Strategic Substitutes and Potential Games

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Abstract

We show that games of strategic substitutes (or complements) with aggregation are "pseudo-potential" games, and therefore possess Nash equilibria in pure strategies. Our notion of aggregation is quite general and enables us to take a unified view of several disparate models.

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1 Introduction

Economic theory is replete with examples of what have come to be called, after Bulow et al (1985), games of strategic substitutes (STS) or strategic complements (STC). They cover phenomena ranging from[1] oligopolistic competition between firms, to the problem of the commons (Dasgupta and Heal (1979)), to macroeconomic coordination failures (Diamond (1992)), to new technology adoption (Katz and Shapiro (1986)), to bank runs (Diamond and Dybvig (1983)). The essential feature of these games is that, when his competitors turn more aggressive, an agent has incentive to become less so (for

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[1]This list is only meant to be indicative.

STS) or more so (for STC). A thorough investigation has already been made into STC games. For instance, the existence of Nash equilibrium[2] (NE) was proved, under very general conditions, in Milgrom and Roberts (1990) and Vives (1996). But a comparable study is lacking for STC games (except for the case of two players where, reversing the order of one player's strategies, any STS game is converted into an STC game). One goal of our paper is to begin to redress this imbalance.

General existence of NE in STS games is probably more difficult to establish than in STC games. A simple example might put the matter in perspective. Suppose each player has a finite subset of real numbers for his strategies, with higher numbers connoting more "aggression." Start with the profile of "lowest" strategies, and require all players to unilaterally deviate, whenever possible, to their best-replies to the profile. Iterate the process. The sequence of profiles so generated will be monotonically increasing in STC games. Thus it will converge to an NE in finitely many steps. No similar argument can be made apriori for STS games. It also does not seem feasible to transport the general techniques and insights developed for STC games to the study of STS games.

In this paper we focus attention on STS games which have one further property: the payoff of a player depends only upon his own strategy, and some kind of "market aggregate" of others' strategies. This property is quite common to many examples, including the most famous of all STS games: Cournot oligopoly. In the Cournot setting, and indeed in many others, it suffices to take the aggregate to be just the sum of agents' actions. However, when the strategic interaction between players is more complex, a broader concept of aggregation can often render the games amenable to our analysis. We motivate and develop such a concept in Section 5.

Once we have aggregation, a very striking thing occurs: in STS (or, for that matter, STC) games, players can be thought of as maximizing one *common* payoff function – the potential – in order to deviate to a best reply. Thus, in terms of the structure of best-reply correspondences (reaction functions), STS or STC games with aggregation are potential games[3] (see Theorem 2).

[2] Throughout, we confine ourselves to pure strategies; so NE will always mean "pure-strategy NE."

[3] Monderer and Shapley (1996) defined (ordinal) potential games with a more strigent requirement: *all* unilateral deviations – not just to best replies – must be rank-ordered by the potential. (See also Shapley (1990-).) For this reason, we have dubbed our games as "pseudo-potential" in the main text.

This has some important ramifications. First, NE always exist (see Theorem 1). This is so even for non-convex strategy sets, which are bound to arise when indivisibilities are present in the economic model. In particular, for "discrete Cournot", where each firm can produce finitely many levels of output, NE still exist. This remarkable result was proved by Shapley (1990-) for the linear Cournot model. It follows from our analysis in somewhat greater generality. Indeed we show that indivisibilities pose no problem for the existence of NE in STS games, once there is aggregation. (The analogous result for STC games follows from the work of Milgrom and Roberts (1990) and Vives (1996), even without any appeal to aggregation.) Furthermore, if the games have finite strategy sets then, for generic payoffs, sequential best-replies converge to NE (see Remark 1). This may have implications for devising algorithms to compute NE, though we do not pursue that line of inquiry here.

Our analysis also applies to the standard Cournot model with convex strategy sets and monotonic (possibly discontinuous) reaction functions, as in the scenario in Novshek (1985). His existence theorem is obtained as a byproduct of our analysis (see Remark 6). More generally, strategy sets can be a mix of discrete and continuous parts, without disturbing the existence of NE.

But our approach obviously goes far beyond Cournot. As was said, it brings together and unifies several disparate STS and STC games that have been analyzed in the literature. A decisive role is played here by the potential and by our general concept of aggregation.

The paper is organized as follows. In Section 2 we introduce the notion of STS games with simple (additive) aggregation, and state Theorem 1 on the existence of NE. Pseudo-potential games are introduced in Section 3. It is shown in Theorems 2 and 3 that pseudo-potential games include STS games, and always possess NE. Section 4 contains proofs of the theorems. We develop the concept of general (non-additive) aggregation in Section 5, and verify that our results remain intact. This implies in particular that our analysis can be carried over en toto to STC games (see Remark 3). Finally, in Section 6, we discuss how our approach extends to include discontinuous reaction functions, yielding the result of Novshek (1985) as a special case.

