A DYNAMIC ECONOMY WITH SHARES, FIAT, BANK AND ACCOUNTING MONEY.

J.J.M. Evers and M. Shubik

Research memorandum

TILBURG UNIVERSITY
DEPARTMENT OF ECONOMICS
Hogeschoollaan 225 Tilburg (Holland)

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1. INTRODUCTION.

This paper is aimed at exposition and modeling several of the extremely detailed but necessary aspects of a closed competitive economy without a terminal time point, i.e. an economy which is closed with regard to trade and competitors at any point of time but is an ∞-horizon economy or is open ended with respect to time.

Particular attention is paid to invariant competitive equilibria, or in other words: competitive equilibria which can repeat themselves over time.

A number of simple models are studied which have just enough ingredients to expose the meaning of a couple of crucial assumptions.

Our choice criterion concerning modeling the monetary institutions is quite rigorous based on the rule: minimize complexity while maintaining essential aspects of economic relevance.

The results concerning "accounting money" and "negotiable shares" may be considered as illustrations of more general results already obtained by Evers* and indicated by Shubik.**

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* Evers (1975).

** Shubik (1973).

Our approach is related to but different in emphasis from the considerable amount of work being done in what can be called "Temporary General Equilibrium Theory". A detailed survey of this work has been presented elsewhere by Grandmont.* We do not attempt to summarize this survey here but rather try to indicate where our approach is similar and where it differs.

We believe that many of the phenomena associated with money and financial institutions cannot be fully appreciated without a clear specification of the dynamic features of an economy in disequilibrium. Furthermore we believe that when both exogenous uncertainty and bank money are present in an economy even the specification of stationary equilibrium conditions involves details concerning the method of issue of bank money and the possibility of bankruptcy and even bank failure. In short the minimal description of the dynamics calls for a specification of rules which amount to a <u>Mathematical Institutional Economics</u>** as the rules which specify the limitations on process amount to a description of rudimentary financial instruments and institutions.

Because, in this paper we are primarily concerned with invariant equilibria and we rule out exogenous uncertainty we obscure many of the features of money and financial institutions which appear clearly only in disequilibrium. However even for a carefully defined stationary equilibrium far more detailed modeling is required than is usually used. This discrepancy is easily explained when we observe that in a stationary equilibrium much of the financial apparatus lies dormant and in effect "disappears" to the casual observer.

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* Grandmont (1975).
** Shubik (1975).

2. ON MONEY AND SHARES

2.1. On Three Types of Money.

In much of the literature and popular debate on monetary control "the amount of money" in the economy is frequently referred to. Before this can be meaningfully discussed we must specify what is meant by "money" and who creates it and how it is destroyed.

There are many shades of meaning and fine distinctions which can be made in measuring the "moneyness" of many different items in an economy. We offer a simplification into three classes which we define and discuss below.

(1) Accounting money = "inside" interpersonal money or instant trust. It includes clearing house operations where no bank or government money changes hands. It is generally interest free. It includes casual loans among friends; 30 day credits to purchasers; intra firm transfers, intra agency transfers. All trade where the exchange is an "on faith" crediting and debiting.

(2) Bank money = money issued by distinguished or special individuals. They can be "inside" or "outside" of the private sector. If they are inside then the rules for the spending of profits of the banking system must be specified.

A convention of use has bank money accepted in trade: i.e. even if trader i will not take j's accounting money he accepts from j a debt instrument on bank B.

It is important to note that bank money is bank debt. It may come into circulation when individual j exchanges claims with bank B. I.e. j gives B his note or "paper" (which may or may not be negotiable) and

B gives A its paper (usually in the form) of a drawing account or sometimes it may give cash.

(3) Government money = fiat money = "outside" money and is issued and controlled by the government. It includes coins and notes, often referred to as cash. It may also include an array of short term governmental debt instruments bearing various interest rates.

The full meaning of all the "monies" noted above can only be given by fully specifying their rules of operation, or laws.

2.2. On shares.

Shares, as they appear in our models are negotiable certificates of ownership. The details concering voting rights, dividend entitlements and so forth do make a considerable difference among these instruments and it is easy to construct instances where the very existence of any economic equilibrium depends upon the details of the specification of corporate law concerning voting rights.

Corporate shares are a part of the broader class of financial instruments which we may term as "ownership paper".* This includes for example, house deeds, automobile ownership paper and other evidences of ownership for durables. Features such as whether the item is owned singly or jointly and what are the conditions on the negotiability of the instrument must be specified in order to describe its use.

In this paper we make the same gross simplification as Arrow and Debreu** and others by ignoring the voting aspects of shares and assuming

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* Shubik (1975).

** Arrow and Debreu (1954).

that short term profits in the dynamic context are well defines and are paid out to stockholders in proportion to their shares.

3. THE PHYSICAL ECONOMIC ASPECTS OF THE MODELS.

In the remainder of this paper we work with a number of simple examples all of which have the same nonmonetary economic background. They differ only in their monetary and financial aspects. In this section the nonmonetary aspects of the models are described. We assume that economic activities take place at a sequence of "periods" with equal duration, numbered $t = 0,1,2,\dots$. The initial period is numbered 0. The moments of period changing are called "time-points". We refer to the time-points as "the start of period t", or "the end of period t". The total number of periods over which the activities take place is not specified. We cover this aspect by assuming an "infinite horizon".

There are two types of commodities: "labor" and a simple consumer good-say "wheat". Quantities of labor and wheat will be represented by non-negative scalars, which are sometimes endowed with a sub-index referring to a time-point.

In the model we have three agents: two "individuals" and one "firm". The activities of the individuals are characterized by consumption of wheat, supply of labor, and by financing of the firm. The latter will be specified later. For each period, firm's activities are characterized by taking inputs (i.e. labor and wheat) at the start of that period and transforming these into outputs (i.e. wheat) which become available at the end of that period. The productive process takes exactly one period.

We assume that only the firm is able to carry out production. Furthermore exchanges of commodities between agents takes place at the moment of period change.

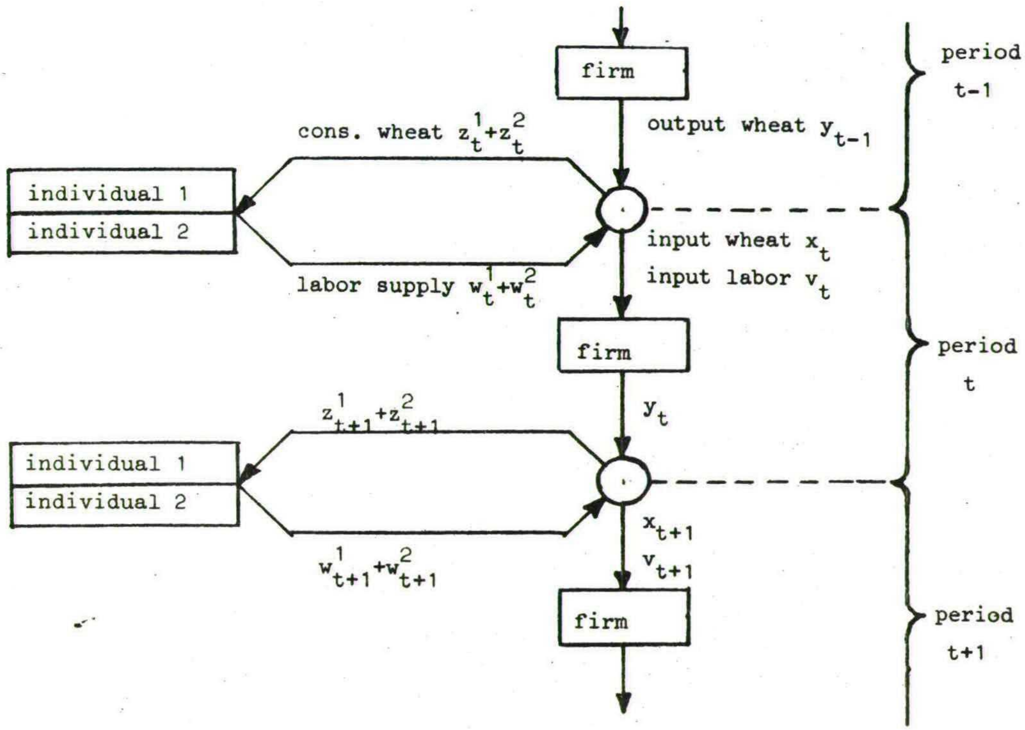

FIGURE 1: Flow of Commodities.

Under these assumptions, the flow of commodities may be represented by Figure 1. The action plans of the individuals are described sequences of scalars $\{(z_t^1,w_t^1)\}_{t=1}^{\infty}$, $\{(z_t^2,w_t^2)\}_{t=1}^{\infty}$, where z_t^i (i is 1 or 2) stands for the consumption of wheat at the beginning of period t by individual i, and where w_t^i represents its labor-supply at that time-point. Firm's action plan is described by the sequence $\{(x_t, v_t, y_t)\}_{t=1}^{\infty}$; where: x_t is the wheat-input at the beginning of t, v_t is the labor-input at the beginning of t, and where y_t is the output of wheat which becomes available at the end of t.

Under the assumption of a closed economy and of free disposal, the balance of goods is formulated by:

$$(3.1) \quad \left.\begin{array}{r} z_t^1 + z_t^2 + x_t \leqq y_{t-1} \\ v_t \leqq w_t^1 + w_t^2 \end{array}\right\} t = 1,2,\ldots,$$

where y_0 represents a given amount of output which is an initial condition (the result of production in a period prior to the start of this model).

Individual's consumption-labor supply possibilities are supposed invariant over time, and given by:

$$(3.2) \quad \left.\begin{array}{r} w_t^i \leqq \bar{w}^i \\ z_t^i, w_t^i \geqq 0 \end{array}\right\} \quad i = 1,2, \quad t = 1,2,\ldots, \text{ with } \bar{w}^1, \bar{w}^2 > 0.$$

For all periods, firm's production possibilities are represented by:

$$(3.3) \quad \left.\begin{array}{r} y_t \leqq f(x_t, v_t) \\ y_t, x_t, v_t \geqq 0 \end{array}\right\} t = 1,2,\ldots$$

For simplicity reasons, we here assume that the production function is a neo-classic.

A <u>path</u> $\{(z_t^1, w_t^1, z_t^2, w_t^2, x_t, v_t, y_t)\}_t^{\infty}$ of consumptions, labor supplies, and inputs-outputs will be called <u>feasible</u> if it satisfies the physical constraints (3.1), (3.2), and (3.3). Under the assumptions mentioned above, we have the following property:

<u>Property 3.4.:</u> For every initial state y_0, there is a number M such that every feasible path $\{(z_t^1, w_t^1, z_t^2, w_t^2, x_t, v_t, y_t)\}_{t=1}^{\infty}$ satisfies: $z_t^1, w_t^1, z_t^2, w_t^2, x_t, v_t, y_t \leqq M$, $t = 1,2,\ldots$ ([1] , th. 1.8.2.)

In that context $\{(z_t^i, w_t^i)\}_{t=1}^{\infty}$ is called a <u>feasible action plan of individual i</u>, if (3.2) is satisfied and if, in addition, this sequence is bounded. In a similar sense we shall use the term: <u>feasible action plan of the firm.</u>

In this study invariant paths
$(z_t^1, w_t^1, z_t^2, w_t^2, x_t, v_t, y_t) := (z^1, w^1, z^2, w^2, x, v, y)$, $t = 1,2,\ldots$
with initial state $y_0 := y$, take a central place. Clearly, in that context the physical conditions (3.1), (3.2), and (3.3) take the form:

$$
(3.5)\qquad
\begin{aligned}
z^1 + z^2 + x - y &\leqq 0 \\
v - w^1 - w^2 &\leqq 0 \\
w^i &\leqq \bar{w}^i, \ i = 1,2. \\
y &\leqq f(x,v) \\
z^1, z^2, w^1, w^2, x, v, y &\geqq 0
\end{aligned}
$$

Then, under the assumptions mentioned above, we have:

Property 3.6.: The solution set of (3.5) is bounded.*

To complete the "non-value" part of our model, we assume that individual's choice criterion can be expressed by:

$$(3.6) \qquad \sum_{t=1}^{\infty} (\pi_i)^t \cdot \varphi_i(z_t^i), \quad i = 1,2,$$

where $0 < \pi_i < 1$ is the time-discount factor of individual i, and where φ_i is his single-period utility function on (for simplicity reasons) "wheat"-consumption, only. We assume that these single period utility functions are continuous, concave, increasing, and finally: $\varphi_i(0) = 0$. Under these assumptions, boundedness of feasible consumption-supply paths implies that the infinite-horizon utility functions (3.6) are well-defined.

4. MODEL 1: ACCOUNTING MONEY AND NEGOTIABLE SHARES.

All expenditures and earnings of the agents are expressed in units of values; i.e. as products of prices and quantities, representing only a bookkeeping reality. In addition, prices and dividends (which are defined later) constitute the only information, concerning the system as a whole, agents use by use by choosing their action plans. The prices of "wheat" and "labor" at the beginning of a period t are denoted by non-negative numbers p_t and q_t, respectively.

