A note on the computation of an actuarial Waring formula in the finite-exchangeable case*

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Abstract

We present in this paper the actuarial Waring formula, which is used in several fields, like life-insurance or credit risk. In a particular framework where considered random variables are exchangeable, we show that some problems can occur when using this formula. We propose alternative recursions in order to improve the complexity of the calculations, and to cope with the numerical instability of the formula.

1 Introduction

1.1 The considered actuarial Waring formula

Let $n \in \mathbb{N}$ and $\Omega = \{1, \dots, n\}$. Consider a sequence of Bernoulli random variables $\{X_i\}_{i\in\Omega}$, $X_i \in \{0, 1\}$, $i \in \Omega$, non necessarily an iid sequence. The actuarial Waring formula gives the law of $X_1 + \cdots + X_m$ as a linear combination of some coefficients μ_k, which are sums of probabilities of kind $\mathrm{P}[X_{i_1} = 1 \cap \cdots \cap X_{i_k} = 1]$, $i_1, \dots, i_k \in \Omega$. These coefficients are quite easy to express in several cases, such as the independent case (not necessarily identically distributed), or such as the exchangeable case.

The Waring formula is based on the inclusion-exclusion principle (see Feller, 1968, chapter IV.3, page 106). The reference of this formula as "Waring formula" is given in some actuarial writing, like in Gerber (1995), ch.8.6, in Tupper (1981) or in some probability books, like in Bizley (1957), p74. A reference of this formula as "Waring's Theorem" is also given in MacDonald (2004).

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This formula is known in the life-insurance field under the name of "Z-method"(cf. Scott, 1992; Neill, 1977, section 7.4). It appears in some early references, like in a book written by George King and published in 1902 by the U.K. Institute of Actuaries (see King, 1902). We found an earlier mention of this formula in Baily (1810), p.48, §65, which refers to the original writing of Waring (1792) at the end of the 18th century. In order to distinguish this formula with other famous formulas from Edward Waring, we will call it the "actuarial" Waring formula. This actuarial Waring formula corresponds to a particular case of the more recent Schuette-Nesbitt formula (see Gerber, 1995, ch 8.6, p.89).

First, let us define first some coefficients that will be useful to write the Waring formula. Let $m \in \mathbb{N}$, $m \leqslant n$. We introduce the coefficients of order k, $k \leqslant m$, for the set $\{X_i,\ i \in \Gamma_m\}$ where $\Gamma_m \subset \Omega$ is such as $card(\Gamma_m) = m$:

Definition 1.1 (μ_k coefficients) *Let $\Gamma_m \subset \Omega$, where $m = card(\Gamma_m)$. The coefficient of order k ($k \leq m$) for the set $\{X_i,\ i \in \Gamma_m\}$, denoted $\mu_k(\Gamma_m)$, is defined as*

$$\mu_k(\Gamma_m) = \frac{1}{\mathrm{C}_m^k} \sum_{\substack{j_1<j_2<..<j_k \\ j_1,\ldots,j_k \in \Gamma_m}} \mathrm{P}\left[X_{j_1} = 1 \cap \ldots \cap X_{j_k} = 1\right], \quad 1 \leq k \leq m,$$

$$\mu_0(\Gamma_m) = 1 \ \textit{(including if } \Gamma_m = \emptyset).$$

The $\sum\limits_{\substack{j_1<j_2<..<j_k \\ j_1,\ldots,j_k \in \Gamma_m}}$ *symbol means to sum all C_m^k possible choices of k different elements taken among the elements of the set Γ_m. $\mathrm{C}_n^k = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient, $0 \leq k \leq n,\ n \in \mathbb{N}$.*

Theorem 1.1 (The actuarial Waring formula) *Let $X_1, \ldots, X_n$ be n dependent Bernoulli random variables and let Γ_m be a subset of Ω such that $card(\Gamma_m) = m \leq n$. Then*

$$W_\mu(k, \Gamma_m) = \mathrm{P}\left[\sum_{i \in \Gamma_m} X_i = k\right] = \mathbb{1}_{k \leq m} \mathrm{C}_m^k \sum_{j=0}^{m-k} \mathrm{C}_{m-k}^j (-1)^j \mu_{j+k}(\Gamma_m).$$

