribution $\mathbf{v}$ by means of a single favourable permutation and the proof is complete. If $\mathbf{u} \neq \mathbf{w}$, then we apply the above reasoning to the distributions $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ obtained by deleting all indices h such that $u_h = w_h$. It is possible that no such indices exist in which case $\hat{\mathbf{u}} = \mathbf{u}$ and $\hat{\mathbf{w}} = \mathbf{w}$. By successive permutations of the kind described above, we finally obtain a distribution $\mathbf{z}$ such that $z_i = v_j = u_i$. Given $i < j$, we need at most $j - i$ favourable permutations in order to obtain $\mathbf{z}$ starting from $\mathbf{v}$, which gives a maximum of $(n-1)$ permutations in the case where $i = 1$ and $j = n$. At the next stage, we would need at most $(n-2)$ favourable permutations since the algorithm involves distributions with at most $(n-1)$ elements. Therefore, we obtain distribution $\mathbf{u}$ from distribution $\mathbf{v}$ by means of a finite sequence of at most $n\,(n-1)/2$ favourable permutations. □

Theorem 3.1 has interesting implications for the ranking of the permutations of any arbitrary distribution. Given a distribution $\mathbf{u} \in \mathscr{D}^n$, we indicate by $\mathbf{u}_{[\ ]} := (u_{[1]}, u_{[2]}, \ldots, u_{[n]})$ its non-increasing rearrangement defined by $u_{[1]} \geqq u_{[2]} \geqq \cdots \geqq u_{[n]}$. Denoting by $\mathscr{P}_n$ the set of permutation matrices $P \equiv [p_{ij}]$ of order n, one establishes using a similar argument to that used in the first part of the proof of Theorem 3.1 the following result:

Proposition 3.1. *Let $\mathbf{u} \in \mathscr{D}^n$ be an arbitrary heterogeneous distribution. Then, we have:*

$$\mathbf{u}_{(\ )} \geq_{SRO} P\,\mathbf{u} \geq_{SRO} \mathbf{u}_{[\ ]},\ \forall\ P \in \mathscr{P}_n. \tag{3.23}$$

Since condition (3.19) holds whatever the choice of the distribution $\mathbf{u}$, Proposition 3.1 provides implicitly a means of ranking permutation matrices.

4. Three Applications

Matching and inequality We consider two populations of the same size n – men and women – where each population is characterised by a distribution of income. We denote by $\mathbf{u} := (u_1, \ldots, u_n)$ and $\mathbf{v} := (v_1, \ldots, v_n)$ the distributions of income for men and women, respectively, and we assume for simplicity that no two incomes within distributions $\mathbf{u}$ and $\mathbf{v}$ are the same. We are interested in pairing men and women in such a way that aggregate income inequality is minimised. In order to compare distributions on the basis of inequality, we appeal to the standard Lorenz criterion. More precisely, given two distributions $\mathbf{u} := (u_1, \ldots, u_n)$ and

$\mathbf{v} := (v_1, \ldots, v_n)$ such that $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$, we say that $\mathbf{u}$ *Lorenz dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_L \mathbf{v}$, if and only if:

$$\sum_{j=1}^{k} u_{(j)} \geqq \sum_{j=1}^{k} v_{(j)}, \ \forall\, k = 1, 2, \ldots, n-1. \tag{4.1}$$

We assume without loss of generality that the distribution of women's incomes is fixed and that their incomes are non-decreasingly arranged. Consider the situation where the richest woman is matched with the richest man, the second richest woman with the second richest man and so on, which implies the following aggregate income distribution:

$$(\mathbf{u}_{(\ )} + \mathbf{v}_{(\ )}) := (u_{(1)} + v_{(1)}, u_{(2)} + v_{(2)}, \ldots, u_{(n)} + v_{(n)}). \tag{4.2}$$

Suppose now that women i and j switch men, which results in the aggregate income distribution:

$$(\tilde{\mathbf{u}} + \mathbf{v}_{(\ )}) := (u_{(1)} + v_{(1)}, \ldots, u_{(j)} + v_{(i)}, \ldots, u_{(i)} + v_{(j)}, \ldots, u_{(n)} + v_{(n)}), \tag{4.3}$$

where $\tilde{\mathbf{u}} := (u_{(1)}, \ldots, u_{(j)}, \ldots, u_{(i)} \ldots, u_{(n)})$. This exchange of partners between women i and j can be interpreted as a favourable permutation for women, and it follows from Theorem 3.1 that $\tilde{\mathbf{u}} \geq_{SRO} \mathbf{u}_{(\ )}$. By Lemma 2.3 we know that this is equivalent to