We assume that, at the end of each period, the firm supplies its total outputs to the commodity market. Next, the inputs with respect to the succeeding period are completely financed by the individuals. Consequently, the yields of the outputs at the end of that period - say

*) Evers, J.J.M., (1975)

period t - are distributed among the individuals in the same proportion as each of them contibutes in financing the inputs at the start of period t. These contributions, from now on to be called shares, will be represented by a sequence of non-negative scalars $\{s_t^i\}_{t=0}^{\infty}$, i = 1,2, where s_t^i stands for the contribution of the i^{th} individual at the beginning of period t. Now, given prices and shares, the budget constraints of the firm is formulated: $p_t \cdot x_t + q_t \cdot v_t \leqq s_t^1 + s_t^2$, t = 1,2,... and, consequnetly, his economic behavior is characterized by a sequence of programs:

$$
(4.1) \qquad \left.\begin{array}{l} \max\ p_{t+1} \cdot y_t : \text{ over } x_t,\ y_t,\ v_t \geqq 0 \\ \text{subject to: } y_t \leqq f(x_t,\ v_t) \\ \qquad\qquad p_t \cdot x_t + q_t \cdot v_t \leqq s_t^1 + s_t^2 \end{array}\right\} t = 1,2,\ldots
$$

Denoting optimal solutions by sequences $\{(\hat{x}_t,\ \hat{v}_t,\ \hat{y}_t)\}_{t=1}^{\infty}$ (provided they exist), one can interpret a sequence $\{d_t\}_{t=1}^{\infty}$, satisfying

$$
(4.2) \qquad p_{t+1} \cdot \hat{y}_t = d_{t+1} \cdot (s_t^1 + s_t^2), \quad t = 0,1,\ldots,
$$

as a sequence of <u>dividend-factors</u> or as liquidating dividends.
With this definition, the liquidating values which become available to the individuals at the end of each period t, can be expressed by $d_{t+1} \cdot s_t^i$, i = 1,2. In order to cover the case of $s_t^1 + s_t^2 = 0$ for some period t, the definition must be refined *.
However, in this particular example the simplifying assumptions allow us to ignore the zero-budget case of the firm.

Focusing our attention to invariant prices and shares $(p_t,\ q_t,\ s_t^1,\ s_t^2) := (p,\ q,\ s^1,\ s^2)$, t = 1,2,..., the corresponding

*) Evers, J.J.M., (1975)

economic behavior of the firm can be expressed by:

(4.3) $\max p.y$, over $x,y,v \geqq 0$,

subject to: $y \leqq f(x,\mathbf{y})$, $p.x + q.v \leqq s^1 + s^2$.

The corresponding dividend-factor d has to satisfy:

(4.4) $p.\hat{y} = d.(s^1 + s^2)$,

provided $\hat{y}$ is optimal.

With respect to the budget constraints of the individuals, the effect of buying shares, and earning the profits one period later, is expressed as follows:

(4.5) $p_t.z_t^i - q_t.w_t^i + s_t^i - d_t.s_{t-1}^i \leqq 0$, $t = 1,2,\ldots,$

where individuals income is obtained from the sale of his labor-supply and the receipt of liquidating dividends $(q_t.w_t + d_t.s_{t-1}^i)$ and where expenditures consist of consumption and buying new shares $(p_t.z_t^i + s_t^i)$. Thus, given the prices of "wheat" and "labor" $\{(p_t,q_t)\}_{t=1}^{\infty}$, and given the dividend-factors $\{d_t\}_{t=1}^{\infty}$, the economic behavior of the individuals is characterized by:

(4.6) $\max \sum_{t=1}^{\infty} (\pi_i)^t.\varphi_i(z_t^i)$, over z_t^i, w_t^i, $s_t^i \geqq 0$, $t = 1,2,\ldots,$

subject to: $w_t^i \leqq \bar{w}_t^i$, $p_t.z_t^i - q_t.w_t^i + s_t^i - d_t.s_{t-1}^i \leqq 0$, $t = 1,2,\ldots,$

where the initial shares s_0^i are the given result of the past.

In the case of invariant prices and dividends:

$(p_t,q_t,d_t) := (p,q,d)$, $t = 1,2,\ldots$, these programs take the form:

$$(4.7)\qquad \max \sum_{t=1}^{\infty} (\pi_i)^t.\varphi_i(z_t^i),\ \text{over } z_t^i,w_t^i,s_t^i \geqq 0,\ t = 1,2,\ldots,$$

$$\text{subject to: } w_t^i \leqq \bar{w}^i,\ p.z_t^i - q.w_t^i + s_t^i - d.s_{t-1}^i \leqq 0,\ t = 1,2,\ldots$$

In connection with the total balance goods (3.1) we mentioned already that we may restrict ourselves to bounded action plans. For invariant prices this implies that firm's demand for shares is bounded, as well. Thus, without loss of generality we may limit ourselves to bounded action plans $\{(z_t^i,w_t^i,s_t^i)\}_{t=1}^{\infty}$, $i = 1,2$, i.e. to action plans subject to:

$$(4.8)\qquad z_t^i,w_t^i \leqq N_1,\ t = 1,2,\ldots,\ i = 1,2,$$

$$(4.9)\qquad s_t^i \leqq N_2,\ t = 1,2,\ldots,\ i = 1,2,$$

provided the constants N_1, N_2 are chosen large enough.

For invariant optimal action plans, it appears that the ∞-horizon decision processes, described by (4.7), (4.8), and (4.9), can be reduced to the following single-period decision processes:

$$(4.10)\qquad \left.\begin{array}{l} \max \varphi_i(z^i),\ \text{over } z^i,w^i,s^i \geqq 0,\ \text{subject to:} \\ w^i \leqq \bar{w}^i,\ p.z^i-q.w^i + (1-\pi_i.d).s^i \leqq (1-\pi_i).d.s_0^i \end{array}\right\} i = 1,2.$$

More precisely, under general assumptions (satisfied in our model), we have the following properties:

<u>Proposition 4.11.:</u> If, for any initial amount of shares s_0^i, $\{(z_t^i,w_t^i,s_t^i)\}_{t=1}^{\infty}$

is feasible with respect to (4.7), (4.8), (4.9) then, for the same initial shares, $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i) := ((1-\pi_i)/\pi_i).\Sigma_{t=1}^{\infty} (\pi_i)^t.(z^i, w^i, s^i)$ is a feasible solution of (4.10). In addition we have:
$\varphi_i(\tilde{z}^i) \geqq ((1-\pi_i)/\pi_i).\Sigma_{t=1}^{\infty}(\pi_i)^t.\varphi_i(z_t^i)$. ([1] , th. 3.4.2. and 3.4.4.).

Proposition 4.12.: If, for any initial amount of shares s_0^i, $(\hat{z}^i, \hat{w}^i, \hat{s}^i)$ is optimal with respect to the single-period program (4.10) such that $\hat{s}^i = s_0^i$, then $(z_t^i, w_t^i, s_t^i) := (\hat{z}^i, \hat{w}^i, \hat{s}^i)$, t = 1,2,... is an optimal solution for the ∞-horizon program defined by (4.7), (4.8), (4.9), provided $s_0^i := \hat{s}^i$. ([1] , th. 3.4.5.)

Proposition 4.11 states that every feasible solution of the ∞-horizon problem can be identified with a feasible solution of the corresponding single-period program.

Proposition 4.12 says that invariant optimal ∞-horizon action plans can be found as optimal solutions of the single period program by chosing appropriate initial shares. We observe that the opposite is not stated; i.e. an optimal ∞-horizon action plan which is invariant does not necessarily generate an optimal action with respect to the corresponding single-period program.

However, the properties mentioned above ensure that the "best" invariant ∞-horizon action plans will be selected by the single-period programs with appropriate initial shares. For that reason we adopt the single-period programs as the adequate description of individual's economic behavior under invariant prices and dividends. Obviously, the most important advantageof the single period approach is

that the influence of prices and dividend-factors on individual's invariant optimal action plans can be read off very easily.
Describing an individual's economic behavior in a single-period decision process is possibly more realistic than assuming multi-period (or even ∞-horizon) decision procesess. For the latter implicitely is based on the assumption that individuals possess, and actually use, price information over the whole time-horizon.

Now, starting from the economic behavior of individuals and the firm as described above, we define an <u>invariant competitive equilibrium</u> (briefly <u>I.C.E.</u>) for this model, as a combination of invariant prices, dividend-factors, and invariant action plans
$(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i)\}_{i=1}^2, (\hat{x},\hat{v},\hat{y}))$ with $\hat{s}^1+\hat{s}^2 > 0$ such that:

(a) $(\hat{z}^i,\hat{w}^i,\hat{s}^i)$, $i = 1,2$, is optimal with respect to the single-period programs (4.10), with $(p,q,d,s_0^i) := (\hat{p},\hat{q},\hat{d},\hat{s}^i)$

(b) $(\hat{x},\hat{v},\hat{y})$ is optimal with respect to firm's single-period program (4.3), with $(p,q,s^1,s^2) := (\hat{p},\hat{q},\hat{s}^1,\hat{s}^2)$.

(c) The dividend-factor $\hat{d}$ satisfies $\hat{p}.\hat{y} = \hat{d}.(\hat{s}^1 + \hat{s}^2)$

(d) Total demand and total supply of "wheat" and "labor" are equal; i.e. $\hat{z}^1 + \hat{z}^2 + \hat{x} = \hat{y}$, $\hat{w}^1 + \hat{w}^2 = \hat{v}$.

Under general assumptions,covering our model, it can be shown that such an competitive equilibrium exists.*

Considering individual's single-period decision process (4.10) in the context of the I.C.E. conditions (a) and (d) the following properties can be deduced as necessary conditions for the existence of optimal action

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*) Evers, J.J.M., (1975)

plans or as necessary conditions for optimality feasible actions:

Proposition 4.13: $\hat{p} > 0$. Argumentation: individuals utility function is increasing. With $\hat{p} = 0$, the individuals always are able to increase their utility by increasing their consumption.

Proposition 4.14: $\hat{d} \leqq 1/\pi_i$. Argumentation: with $\pi_i.\hat{d} > 1$, individuals always are able to increase their utility by increasing their amount of shares and their consumption.

Proposition 4.15: $\pi_i.\hat{d} < 1$ implies $\hat{s}^i = 0$. To be deduced as a necessary condition for optimality, under $\hat{p} > 0$.

Proposition 4.16: Defining $\pi^* := \max(\pi_1,\pi_2)$, $\hat{s}^1 + \hat{s}^2 > 0$ implies $\hat{d} = 1/\pi^*$. Direct consequence of 4.14 and 4.15.

Proposition 4.17: $\hat{q} > 0$ implies $\hat{w}^1 = \bar{w}^1$, $\hat{w}^2 = \bar{w}^2$. To be deduced as a necessary condition for optimality, under $\hat{p} > 0$.

Proposition 4.18: $\hat{p}.\hat{z}^i - \hat{q}.\hat{w}^i + (1-\pi_i.\hat{d}).\hat{s}^i = (1-\pi_i).\hat{d}.\hat{s}^i$, $i = 1,2$, To be deduced as a necessary condition for optimality, under $\hat{p} > 0$.

In a similar manner firm's single-period decision process (4.3) gives rise to be following properties concerning an I.C.E. $(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i)\}_{i=1}^2, (\hat{x},\hat{v},\hat{y}))$:

Proposition 4.19.: $\hat{s}^1 + \hat{s}^2 > 0$ and $\hat{p} > 0$ imply: $\hat{q} > 0$. To be deduced as a necessary condition for the existence of an optimal solution.

Proposition 4.20.: $\hat{p} > 0$, $\hat{q} > 0$ implies: $\hat{p}.\hat{x} + \hat{q}.\hat{v} = \hat{s}^1 + \hat{s}^2$, $\hat{y} = f(\hat{x},\hat{v})$. To be deduced as a necessary condition for optimality.

In the numerical example the role of these properties is illustrated. Further, the definition of an I.C.E. implies the following homogenity property:

Proposition 4.21.: If $(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i)\}_{i=1}^2, (\hat{x},\hat{v},\hat{y}))$ is an I.C.E. then, for every $\lambda > 0$, $(\lambda.\hat{p},\lambda.\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\lambda.\hat{s}^i)\}_{i=1}^2, (\hat{x},\hat{v},\hat{y}))$ is an I.C.E., as well.

Turning our attention to the underlying dynamic character of an I.C.E. $(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i)\}_{i=1}^2, (\hat{x},\hat{v},\hat{y}))$, the relations between the ∞-horizon decision processes and the single-period programs imply:

Proposition 4.22.: $(z_t^i, w_t^i, s_t^i) := (\hat{z}^i, \hat{w}^i, \hat{s}^i)$, $t = 1,2,\ldots$, is optimal with respect to the ∞-horizon program defined by (4.6), (4.8), and (4.9) with $(p_t, q_t, d_t) := (\hat{p}, \hat{q}, \hat{d})$, $t = 1,2,\ldots$ and $s_0^i := \hat{s}^i$.