Proof: A proof can be found in Feller (1968) (chapter IV.3, page 106). This formula can also be seen as a particular case of the Schuette-Nesbitt Formula often used in actuarial science (see White and Greville, 1959; Schuette and Nesbitt, 1959; Gerber, 1979; Buchta, 1994; Gerber, 1995, chapter 8.6, page 89 for some proofs). □

This actuarial Waring formula was originally used in the life-insurance field, where random lifetimes and death indicators are often considered to be independent but not identically distributed (due to the difference of ages among considered people): in these circumstances, coefficients μ_k are easy to compute, and the law of a number of death among a group of independent but heterogeneous people is easy to obtain. This formula is also useful when we consider finite-exchangeable random variables, even without an underlying random factor. We previously used it in order to compute a number of defaults into a credit-risk model (see Cousin, Dorobantu, Rullière, 2010).

Remark 1.1 (Exchangeable case) *If the random variables $\{X_i,\ i \in \Omega\}$ are exchangeable, then for all $\Gamma_m \subset \Omega$ we have*

$$\mu_k(\Gamma_m) = \mu_k = \mathrm{P}\left[X_1 = 1 \cap ... \cap X_k = 1\right],\ 1 \leq k \leq m.$$

Remark 1.2 (Independent case, not i.d.) *If the random variables $\{X_i,\ i \in \Omega\}$ are independent, but not identically distributed, then for all $\Gamma_m \subset \Omega$ we have*

$$\mu_k(\Gamma_m) = \frac{1}{\mathrm{C}_m^k} \sum_{\substack{j_1 < j_2 < .. < j_k \\ j_1, \dots, j_k \in \Gamma_m}} \mathrm{P}\left[X_{j_1} = 1\right] \dots \mathrm{P}\left[X_{j_k} = 1\right], \quad 1 \leq k \leq m,$$

Remark 1.3 (Infinite-exchangeable case) *If the random variables $\{X_i,\ i \in \Omega\}$ are infinite exchangeable, then due to De Finetti's Theorem, there exists an underlying random factor Θ such that $\{X_i,\ i \in \Omega\}$ are independent given the value of a common probability Θ (see De Finetti, 1931). In this case,*

$$\mathrm{P}\left[\sum_{i \in \Gamma_m} X_t^i = k\right] = C_m^k E[\Theta^k (1-\Theta)^{m-k}],$$

and Waring formula correspond to the power development of this expression, with $\mu_k = \mathrm{E}\left[\Theta^k\right]$. In this last case, these probabilities may also be computed by numerical integration, given the distribution of the underlying factor Θ. Nevertheless, when Ω is finite, stating that the sequence $X_1, X_2, \dots$ is finite-exchangeable does not involve the existence of an underlying common factor Θ, since De Finetti's theorem does not apply in this case.

Remark 1.4 (Analytic solutions) *In some cases, $W_\mu(k, \Gamma_m)$ has an analytical simple expression. As an example, with a beta-distributed underlying factor Θ, $W_\mu(k, \Gamma_m)$ can be expressed easily using Euler Gamma function (see Cousin, Dorobantu, Rullière, 2011)*

1.2 The problem

We consider here the exploitation of the Waring formula in the finite-exchangeable case. We used such an assumption in a previous work modeling dependencies and infections in a credit-risk model (see Cousin, Dorobantu, Rullière, 2010).

Suppose that $\{X_i\}$, $i \in \Omega$ are exchangeable random variables, so that for all $\Gamma_m \subset \Omega$, $\mu_k(\Gamma_m) = \mu_k$. In this case, all $W_\mu(k,\Gamma_m)$ are equals for any subset of cardinal m, so that we will simply write $W_\mu(k,m)$ instead of $W_\mu(k,\Gamma_m)$.