2 Strategic Substitutes with Aggregation

Consider a set of players $N = \{1, 2, ...n\}$. Each $i \in N$ has a set of pure strategies $S^i \subset R_+$, which is a finite union of closed, bounded, and w.l.o.g. disjoint intervals. (The intervals could all have zero length, in which case S^i is a finite set). Put $S \equiv S^1 \times ... \times S^n$. For any $s = (s^1, ..., s^n) \in S$ and $t \in S^i$, denote $(s^1, ..., s^{i-1}, t, s^{i+1}, ..., s^n)$ by $(s \mid_i t)$; $(s^1, ..., s^{i-1}, s^{i+1}, ..., s^n)$ by s_{-i}; and $\sum_{j \in N \setminus \{i\}} s^j$ by $\overline{s}_{-i}$. The payoff function $\pi^i : S \rightarrow R$ of player i depends only upon his own strategy s^i and the aggregate[4] $\overline{s}_{-i}$ of others' strategies. So, with a slight abuse of notation, we will write $\pi^i(s_i, \overline{s}_{-i})$ for $\pi^i(s)$, and view π^i as defined on the domain $S^i \times \overline{S}_{-i}$, where $\overline{S}_{-i} \equiv \sum_{j \in N \setminus \{i\}} S^j$.

For any choice $s_{-i} \in \prod_{j \in N \setminus \{i\}} S^j$ of others' strategies, the set $\beta^i(\overline{s}_{-i})$ of *best replies* of player i is given by

$$\beta^i(\overline{s}_{-i}) = \arg\max_{t \in S^i} \pi^i(t, \overline{s}_{-i}).$$

Recall that $s = (s^1, ..., s^n) \in S$ is a *Nash equilibrium* (NE) if

$$s^i \in \beta^i(\overline{s}_{-i})$$

for all $i \in N$.

Finally, let us recall the notion of strategic substitutes. We shall present it in a slightly more general form than is standard[5], allowing for multiplicity of best replies.

We say that $\Gamma = (N, S^1, ..., S^n, \pi^1, ..., \pi^n)$ is a game of *strategic substitutes* (STS) with aggregation if, for every $i \in N$, there exists a continuous[6] and nonincreasing best-reply selection $b^i : \overline{S}_{-i} \rightarrow S^i$, i.e.,

(i) $b^i(x) \in \beta^i(x)$ for all $x \in \overline{S}_{-i}$,

(ii) b^i is continuous on $\overline{S}_{-i}$,

and

(iii) $b^i(x) \leq b^i(y)$ whenever $x > y$.

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[4]For a more general notion of aggregation see Section 5.

[5]The term "strategic substitutes" was introduced by Bulow et al (1985) to refer to games in which the best reply functions of the players are downward slopping. (See also Fudenberg and Tirole (1986).) This property can, in turn, be derived from a more primitive submodularity assumption on payoffs in the game (see Milgrom and Roberts (1990)).

[6]If S^i is finite, the requirement of continuity is vacuous. For relaxations of continuity, see Section 6.

Our main result is

Theorem 1. *A game of strategic substitutes with aggregation has a Nash equilibrium.*

Let us immediately remark that, if we define strategic complements (STC) exactly as STS above, except for replacing "$x > y$" by "$x < y$" in (iii), then Theorem 1 will hold for STC games with aggregation, as indeed will the rest of our analysis. (See Remark 3 for an explanation). But since existence results are already known for STC games, we keep STS games in the forefront.

3 Pseudo-Potential Games

We develop the notion of a pseudo-potential game, which will be crucial for establishing Theorem 1.

Consider a game $\widetilde{\Gamma} = (N, S^1, ..., S^n, \widetilde{\pi}^1, ..., \widetilde{\pi}^n)$ in which the players and their strategy-sets are as before, but payoff functions $\widetilde{\pi}^i : S \to R$ are allowed to take a general form. We say that $\widetilde{\Gamma}$ is a *pseudo-potential game* if there exists a continuous function $P : S \to R$ such that, for all $i \in N$ and all $s \in S$,

$$\arg\max_{t \in S^i} \widetilde{\pi}^i(s \mid_i t) \supset \arg\max_{t \in S^i} P(s \mid_i t).$$

In other words, each player's best reply correspondence in the game $\Gamma^* = (N, S^1, ..., S^n, P, ..., P)$ is included in that of $\widetilde{\Gamma}$: it suffices for a player to maximize the potential P, rather than his real payoff $\widetilde{\pi}^i$, in order to get to *some* best reply. One may therefore think of the potential P as a convenient common proxy for all the different payoff functions $\widetilde{\pi}^i$, $i \in N$, in the analysis of NE of $\widetilde{\Gamma}$. (For, as is evident, NE of Γ^* are afortiori NE of $\widetilde{\Gamma}$.)

The following two results immediately imply Theorem 1, but may be of independent interest.

Theorem 2. *A game of strategic substitutes with aggregation is a pseudo-potential game.*

Theorem 3. *A pseudo-potential game has a pure strategy NE.*

Remark 1 (Generic Convergence of Sequential Best Replies) In the light of theorem 2, this remark is obvious, but seems worth putting on record. Consider an STS game with aggregation and with finite strategy sets. Then, for generic payoffs, all best reply correspondences will be single-valued. Assume this is the case. Start with an arbitrary strategy profile, and let each player, one at a time, unilaterally deviate to his unique best reply, if he does not happen to be there already. More precisely, let $\{s(t)\}_{t=1}^{\infty} = \{(s^1(t), ..., s^n(t))\}_{t=1}^{\infty}$ be a sequence of strategy-profiles with the following property. At every stage t, if $s^j(t-1) \neq \beta^j(\overline{s}_{-j}(t-1))$ for a nonempty set of players j, one of them (say i) is required to choose his best reply to $s_{-i}(t-1)$ at stage t, i.e., $s^i(t) = \beta^i(\overline{s}_{-i}(t-1))$; all the other players stay put at their previous strategies, i.e., $s_{-i}(t) = s_{-i}(t-1)$. Then the sequence becomes stationary in finite time, since the game is pseudo-potential, and since the monotone sequence $\{P(s(t))\}_{t=1}^{\infty}$ cannot have infinitely many strict increases on its finite domain S. The stationary profile is clearly an NE of $\widetilde{\Gamma}$.