$$\sum_{i=1}^{n} \psi_i(\tilde{u}_i) \geqq \sum_{i=1}^{n} \psi_i(u_{(i)}), \tag{4.4}$$

for all n-tuples $\boldsymbol{\psi} := (\psi_1, \ldots, \psi_n) \in \boldsymbol{\Psi}$, and in particular for those n-tuples $\boldsymbol{\psi} := (\psi_1, \ldots, \psi_n)$ such that

$$\psi_i(s) := \phi(s + v_{(i)}), \ \forall\, s > 0, \ \forall\, i = 1, 2, \ldots, n, \tag{4.5}$$

where ϕ is concave. Hence,

$$\sum_{i=1}^{n} \phi(\tilde{u}_i + v_{(i)}) \geqq \sum_{i=1}^{n} \phi(u_{(i)} + v_{(i)}), \ \forall\, \phi \in \boldsymbol{\Phi}, \tag{4.6}$$

which, upon invoking the Hardy, Littlewood, and Pólya (1934) theorem, is equivalent to

$$(\tilde{\mathbf{u}} + \mathbf{v}_{(\ )}) \geq_L (\mathbf{u}_{(\ )} + \mathbf{v}_{(\ )}). \tag{4.7}$$

Invoking Proposition 3.1, one establishes using a similar argument that

$$(\mathbf{u}_{[\ ]} + \mathbf{v}_{(\ )}) \geq_L (P\,\mathbf{u} + \mathbf{v}_{(\ )}) \geq_L (\mathbf{u}_{(\ )} + \mathbf{v}_{(\ )}), \ \forall\, \mathbf{u}, \mathbf{v} \in \mathscr{D}^n, \ \forall\, P \in \mathscr{P}_n. \tag{4.8}$$

Thus, assuming that the distribution of income among men and women are given, the most effective way to reduce income inequality among couples is to match the richest woman with the poorest man, the second richest woman with the second poorest man, and so on.

Intertemporal choice under impatience Consider an individual who has to decide which occupation to choose on the basis of the incomes these occupations generate in each period of her lifetime. Suppose that all that matters for the individual is the incomes she receives in the different periods and that no uncertainty is attached to these incomes. An *income profile* is a list $\mathbf{u} := (u_1, u_2, \ldots, u_t, \ldots, u_T)$, where $u_t \in \mathscr{D}$ is the income the individual receives in period t and T the finite horizon. In order to evaluate the different income profiles available to her, the individual uses an *intertemporal utility function*

$$V(\mathbf{u}) \equiv V(u_1, u_2, \ldots, u_T). \tag{4.9}$$

It is generally considered that individuals, when they have to make choices involving future prospects, exhibit a preference for the present (see, e.g., Ekern (1981), Bohren and Hansen (1980)). To be more precise, we assume here that the individual is *impatient* in the sense that if she is given the possibility of choosing between getting an extra dollar today or receiving it tomorrow and if she has the same income in both periods, then she always prefers to have it today. Formally, this amounts to requiring that the intertemporal utility function satisfies:

$$V(u_1, \ldots, u_r + \Delta, \ldots, u_s, \ldots, u_T) \geqq V(u_1, \ldots, u_r, \ldots, u_s + \Delta, \ldots, u_T), \ \forall\ \mathbf{u},\ \forall\ \Delta > 0, \tag{4.10}$$

whenever $u_r = u_s$ and $r < s$. In the particular case where the intertemporal utility function is additively separable, that is $V(u_1, \ldots, u_T) = \sum_{t=1}^{T} \psi_t(u_t)$, this condition reduces to

$$\psi_r'(u) \geqq \psi_s'(u),\ \forall\ u,\ \forall\ r < s, \tag{4.11}$$

where we assume for simplicity that the *temporal utility functions* $\psi_t(u)$ $(t = 1, 2, \ldots, T)$ are differentiable. It follows from Lemma 2.3 that a necessary and sufficient condition for profile $\mathbf{u}$ to be preferred to profile $\mathbf{v}$ by all additively separable intertemporal utility functions with impatience is that $\mathbf{u} \geq_{SRO} \mathbf{v}$. Invoking Theorem 3.1, this is in turn equivalent to the fact that $\mathbf{u}$ can be derived from $\mathbf{v}$ by a finite sequence of favourable permutations, which in the present context consists in permuting incomes from some period in the future to a more recent period.