In this case,

$$W_\mu(k,m) \quad = \quad \mathrm{P}\left[X_1 + \cdots + X_m = k\right] = \mathbb{1}_{k \le m} \mathrm{C}_m^k \sum_{j=0}^{m-k} \mathrm{C}_{m-k}^j (-1)^j \mu_{j+k},$$

where

$$\begin{aligned} \mu_k &= \mathrm{P}\left[X_1 = 1 \cap ... \cap X_k = 1\right], \quad 1 \le k \le m, \\ \mu_0 &= 1\,. \end{aligned}$$

Waring formula is useful when

- Coefficients $\{\mu_k\}_{k\in\Omega}$ are given.
- We want to compute the law of $\sum_{i=1}^m X_i$, for each $m \in \Omega$.

Suppose that coefficients μ_k are given, and that we want to compute all $W_\mu(k,m)$, $k,m \in \Omega$.

$$W_\mu(k,m) = \mathbb{1}_{k \le m} \mathrm{C}_m^k \sum_{j=0}^{m-k} \mathrm{C}_{m-k}^j (-1)^j \mu_{j+k}.$$

One can then make two remarks:

- First, the loops number for a given couple $(k,m) \in \Omega^2$, $k \leqslant m$, is of order $m-k$, so that computing this formula for any couple (k,m) involves approximatively $n^3/6$ loops, which may be important when n is large.
- Second, binomial coefficients C_m^k and C_{m-k}^j may become extremely large when n is large, so that numerically, we are then adding some huge quantities to get a final probability belonging to $[0,1]$. Even when writing binomial coefficients as exponentials of log-binomial coefficients, due to computer arithmetic, we can not ensure that numerically $(\exp(\gamma)+1) - (\exp(\gamma))$ equals one for large γ. We will investigate more precisely these issues in section 3.2.

In the following, we will try to propose some solutions for these two problems, by using some recursions.

2 Proposed recursions in the exchangeable case

2.1 Basic recursion

Theorem 2.1 (First recursion) *Coefficients $W_\mu(k,m)$, can be computed recursively, for m varying from 1 to n, for k varying from $m-1$ to 0, using:*

$$\begin{aligned} W_\mu(m,m) &= \mu_m \,, \\ W_\mu(k,m) &= \frac{m}{m-k} W_\mu(k,m-1) - \frac{k+1}{m-k} W_\mu(k+1,m) \,. \end{aligned}$$

with the value when $m=0$, $W_\mu(0,0)=1$.

Proof: Consider $k,m \in \mathbb{N}$, $m \geqslant 1$, $m-k \geqslant 2$. First write

$$\begin{aligned} W_\mu(k,m)/\mathrm{C}_m^k &= \sum_{j=0}^{m-k} \mathrm{C}_{m-k}^j (-1)^j \mu_{j+k} \\ &= \mu_k + (-1)^{m-k}\mu_m + \sum_{j=1}^{m-k-1} \mathrm{C}_{m-k}^j (-1)^j \mu_{j+k} \end{aligned}$$

Using $\mathrm{C}_{m-k}^j = \mathrm{C}_{m-k-1}^j + \mathrm{C}_{m-k-1}^{j-1}$,

$$\begin{aligned} W_\mu(k,m)/\mathrm{C}_m^k &= \sum_{j=0}^{m-k-1} \mathrm{C}_{m-k-1}^j (-1)^j \mu_{j+k} + \sum_{j=1}^{m-k} \mathrm{C}_{m-k-1}^{j-1} (-1)^j \mu_{j+k} \\ &= W_\mu(k,m)/\mathrm{C}_{m-1}^k + (-1) \sum_{i=0}^{m-k-1} \mathrm{C}_{m-k-1}^i (-1)^i \mu_{i+1+k} \\ &= W_\mu(k,m)/\mathrm{C}_{m-1}^k - W_\mu(k+1,m)/\mathrm{C}_m^{k+1}. \end{aligned}$$

and checking that the result is also true for $m-k=1$, the recursion holds. □

Remark 2.1 (Symmetrical recursion) *By symmetry, working on $\bar{X} = 1-X$ and applying the same recursion, we also get for $k \in \{1,\ldots,n\}$:*

$$W_\mu(k,n) = \frac{n}{k} W_\mu(k-1,n-1) - \frac{n-k+1}{k} W_\mu(k-1,n) \,,$$

Remark 2.2 (Local numerical problem) *Suppose $\mu_n = p^n$, corresponding to the iid case. Since for $k=n$, $W_\mu(n-1,n) = n(\mu_{n-1}-\mu_n)$, applying this recursion with $p = 1-\epsilon$ leads to $(1-\epsilon)^{n-1}\epsilon = (1-\epsilon)^{n-1} - (1-\epsilon)^n$, which can cause local numerical problems when ϵ is small. These problems will be studied in Section 3.1 and 3.2.*