4 Proofs

4.1 Proof of Theorem 2

Let $\Gamma = (N, S^1, ..., S^n, \pi^1, ..., \pi^n)$ be an STS game with aggregation; and, for all $i \in N$, let $b^i : \overline{S}_{-i} \to S^i$ be a continuous and nonincreasing best-reply selection.

Denote by Σ_{-i} the convex hull of $\overline{S}_{-i}$. We extend b^i, in a piecewise-linear fashion, to a function τ^i, defined on the entire domain Σ_{-i}. (Thus τ^i coincides with b^i on $\overline{S}_{-i}$, and if $x, y \in \overline{S}_{-i}$ are such that $(x, y) \subset \Sigma_{-i} \backslash \overline{S}_{-i}$, then τ^i is an affine function on $[x, y]$.) Furthermore, we enhance the domain of τ^i to include the interval $[0, \min \Sigma_{-i}]$, by setting: $\tau^i(0) = \max S^i$, $\tau^i(\min \Sigma_{-i}) = b^i(\min \Sigma_{-i})$, and then extending τ^i linearly on $(0, \min \Sigma_{-i})$. Figure 1 illustrates the construction of τ^i (the bold line represents b^i and the broken line represents the rest of τ^i; bold intervals and dots on the vertical and horizontal axes represent the sets S^i, $\overline{S}_{-i}$ respectively).

Insert Figure 1

Notice that τ^i inherits continuity, and the property of being nonincreasing, from b^i. For every $i \in N$, now define $F_i : S^i \to R$ by

$$F_i\left(s_i\right) = \int_0^{\max(\Sigma_{-i})} \min(\tau^i(x), s_i) dx.$$

Consider the continuous function[7] $P : S^1 \times ... \times S^n \to R$ given by

$$P\left(s^1, ..., s^n\right) = -\sum_{i<j} s^i s^j + \sum_i F_i\left(s_i\right). \qquad (1)$$

We claim that P renders Γ into a pseudo-potential game. To check this, fix $s \in S$, and suppose that $s^i \in \arg\max_{t \in S^i} P(s \mid_i t)$. Note that for any $t \in S^i$

$$P\left(s \mid_i t\right) = \left[-t\overline{s}_{-i} + F_i\left(t\right)\right] + \left[\sum_{\substack{j<k \\ j,k \neq i}} s^j s^k + \sum_{j \neq i} F_j\left(s_j\right)\right]. \qquad (2)$$

Since the second (bracketed) term in (2) is not a function of t, the first term is maximized at s^i (for the given s_{-i}). We will deduce from this that $s^i = \tau^i\left(\overline{s}_{-i}\right)$ $\left(= b^i\left(\overline{s}_{-i}\right)\right)$.

Note first that if $t \leq \tau^i\left(\overline{s}_{-i}\right)$, then the first term in (2) is equal to the area $A\left(t\right)$ of the region (shown shaded in Figure 2) which is bounded by: the graph of τ^i and the horizontal line $y = t$ from above,[8] the horizontal line $y = 0$ from below, the vertical line $x = \overline{s}_{-i}$ from the left, and the vertical line $x = \max \Sigma_{-i}$ from the right.

Insert Figure 2

[7]A function of this form first came to our attention in Huang (2002). He, however, defined it under more restrictive assumptions on best-reply functions, in the context of certain Cournot oligopoly games with convex strategy sets, in order to study properties of fictitious play.

[8]Where x and y denote the horizontal and vertical coordinates as usual.

On the other hand, if $t \geq \tau^i(\overline{s}_{-i})$ $(= b^i(\overline{s}_{-i}))$, then the first term in (2) is equal to the difference $A(b^i(\overline{s}_{-i})) - B(t)$, where $B(t)$ is the area of the region (shown shaded in Figure 3), which is bounded by: the graph of τ^i from below, the horizontal line $y = t$ from above, and the vertical line $x = \overline{s}_{-i}$ from the right.

Insert Figure 3

Since $A(t)$ is strictly increasing from $\min S^i$ to $\tau^i(\overline{s}_{-i})$, and since $B(t) > 0$ for $t > \tau^i(\overline{s}_{-i})$, we conclude that the first term in (2) has a unique maximum which is attained at $t = \tau^i(\overline{s}_{-i})$ $(= b^i(\overline{s}_{-i}))$, for the given s_{-i}. Thus,

$$\{s^i\} = \arg\max_{t \in \Sigma^i} P(s \mid_i t) = \{b^i(\overline{s}_{-i})\}.$$

Therefore b^i is indeed the best-reply (single-valued) correspondence of i in the game $(N, S^1, ..., S^n, P, ..., P)$. Since b^i, to begin with, was a selection from the best reply correspondence of $\Gamma = (N, S^1, ..., S^n, \pi^1, ..., \pi^n)$, we conclude that Γ is a pseudo-potential game.■

4.2 Proof of Theorem 3

Let $\widetilde{\Gamma} = (N, S^1, ..., S^n, \widetilde{\pi}^1, ..., \widetilde{\pi}^n)$ be a pseudo-potential game with potential P. Suppose $s = (s^1, ..., s^n) \in \arg\max_{(t^1,...,t^n) \in S} P(t^1, ..., t^n)$ (such an s exists because P is continuous and S is compact). If s is not an NE of $\Gamma^* = (N, S^1, ..., S^n, P, ..., P)$, then $P(s \mid_i t) > P(s)$ for some $t \in S^i$, contradicting that s maximizes P.