Proposition 4.23.: $(x_t, v_t, y_t) := (\hat{x}, \hat{v}, \hat{y})$, $t = 1,2,\ldots$ is optimal with respect to the sequence of the programs (4.1) with $(p_t, q_t, s_t^1, s_t^2) := (\hat{p}, \hat{q}, \hat{s}^1, \hat{s}^2)$, $t = 1,2,\ldots$

Now consider, for any sequence of positive numbers $\{\delta_t\}_{t=0}^{\infty}$ with $\delta_0 := 1$, a price-system $(p_t,q_t) := \delta_t.(\hat{p},\hat{q})$, $t = 1,2,\ldots$ Then the structure of the ∞-horizon decision processes implies the following properties with respect to the I.C.E.:

Proposition 4.24.: $(z_t^i,w_t^i,s_t^i) := (\hat{z}^i,\hat{w}^i,\delta_t.\hat{s}^i)$, $t = 1,2,\ldots$ is optimal with respect to the ∞-horizon program (4.6), with

$(p_t, q_t, d_t) := (\delta_t.\hat{p}, \delta_t.\hat{q}, (\delta_t/\delta_{t-1}).\hat{d}))$, $t = 1,2,\ldots$, and $s_0^i = \hat{s}^i$.

Proposition 4.25.: $(x_t, v_t, y_t) := (\hat{x}, \hat{v}, \hat{y})$, $t = 1,2,\ldots$ is optimal with respect to (4.1) with $(p_t, q_t, s_t^1, s_t^2) := \delta_t.(\hat{p}, \hat{q}, \hat{s}^1, \hat{s}^2)$, $t = 1,2,\ldots$
In addition, the sequence of dividend-factors defined by
$d_t := (\delta_t/\delta_{t-1}).\hat{d}$, $t = 1,2,\ldots$ (viz. 4.24) satisfies the relation:
$p_{t+1}.y_t = d_{t+1}.(s_t^1 + s_t^2)$, $t = 0,1,\ldots$, with $y_0 := \hat{y}$, $s_0^1 + s_0^2 = \hat{s}^1 + \hat{s}^2$.
([1], th. 5.1.3.)

Interpreting the sequence of positive numbers $\{\delta_t\}_{t=0}^{\infty}$ as inflation or deflation ratios, it should be clear that these statements can be taken as: "the physical part of an I.C.E. is independent with respect to any degree of inflation or deflation".

A next topic in the dynamic context of an I.C.E. is the question of Pareto efficiency. Given an initial state y_0, we introduce two different optimality criteria:

Definition strict efficiency: 4.26.: A feasible path $\{(z_t^1, w_t^1, z_t^2, w_t^2, x_t, v_t, y_t)\}_{t=1}^{\infty}$ is called strictly efficient if no feasible path $\{(\tilde{z}_t^1, \tilde{w}_t^1, \tilde{z}_t^2, \tilde{w}_t^2, \tilde{x}_t, \tilde{v}_t, \tilde{y}_t)\}_{t=1}^{\infty}$ exists such that
$\Sigma_{t=1}^{\infty} (\pi_i)^t.\varphi_i(\tilde{z}_t^i) \geqq \Sigma_{t=1}^{\infty} (\pi_i)^t.\varphi_i(z_t^i)$, $i = 1,2$, with strict inequality for at least one i.

Definition weak efficiency 4.27.: In this concept, the optimality criterion of 4.26 is replaced by: $\varphi_i(\tilde{z}_t^i) \geqq \varphi_i(z_t^i)$, $i = 1,2$, $t = 1,2,\ldots$, with strict inequality for at least one pair (i,t).

Clearly, strict efficiency is based on a complete ordering over the periods and weak efficiency on a partial ordering. Evidently, strict efficiency implies weak efficiency.

Under much more general assumption than imposed on our model it can be shown that every invariant path generated by the physical part of an I.C.E. is weakly efficient. If, in addition, the time-discount factor are equal (i.e. $\pi_1 = \pi_2$) then such a path is strictly efficient. ([1], th. 5.2.4.)

5. Model I with a Cobb-Douglas production function.

Restricting ourselves to the case where $\hat{s}^1 + \hat{s}^2 > 0$, and assuming that $\pi_1 \geqq \pi_2$, we summarize the properties 4.13 to 4.20 concerning an I.C.E. $(\hat{p}, \hat{q}, \hat{d}, \{(\hat{z}^i, \hat{w}^i, \hat{s}^i)\}_{i=1}^2, (\hat{x}, \hat{v}, \hat{y}))$:

(1) $\hat{p} > 0$, $\hat{q} > 0$.

(2) $\hat{d} = 1/\pi_1$.

(3) $\hat{w}^1 = \bar{w}^1$, $\hat{w}^2 = \bar{w}^2$.

(4) $\hat{v} = \bar{w}^1 + \bar{w}^2$.

(5) $\hat{y} = f(\hat{x},\hat{v})$.

(6) $1/\hat{d}.\ \hat{p}.\hat{y} = \hat{p}.\hat{x} + \hat{q}.\hat{v} = \hat{s}^1 + \hat{s}^2$

Now, we consider the following maximization problem:

(5.1) $\max \hat{p}.f(x,y)$ over $x,y \geqq 0$, s.t. $\hat{p}.x + \hat{q}.y \leqq \hat{s}^1 + \hat{s}^2$.

Using Lagrange multiplier technics, one can deduce the following necessary condition for $(\hat{x},\hat{v})$ to be optimal w.r.t. (5.1):

(5.2) $f_x(\hat{x},\hat{v})/f_v(\hat{x},\hat{v}) = \hat{p}/\hat{q}$, where f_x and f_v are the partial derivatives of f(x,y) with respect to x and v resp.

Further, by the relations (5) and (6) we have:

(5.3) $\quad (1./\hat{d}).\hat{p}.f(\hat{x},\hat{v}) = \hat{p}.\hat{x} + \hat{q}.\hat{v}.$

Defining:

(5.4) $\quad \tilde{x} := \hat{x}/\hat{v}, \qquad \tilde{p} := \hat{p}/\hat{q},$

and using the linear homogenity property of the non-classical production function, (5.3) and (5.4) can be reduced to:

(5.5) $$\begin{cases} f_x(\tilde{x},1)/f_v(\tilde{x},1) = \tilde{p} \\ (1/\hat{d}).\tilde{p}.f(\tilde{x},1) = \tilde{p}.\tilde{x} + 1. \end{cases}$$

(Note, the relations (1) and (4) imply that $\tilde{x}$ and $\tilde{v}$ are well defined).

With the help of system (5.3) it is possible to express $\tilde{x}$ and $\tilde{p}$ as a function of $\hat{d}$. To be specific, let us assume that f(x,v) is a Cobb-Douglas production function of the form $\rho.x^u.v^\nu$, with $\rho,\mu,\nu \geqq 0$, $\mu + \nu = 1$, Then (5.3) implies the relation $\tilde{p}.\tilde{x} = \mu/\nu$ and $(1/\hat{d}).\tilde{p}.x = (\mu/\nu)+1$ which can be reduced to:

(5.6) $$\begin{cases} \tilde{p} = (\mu/\nu).(\mu.\rho/\hat{d})^{-1/\nu} \\ \tilde{x} = (\mu.\rho/\hat{d})^{1/\nu}. \end{cases}$$

From (5.4), (5.6), and from the relations $\hat{y} = \rho.\hat{x}^\mu.\hat{v}^\nu$ and $\hat{s}^1 + \hat{s}^2 = \hat{p}.\hat{x} + \hat{q}.\hat{v}$, one can deduce:

$$
(5.7)\quad
\begin{aligned}
\hat{p} &= (\mu/\nu).(\mu.\rho/\hat{d})^{-1/\nu}.\hat{q} \\
\hat{x} &= (\mu.\rho/\hat{d})^{1/\nu}.\hat{v} \\
\hat{y} &= \rho.(\mu.\rho/\hat{d})^{\mu/\nu}.\hat{v} \\
\hat{s}^1 + \hat{s}^2 &= (1/\nu).\hat{q}.\hat{v} \\
\hat{p}.\hat{x} &= (\mu/\nu).\hat{q}.\hat{v} \\
\hat{p}.\hat{y} &= (\hat{d}/\nu).\hat{q}.\hat{v}.
\end{aligned}
$$

Further, defining $\omega_i := \bar{w}^i/(\bar{w}^1+\bar{w}^2)$, $\hat{\gamma}_i := \hat{s}^i/(\hat{s}^i+\hat{s}^2)$, the relations $\hat{v} = \bar{w}^1+\bar{w}^1$, $\hat{d} = 1/\pi_1$, $\hat{s}^1+\hat{s}^2 = (1/\beta).\hat{q}.\hat{v}$, and property 4.18 imply:

$$
(5.8)\qquad \hat{z}^i/\hat{v} = (\rho.\pi_1)^{1/\nu}.\mu^{\mu/\nu}.[\,\nu.\omega_i+(\frac{1}{\pi_i}-1).\hat{\gamma}_i] , \quad i = 1,2.
$$

One may verify that $\hat{z}^1 + \hat{z}^2 + \hat{x} = \hat{y}$, implying that the total demand of "wheat" equals total supply (viz. equilibrium condition d).

With respect to share holding, we distinguish two cases: (1) $\pi_2 < \pi_1$, and (2) $\pi_2 = \pi_1$. In both cases we have $\hat{d} = 1/\pi_1$.

In the case that $\pi_2 < \pi_1$, we have $\pi_2.\hat{d} < 1$, implying (by 4.15) that $\hat{s}^2 = 0$. Consequently, $\hat{s}^1 = \hat{q}.\hat{v}/\nu$.

In the case that $\pi_2 = \pi_1$ we have $\pi_2.\hat{d} = 1$, $\pi_1.\hat{d} = 1$, implying that every share distribution $\hat{s}^1$, $\hat{s}^2$ is compatible with an I.C.E., provided $\hat{s}^1 + \hat{s}^2 = \hat{q}.\hat{v}/\nu$.

6. Model II-a: Fiat money and negotiable shares.

Starting from the same physical structure as described in § 3, we now assume that all payments have to be carried out with the help of a legal means of payment, to be called "fiat-money". Fiat money is characterized by the following assumptions: (1) The value of one unit is one. (2) It cannot be produced, it is not subject to attrition, agents can not destroy it, (3) Stock holding of fiat money or "hoarding" is permitted, (4) In accordance with the assumption that the exchange of commodities takes place at the moments of period change, we assume that all payments take place at these time-points in such a manner that, at each time-point, all transactions must be covered completely by payments in fiat money.

The order of transaction and payments can be specified in several ways. We shall study three different cases; in all of them we assume that the order of payments is invariant over the time-points. Our first approach is represented by the following diagram:

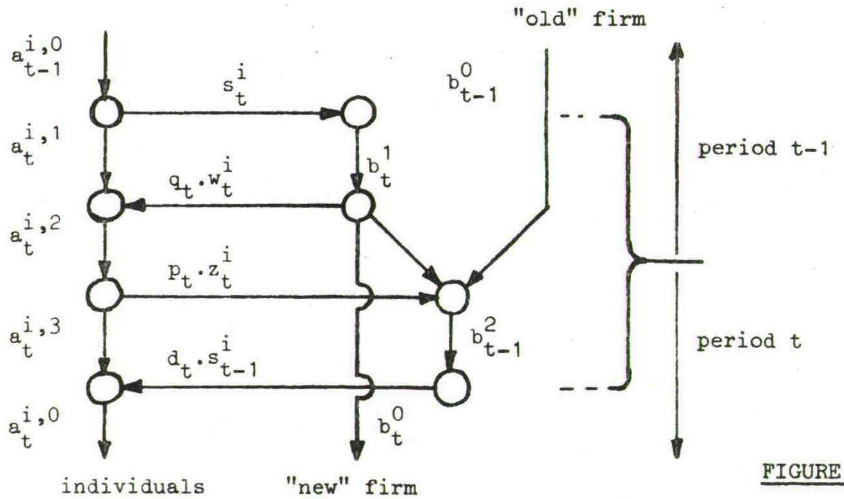

FIGURE 2.

In this, the amounts of fiat money owned by individual i at the different stages of transactions and payments at the end of period t-1 are represented by non-negative reals $a_t^{i,1}$, $a_t^{i,2}$, $a_t^{i,3}$. The amounts of fiat money he owns during period t-1 and period t are expressed by $a_{t-1}^{i,0}$ and $a_t^{i,0}$ resp.

Concerning the firm, the diagram is based on the assumption that the life time of the firm is exactly one period; i.e. the firm acting during a period t has to be estabilshed at the beginning of t and has to be liquidated at the end of that period. Since in the diagram, the "new" firm buys its "wheat" input from the "old" firm, the "new" firm must be established just before the liquidation point of the "old" firm. The amounts of money, owned by the firm acting over period t, is denoted $b_t^0, b_t^1, b_t^2 \geqq 0$. Further, the money streams between individuals and firms are represented by the horizontal arrows.