2.2 Normalized recursions

We can remark that if the algorithm complexity has been reduced, there remain some numerical problems. We can first imagine changing the recursion, since this recursion is simplified in some case. A first investigation consists in looking for another recursion that would have a better stability. A way to propose other recursions is to express each $W_\mu(k,n)$ as a function of a given value $W_0(k,n)$ and some residual terms, obtained by recursion, that would have a better stability.

Proposition 2.2 (Additive normalization) *Suppose we have a quantity $W_0(k,n)$ such that for $n \geqslant 1$, $k \leqslant n$:*

$$W_0(k,n) = \frac{n}{n-k} W_0(k,n-1) - \frac{k+1}{n-k} W_0(k+1,n)\,,$$

And suppose furthermore that this quantity is numerically easy to obtain. If we write $\bar{W}(k,n) = W_\mu(k,n) - W_0(k,n)$, then

$$\bar{W}(k,n) = \frac{n}{n-k} \bar{W}(k,n-1) - \frac{k+1}{n-k} \bar{W}(k+1,n)\,,$$

so that $\bar{W}(k,n)$ may be computed by recurrence, and then $W_\mu(k,n)$ deduced from $\bar{W}(k,n)$.

As an example, following quantities may be used for additive normalization:

- $W_0(k,n) = \mathrm{C}_n^k p^k (1-p)^{n-k}$ (iid case)
- $W_0(k,n) = \mathrm{P}\left[N_n = k\right]$, where N_n is a beta-mixed binomial law

In the particular iid case or beta-mixed binomial case, one can thus find W_0 such that all $\bar{W}(k,n)$ are equal to zero, so that the recursion remains stable.

Proposition 2.3 (Multiplicative normalization) *Suppose we have a quantity $W_0(k,n)$ such that for $n \geqslant 1$, $k \leqslant n$, $W_0(k,n) \neq 0$. If we write $\tilde{W}(k,n) = W_\mu(k,n)/W_0(k,n)$, then*

$$\tilde{W}(k,n) = \alpha_k^n \tilde{W}(k,n-1) + \beta_k^n \tilde{W}(k+1,n)\,,$$
$$\textit{with } \alpha_k^n = \tfrac{n}{n-k} \tfrac{W_0(k,n-1)}{W_0(k,n)} \textit{ and } \beta_k^n = -\tfrac{k+1}{n-k} \tfrac{W_0(k+1,n)}{W_0(k,n)}\,.$$

so that $\tilde{W}(k,n)$ may be computed by recurrence, and then $W_\mu(k,n)$ deduced from $\tilde{W}(k,n)$.
We can also show that $\beta_k^n = 1 - \alpha_k^n$ when $W_0(k,n)$ is such that

$$W_0(k,n) = \frac{n}{n-k} W_0(k,n-1) - \frac{k+1}{n-k} W_0(k+1,n)\,.$$

Remark 2.3 (Normalization example) *One can think about finding one W_0 that would lead to a greater stability of the recursion. Using for example*

$$\begin{aligned}&\tilde{W}(k,n) = \frac{1}{1-p}\tilde{W}(k,n-1) - \frac{p}{1-p}\tilde{W}(k+1,n)\,,\\ &\textit{when } W_0(k,n) = \mathrm{C}_n^k p^k (1-p)^{n-k}\,.\end{aligned}$$

In the particular iid case, all $\tilde{W}$ are here equal to one, so that the recursion $1 = 1/(1-p) - p/(1-p)$ remains very stable (except for p close to 1). In the general case, we will see that there remain some numerical problems (see Section 3.1).