But any NE of Γ^* is afortiori an NE of $\widetilde{\Gamma}$, since best replies in Γ^* are by definition best replies in $\widetilde{\Gamma}$.■

5 Non-Additive Aggregation

Adding up players' strategies is but one way of aggregating them. It suits most of the examples we have cited, starting with Cournot oligopoly. However, there are also many kinds of strategic interaction which, at first glance, look alien to our framework. It is only when appropriate aggregators $\alpha : S_{-i} \to R$ are constructed for them, that their hidden structure is unmasked, and they fit into our framework, with $\overline{s}_{-i}$ replaced by $\alpha(s_{-i})$ and $\overline{S}_{-i}$ by $\alpha(S_{-i})$.

To define a general class of aggregators, denote

$$s^*_{-i}(k) \equiv \sum_{\substack{i_1<i_2<...<i_k \\ i_1,i_2,...,i_k \neq i}} s_{i_1}...s_{i_k}$$

for $1 \leq k \leq n-1$ (i.e., $s^*_{-i}(k)$ is the sum of all possible products of k distinct strategies picked from $s_{-i} = (s^1, ..., s^{i-1}, s^{i+1}, ..., s^n)$; and so for $k = 1$ we get $s^*_{-i}(1) = \overline{s}_{-i}$.) Let $a_1, ..., a_{n-1}$ be scalars, and define

$$\alpha(s_{-i}) \equiv \sum_{k=1}^{n-1} a_k s^*_{-i}(k).$$

For the moment assume, by way of simplicity, that the scalars a_k are such that $\alpha(s_{-i}) \geq 0$ for all $i \in N$ and all $s_{-i} \in S_{-i}$. (This restriction can be dropped, see Remark 2.) Notice that the aggregator $\alpha(s_{-i})$ is the *same* linear combination of $\left\{s^*_{-i}(k)\right\}_{k=1}^{n-1}$ for all $i \in N$.

Our aggregators are seemingly abstruse. We shall now give three examples to illustrate how they might arise in a natural manner. In all the examples, each player i chooses effort level $s^i \in [0, B^i]$ to apply to the personal task faced by him. This gives rise to the probability $p_i(s^i)$ of "success" in his task, where $p_i : [0, B^i] \to [0, 1]$ is a strictly increasing function. By relabeling effort levels if necessary, we take $B^i = 1$ and $p_i(s^i) = s^i$. The events of individual success are assumed to be independent across different players. Furthermore, for ease of calculation, we suppose there are three players ($N = \{1, 2, 3\}$), and that each i incurs quadratic cost $c_i(s^i)^2$, on account of his effort s^i, for some constant $c_i > 0$.

Example 1 (Team Projects with Complementary Tasks) Each player's task is critical to the success of the team's project. Thus $s^1s^2s^3$ is the probability that the project will succeed. Suppose $r_i > 0$ is the utility to player i of a successful project. This yields the payoff function

$$\pi^i\left(s^1, s^2, s^3\right) = r_i s^1 s^2 s^3 - c_i\left(s^i\right)^2 = r_i s^i \alpha(s_{-i}) - c_i\left(s^i\right)^2,$$

where $\alpha(s_{-i})$ is the aggregator $s^*_{-i}(2) = s^j s^k$ (where we have denoted $N \backslash \{i\} = \{j,k\}$). Then i's best reply is

$$\beta^i(\alpha(s_{-i})) = \left\{ \min\left(\frac{r_i \alpha(s_{-i})}{2c_i}, 1 \right) \right\},$$

which is a nondecreasing function of $\alpha(s_{-i})$; and shows that we have an STC game with aggregation (when $\overline{s}_{-i}$ is replaced by $\alpha(s_{-i})$).■

Example 2 (Team Projects with Substitutable Tasks) Here we suppose that each player by himself can make the project successful. Then the probability that the project is successful is

$$\begin{aligned} f\left(s^1, s^2, s^3\right) &= 1 - \left(1 - s^1\right)\left(1 - s^2\right)\left(1 - s^3\right) \\ &= s^1 + s^2 + s^3 - s^1 s^2 - s^1 s^3 - s^2 s^3 + s^1 s^2 s^3, \end{aligned}$$

and the payoff to player i is

$$\pi^i\left(s^1, s^2, s^3\right) = r_i f\left(s^1, s^2, s^3\right) - c_i \left(s^i\right)^2 = r_i s^i \left[1 - \alpha(s_{-i})\right] + \alpha(s_{-i}) - c_i \left(s^i\right)^2,$$

where $\alpha(s_{-i})$ is the aggregator $s^*_{-i}(1) - s^*_{-i}(2)$. Thus

$$\beta^i(\alpha(s_{-i})) = \left\{ \min\left(\frac{r_i \left[1 - \alpha(s_{-i})\right]}{2c_i}, 1 \right) \right\}$$

is a nonincreasing function of $\alpha(s_{-i})$, and so this example describes an STS game with aggregator α.■