Exchange mobility We consider a society composed of n dynasties ($n \geqq 2$), where each dynasty consists of one father and one son. A *situation* for the society is a $n \times 2$ matrix

$$\mathbf{x} \equiv (\mathbf{x}^F; \mathbf{x}^S) := \begin{bmatrix} x_1^F & x_1^S \\ \vdots & \vdots \\ x_h^F & x_h^S \\ \vdots & \vdots \\ x_n^F & x_n^S \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_h \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \tag{4.12}$$

such that $\mathbf{x}_h = (x_h^F, x_h^S)$ is the *intergenerational income distribution* of dynasty h, $x_h^F \in \mathscr{S}$ and $x_h^S \in \mathscr{S}$ are respectively the father's and son's incomes of dynasty h, and $\mathscr{S} := \{s_1, s_2, \ldots, s_m\} \subset \mathscr{D}$ is the set of possible incomes with $s_1 < s_2 < \cdots < s_m$. To simplify things, we let $m = n$ and $x_h^F = s_h$, for all $h = 1, 2, \ldots, n$, which implies that $x_1^F < x_2^F < \cdots < x_n^F$. We also assume that the incomes of children are permutations of parents' incomes which implies that no two children can have the same income. Given the conventions above, two situations $\mathbf{x} \equiv (\mathbf{x}^F; \mathbf{x}^S)$ and $\mathbf{y} \equiv (\mathbf{y}^F; \mathbf{y}^S)$ can only differ to the extent that $\mathbf{x}^S$ and $\mathbf{y}^S$ are permutations of each other. Following Atkinson (1981), we say that $\mathbf{x}$ *exhibits more mobility than* $\mathbf{y}$ if and only if

$$\sum_{h=1}^{n} V(x_h^F, x_h^S) \geqq \sum_{h=1}^{n} V(y_h^F, y_h^S), \ \forall\ V \in \mathscr{V}, \tag{4.13}$$

where $\mathscr{V} := \{V : \mathscr{D} \times \mathscr{D} \to \mathbb{R} \mid V_{12}(u,v) \leqq 0,\ \forall\ (u,v) \in \mathscr{D} \times \mathscr{D}\}$ is the set of admissible *dynasty utility functions*. It follows from Lemma 2.3 that, if $\mathbf{x}$ is more mobile than $\mathbf{y}$, then $\mathbf{x}^S \geq_{SRO} \mathbf{y}^S$, and conversely. Appealing next to Theorem 3.1, this is equivalent to saying that $\mathbf{x}^S$ is obtained from $\mathbf{y}^S$ by means of a finite sequence of favourable permutations.

As it is suggested for instance by Shorrocks (1980, Section 4), we note that, the more mobile the society is, the more equally distributed are the dynasties' aggregate incomes. Indeed, consider the class of dynasty utility functions $V(u,v) := \phi(f(u)+g(v))$, where $f(u)+g(v)$ can be interpreted as the *net present value* of the dynasty's intergenerational income distribution and where ϕ measures the value attached to it. Assuming that f and g are increasing and that ϕ is concave, we have

$$V_{12}(u,v) = \phi''(f(u)+g(v))\, f'(u)\ g'(u) \leqq 0,\ \forall\ (u,v) \in \mathscr{D} \times \mathscr{D}, \tag{4.14}$$

hence $V(u,v) := \phi(f(u)+g(v)) \in \mathscr{V}$. Then, we deduce from (4.13) that

$$\sum_{h=1}^{n} \phi(f(x_h^F) + g(x_h^S)) \geqq \sum_{h=1}^{n} \phi(f(y_h^F) + g(y_h^S)),\ \forall\ \phi \in \mathbf{\Phi}. \tag{4.15}$$

Letting $f(\mathbf{z}^F) := (f(z_1^F), \ldots, f(z_n^F))$ and $g(\mathbf{z}^S) := (g(z_1^S), \ldots, g(z_n^S))$, for $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}$, and invoking the Hardy *et al.* (1934) theorem, this is equivalent to

$$(f(\mathbf{x}^F) + g(\mathbf{x}^S)) \geq_L (f(\mathbf{y}^F) + g(\mathbf{y}^S)). \tag{4.16}$$

Therefore, whatever the way we compute the net present values of the intergenerational income distributions – provided that fathers' and sons' incomes contribute positively to it and that their distributions of income are permutations of each other – inequality decreases as mobility increases.

5. Concluding Remarks

We have shown in this note that, if one distribution of two variables sequential rank order dominates another, then it can be derived from the dominated distribution by means of a finite sequence of favourable permutations, and conversely. We have provided three examples that involve more or less explicitly favourable permutations and where the application of the sequential rank order criterion proves to be relevant.

An obvious limitation of the present note is the strong assumption that there is one and only one individual for each category of the variable whose marginal distribution is fixed. While there is a high presumption that our result generalises to the case where there is more than one individual per category of the conditioning variable, it is fair to admit that this is still something that has to be proven.

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