Theorem 2.4 (Simpliest proposed recursion) *Setting $W'(k,n) = W(k,n)/C_n^k$, one have for $n \geqslant 1$ and $k < n$:*

$$\begin{aligned}W'(n,n) &= \mu_n\,,\\ W'(k,n) &= W'(k,n-1) - W'(k+1,n)\,,\end{aligned}$$

So that we can build W' recursively, with $W'(0,0) = 1$. Waring coefficients are then given by $W(k,n) = \mathrm{C}_n^k W'(k,n)$.

Proof: This result is a direct consequence of Proposition 2.3 with $W_0(k,n) = \mathrm{C}_n^k$. Once written, we can check that in the particular finite-exchangeable case, it is also deriving directly from the fact that

$$\begin{aligned}W(k,n) &= \sum_{x_1+\cdots+x_n=k} \mathrm{P}\left[X_1 = x_1, \ldots, X_n = x_n\right]\\ &= \mathrm{C}_n^k \mathrm{P}\left[X_1 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_n = 0\right].\end{aligned}$$

If $W'(k,n) = \mathrm{P}\left[X_1 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_n = 0\right]$, then by the same kind of arguments as in Kendall (1967),

$$\begin{aligned}W'(k,n) = \;& \mathrm{P}\left[X_1 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_{n-1} = 0\right]\\ & - \mathrm{P}\left[X_1 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_{n-1} = 0, X_n = 1\right],\end{aligned}$$

And using exchangeability we get:

$$W'(k,n) = W'(k,n-1) - W'(k+1,n)\,,$$

□

Considering the complexity problem, computing all $W_\mu(k,n)$ has now a complexity of order n^2 rather than n^3, meaning the calculation of order n times faster, which may be important when n is large. Furthermore, all involved operations are very basic operations, directly available in the arithmetic coprocessor Floating Point Unit, allowing very low level programming of the recurrence.

Considering the numerical stability problem, all W' are probabilities, which is likely to help controlling the local errors. Nevertheless, we will see that numerical stability is not ensured, so that we may have to use a greater arithmetic precision than the 64-bit IEEE754 standard. One advantage of such a recurrence is that it involves only basic operations for which many arbitrary precision floating point libraries exist.

3 The residual problem

3.1 Local errors and propagation errors

In proposed recursions, there are essentially two kind of errors involving the floating point arithmetic precision:

- Local calculation errors, for a fixed couple (k, n).
- Propagation errors, linked to the propagation (and emphasis) of previous errors.

We present here some situations where local errors or where propagation errors may be involved. These errors will not be here precisely quantified, since we will see in a further section that it is not possible to eliminate these errors due to the sensitivity of the formula on initial parameters.

Local errors Local errors are those for which, starting with slightly modified values in the recurrence, for a given couple (k, n), one may get large errors on $W_\mu(k, n)$. Consider for example (see Remark 2.3)

$$\tilde{W}(k,n) = \frac{1}{1-p}\tilde{W}(k,n-1) - \frac{p}{1-p}\tilde{W}(k+1,n)\,,$$

due to the normalization method, the $\tilde{W}$ are here a ratio of two probabilities and may be huge. We can then obtain situations where a small quantity is equal to the difference of two huge quantities, causing a large local error. Another example of local error is given in Remark 2.2.

Propagation errors Propagation errors are due to the iteration process: starting with slightly modified values in the recurrence, for a given couple (k, n), one get relatively small errors on $W_\mu(k, n)$. But the error is due to the iteration process, which emphasis at each step the small error on $W_\mu(k, n)$.

Consider for example

$$W(k,n) = \frac{n}{n-k}W(k,n-1) - \frac{k+1}{n-k}W(k+1,n)\,,$$

If $W(k+1,n)$ is replaced by $W(k+1,n)+\epsilon$, then $W(k,n)$ will be translated by a quantity $\frac{k+1}{n-k}\epsilon$, that is for example $n\epsilon$ in the case $k=n-1$. If ϵ is small, $n\epsilon$ will be locally an acceptable error. Nevertheless, during the first steps the error will be multiplied at each step by a factor near n. For $n=100$, an initial error of 10^{-16} becomes too large after only ten iterations for example. The initial error may be smaller if μ_n is very small, but in the case where μ_n is too large, propagation errors may occur.