Example 3 (Tournaments) Assume that a reward of r dollars is shared by the group of players who succeed. If only one player succeeds, he gets r for sure; if exactly two succeed, each gets r with probability $\frac{1}{2}$; if all three succeed, each gets r with probability $\frac{1}{3}$. By rescaling utilities, we may assume w.l.o.g. that r dollars yield r utiles to each player. Then the the expected value of the reward to i is

$$\begin{aligned} & r s^i \left(1 - s^j\right)\left(1 - s^k\right) + \frac{r}{2} s^i s^j \left(1 - s^k\right) + \frac{r}{2} s^i s^k \left(1 - s^j\right) + \frac{r}{3} s^i s^j s^k \\ &= r s^i \left[1 - \frac{1}{2} s^j - \frac{1}{2} s^k + \frac{1}{3} s^j s^k\right] = r s^i \left[1 - \alpha(s_{-i})\right], \end{aligned}$$

where $\alpha(s_{-i}) = \frac{1}{2}s^*_{-i}(1) - \frac{1}{3}s^*_{-i}(2)$. Therefore each player's payoff function is

$$\pi^i\left(s^1, s^2, s^3\right) = rs^i\left[1 - \alpha(s_{-i})\right] - c_i\left(s^i\right)^2.$$

Consequently,

$$\beta^i(\alpha(s_{-i})) = \left\{\min\left(\frac{r\left[1 - \alpha(s_{-i})\right]}{2c_i}, 1\right)\right\}$$

is a nonincreasing function of $\alpha(s_{-i})$, and therefore tournaments are also STS games with aggregator α.$\blacksquare$

As we said, our results remain intact if we postulate that the payoff to any player i depends only upon his own strategy s^i and the aggregate $\alpha(s_{-i})$ of others' strategies. Other than the obvious change of notation ($\overline{s}_{-i}$ replaced by $\alpha(s_{-i})$ and $\overline{S}_{-i}$ by $\alpha(S_{-i})$), the only variation needed is in the proof of Theorem 2. We redefine P (which was defined for the additive aggregator in (1)) as follows:

$$P\left(s^1, ..., s^n\right) = -\sum_{k=1}^{n-1} a_k \cdot \sum_{i_1<i_2<...<i_{k+1}} s_{i_1}...s_{i_{k+1}} + \sum_i F_i(s_i).$$

Note that for any $i \in N$,

$$P(s\,|_i\,t) = \left[-t\sum_{k=1}^{n-1} a_k s^*_{-i}(k) + F_i(t)\right] + \left[-\sum_{k=1}^{n-2} a_k s^*_{-i}(k+1) + \sum_{j\neq i} F_j(s_j)\right]$$

$$= \left[-t\alpha(s_{-i}) + F_i(t)\right] + \left[-\sum_{k=1}^{n-2} a_k s^*_{-i}(k+1) + \sum_{j\neq i} F_j(s_j)\right].$$

The above equality replaces (2) in the proof of Theorem 2, and the rest of the arguments hold exactly as before.

Remark 2 (Aggregation without the Positivity Requirement) The requirement that aggregation be nonnegative can be dropped. If $\alpha(s_{-i})$ is negative for some $s_{-i} \in S_{-i}$, we can define another aggregator $\widetilde{\alpha}$ by $\widetilde{\alpha}(s_{-i}) \equiv \alpha(s_{-i}) + a$ for all $i \in N$ and all $s_{-i} \in S_{-i}$. Clearly, for large enough a, $\widetilde{\alpha}(s_{-i})$ is always nonnegative. It is obvious that if i's payoff is a function of s_i and $\alpha(s_{-i})$, then it is representable also as a function of s_i and $\widetilde{\alpha}(s_{-i})$, and any nonincreasing and continuous best-reply selection remains such after this change of variables. Our analysis holds with these "non-homogeneous"aggregations just as well. One only has to add $s^*_{-i}(0) \equiv 1$ to the set $\left\{s^*_{-i}(k)\right\}_{k=1}^{n-1}$, and allow aggregators $\alpha(s_{-i})$ to be linear combinations of $\left\{s^*_{-i}(k)\right\}_{k=0}^{n-1}$, not just $\left\{s^*_{-i}(k)\right\}_{k=1}^{n-1}$.

Remark 3 (STC Games with Aggregation) The previous remark also enables us to include STC games with aggregation in our approach. Indeed, given an STC game with aggregator α, the payoff function of each player i can be obviously redefined to depend on s^i and $\widetilde{\alpha}(s_{-i}) \equiv -\alpha(s_{-i})$, instead of s^i and $\alpha(s_{-i})$. Consequently, if a best-reply selection b^i is a nondecreasing function of α (which is the case for STC), it turns into $\widetilde{b^i}(\widetilde{\alpha}) \equiv b^i(-\widetilde{\alpha})$, a nonincreasing function of $\widetilde{\alpha}$, and our analysis goes through by Remark 2. Note that this trick is purely technical, and does not change the STC character of the game: while $\widetilde{b^i}$ is a nonincreasing function of $\widetilde{\alpha}$, it remains nondecreasing in the underlying basic variable s_{-i}.

6 Discontinuous Best Reply Selections

We do not know if continuity of our best reply selections is necessary for the validity of Theorem 1. However, Theorem 1 stays intact even if discontinuities are allowed, provided *one* of the following assumptions is made:

(i) Γ is a game of *strict* strategic substitutes, i.e., for every $i \in N$ there exists a best reply selection b^i which is a strictly decreasing function[9] of $\overline{s}_{-i}$ (it does not have to be continuous);

(ii) for every $i \in N$ there exists a best reply selection b^i which is nonincreasing and right-continuous;

(iii) for every $i \in N$ there exists a best reply selection b^i which is nonincreasing and left-continuous.