With these assumptions the amounts of fiat money held by individuals and firms during the periods has to satisfy:

$$(6.1) \qquad a_t^{1,0} + a_t^{2,0} + b_t^0 \leqq a_{t-1}^{1,0} + a_{t-1}^{2,0} + b_{t-1}^0, \quad t = 1,2,\ldots,$$

where $a_0^{1,0}$, $a_0^{2,0}$, b_0^0 are the given initial amounts of fiat money.

Now, the budget constraints of the individuals (see (4.5)) are replaced by the following balances of payments:

$$(6.2)\quad \left.\begin{aligned} a_t^{i,1} - a_{t-1}^{i,0} + s_t^i &\leqq 0\\ a_t^{i,2} - a_t^{i,1} - q_t.w_t^i &\leqq 0\\ a_t^{i,3} - a_t^{i,2} + p_t.z_t^i &\leqq 0\\ a_t^{i,0} - a_t^{i,3} - d_t.s_{t-1}^i &\leqq 0 \end{aligned}\right\}\quad t = 1,2,\ldots$$

Firm's balances of payments can be summarized by:

$$(6.3)\quad \left.\begin{aligned} p_t.x_t + q_t.v_t + b_t^0 &\leqq s_t^1 + s_t^2\\ d_{t+1}.(s_t^1 + s_t^2) &= p_{t+1}.y_t + b_t^0 \end{aligned}\right\}\quad t = 1,2,\ldots$$

With $v_t \leqq w_t^1 + w_t^2$, $z_t + x_t \leqq y_{t-1}$, $t = 1,2,\ldots$, the relations (6.2) and (6.3) imply (6.1).

Starting from invariant prices (p,q) and an invariant dividend factor d, an individual's economic behavior is characterized by:

$$(6.4)\quad \max \sum_{t=1}^{\infty} (\pi_i)^t.\varphi_i(z_t^i),\ \text{over } z_t^i,\ w_t^i,\ s_t^i,\ a_t^{i,0} \geqq 0,\ t = 1,2,\ldots$$

$$\text{subject to:}\quad \left.\begin{aligned} w_t^i &\leqq \bar{w}^i\\ s_t^i - a_{t-1}^{i,0} &\leqq 0\\ p.z_t^i - q.w_t^i + s_t^i - a_{t-1}^{i,0} &\leqq 0\\ a_t^{i,0} + p.z_t^i - q.w_t^i + s_t^i - d.s_{t-1}^i - a_{t-1}^{i,0} &\leqq 0 \end{aligned}\right\}\quad t = 1,2,\ldots,$$

where the initial amount of money and shares, $a_0^{i,0}$ and s_0^i resp., are the given result of the initial period $t := 0$. Further, proposition

(3.4) and the inequalities (6.1) imply that we may restrict ourselves to bounded action plans; i.e. to action plans which satisfy:

$$(6.5) \qquad z_t^i, w_t^i \leqq N_1, \qquad t = 1,2,\ldots,$$

$$(6.6) \qquad s_t^i \leqq N_2, \qquad t = 1,2,\ldots,$$

$$(6.7) \qquad a_t^{i,0} \leqq N_3, \qquad i = 1,2, \quad t = 1,2,\ldots,$$

provided the constants N_1, N_2, N_3 are chosen large enough.

For invariant optimal action plans it can be shown that the ∞-horizon decision processes, defined by (6.4) to (6.7) can be reduced to the single-period decision processes:

$$(6.8) \qquad \max \varphi_i(z^i), \text{ over } z^i,\ w^i,\ s^i,\ a^{i,0} \geqq 0$$

$$\left.\begin{aligned} \text{subject to:} \qquad w^i &\leqq \bar{w}^i \\ s^i - \pi_i . a^{i,0} &\leqq (1-\pi_i).a_0^{i,0} \\ p.z^i - q.w^i + s^i - \pi_i . a^{i,0} &\leqq (1-\pi_i).a_0^{i,0} \\ (1-\pi_i).a^{i,0} + p.z^i - q.w^i + (1-\pi_i.d).s^i &\leqq (1-\pi_i).(d.s_0^i + a_0^{i,0}) \end{aligned}\right\}$$

Analogous to proposition 4.12, we have:

<u>Proposition 6.9.:</u> If, for any initial $(s_0^i, a_0^{i,0})$, $(\hat{z}^i, \hat{w}^i, \hat{s}^i, \hat{a}^{i,0})$ is optimal with respect to (6.8) such that $(\hat{s}^i, \hat{a}^{i,0}) = (s_0^i, a_0^{i,0})$, then $(z_t^i, w_t^i, s_t^i, a_t^{i,0}) := (\hat{z}^i, \hat{w}^i, \hat{s}^i, \hat{a}^{i,0})$, $t = 1,2,\ldots$ is optimal with respect to the ∞-horizon program defined by (6.4) to (6.7), provided $(s_0^i, a_0^{i,0}) := (\hat{s}^i, \hat{a}^{i,0})$ and provided the bounds appearing in (6.5) to

(6.7) are chosen large enough.

Briefly: invariant optimal ∞-horizon action plans can be found by solving (6.8) with appropriate initial states $(s_0^i, a_0^{i,0})$.

The simplicity of max. problem (6.8) allows us to deduce the following properties:

Proposition 6.10.: The following conditions are necessary for max. problem (6.8) in order to possess an optimal solution:

(1) $p > 0$.

(2) $d \leqq (1/\pi_i)^2$.

Proposition 6.11.: If, for some $(p.q.d.s_0^i, a_0^{i,0})$ with $p > 0$, $q > 0$, and with $d \leqq (1/\pi_i)^2$, an action $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i, \tilde{a}^{i,0})$ is optimal for (6.8), then:

(1) $p.\tilde{z}^i - q.\tilde{w}^i + (1-\pi_i.d).\tilde{s}^i + (1-\pi_i).\tilde{a}^{i,0} = (1-\pi_i).(d.s_0^i + a_0^{i,0})$.

(2) $\tilde{w}^i = \bar{w}^i$.

(3) $d < (1/\pi_i)^2$ implies: $\tilde{s}^i = 0$.

Proposition 6.12.: Consider max. problem (6.8) with $p > 0$, $q > 0$, $d \leqq (1/\pi_i)^2$. For such a max. problem, an action $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i, \tilde{a}^{i,0})$ satisfying $\tilde{s}^i = s_0^i$, $\tilde{a}^{i,0} = a_0^{i,0}$, is optimal if and only if:
$(1-(\pi_i)^2.d).s_0^i = 0$, $a_0^{i,0} = d.s_0^i$, $\tilde{w}^i = \bar{w}^i$, $p.\tilde{z}^i = q.\bar{w}^i + (d-1).\bar{s}_0^i$.

Turning our attention to the economic behavior of the firm, the possibility of hoarding of money gives rise to the following max. problems:

$$\left.\begin{aligned} &\max\ p_{t+1}\cdot y_t + b_t^0, \quad \text{over} \quad x_t, v_t, y_t, b_t^0 \geqq 0 \\ (6.13) \quad &\text{subject to: } y_t \leqq f(x_t, v_t) \\ &\qquad p_t\cdot x_t + q_t\cdot v_t + b_t \leqq s_t^1 + s_t^2. \end{aligned}\right\} \quad t = 1,2,\ldots$$

Consequently, the dividend-factors $\{d_t\}_1^\infty$ have to satisfy:

(6.14) $\quad p_{t+1}\cdot\tilde{y}_t + \tilde{b}_t^0 = d_{t+1}\cdot(s_t^1 + s_t^2), \quad t = 0,1,2,\ldots,$

provided $\{(y_t, b_t^0)\}_0^\infty$ is a part in a sequence of optimal solutions $\{(x_t, v_t, y_t, b_t^0)\}_0^\infty$.

With invariant prices (p,q) we obtain the max. problem:

$$\begin{aligned} &\max\ p.y + b^0, \quad \text{over} \quad x,v,y,b^0 \geqq 0, \\ (6.15) \quad &\left.\begin{aligned} &\text{subject to: } y \leqq f(x,v), \\ &p.x + q.v + b^0 \leqq s^1 + s^2. \end{aligned}\right\} \end{aligned}$$

Evidently, we have the following properties:

Proposition 6.16.: If $p > 0$, $s^1 + s^2 > 0$, then a necessary condition for max. problem (6.14) in order to possess an optimal solution is: $q > 0$. (Implied by the assumption that f is neo-classic).

Proposition 6.17.: If, for some $p > 0$, $q > 0$, $s^1 + s^2 > 0$, the action $(\tilde{x}, \tilde{v}, \tilde{y}, \tilde{b}^0)$ is optimal with respect to (6.14), then:

(1) $\quad p.\tilde{x} + q.\tilde{v} + \tilde{b}^0 = s^1 + s^2$

(2) $\quad \tilde{y} = f(\tilde{x}, \tilde{v})$

(3) $\quad p.\tilde{y} + \tilde{b}^0 > s^1 + s^2$ implies: $\tilde{b}^0 = 0$.

Now, let $M > 0$ be the initial amount of fiat money in this economy (i.e. $M := a_0^{1,0} + a_0^{2,0} + b_0^0$). Then, starting from the economic behavior of the individuals and the firm (viz. 6.8 and 6.15 resp.) we define an <u>invariant competitive equilibrium</u> for this model as a combination $(\hat{p},\hat{q},\hat{d}, \{(\hat{z}^i,\hat{w}^i,\hat{s}^i,\hat{a}^{i,0})\}_{i=1}^2, (\hat{x},\hat{v},\hat{y},\hat{b}^0))$, with $\hat{s}^1 + \hat{s}^2 > 0$, such that, simultaneously:

(a) For each individual i, $(\hat{z}^i,\hat{w}^i,\hat{s}^i,\hat{a}^{i,0})$ is optimal with respect to (6.8) with $(p,q,d,s_0^i,a_0^{i,0}) := (\hat{p},\hat{q},\hat{d},\hat{s}^i,\hat{a}^{i,0})$.

(b) $(\hat{x},\hat{v},\hat{y},\hat{b}^0)$ is optimal with respect to (6.15) with $(p,q,s^1,s^2) := (\hat{p},\hat{q},\hat{s}^1,\hat{s}^2)$.

(c) The dividend-factor $\hat{d}$ satisfies $\hat{p}.\hat{y} + \hat{b}^0 = \hat{d}.(\hat{s}^1+\hat{s}^2)$.

(d) Total demand and total supply of "wheat" and "labor" are equal; i.e. $\hat{z}^1 + \hat{z}^2 + \hat{x} = \hat{y}$, $\hat{w}^1 + \hat{w}^2 = \hat{v}$.

(e) The total amount of fiat money hoard by the agents is equal to the initial amount of fiat money; i.e. $\hat{a}^{1,0} + \hat{a}^{2,0} + \hat{b}^0 = M$.

By virtue of the properties 6.10, 6.11, 6.12, 6.16 and 6.17, and by virtue of the equilibrium conditions, one can deduce:

<u>Proposition 6.17.:</u> If $(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i,\hat{a}^{i,0})\}_{i=1}^2, (\hat{x},\hat{v},\hat{y},\hat{b}^0))$ is an I.C.E. with $\hat{s}^1+\hat{s}^2 > 0$, then:

(1) $\hat{p} > 0$, $\hat{q} > 0$, $\hat{w}^1 = \bar{w}^1$, $\hat{w}^2 = \bar{w}^2$.

(2) $(1/\pi_i)^2 > \hat{d}$ implies: $\hat{s}^i = 0$, $\hat{a}^{i,0} = 0$.

(3) Defining $\pi^* := \max(\pi_1,\pi_2)$, we have $\hat{d} = (1/\pi^*)^2$.

(4) $\hat{p}.\hat{z}^i - q.\hat{w}^i - (\hat{d}-1).\hat{s}^i = 0$, $\hat{a}^{i,0} = \hat{d}.\hat{s}^i$, $i = 1,2$.

(5) $\hat{b}^0 = 0$, $\hat{p}.\hat{x} + \hat{q}.\hat{v} = \hat{s}^1+\hat{s}^2$, $\hat{a}^1+\hat{a}^2 = M$.

(6) $\hat{y} = f(\hat{x},\hat{v})$.

(7) $(\pi^*)^2.\hat{p}.\hat{q} = \hat{s}^1+\hat{s}^2$, where $\pi^* := \max(\pi_1,\pi_2)$.