3.2 Numerical sensitivity

Let us start with the Waring formula,

$$W_\mu(k,m) = \mathbb{1}_{k\leq m}\mathrm{C}_m^k \sum_{j=0}^{m-k} \mathrm{C}_{m-k}^j (-1)^j \mu_{j+k},$$

and consider two set of possible initial coefficients $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$, differing slightly (due to the computer limited arithmetic precision for example). Will the quantities $W_\mu(k,m)$ and $W_{\tilde{\mu}}(k,m)$ be very different ?

Consider the particular value

$$W_\mu(m,2m) = \mathrm{C}_{2m}^m \sum_{j=0}^{m} \mathrm{C}_m^j (-1)^j \mu_{j+m}.$$

Consider for example that $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$ are differing only on one value:

$$\begin{aligned} \tilde{\mu}_k &= \mu_k, \quad k < 2m\,, \\ \tilde{\mu}_{2m} &= \mu_{2m} + \epsilon\,. \end{aligned}$$

After some basic calculations, and even supposing all calculations are done with an infinite precision, we can easily show that

$$W_{\tilde{\mu}}(m,2m) = W_\mu(m,2m) + (-1)^m \mathrm{C}_{2m}^m \epsilon\,.$$

Suppose for example that $\epsilon = \mu_{2m}10^{-16}$, due to the numerical uncertainty on the μ_{2m} quantity. The absolute error $\Delta = |W_{\tilde{\mu}}(m,2m) - W_\mu(m,2m)|$ is in this case:

$$\Delta = \mathrm{C}_{2m}^m \mu_{2m} 10^{-16}.$$

This error can be huge, and obviously greater than one, when m is large and μ_{2m} not enough close to zero. When $\mu_{2m} = p^{2m}$ (iid case), this can occur only when p is close to one, but in the general exchangeable case, this can occur when an important dependency structure among indicators $X_1, X_2, \ldots$ cause the higher moments μ_k to be important. Since all $W_\mu(k,m)$ are probabilities and should belong to $[0,1]$, situations where Δ is greater than one are not acceptable. In the exchangeable case, this reveals that in some particular cases, the final law of $X_1 + \cdots + X_m$ may be too highly sensitive on the input moments $\mu_k = \mathrm{E}\left[\Theta^k\right]$ of the underlying factor Θ.

4 Conclusion

We have investigated the calculation of the sum of indicators random variables $X_1+\cdots+X_m$, when they were not necessarily iid, based on the Waring formula.

We have seen some basic recursions which allow to compute very simply the Waring formula in the exchangeable case, and to reduce the complexity of the formula when computing the whole law of $X_1+\cdots+X_m$.

Despite their simplicity we have seen that all proposed recursion were facing either local arithmetic errors, either propagation errors, and that it was not possible to reduce these errors in all cases, since the underlying formula was in some cases too sensitive on some input parameters.

As a conclusion three solutions seem to us possible:

- Checking that input parameters are not leading to unacceptable errors. In particular, initial coefficients μ_n should be small enough for large n. Since initial moments of $X_1+\cdots+X_m$ are usually easy to get, one can a posteriori check that the law of $X_1+\cdots+X_m$ leads to the correct moments, including the sum of probability equals to one.
- Using a greater computer arithmetic floating point precision, which might be easy since all operations involved in the recursions are very basic operations, for which many arbitrary precision arithmetic exists. Investigations could also be developed in order to bound each quantity in the recursion, using interval arithmetic for example.
- Changing input parameters $\{\mu_k\}$, and using another formula in problematic cases. As an example, in the infinite-exchangeable framework with an underlying factor Θ, according to remark 1.3, numerical integration can be used to compute $W_\mu(k,m)$, without starting from $\{\mu_k\}$ coefficients. Another example is getting coefficients $\ln W_\mu(k,m)$ as a function of coefficients $\ln\mu_n$, which might also reduce the required floating point precision in some cases.

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