[9] For ease of notation, we revert from $\alpha(s_{-i})$ to $\overline{s}_{-i}$ (though the argument holds replacing $\overline{s}_{-i}$ by $\alpha(s_{-i})$ throughout, provided we assume that $\alpha(s_{-i})$ is strictly increasing in s_{-i}).

To see this, construct τ^i and P exactly as in the proof of Theorem 2. P is continuous as before, but this time

$$\arg\max_{t\in S^i} P(s\mid_i t) = \left[\lim_{x\downarrow\overline{s}_{-i}} \tau^i(x), \lim_{x\uparrow\overline{s}_{-i}} \tau^i(x)\right] \cap S^i \tag{3}$$

for all $i \in N$ and all $s \in S$. In particular, $\arg\max_{t\in S^i} P(s\mid_i t)$ need not be single-valued, if τ^i is discontinuous at $\overline{s}_{-i}$. (This is why P may fail to be a pseudo-potential function for the given game.) However, it follows from (3) that

$$b^i(\overline{s}_{-i}) \in \arg\max_{t\in S^i} P(s\mid_i t), \tag{4}$$

as before.

Now consider some $s = (s^1, ..., s^n) \in \arg\max_{(t^1,...,t^n)\in S} P(t^1, ..., t^n)$. Suppose first that assumption (i) is satisfied for every b^i. Note that if there is $i \in N$ (say, $i = 1$) such that $s^1 \neq b^1(\overline{s}_{-1})$, then by (4), $s' \equiv (b^1(\overline{s}_{-1}), s^2, ..., s^n) \in \arg\max_{(t^1,...,t^n)\in S} P(t^1, ..., t^n)$. Obviously, since s and s' maximize P, $s^i \in \arg\max_{t\in S^i} P(s\mid_i t)$ and $(s')^i \in \arg\max_{t\in S^i} P(s'\mid_i t)$ for all $i \in N$. Then (3) implies

$$s^2 \in \left[\lim_{x\downarrow\overline{s}_{-2}} \tau^2(x), \lim_{x\uparrow\overline{s}_{-2}} \tau^2(x)\right] \cap \left[\lim_{x\downarrow\overline{s'}_{-2}} \tau^2(x), \lim_{x\uparrow\overline{s'}_{-2}} \tau^2(x)\right].$$

But clearly $\overline{s}_{-2} \neq \overline{s'}_{-2}$, and so, from the fact that τ^2 is strictly decreasing, the intersection of the above two intervals must be empty. This is a contradiction, so $s^i = b^i(\overline{s}_{-i})$ for all $i \in N$, and s is an NE of Γ.

Next suppose that assumption (ii) holds. Then, since every b^i (and τ^i) is nonincreasing and right-continuous, it follows from (3) that

$$b^i(\overline{s}_{-i}) = \min\left[\arg\max_{t\in S^i} P(s\mid_i t)\right] \tag{5}$$

for all $i \in N$. If (say) $s^1 \notin \arg\max_{t\in S^1} \pi^1(s^1, \overline{s}_{-1})$, then $b^1(\overline{s}_{-1}) < s^1$ by (5) and the fact that $s^1 \in \arg\max_{t\in S^1} P(s\mid_1 t)$. Thus for all $j \neq 1$

$$\overline{s'}_{-j} < \overline{s}_{-j}, \tag{6}$$

where (recall) $s' \equiv (b^1(\overline{s}_{-1}), s^2, ..., s^n) \in \arg\max_{(t^1,...,t^n)\in S} P(t^1, ..., t^n)$. Since

$$s^j \in \arg\max_{t\in S^j} P(s\mid_j t) \cap \arg\max_{t\in S^j} P(s'\mid_j t)$$

for all $j \neq 1$, the conjunction of (3), (6), and the fact that τ^j is nonincreasing, yields

$$s^j = \min \left[\arg\max_{t \in S^j} P(s' \mid_j t) \right]. \tag{7}$$

The right-hand side of (7) is equal to $b^j\left(\overline{s'}_{-j}\right)$ by (5). Thus $(s')^j = s^j = b^j\left(\overline{s'}_{-j}\right)$ for all $j \neq 1$. Since $(s')^1 = b^1\left(\overline{s}_{-1}\right) = b^1\left(\overline{s'}_{-1}\right)$ by definition, s' is an NE of Γ.

Finally, when assumption (iii) holds, the analysis is similar as for assumption (ii).

Remark 4 (Novshek's Existence Theorem for Cournot Oligopoly) If the best-reply correspondence of every player in Γ is nonempty-valued, upper hemi-continuous, and nonincreasing in the sense that[10] $\max \beta^i(x) \leq \min \beta^i(y)$ whenever $x > y$, then Γ satisfies both (ii) and (iii) above. Indeed,

$$b_r^i(x) = \min \beta^i(x) \text{ for all } x \in \overline{S}_{-i}$$

defines a nonincreasing best-reply selection which is right-continuous, and

$$b_l^i(x) = \max \beta^i(x) \text{ for all } x \in \overline{S}_{-i}$$

defines a nonincreasing best-reply selection which is left-continuous.

This observation can be quite useful. For instance, Novshek (1985) showed that, under quite general conditions, best-reply correspondences in Cournot oligopoly are nonempty, upper hemi-continuous and nonincreasing. Thus our analysis implies Novshek's result on the existence of NE.