By virtue of 6.17 and 6.18, it is possible to identify invariant competitive equilibria of model I with invariant competitive equilibria of this model with fiat money. More preciesly, we compare this model with fiat money specified by the quantities $(\pi_1, \pi_2, \bar{w}^1, \bar{w}^2, M)$, the utility functions φ_1, φ_2, and by the production function f, with model I where the time-discount factors are modified such that $\tilde{\pi}_1 := (\pi_1)^2$, $\tilde{\pi}_2 := (\pi_2)^2$. Then we have the following relation:

Proposition 6.19.: $(\hat{p}, \hat{q}, \hat{d}, \{(\hat{z}^i, \hat{w}^i, \hat{s}^i)\}_1^2, (\hat{x}, \hat{v}, \hat{y}))$, $\hat{s}^1 + \hat{s}^2$ being positive, is an I.C.E. of model I with time-discount factors $(\tilde{\pi}_1, \tilde{\pi}_2)$ as defined above, if and only if, for $\lambda := M/(\hat{d}.(\hat{s}^1 + \hat{s}^2))$, for $\hat{b}^0 := 0$ and for $(\hat{a}^{1,0}, \hat{a}^{2,0}) := (\hat{d}.\hat{s}^1, \hat{d}.\hat{s}^2)$, the combination $(\lambda.\hat{p}, \lambda.\hat{q}, \hat{d}, \{(\hat{z}^i, \hat{w}^i, \lambda.\hat{s}^i, \lambda.\hat{a}^{i,0})\}_1^2, (\hat{x}, \hat{v}, \hat{y}, \hat{b}^0))$ is an I.C.E. for the model described in this section.

Clearly, replacing the time-discount factors (π_1, π_2) appearing in section 5 by $(\pi_1)^2$, $(\pi_2)^2$, the result of this section are fully applicable on the model with fiat money. We observe that the effect on the time-discount factors is cause by the fact that the profits on shares can be effectuated two periods (instead of one in model I) after the point of investment.

Further, it should be clear (viz. 6.18-(4)) that the property concerning inflation, as described in 4.25, is not valid for this model. Finally we observe that infinite horizon action plans generated by this I.C.E. are <u>not</u> Pareto efficient (viz. definition 4.26 and 4.27). Actually, a counter example may be constructed in the setting of § 5.

7. Model II-b: Fiat money and negotiable shares.

In our second model concerning fiat money, the order of transactions and payments is represented by the following diagram:

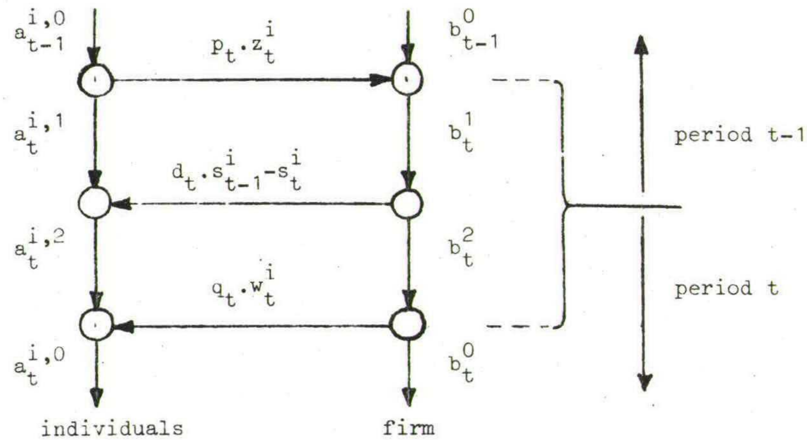

FIGURE 3.

Again the amounts of fiat money owned by the individuals and the firm are represented by $\{(a_t^{i,0}, a_t^{i,1}, a_t^{i,2})\}_{t=1}^{\infty}$, i = 1,2, and $\{(b_t^0, b_t^1, b_t^2)\}_{t=1}^{\infty}$ resp. The initial state is given by $(a_0^{1,0}, a_0^{2,0}, b_0^0)$. In this scheme, the exchange of shares and dividends takes place simultaneously, implying a "on going" character of the firm. Further, we maintain all assumptions concerning fiat money.

With invariant prices (p,q) and an invariant dividend factor d, we now arrive at the following characterization of individual's economic behavior:

$$(7.1) \quad \max \sum_{t=1}^{\infty} (\pi_i)^t.\varphi_i(z_t^i), \text{ over } z_t^i, w_t^i, s_t^i, a_t^{i,0} \geqq 0,\ t = 1,2,\ldots$$

$$\left.\begin{array}{r} w_t^i \leqq \bar{w}^i \\ p.z_t^i - a_{t-1}^{i,0} \leqq 0 \\ p.z_t^i + s_t^i - d.s_{t-1}^i - a_{t-1}^{i,0} \leqq 0 \\ p.z_t^i - q.w_t^i + s_t^i - d.s_{t-1}^i + a_t^{i,0} - a_{t-1}^{i,0} \leqq 0 \end{array}\right\} \quad t = 1,2,\ldots$$

where $(s_0^{i,0}, a_0^{i,0})$ is the initial state. As discussed earlier, we may restrict ourselves to action plans which satisfy:

(7.2) $z_t^i,\ w_t^i \leqq N_1,\ t = 1,2,\ldots,$

(7.3) $s_t^i \leqq N_2,\ t = 1,2,\ldots,$

(7.4) $a_t^{i,0} \leqq N_3,\ t = 1,2,\ldots,$

provided the constraints are chosen large enough.

In the same manner as described in proposition 6.9., invariant optimal action plans can be found by the following single-period decision process:

$$(7.5) \quad \max \varphi_i(z^i), \text{ over } z^i, w^i, s^i, a^{i,0} \geqq 0,$$

$$\text{subject to:} \left.\begin{array}{r} w^i \leqq \bar{w}^i, \\ p.z^i - \pi_i.a^{i,0} \leqq (1-\pi_i).a_0^{i,0} \\ p.z^i + (1-\pi_i.d).s^i - \pi_i.a^{i,0} \leqq (1-\pi_i).(d.s_0^i + a_0^{i,0}) \\ p.z^i - q.w^i + (1-\pi_i.d).s^i + (1-\pi_i).a^{i,0} \leqq (1-\pi_i).(d.s_0^i + a_0^{i,0}) \end{array}\right\}$$

With the help of duality methods the following properties can be deduced:

<u>Proposition 7.6.:</u> Necessary conditions for max. problem (7.5), in order to possess an optimal solutions are:

(1) $p > 0$.

(2) $d \leqq 1/\pi_i$.

Proposition 7.7.: If, for some (p,q,d,s_0^i,a_0^i) with $p > 0$, $q > 0$, and with $d \leqq 1/\pi_i$, the action $(\tilde{z}^i,\tilde{w}^i,\tilde{s}^i,\tilde{a}^{i,0})$ is optimal for (7.5), then:

(1) $p.\tilde{z}^i - q.\tilde{w}^i + (1-\pi_i.d).\tilde{s}^i + (1-\pi_i).\tilde{a}^{i,0} = (1-\pi_i).d.s_0^i + a_0^{i,0})$.

(2) $\tilde{w}^i = \bar{w}^i$.

(3) $d < 1/\pi_i$ implies: $\tilde{s}^i = 0$.

Proposition 7.8.: Consider max. problem (7.5) with $p > 0$, $q > 0$, and with $d \leqq 1/\pi_i$. For such a max. problem, an action $(\tilde{z}^i,\tilde{w}^i,\tilde{s}^i,\tilde{a}^{i,0})$ satisfying $\tilde{s}^i = s_0^i$, $\tilde{a}^{i,0} = a_0^{i,0}$, is optimal if and only if:

$(1-\pi_i.d).s_0^i = 0$, $a_0^{i,0} = q.\bar{w}^i + (d-1).s_0^i$, $\tilde{w}^i = \bar{w}^i$, $p.\tilde{z} = a_0^{i,0}$.

The economic behavior of the firm can be described in the same way; i.e. by (6.12) and, under invariant prices and shares by (6.14). Further, replacing individuals optimization proces (6.8) by (7.5), we can maintain the same I.C.E. concept. Now starting from the proposition 7.6, 7.7, 7.8, 6.15, and 6.16, one will find the following properties:

Proposition 7.8.: If $(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i,\hat{a}^{i,0})\}_1^2, (\hat{x},\hat{v},\hat{y},b^0))$ is an I.C.E. with $\hat{s}^i + \hat{s}^1 > 0$, then:

(1) $\hat{p} > 0$, $\hat{q} > 0$.

(2) $\hat{p}.\hat{z}^i - \hat{q}.\hat{w}^i - (\hat{d}-1)\hat{s}^i = 0$

(3) $\hat{a}^{i,0} = p.\hat{z}^i$, implying: $p.(\hat{z}^1+\hat{z}^2) = M$.

(4) $1/\pi_i > \hat{d}$ implies: $\hat{s}^i = 0$.

(5) Defining $\pi^* := \max(\pi_1,\pi_2)$: $\hat{d} = 1/\pi^*$.

(6) $\hat{w}^1 = \bar{w}^1,\ \hat{w}^2 = \bar{w}^2,\ \hat{v} = \bar{w}^1 + \bar{w}^2.$

(7) $\hat{b}^0 = 0,\ \hat{p}.\hat{x} + \hat{q}.\hat{v} = \hat{s}^1 + \hat{s}^2,\quad \hat{y} = f(\hat{x},\hat{v}).$

(8). $(\pi^*).\hat{p}.\hat{y} = \hat{s}^1 + \hat{s}^2$, where $\pi^* := \max(\pi_1,\pi_2)$.

Comparing the properties 4.13 to 4.20 of model I with the properties mentioned above, it should be clear that invariant competitive equilibria of these models are related as follows:

Proposition 7.10.: $(\hat{p},\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\hat{s}^i)\}_1^2,\ (\hat{x},\hat{v},\hat{y}))$ - $\hat{s}^1 + \hat{s}^2$ being positive - is an I.C.E. of model I, if and only if, for $\lambda := M/(p.(\hat{z}^1+\hat{z}^2))$, for $\hat{b}^0 := 0$, and for $(\hat{a}^{1,0},\hat{a}^{2,0}) := (p.\hat{z}^1,\ p.\hat{z}^2)$, the combination $(\lambda.\hat{p},\lambda.\hat{q},\ \hat{d},\{(\hat{z}^i,\hat{w}^i,\lambda.\hat{s}^i,\lambda\hat{a}^{i,0})\}_1^2,\ (\hat{x},\hat{v},\hat{y},\hat{b}^0))$ is an I.C.E. for model II-b.

This ensures the existence of an I.C.E. under the same conditions as mentioned in model I. Further, it should be clear that the property concerning inflation (viz. 4.24 and 4.25) is not applicable with respect to model II-b.

8. Model II-C: Fiat money and negotiable shares.

In the third model with fiat money the order of transactions and payments is specified as follows:

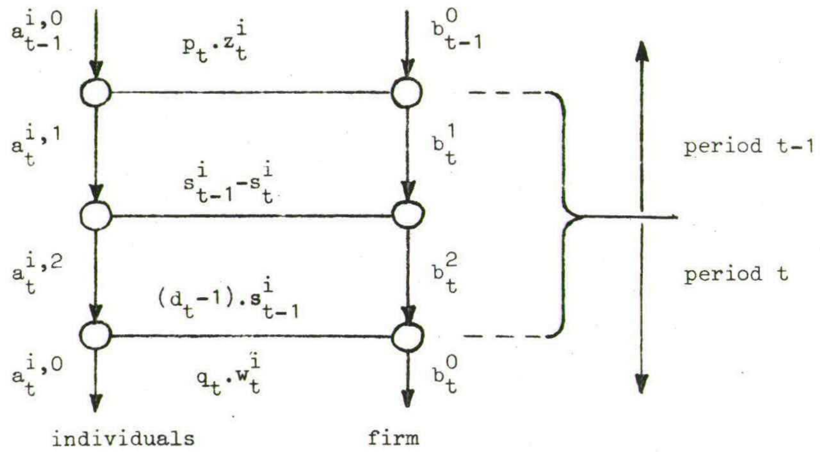

FIGURE 4.

The only difference with respect to model II-b is that the real dividend (i.e. $(d_t-1).s^i_{t-1}$) is paid off after the point where shares are exchanged. It will appear that this affects the nature of the I.C.E. substantially.

Under this scheme, the economic behavior of the individuals under invariant prices and dividend-factors is described by:

$$(8.1) \qquad \max \sum_{t=1}^{\infty} (\pi_i)^t . \varphi_i(z_t^i), \text{ over } z_t^i, w_t^i, s_t^i, a_t^{i,0} \geqq 0, \quad t = 1,2,\ldots,$$

$$\text{subject to: } \left.\begin{array}{r} w_t^i \leqq \bar{w}^i \\ p.z_t^i - a_{t-1}^{i,0} \leqq 0 \\ p.z_t^i + s_t^i - s_{t-1}^i - a_{t-1}^{i,0} \leqq 0 \\ p.z_t^i - q.w_t^i + s_t^i - d_t . s_{t-1}^i + a_t^{i,0} - a_{t-1}^{i,0} \leqq 0 \end{array}\right\} t = 1,2,\ldots,$$

where $(s_0^{i,0}, a_0^{i,0})$ is the given initial state. In the same manner as described in 6.9, invariant optimal action plans can be found by the single-period decision process:

$$(8.2) \qquad \left.\begin{array}{l} \max \varphi_i(z^i), \text{ over } z^i, w^i, s^i, a^{i,0} \geqq 0, \\ \text{subject to: } \quad w^i \leqq \bar{w}^i \\ \qquad p.z^i - \pi_i . a^{i,0} \leqq (1-\pi_i).a_0^{i,0} \\ \qquad p.z^i + (1-\pi_i).s^i - \pi_i . a^{i,0} \leqq (1-\pi_i).(s_0^i + a_0^{i,0}) \\ \qquad p.z^i - q.w^i + (1-\pi_i . d).s^i + (1-\pi_i).a^{i,0} \leqq (1-\pi_i).(d.s_0^i + a_0^{i,0}) \end{array}\right\}$$

For this max problem we can deduce:

<u>Proposition 8.3.:</u> The following conditions are necessary for (8.2) in order to possess an optimal solution:

(1) $\qquad p > 0.$

(2) $\qquad d \leqq (1/\pi_i) + ((1-\pi_i)/\pi_i)^2.$

<u>Proposition 8.4.:</u> If, for some $(p, q, d, s_0^i, a_0^{i,0})$ with $p > 0$, $q > 0$, and with $d \leqq (1/\pi_i) + ((1-\pi_i)/\pi_i)^2$, an action $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i, \tilde{a}^{i,0})$ is optimal with respect to (8.2) then:

(1) $\qquad p.\tilde{z}^i - q.\tilde{w}^i + (1-\pi_i . d).\tilde{s}^i + (1-\pi_i).\tilde{a}^{i,0} = (1-\pi_i).(d.s_0^i + a_0^{i,0}).$

(2) $\tilde{w}^i = \bar{w}^i$

(3) $d < 1/\pi_i$ implies: $\tilde{s}^i = 0$.

Proposition 8.5.: Consider max. problem (8.2) with $p > 0$, $q > 0$, and with $d \leqq (1/\pi_i) + ((1-\pi\ )/\pi_i)^2$. For such a max. problem, $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i, \tilde{a}^{i,0})$ satisfying $\tilde{s}^i = s_0^i$, $\tilde{a}^{i,0} = a^{i,0}$, is optimal if and only if: $(1-\pi_i.d).s_0^i = 0$, $a_0^{i,0} = q.\bar{w}^i + (d-1).s_0^i$, $\tilde{w}^i = \bar{w}^i$, $p.\tilde{z}^i = a_0^{i,0}$.

Comparing this result with the properties 6.10-(2), 6.11-(4) of model II-a and the properties 7.6-(2), 7.7-(2), we observe a suprising difference. Namely, in (6.8) an optimal solution with $\tilde{s}^i > 0$ is compatible with a single dividend-factor $d := (1/\pi_i)^2$, and, in (7.5) such an optimal solution is compatible with $d = (1/\pi_i)$ only. However, proposition 8.5 shows that an optimal solution of (8.2) with $\tilde{s}^i > 0$ is compatible with every dividend-factor d in the closed interval $[1/\pi_i, 1/\pi_i + ((1-\pi_i)/\pi_i)^2]$.

Starting from the economic behavior of the firm as described by (6.4), and replacing individuals optimization procedure (6.8) by (8.2) we maintain the same I.C.E. concept as defined for model II-a.
Then by virtue of 8.3 to 8.5 and of 6.15 and 6.16, the following properties can be deduced:

Proposition 8.6.: If $(\hat{p}, \hat{q}, \hat{d}, \{(\hat{z}^i, \hat{w}^i, \hat{s}^i, \hat{a}^{i,0})\}_1^2, (\hat{x}, \hat{v}, \hat{y}, \hat{b}^0))$ is an I.C.E. with $\hat{s}^1 + \hat{s}^2 > 0$, then:

(1) $\hat{p} > 0$, $\hat{q} > 0$

(2) $\hat{p}.\hat{z}^i - \hat{q}.\hat{w}^i - (\hat{d}-1).\hat{s}^i = 0$.

(3) $\hat{a}^{i,0} = p.\hat{z}^i$, implying $p.(\hat{z}^1 + \hat{z}^2) = M$.

(4) $\hat{d} < 1/\pi_i$ implies: $\hat{s}^i = 0$.

(5) Defining $\pi^* := \max(\pi_1,\pi_2)$: $\hat{d} \in [\,1/\pi^*,\ 1/\pi^* + ((1-\pi^*)/\pi^*)^2]$.

(6) $\hat{w}^1 = \bar{w}^1$, $\hat{w}^2 = \bar{w}^2$, $\hat{v} = \bar{w}^1 + \bar{w}^2$.

(7) $\hat{b}^0 = 0$, $\hat{p}.\hat{x} + \hat{q}.\hat{v} = \hat{s}^1 + \hat{s}^2$, $\hat{y} = f(\hat{x},\hat{v})$.

By virtue of the propositions 8.5 and 8.6., it is possible to identify invariant competitive equilibria of model I with invariant competitive equilibria of this model. In a similar way as for model II-a, we compare model II-c, specified by the quantities $(\pi_1,\pi_2,\bar{w}^1,\bar{w}^2,M)$, the utility functions φ_1,φ_2, and by the production function f, with model I where the time-discount factors $\tilde{\pi}_1,\tilde{\pi}_2$ can be chosen, such that: $\tilde{\pi}_i \in [\,(\pi_i)^2/(\pi_i + (1-\pi_i)^2),\ \pi_i]$, $i = 1,2$. Then we can deduce the following relation:

<u>Proposition 8.7.:</u> $(\hat{p},\hat{q},\hat{d},\ (\hat{z}^i,\hat{w}^i,\hat{s}^i)_1^2\ (\hat{x},\hat{v},\hat{y}))$ – $(\hat{s}^1+\hat{s}^2)$ positive – is an I.C.E. of model I, with time-discount factors as mentioned above, if and only if, for $\hat{b}^0 := 0$, for $\lambda := M/(p.\hat{z}^1 + p.\hat{z}^2)$ and for $(\hat{a}^{1,0},\hat{a}^{2,0}) := (p.\hat{z}^1,\ p.\hat{z}^2)$, the combination $(\lambda.\hat{p},\lambda.\hat{q},\hat{d},\{(\hat{z}^i,\hat{w}^i,\lambda.\hat{s}^i,\lambda.\hat{a}^{i,0})\}_1^2,\ (\hat{x},\hat{y},\hat{v},\hat{b}^0))$ is an I.C.E. for model II-c.

Clearly, this ensures the existence of such an I.C.E. Further, we observe that, in this case, the dividend factor is not uniquely determined.

9. Model III: Fiat money, banking, and negotiable shares.

In this section we extend model II-c by adding an inside bank, which includes the possibility of borrowing and lending money from the bank, the possibility of holding bank-shares, and the possibility of paying with bank-cheques. The latter is based on the assumption that the individuals and the firm are allowed to have a checking account. Dept on checking accounts are not permitted.

For simplicity reasons, we assume that credit and saving transactions are available only, for the individuals. We shall discuss the details with the help of the following diagram of payments and transactions:

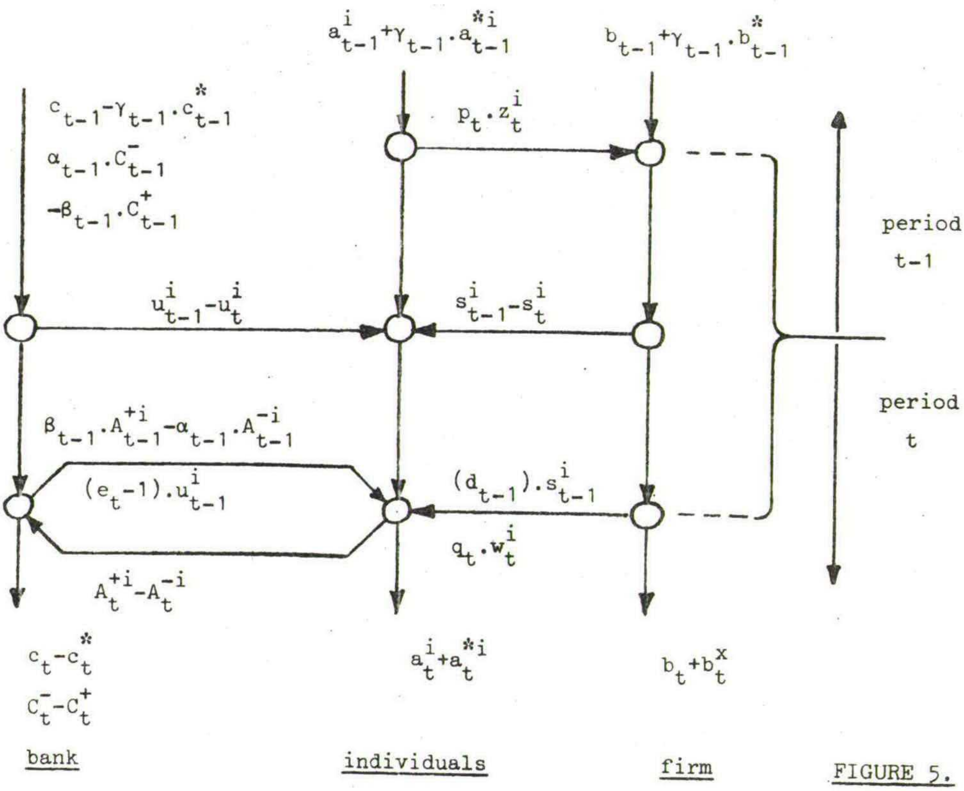

FIGURE 5.

with individual i, <u>during</u> a period t, we associate the following monetary quantities: $a_t^i \geqq 0$: hoarding of fiat money, $a_t^{*i} \geqq 0$: balance on his checking account, $A_t^{+i} \geqq 0$ his savings deposits, $A_t^{-i} \geqq 0$: his bank credit, $u_t^i \geqq 0$: his bank shares.

Concerning the firm, <u>during</u> a period t, we have: $b_t \geqq 0$: the quantity of fiat money, $b_t^* \geqq 0$: its balance on his checking account.

For the bank, <u>during</u> a period t, we introduce: $c_t \geqq 0$: the quantity of fiat money, $c_t^* \geqq 0$: the total balance on the checking accounts, $C_t^+ \geqq 0$: total saving deposits, $C_t^- \geqq 0$ total outstanding credits.

Finally, at the <u>beginning</u> of a period t, we have: e_t: the bank dividend-factor (to be defined later in a similar manner as firm's dividend factor) $\gamma_t \geqq 1$: interest-factor on checking accounts, $\alpha_t \geqq 1$: interest factor on credits, $\beta_t \geqq 1$: interest factor on saving deposits.
Note: the interest rate which corresponds with an interest-factor - say γ_t - is: $(\gamma_t - 1)$.

Obviously, the diagram is based on the simplifying convention that credit and saving transactions are concluded at the final stage, only. In this, a crucial assumption is that such a contract is terminated on exactly one period after it is initiated.
The difference between saving and checking accounts is due to the fact that the balances on checking accounts are available at each stage, this being a necessary condition for paying with bank cheques.

We shall discuss the economic behavior of the individuals, the firm, and the bank, successively.

Under invariant prices, dividend-factors, and interest-factors, $(p,q,d,e,\alpha,\beta,\gamma)$, individual's economic behavior is characterized by:

(9.1) $\max \sum_{t=1}^{\infty} (\pi_i)^t . \varphi_i(z_t^i)$,

over $z_t^i, w_t^i, s_t^i, u_t^i, a_t^i, a_t^{*i}, A_t^{+i}, A_t^{-i} \geqq 0, \quad t = 1,2,\ldots,$

subject to:

$$\left.\begin{array}{r} w_t^i \leqq \bar{w}_t^i \\ p.z_t^i - a_{t-1}^i - \ .a_{t-1}^{*i} \leqq 0 \\ p.z_t^i + s_t^i - s_{t-1}^i + u_t^i - u_{t-1}^i - a_{t-a}^i - \gamma.a_{t-1}^{*j} \leqq 0 \\ p.z_t^i - q.w_t^i + s_t^i - d.s_{t-1}^i + u_t^i - e.u_{t-1}^i + A_t^{+i} - \beta.A_{t-1}^{+i} - \\ -A_t^{-i} + \alpha.A_{t-1}^{-i} + a_t^i + a_t^{*i} - a_{t-1}^i - \gamma.a_{t-1}^{*i} \leqq 0 \end{array}\right\} \quad t = 1,2,\ldots,$$

To the budget restriction we add the "credit limit":

(9.2) $\xi_1(\alpha-1).A_t^{-i} - \xi_1.q.w_t^i - \xi_2.(s_t^i + u_t^i) \leqq 0, \quad t = 1,2,\ldots,$

where $0 \leqq \xi_1 < 1$, $0 \leqq \xi_2 < (\alpha-1)$. Without loss of generality we may restrict ourselves to bounded action plans; such that:

(9.3) $z_t^i, w_t^i \leqq N_1, \quad s_t^i \leqq N_2, \quad a_t^i \leqq N_3, \quad t = 1,2,\ldots,$

(9.4) $a_t^{*i},\ A_t^{+i},\ A_t^{-i} \leqq M_1, \quad u_t^i \leqq M_2, \quad t = 1,2,\ldots,$

provided the constants are chosen large enough. The meaning of (9.4) will be clarified later (viz. 9.23). Further, the initial state $(s_0^i, u_0^i, a_0^i, a_0^{*i}, A_0^{+i}, A_0^{-i})$ is suppose to be a given result of the initial period $t := 0$.

In a similar manner as indicated in proposition 4.12., invariant optimal solutions of programs defined by (9.1) to (9.4) can be found by the single-period decision problem:

(9.5) $\max \varphi_i(z_i)$, over $z^i, w^i, s^i, u^i, a^i, a^{*i}, A^{+i}, A^{-i} \geqq 0$,

subject to:

$$w^i \leqq \bar{w}^i,$$

$$p.z^i - \pi_i.(a^i + \gamma.a^{\;i}) \leqq (1-\pi_i).(a_0^i + \gamma.a_0^{*i}),$$

$$p.z^i + (1-\pi_i).(s^i + u^i) - \pi_i.(a^i + \gamma.a^{*i}) \leqq (1-\pi_i).(a_0^i + \gamma.a_0^{*i} + s_0^i + u_0^i),$$

$$p.z^i - q.w^i + (1-\pi_i.d).s^i + (1-\pi_i.e).u^i + (1-\pi_i.\beta).A^{+i} -$$

$$-(1-\pi_i.\alpha).A^{-i} + (1-\pi_i).a^i + (1-\pi_i.\gamma).a^{*i} \leqq$$

$$\leqq (1-\pi_i).(a_0^i + \gamma.a_0^{\;i} + d.s_0^i + e.u_0^i + \beta.A_0^{+i} - \alpha.A_0^{-i}),$$

$$(\alpha-1).A^{-i} - \xi_1.q.w^i - \xi_2.(s^i + u^i) \leqq 0.$$

With respect to this optimization process one can deduce the following properties, with the help of duality methods:

Proposition 9.6.: Necessary conditions for max. problem (9.5) with $\gamma \geqq 1$, $\alpha \geqq \max(\beta,\gamma,d,e)$, in order to possess an optimal solution, are:

(1) $p > 0$. (2) $\beta \leqq 1/\pi_i$. (3) $\gamma \leqq 1/\pi_i$.

(4) $d \leqq (\gamma-1)/(\pi_i.\gamma)^2 + 1/(\pi_i.\gamma) + (1-\pi_i.\gamma)^2/(\pi_i.\gamma)^2$.

(5) $e \leqq (\gamma-1)/(\pi_i.\gamma)^2 + 1/(\pi_i.\gamma) + (1-\pi_i.\gamma)^2/(\pi_i.\gamma)^2$.

Proposition 9.7.: Consider max. problem (9.5), where $p > 0$, $q > 0$, $\gamma \in [1, 1/\pi_i]$, $\beta \leqq 1/\pi_i$, $d \leqq (\gamma-1)/(\pi_i.\gamma)^2 + 1/(\pi_i.\gamma) + (1-\pi_i\gamma)^2/(\pi_i.\gamma)^2$ $e \leqq (\gamma-1)/(\pi_i.\gamma)^2 + 1/(\pi_i.\gamma) + (1-\pi_i.\gamma)^2/(\pi_i.\gamma)^2$, $\alpha > \max(\beta,\gamma,d,e,1)$,

$0 \leqq \xi_1 < 1$, $0 \leqq \xi_2 < (\alpha-1)$, and where $(\alpha-1).A_0^{-i} \leqq \xi_1.q.\bar{w}^i + \xi_2.(s_0^i+u_0^i)$. If, for such a problem, $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i, \tilde{u}^i, \tilde{a}^i, \tilde{a}^{*i}, \tilde{A}^{+i}, \tilde{A}^{-i})$ is an optimal action then:

(1) $p.\tilde{x}^i - q.\tilde{w}^i + (1-\pi_i.d).\tilde{s}^i + (1-\pi_i.e).\tilde{u}^i + (1-\pi_i.\beta).\tilde{A}^{+i} - (1-\pi_i.\alpha).\tilde{A}^{-i} +$
$+ (1-\pi_i).\tilde{a}^i + (1-\pi_i.\gamma).\tilde{a}^{*i} = (1-\pi_i).(a_0^i + \gamma.a_0^{\,i} + d.s_0^i + e.u_0^i + \beta.A_0^{+i} - \alpha.A_0^{-i})$.

(2) $\tilde{w}^i = \bar{w}^i$

(3) $\gamma > 1$ implies $a^i = 0$

(4) $d < \max(1/\pi_i, e)$ implies: $\tilde{s}^i = 0$.

(5) $e < \max(1/\pi_i, d)$ implies: $\tilde{u}^i = 0$.

(6) $\beta < 1/\pi_i$ implies: $\tilde{A}^{+i} = 0$.

(7) $\alpha > 1/\pi_i$ implies: $A^{-i} = 0$.

(8) If $\alpha < 1/\pi_i$ then: $\tilde{A}^{-i} = \xi_1.q.\bar{w}^i/(\alpha-1) + \xi_2.(\tilde{s}^i+\tilde{u}^i)/(\alpha-1)$.

<u>Proposition 9.8.:</u> Consider a problem (9.5) as specified in proposition 9.7. For such a max. problem, an action $(\tilde{z}^i, \tilde{w}^i, \tilde{s}^i, \tilde{u}^i, \tilde{a}^i, \tilde{a}^{*i}, \tilde{A}^{+i}, \tilde{A}^{-i})$ satisfying $\tilde{s}^i = s_0^i$, $\tilde{u}^i = u_0^i$, $\tilde{a}^i = a_0^i$, $\tilde{a}^{*i} = a_0^{*i}$, $\tilde{A}^{+i} = A_0^{+i}$, $\tilde{A}^{-i} = A_0^{-i}$, is optimal if and only if, simultaneously:

(1) For $\gamma > 1$: $a_0^i = 0$. (2) For $d < \max(1/\pi_i.e)$: $s_0^i = 0$.

(3) For $e < \max(1/\pi_i, d)$: $u_0^i = 0$. (4) For $\beta < 1/\pi_i$: $A_0^{+i} = 0$.

(5) For $\alpha > 1/\pi_i$: $A_0^{-i} = 0$. (6) For $\alpha < 1/\pi_i$: $(\alpha-1).A_0^{-i} = \xi_1.q.\bar{w}^i + \xi_2.(\hat{s}^i+\hat{u}^i)$, and
For $\alpha = 1/\pi_i$: $(\alpha-1).A_0^{-i} \leqq \xi_1.q.\bar{w}^i + \xi_2.(\hat{s}^i+\hat{u}^i)$.

(7) $a_0^i + a_0^{*i} = q.\bar{w}^i + (d-1).s_0^i + (e-1).u_0^i + (\beta-1).A_0^{+i} - (\alpha-1).A_0^{-i}$.

(8) $\tilde{w}^i = \bar{w}^i$. (9) $p.\tilde{z}^i = a_0^i + \gamma.a_0^{*i}$.

Under invariant prices and interest rates, the economic behavior of the firm is characterized by the max. problem:

(9.10) $\max\ p.y + b + \gamma.b^*$, over $x,v,y,b,b^* \geqq 0$,

subject to: $y \leqq f(x,v)$,

$$p.x + q.v + b + b^* \leqq s^1 + s^2 \quad .$$

For this program the following properties hold:

Proposition 9.11.: If $p > 0$, $s^1+s^2 > 0$, then a necessary condition for the problem (9.10) in order to possess an optimal solution is: $q > 0$. (Implied by the assumption that f is neo-classic).

Proposition 9.12.: If, for some $p > 0$, $q > 0$, $\gamma \geqq 1$, $s^1+s^2 > 0$, the action $(\tilde{x},\tilde{v},\tilde{y},\tilde{b},\tilde{b}^*)$ is optimal with respect to (9.10), then:

(1) $p.\tilde{x} + q.\tilde{v} + \tilde{b} + \tilde{b}^* = s^1 + s^2$.

(2) $\tilde{y} = f(\tilde{x},\tilde{v})$

(3) $\gamma > 1$ implies $\tilde{b} = 0$

(4) $p.\tilde{y} + \tilde{b} + \tilde{b}^* > \gamma.(s^1+s^2)$ implies: $\tilde{b} = 0$, $\tilde{b} = 0$.

Turning our attention to the bank and to the total demand and total supply of fiat money, deposits, and credits, we arrive at following requirements:

(9.13) $c_t + a_t^1 + a_t^2 + b_t \leqq c_{t-1} + a_{t-1}^1 + a_{t-1}^2 + b_{t-1}$, $t = 1,2,\ldots$,

(9.14) $c_t^* = a_t^{*1} + a_t^{*2} + b_t^*$, $t = 1,2,\ldots$,

(9.15) $C_t^+ = A_t^{+1} + A_t^{+2}$, $t = 1,2,\ldots$,

(9.16) $C_t^- = A_t^{-1} + A_t^{-2}$, $t = 1,2,\ldots.$

Starting from the simplifying assumption that bank transactions do not require labor, the balance restrictions on the activities of the bank can be formulated:

$$c_t - c_t^* - C_t^+ + C_t^- \leqq u_t^1 + u_t^2, \quad t = 1,2,... \tag{9.17}$$

Further, we assume that an "outside" agent (i.e. the government or a central "outside" bank) imposes the following conditions concerning solvability and liquidity resp.:

$$\varkappa_1 . C_t^- \leqq u_t^1 + u_t^2, \quad t = 1,2,..., \tag{9.18}$$

$$\varkappa_2 . c_t^* - c_t \leqq 0, \quad t = 1,2,..., \tag{9.19}$$

where the given constants $\varkappa_1, \varkappa_2$ are positive and smaller than one.

Concerning the economic behavior of the bank, we assume a competitive situation; i.e.: we assume that the only information concerning the money-markets as a whole is constituted by the interest rates. Of course, such an assumption makes sense for an economy with two or more non-coöperative inside banks. However, if all banks in such an economic system work under the same conditions it can be shown (viz. Shubik) that the aggregate results can be found in model with only one bank in the competitive setting as mentioned above. For such a bank, the economic behavior, under given invariant external conditions constituted by interest factors $\alpha > \beta \geqq \gamma \geqq 1$ and outstanding bankshares u^1, u^2, is characterized by the single-period decision problem:

$$\Psi := \max \alpha . C^- - \beta . C^+ - \gamma . c^* + c, \tag{9.20}$$

over $C^-, C^+, c^*, c \geqq 0$,

subject to:

$$\left.\begin{aligned} c-c^* - C^+ + C^- &\leqq u^1+u^2 \\ \varkappa_1 . C^- &\leqq u^1+u^2 \\ \varkappa_2 . c^* - c &\leqq 0 \end{aligned}\right\}$$

For $u^1+u^2 > 0$, the corresponding bank dividend-factor is defined by:

(9.21) $\quad e := \Psi/(u^1+u^2)$,

provided there is an optimal solution.

We observe that (9.20) is linear programming problem. Thus, with the help of the corresponding dual problem, we can deduce the following properties:

Proposition 9.22.: For $\varkappa_1, \varkappa_2 > 0$, $\varkappa_1, \varkappa_2 < 1$, and $\alpha > 1$, problem (9.20) possesses an optimal solution if and only if $\beta, \gamma \geqq 1$.

Proposition 9.23.: Let $\alpha > 1$, $\beta \geqq 1$, $\gamma \geqq 1$, and let $\varkappa_1$, $\varkappa_2$ be positive and smaller than one. Further, let $\underline{\gamma} := (\gamma - \varkappa_2)/(1-\varkappa_2)$. Then:

(1) In the case that $\beta > \alpha$, $\underline{\gamma} > \alpha$: action $(\tilde{c}, \tilde{c}^*, \tilde{C}^+, \tilde{C}^-)$ is optimal if and only if : $\tilde{c} = 0$, $\tilde{c}^* = 0$, $\tilde{C}^+ = 0$, $\tilde{C}^- = u^1+u^2$.

(2) In the case that $\beta > \alpha$, $\underline{\gamma} > \beta$: action $(\tilde{c}, \tilde{c}^*, \tilde{C}^+, \tilde{C}^-)$ is optimal if and only if : $\tilde{c} = 0$, $\tilde{c}^* = 0$, $-\tilde{C}^+ + \tilde{C}^- = u^1+u^2$, $\varkappa_2 . \tilde{C}^- = u^1+u^2$.

(3) In the case that $\underline{\gamma} < \alpha$, $\beta > \underline{\gamma}$: action $(\tilde{c}, \tilde{c}^*, \tilde{C}^+, \tilde{C}^-)$ is optimal if and only if : $\tilde{C}^+ = 0$, $\tilde{c} - \tilde{c}^* + \tilde{C}^- = u^1+u^2$, $\varkappa_1 . \tilde{C}^- = u^1+u^2$, $\varkappa_2 . \tilde{c}^* = \tilde{c}$.

(4) In the case that $\beta < \alpha$, $\underline{\gamma} = \beta$: action $(\tilde{c}, \tilde{c}^*, \tilde{C}^+, \tilde{C}^-)$ is optimal if and only if : $\tilde{c} - \tilde{c}^* - \tilde{C}^+ + \tilde{C}^- = u^1+u^2$, $\varkappa_1 . \tilde{C}^- = u^1+u^2$, $\varkappa_2 . \tilde{c}^* = \tilde{c}$.

(5) In the case that $u^1+u^2 > 0$: $e = (\alpha/\varkappa_1) + (1-1/\varkappa_1) . \min(\alpha, \beta, \underline{\gamma})$.

For this model we define an <u>invariant competitive equilibrium</u> as a combination $(\hat{p},\hat{q},\hat{d},\hat{e},\hat{\alpha},\hat{\beta},\hat{\gamma})$, $\{(\hat{z}^i,\hat{w}^i,\hat{s}^i,\hat{u}^i,\hat{a}^i,\hat{a}^{*i},\hat{A}^{+i},\hat{A}^{-i})\}_{i=1}^2$, $(\hat{x},\hat{v},\hat{y},\hat{b},\hat{b}^*)$, $(\hat{c},\hat{c}^*,\hat{C}^+,\hat{C}^-)$, with $\hat{s}^1 + \hat{s}^2 > 0$ and with $\hat{u}^1 + \hat{u}^2 > 0$, such that, simultaneously:

(a) For each individual i, action $(\hat{z}^i,\hat{w}^i,\hat{s}^i,\hat{u}^i,\hat{a}^i,\hat{a}^{*i},\hat{A}^{+i},\hat{A}^{-i})$ is optimal for (9.5) with $(p,q,d,e,\alpha,\beta,\gamma) := (\hat{p},\hat{q},\hat{d},\hat{e},\hat{\alpha},\hat{\beta},\hat{\gamma})$ and with $(s_0^i,u_0^i,a_0^i,a_0^{*i},A_0^{+i},A_0^{-i}) := (\hat{s}^i,\hat{u}^i,\hat{a}^i,\hat{a}^{*i},\hat{A}^{+i},\hat{A}^{-i})$

(b) $(\hat{x},\hat{v},\hat{y},\hat{b},\hat{b}^*)$ is optimal for (9.10) with $(p,q,\gamma,s^1,s^2) := (\hat{p},\hat{q},\hat{\gamma},\hat{s}^1,\hat{s}^2)$.

(c) The dividend-factor $\hat{d}$ satisfies: $\hat{p}.\hat{y} + \hat{b} + \hat{\gamma}.\hat{b}^* = \hat{d}.(\hat{s}^1 + \hat{s}^2)$

(d) $(\hat{c},\hat{c}^*,\hat{C}^+,\hat{C}^-)$ is optimal for (9.20) with $(\alpha,\beta,\gamma,u^1,u^2) := (\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{u}^1,\hat{u}^2)$.

(e) The dividend-factor $\hat{e}$ satisfies: $\hat{\alpha}.\hat{C}^- - \hat{\beta}.\hat{C}^+ - \hat{\gamma}.\hat{c}^* + \hat{c} = \hat{e}.(\hat{u}^1+\hat{u}^2)$.

(f) Total demand and total supply of "wheat" and "labor" are equal; i.e. $\hat{z}^1 + \hat{z}^2 + \hat{x} + \hat{y}$, $\hat{w}^1 + \hat{w}^2 = \hat{v}$.

(g) The total amound of fiat money hoard by the agents is equal to the initial amount of fiat money; i.e. $\hat{a}^1 + \hat{a}^2 + \hat{b} + \hat{c} = M$. (Note $M > 0$).

(h) Total demand and total supply of bank money are equal; i.e. $\hat{a}^{*1} + \hat{a}^{*2} + \hat{b}^* = \hat{c}^*$.

(i) Total demand and total supply of saving deposits are equal; i.e. $\hat{A}^{+1} + \hat{A}^{+2} = \hat{C}^+$.

(j) Total demand and total supply of bank credits are equal; i.e. $\hat{A}^{-1} + \hat{A}^{-2} = \hat{C}^-$.

Using the properties of the underlying optimization problems, we shall construct numerical example of an I.C.E. where $\hat{C}^+ > 0$ and where

$\hat{C}^- > 0$. It will appear that, for such equilibrium, the constants in individual's credit limits (viz. 9.2.) have to be chossen in a particular manner which is related to the constants of the solvability and liquidity restrictions of the bank (viz. 9.18. and 9.19.). Obvious, this implies that an equilibrium exist, for particular values of the constants, only.

In the numerical example we take: $\underline{\pi_1 := 0.9}$, $\underline{\varkappa_1 := 0.5}$, $\underline{\varkappa_2 := 0.5}$. Further we assume $\underline{\pi_2 > \pi_1}$, implying (by 9.8-(4) and by equilibrium condition i) that an equilibrium with $\hat{C}^+ > 0$ is possible, only if $\hat{\beta} = 1/\pi_1$; so $\underline{\hat{\beta} = 1.111}$. Moreover, the assumption $\hat{s}^1 + \hat{s}^2 > 0$, $\hat{u}^1 + \hat{u}^2 > 0$ implies (viz. 9.8-(2) and (3)): $\hat{d} \geqq 1/\pi_1$, $\hat{e} \geqq 1/\pi_1$; i.e. $\hat{d} \geqq 1.111$, $\hat{e} \geqq 1.111$. By virtue of 9.23-(5), the relations $\hat{e} \geqq 1/\pi_1$, $\hat{\beta} = 1/\pi_1$ imply: $\alpha \geqq 1/\pi_1$, and next, by 9.8-(7): $\hat{a}^1 + \hat{a}^{*1} > 0$.
Turning back to the max. problem of the bank, 9.23-(3) shows that $\hat{C}^+ > 0$ is possible, only if $\underline{\gamma} := (\hat{\gamma}-\varkappa_2)/(1-\varkappa_2) \geqq \hat{\beta}$.
With $0 < \varkappa_2 < 1$, this implies $\hat{\gamma} > 1$, and next, by 9.8-(1):
$\underline{\hat{a}^1 = 0}$, $\underline{\hat{a}^2 = 0}$. Clearly, with $\hat{a}^1 + \hat{a}^{*1} > 0$, the latter implies $\hat{a}^{*1} > 0$ and $\hat{c}^* > 0$, as well. From 9.23-(1), (2), (3), and from positivity of $\hat{C}^+$, $\hat{c}^*$, we may conclude $\underline{\gamma} := (\hat{\gamma}-\varkappa_2)/(1-\varkappa_2) = \hat{\beta}$.
Substituting $\varkappa_2 := 0.5$, $\hat{\beta} := 1/0.9$, we find $\underline{\hat{\gamma} = 1.056.}$

Now, by 9.6-(4), (5) and by 9.7-(4), (5), we can deduce that $\hat{e} = \hat{d}$ and may be chosen in the interval [1.111, 1.11697] .
We take $\underline{\hat{d} := 1.116}$, $\underline{\hat{e} := 1.116}$. Then, with the help of 9.23-(5), $\hat{\alpha}$ can be determined as: $\underline{\hat{\alpha} = 1.114.}$

Summarizing, we have:

(9.24) <u>Assumption:</u> $\pi_1 := 0.9$, $\pi_1 > \pi_2$, $\varkappa_1 := 0.5$, $\varkappa_2 := 0.5$, $C^+ > 0$.

(9.25) <u>Results:</u> $\hat{\alpha} = 1.114$, $\hat{\beta} = 1.111$, $\hat{\gamma} = 1.056$, $\hat{d} = \hat{e} = 1.116$, $\hat{a}^1 = 0$, $\hat{a}^2 = 0$.

In order to elaborate the productive activities of the firm we specify the production by:

(9.26) <u>Assumption:</u> $f(x,v) := (0.5).x^{0.75}.v^{0.25}$.

Labor supply is specified by:

(9.27) <u>Assumption:</u> $\bar{w}^1 := 1$, $\bar{w}^2 := 1$.

Clearly, in connection with 9.11, 9.7-(2) and with equilibrium condition f, the latter implies $\underline{\hat{v} = 2.}$ From (5.7) we have: $\hat{p}.\hat{x} = (0,75/0.25).\hat{q}.\hat{v}$, $\hat{p}.\hat{y} = (\hat{d}/0.25).\hat{q}.\hat{v}$, and hence, by equilibrium condition f, $\hat{p}.(\hat{z}^1+\hat{z}^2) = (2.92).\hat{q}$. Since, $\hat{a}^1 = 0$, $\hat{a}^2 = 0$, 9.8-(9) shows: $\hat{p}.(\hat{z}^1+\hat{z}^2) = \hat{\gamma}.(\hat{a}^{*1}-\hat{a}^{*2})$, and therefore: (with $\hat{\gamma} = 1.056$): $\hat{a}^{*1} + \hat{a}^{*2} = (2.765).\hat{q}$. Further, by 9.12-(4) and by $\hat{e} > \hat{\gamma}$, we find $\underline{\hat{b} = 0}$, $\underline{\hat{b}^* = 0}$; implying (viz. equilibrium condition g and h): $\hat{c} = M$, $\hat{c}^* = \hat{a}^{*1} + \hat{a}^{*2}$, and next: $\hat{q} = \hat{c}^*/(2.765)$. Specifying the amount of fiat money by:

(9.28) <u>Assumption:</u> $M := 100$,

we arrive at the following results (viz. 9.23-(4)): $\underline{\hat{c} = 100}$, $\underline{\hat{c}^* = 200}$, $\underline{\hat{q} = 72.30.}$ Now, with the help of (5.7), all quantities concerning the firm can be determined. Summarizing:

(9.29) <u>Results:</u>

$$\hat{v} = 2,\quad \hat{b} = 0,\quad \hat{b}^* = 0,\quad \hat{c} = 100,\quad \hat{c}^* = 200$$
$$\hat{q} = 72.30,\quad \hat{p} = 1.70,\quad \hat{x} = 255,\quad \hat{y} = 378,$$
$$\hat{p}.\hat{x} = 433.20,\quad \hat{p}.\hat{y} = 644.00,\quad \hat{s}^1 + \hat{s}^2 = 577.60.$$

Turning our attention to the saving and credit contracts of the bank, we see (by $\hat{\alpha} > 1/\pi_1$, 9.8-(5)) that $\underline{\hat{A}^{-1} = 0}$, and next (viz. equilibrium

condition j) $\hat{A}^{-2} = \hat{C}^{-}$. By virtue of 9.23, this implies:
$\hat{A}^{-2} \geqq \hat{c}^{*} - \hat{c} = 100$. So, it appears that the constants ξ_1, ξ_2 in individuals credit restriction (9.2) have to be large enough. Specifying:

(9.30) <u>Assumption:</u> $\xi_1 := 0.4$, $0 \leqq \xi_2 < (\hat{\alpha}-1)$, $\pi_2 \leqq 1/\hat{\alpha} = 0.897$.

We shall construct an equilibrium by choosing $\hat{s}^2 = 0$, $\hat{u}^2 = 0$, or choosing $\xi_2 = 0$, or by choosing $\pi_2 = 0.897$. Then, by virtue of 9.8-(6), we may specify <u>$\hat{A}^{-2} := 253.33$</u>, and consequently: <u>$\hat{C}^{-} = 252.33$.</u>
Then, by 9.23-(4), one will find: <u>$\hat{C}^{+} = 26.7$</u>, <u>$\hat{u}^1 + \hat{u}^2 = 126.66$.</u>
Since $\hat{A}^{+2} = 0$ (implied by $\beta < 1/\pi_2$), equilibrium condition i, implies: <u>$\hat{A}^{+} = 26.7$.</u> We observe that share holding by individual 2 is possible. For instance, putting $\bar{\pi}_2 := 0.897$ we have $\pi_2 . \hat{d} = 1.001 > 1$ which is compatible with condition 9.8-(2) and condition 9.8-(3).
Anyway, specifying:

(9.31) <u>Assumption:</u> $\hat{u}^2 := 0$, $\hat{s}^2 := 0$.

the conditions 9.8-(7), (8), (9) (with $a_0^i = 0$) imply: <u>$\hat{z}^1 = 95$</u>, <u>$\hat{z}^2 = 26$</u>.

(9.32) <u>Results:</u>

$$\hat{A}^{+1} = 26.7, \quad \hat{A}^{-1} = 0, \quad \hat{A}^{+2} = 0, \quad \hat{A}^{-2} = 253.33,$$
$$\hat{s}^1 = 577.60, \quad \hat{u}^1 = 126.66,$$
$$\hat{C}^{+} = 26.7, \quad \hat{C}^{-} = 252.33,$$
$$\hat{z}^1 = 95, \quad \hat{z}^2 = 26.$$

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