[10] Note that $\max \beta^i(x)$ and $\min \beta^i(y)$ are well defined by upper hemi-continuity.

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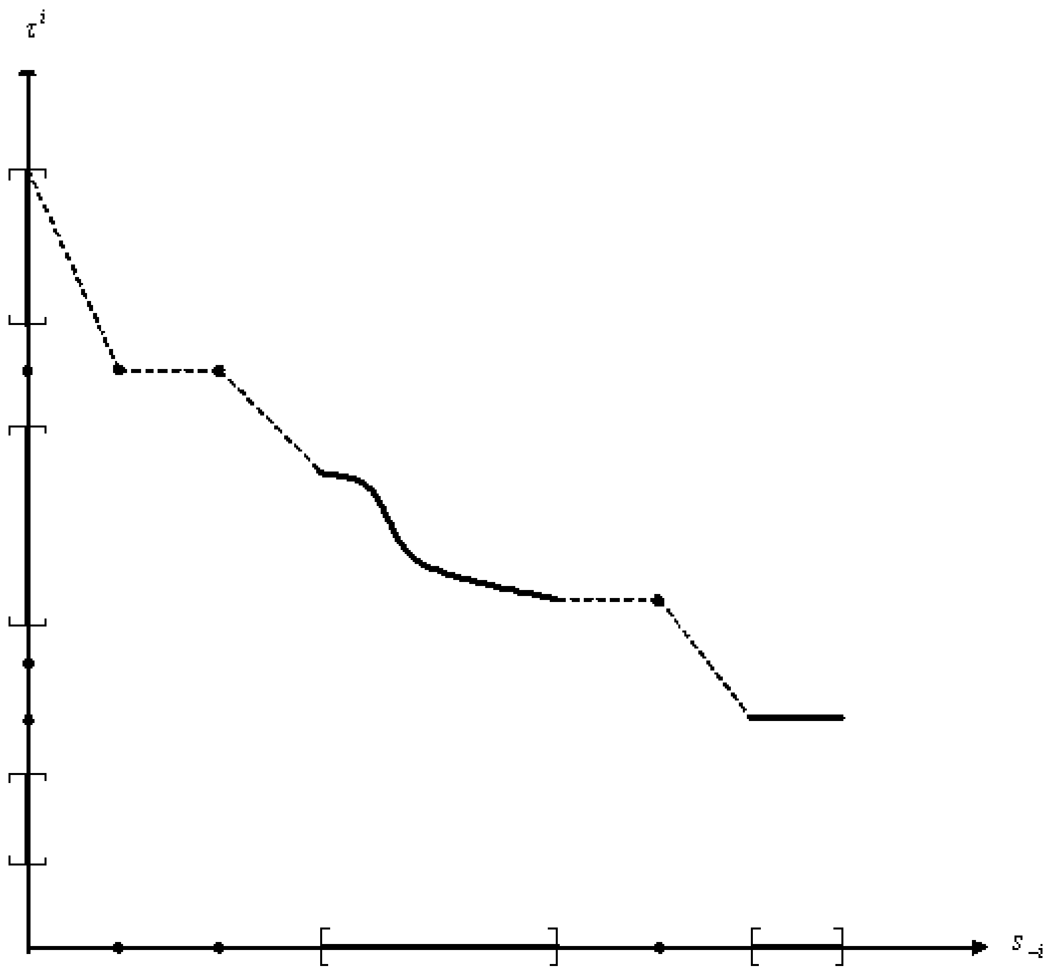

Figure 1:

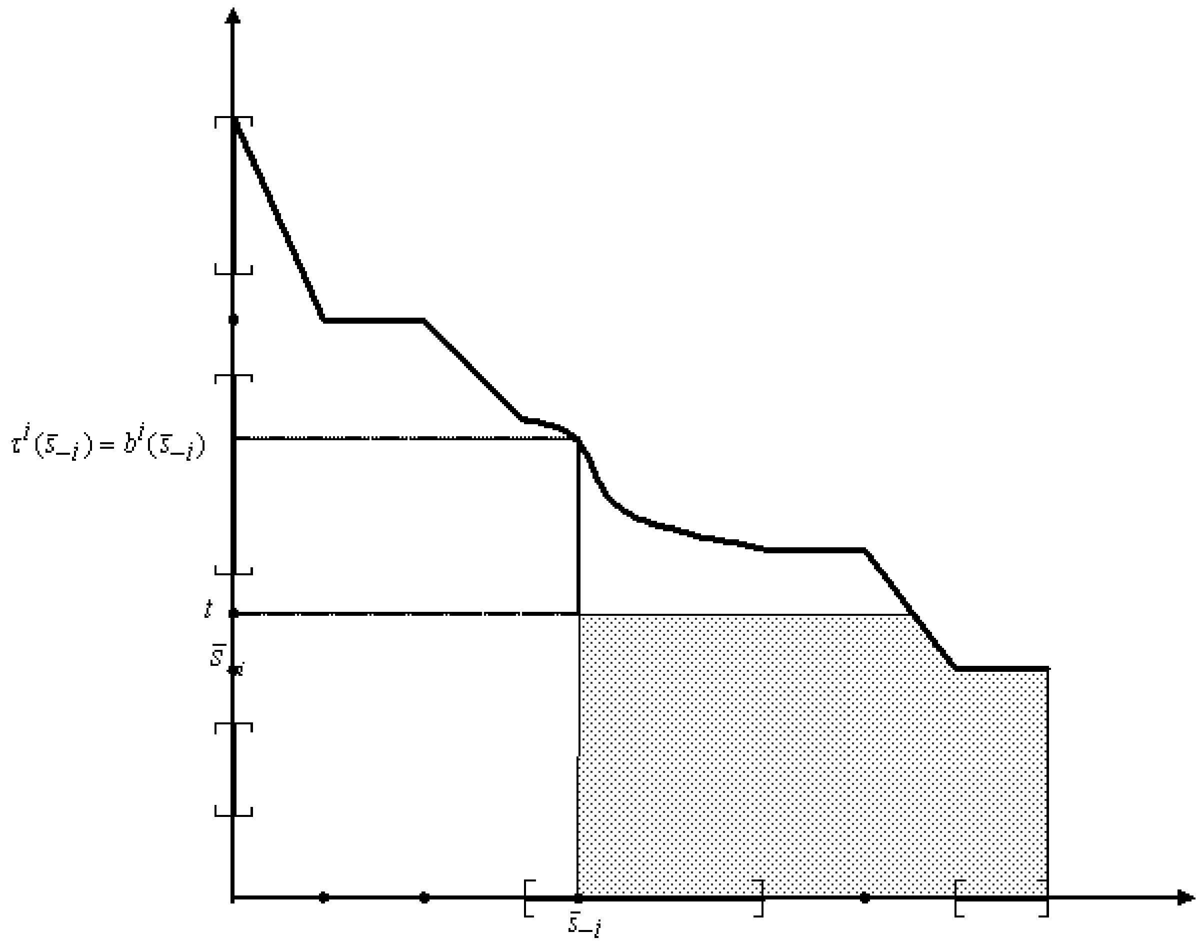

Figure 2:

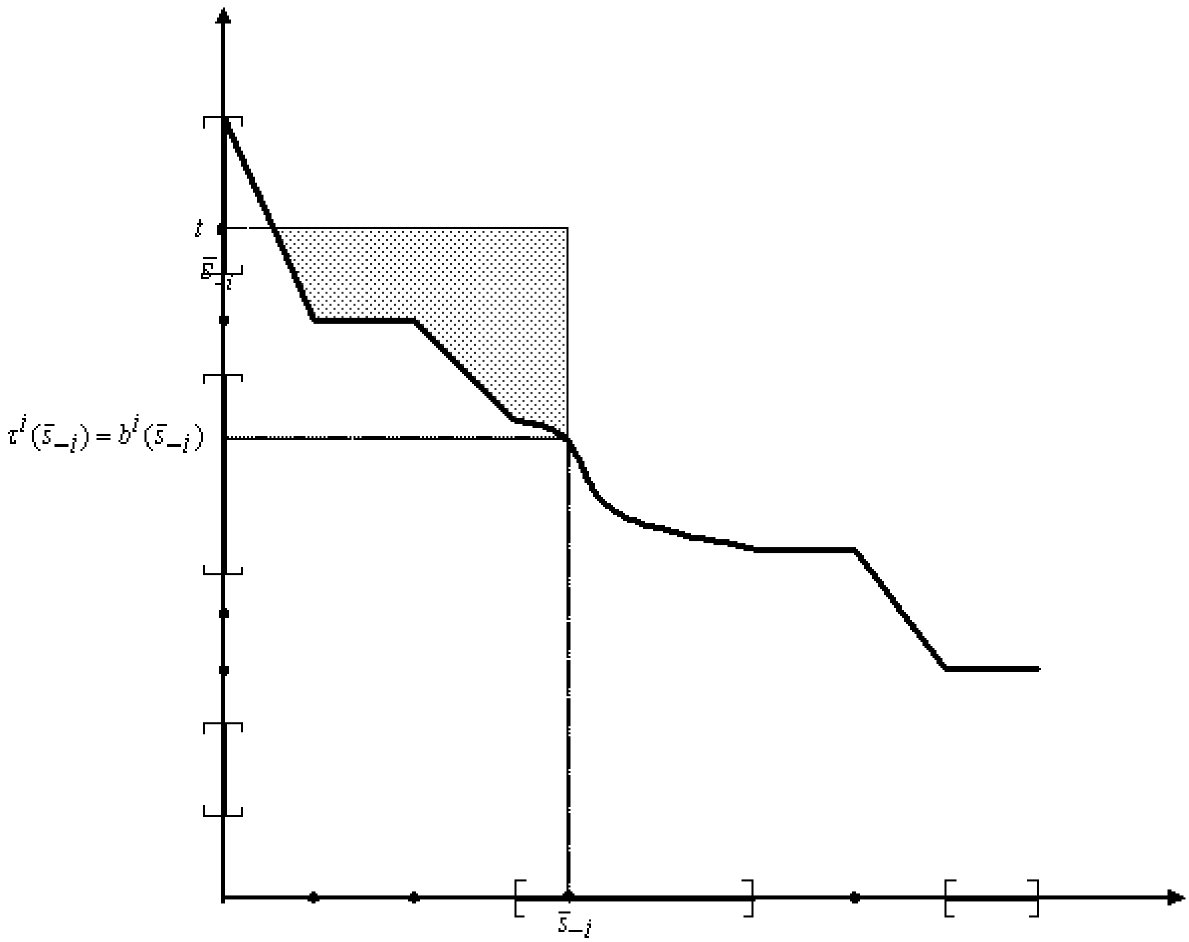

Figure 3: