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RESEARCH MEMORANDUM

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HIERARCHICAL DECENTRALIZED OPTIMAL CONTROL IN ECONOMETRIC POLICY MODELS.

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Abstract

Optimal control methods for policy evaluation in econometric models with a decentralized decision structure, which is also hierarchical or sequential, are developed. Solutions are given for the N-level problem and in case more than one player acts at the same decision level. Also strategies with threats are considered.

Key words

L.Q.-difference games, Stackelberg solutions, (linked) econometric policy models.

Contents

1. Introduction

Econometric policy models are mostly used in the following manner. First one singles out one policy maker and selects more or less controllable (exogenous) variables for this policy maker. Then simulation results for objective variables of this policy maker are compared under different scenarios for the instruments.

Optimal control methods provide an alternative for this analysis. A cost-criterium is formed as a function of the paths of objective variables and the use of control variables. Minimization of this criterium yields the optimal level for the instruments with respect to this criterium.

Advantages of optimal control methods are, that all possible scenarios for the instruments are considered and that one is forced to make the objectives of the policy maker explicit. The second point is at the same time the Achilles tendon of the method: how should one quantify "I want the unemployment rate to go down and I want this more badly than a diminution of the rate of inflation!"? In this paper this problem is skipped. Many contributions to the introduction of optimal control methods in the econometric literature were made by Chow.

A drawback of both simulation and optimal control techniques was made vivid again by Lucas [5]. He states, that the coefficiënts of the behavioral equations are not constant under varying (governmental) policies.

It seems to be an improvement to consider more policy makers in the same system at once. This puts the problem in a game-theoretic setting.

The intersection of optimal control and game theory is mostly referred to as a difference game.

This paper is concerned with optimal control methods in a decentralized decision structure, which is also hierarchical or sequential. In a sense all sort of dominant player (government, dominant firm, dominant country etc.) situations are described. The problem is solved under the following assumptions. The system is linear. The cost functionals are quadratic. The noise on the system is Gaussian. Bellman's principle of optimality is postulated, which gives the problem its stagewise character. All the players have complete, error free information about the present state (the memory) of the system. Under these assumptions the optimization problem is equivalent to the same problem in a deterministic

setting.
Chow [3] describes an iterative procedure, which, in case of convergence, leads to a solution in the covariance stationary equilibrium. This paper chooses a more standard approach and gives solutions to the N-level problem and to situations, where more than one player acts at the same decision level.
Some remarks will be made for the case, where the players have information about the past of the endogenous variables, which contains errors.
In the last section attention is devoted to (history dependent) threat strategies in a deterministic setting.
In the theory of difference games the key word for this approach is Stackelberg solutions.

2. The model

We start from a linear(ized) econometric model in reduced form with N policy makers, $N \in \mathbb{N}$. The error terms are distributed normally and are independent over time. Each policy maker sets target paths for certain objective variables over a planning period. These objective variables are related linearly to the endogenous variables. Now, each player tries to steer the system such that the objective variables follow these target paths closely. Steering will be seen as setting the instrument variables at a level, different from their trend values. The criteria are based on deviations of objective variables and control variables from these target paths and trend paths, respectively. Cost functionals are formed by penalizing these deviations. Different weights represent the (political) choices of the policy maker. The instruments are chosen such as to minimize expected total costs. For technical reasons the cost functionals are assumed to be quadratic.
Mathematically we have:
cost functionals

$$\min_{x_i(.)} J_i(y_0, x_1(.), x_2(.), .., x_N(.)) :=$$

$$E\{ \sum_{t=t_0}^{t_f} \{\frac{1}{2} [z(t) - \hat{z}_i(t)]' Q_i(t) [z(t) - \hat{z}_i(t)] +$$

$$+ \sum_{j=1}^{N} \frac{1}{2}[x_j(t) - \hat{x}_j(t)]' R_{ij}(t) [x_j(t) - \hat{x}_j(t)] \}\}, \; i = 1, 2, .., N$$

subject to the econometric model

$$y(t) = A(t)y(t-1) + \sum_{i=1}^{N} C_i(t)x_i(t) + b_1(t) + u(t),$$

$$y(t_0-1) = y_0,$$

$$z(t) = D(t)y(t) + b_2(t),$$

$$E\{u(t)\} = 0, \; E\{u(t)u'(\tau)\} = \delta(t-\tau)\Sigma(t),$$

$$t,\tau = t_0, t_0+1, .., t_f,$$

where

$y(.) \in \mathbb{R}^n$: endogenous variables,

$x_i(.) \in \mathbb{R}^{s_i}$: control variables,

$z(.) \in \mathbb{R}^m$: objective variables,

$b_1(.) \in \mathbb{R}^{r_1}$: exogenous vector,

$b_2(.) \in \mathbb{R}^{r_2}$: exogenous vector,

$\hat{z}_i(.) \in \mathbb{R}^m$: target path,

$\hat{x}_i(.) \in \mathbb{R}^{s_i}$: trend path.

Naturally the matrices $Q_i(.)$ and $R_{ij}(.)$ are positive semi-definite (nonnegative costs). Without loss of generality they are also symmetric. To avoid singularities the matrices $R_{ii}(.)$ are assumed to be positive definite.

The players decide upon their strategy one after another, starting with player N. Every player expects his followers to behave rationally. This rational behaviour is expressed as a function of the strategies of the players higher in the decision hierarchy. Starting at the bottom of the hierarchy this proces leads to an optimal control problem for player N.

This yields a solution for the game.
Notation: for convenience the time dependency of the data matrices is not written; quadratic terms like $x'Ax$ are written as $\|x\|_A$.

3. Information and the principle of optimality

The hierarchical solution concept implies, that players know the strategies of players higher in the hierarchy. The question rises, what information the players have about the endogenous variables. When the information contains errors, the problem becomes difficult. The players have to estimate the endogenous vector. How can a player use knowledge about the strategies of other players to get insight in the information of these other players? How can a player judge the rational behaviour of other players properly, not knowing the information of these other players or not knowing the estimate of these other players? Attempts to solve this problem so far claim, that it is not possible to deduce information about the endogenous vector from knowledge about the strategies of other players. Castanon and Athans [2] solve the two player problem, assuming that the information of the follower is also known to the leader (a nested information structure). Başar [1] evaluates the stochastical structure of the two player problem. In this paper we will not go into this matter any further. We will assume, that the players have error free complete information about the values of the one step delayed endogenous variables (the state of the system). And the players know at least the cost functionals of the players lower in the hierarchy. Furthermore, we will claim, that the principle of optimality holds. This is generally not the case under the hierarchical solution concept: when the rational behaviour of the first player is substituted in the system equations, the (new) system looses its nonanticipativity (de Zeeuw [10]). Simaan and Cruz [7] also give counterexamples. A technical consequence of the claim is, that we can solve the problem by dynamic programming. Or, the solution can be found stagewise. Another consequence of the claim is, that the solution retains its optimality properties after any suboptimal play. This is an argument in favor of the stagewise solution concept. We conclude this section with the mathematical structure of the problem.
As always, when the method of dynamic programming is used in linear

quadratic frameworks, the basics are backward recursive equations for quadratic value functions.

Quadratic value functions for players 1, 2, .., N, respectively:

$$V_i(.,y) := \frac{1}{2}\|y\|_{K_i(.)} + g_i'(.)y + c_i(.), \; i = 1, 2, .., N. \tag{3.1}$$

Without loss of generality the matrices $K_i(.)$ are symmetric.

Backward recursive equations for the value functions:

$$V_i(t_0,y) = \min_{x_i(t_0)} E\{\frac{1}{2}\sum_{j=1}^{N} \|x_j(t_0) - \hat{x}_j(t_0)\|_{R_{ij}} + \\ + V_i(t_0+1, Ay+\sum_{j=1}^{N} C_j x_j(t_0) + b_1(t_0) + u(t_0))\}, \tag{3.2}$$

$$V_i(t,y) = \min_{x_i(t)} E\{\frac{1}{2}\|Dy + b_2(t-1) - \hat{z}_i(t-1)\|_{Q_i} + \\ + \frac{1}{2}\sum_{j=1}^{N} \frac{1}{2}\|x_j(t) - \hat{x}_j(t)\|_{R_{ij}} + \\ + V_i(t+1, Ay+\sum_{j=1}^{N} C_j x_j(t) + b_1(t) + u(t))\}, \\ t = t_0+1, t_0+2, .., t_f, \tag{3.3}$$

$$V_i(t_f+1, y) = \frac{1}{2}\|Dy + b_2(t_f) - \hat{z}_i(t_f)\|_{Q_i} \tag{3.4}$$

So, at each stage t, $t = t_0, t_0+1, .., t_f$, we have a static optimization problem. Each player observes the value y of the vector of one step delayed endogenous variables $y(t-1)$ and the values x_i of the vectors of actions $x_i(t)$ of players higher in the decision hierarchy.

Define:

$$\hat{J}_i(t, y, x_1, x_2, .., x_N) := E\{\frac{1}{2}\sum_{j=1}^{N}\|x_j - \hat{x}_j(t)\|_{R_{ij}} + \\ + V_i(t+1, Ay + \sum_{j=1}^{N} C_j x_j + b_1(t) + u(t))\}, \\ t = t_0, t_0+1, .., t_f. \tag{3.5}$$

If there exist mappings $f_i : \mathbb{R}^{s_{i+1}} \times \ldots \times \mathbb{R}^{s_N} \to \mathbb{R}^{s_i}$, $i = 1, 2, .., N$, such that for any fixed $(x_{i+1}, \ldots, x_N) \in \mathbb{R}^{s_{i+1}} \times \ldots \times \mathbb{R}^{s_N}$

(i) $\quad (x_1, .., x_{i-1}, f_i((x_{i+1}, .., x_N)), x_{i+1}, .., x_N) \in W_{i-1}$

(ii) $\quad \hat{J}_i(t, y, x_1, .., x_{i-1}, f_i((x_{i+1}, .., x_N)), x_{i+1}, .., x_N) \leqq$

$$\leqq \hat{J}_i(t, y, x_1, .., x_N) \text{ for all } (x_1, .., x_N) \in W_{i-1},$$

where $\quad W_0 := \mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_N}$

$$W_i := \{(x_1, .., x_N) \in W_{i-1} | x_i = f_i((x_{i+1}, .., x_N))\},$$

$$i = 1, 2, .., N-1,$$

and if there exists a $(\tilde{x}_1, .., \tilde{x}_N) \in W_{N-1}$, such that

$$\hat{J}_N(t, y, \tilde{x}_1, .., \tilde{x}_N) \leqq \hat{J}_N(t, y, x_1, .., x_N)$$

for all $(x_1, .., x_N) \in W_{N-1}$, then $(\tilde{x}_1, .., \tilde{x}_N)$ constitutes a solution to the problem.

In fact one optimizes, sequentially over the rational reaction sets W_i, $i = 1, 2, .., N-1$, subsets of the action space W_0. As will be shown in one of the next sections, under the convexity assumptions there is only one solution.

4. Certainty equivalence.

The stochastical structure of the optimization problem is very simple. The optimal strategies are the same as in the deterministic case, where $\Sigma(t) \equiv 0$, $t = t_0, t_0+1, .., t_f$. This "certainty equivalence" property results from the evaluation of the expectations $\hat{J}_i$, $i = 1, 2, .., N$, in (3.5), using (3.1).

$$E\{\frac{1}{2}\sum_{j=1}^{N}\|x_j-\hat{x}_j(t)\|_{R_{ij}} + V_i(t+1,\ Ay + \sum_{j=1}^{N} C_j x_j + b_1(t) + u(t))\} =$$

$$= E\{\frac{1}{2}\sum_{j=1}^{N}\|x_j-\hat{x}_j(t)\|_{R_{ij}} + \frac{1}{2}\|Ay + \sum_{j=1}^{N} C_j x_j + b_1(t)\|_{K_i(t+1)} +$$

$$+ [Ay + \sum_{j=1}^{N} C_j x_j + b_1(t)]'K_i(t+1)u(t) + \frac{1}{2}\|u(t)\|_{K_i(t+1)} +$$

$$+ g_i'(t+1)[Ay + \sum_{j=1}^{N} C_j x_j + b_1(t) + u(t)] + c_i(t+1)\} =$$

$$= \frac{1}{2}\sum_{j=1}^{N}\|x_j-\hat{x}_j(t)\|_{R_{ij}} + \frac{1}{2}\|Ay + \sum_{j=1}^{N} C_j x_j + b_1(t)\|_{K_i(t+1)} +$$

$$+ g_i'(t+1)[Ay + \sum_{j=1}^{N} C_j x_j + b_1(t)] + c_i(t+1) + \frac{1}{2}\,\mathrm{trace}\ \Sigma(t)K_i(t+1),$$

$$i = 1, 2, .., N,\ t = t_0, t_0+1, .., t_f. \qquad (4.1)$$

The term $\frac{1}{2}$ trace $\Sigma(t)K_i(t+1)$ is a constant with respect to the optimization variable. It influences only the total costs of the game (see remark 1, page 11). So, it suffices to restrict ourselves to the purely deterministic case, where $\Sigma(t) \equiv 0$ (or $u(t) \equiv 0$), $t = t_0, t_0+1, .., t_f$.

5. N-level hierarchical solution

The optimal decision of player i at time t can be found by minimizing (4.1) with respect to x_i. The Hessian matrix is positive definite, because the matrices Q_i and R_{ij} are positive semi-definite and the matrices R_{ii} are positive definite (see remark 2 at page 11). So, the first order conditions yield the solution. Remember, that the decision of each player depends upon the decisions of players higher in the hierarchy. This leads to the following set of equations.

$$R_{11}[x_1-\hat{x}_1(t)] + C_1'(K_1(t+1)[Ay + \sum_{j=1}^{N} C_j x_j + b_1(t)] + g_1(t+1)) = 0 \qquad (5.1)$$

$$R_{ii}[x_i-\hat{x}_i(t)] + \sum_{j=1}^{i-1}\frac{\partial x_j}{\partial x_i} R_{ij}[x_j-\hat{x}_j(t)] +$$

$$+ (C_i' + \sum_{j=1}^{i-1} \frac{\partial x_j}{\partial x_i} C_j')(K_i(t+1)[Ay + \sum_{j=1}^{N} C_j x_j + b_1(t)] +$$

$$+ g_i(t+1)) = 0, \; i = 2, 3, .., N, \; t = t_0, t_0+1, .., t_f. \tag{5.2}$$

Notation: the sum term Σ and the product term Π should be understood as follows: if the index is decreasing, then $\Sigma \equiv 0$ and $\Pi \equiv I$, and if not, then as normal.

Theorem:

The solution is given by:

$$x_i^*(t) = -F_i(t)[Ay(t-1) + \sum_{j=i+1}^{N} C_j[x_j^*(t)-\hat{x}_j(t)] + \sum_{j=1}^{N} C_j\hat{x}_j(t) + b_1(t)] +$$

$$+ \sum_{\ell=1}^{i} (-F_i^{(\ell)}(t))g_\ell(t+1) + \hat{x}_i(t), \; i = 1, 2, .., N, \; t = t_0, t_0+1, .., t_f, \tag{5.3}$$

where

$$F_1^{(1)}(t) = (R_{11}+C_1'K_1(t+1)C_1)^{-1}C_1' \tag{5.4}$$

$$F_1(t) = F_1^{(1)}(t)K_1(t+1) \tag{5.5}$$

$$F_i(t) = M_i^{-1}(t)C_i'\{\sum_{j=1}^{i-1}(\prod_{k=j+1}^{i-1}(I-C_kF_k(t)))'F_j'(t)R_{ij}F_j(t)$$
$$(\prod_{k=j+1}^{i-1}(I-C_kF_k(t))) + (\prod_{k=1}^{i-1}(I-C_kF_k(t)))'K_i(t+1)$$
$$(\prod_{k=1}^{i-1}(I-C_kF_k(t)))\} \tag{5.6}$$

$$F_i^{(\ell)}(t) = M_i^{-1}(t)C_i'\{\sum_{j=1}^{i-1}(\prod_{k=j+1}^{i-1}(I-C_kF_k(t)))'F_j'(t)R_{ij}$$
$$(F_j^{(\ell)}(t)-F_j(t)\sum_{m=j+1}^{i-1}(\prod_{k=j+1}^{m-1}(I-C_kF_k(t)))C_mF_m^{(\ell)}(t) +$$
$$+ (\prod_{k=1}^{i-1}(I-C_kF_k(t)))'K_i(t+1)(\sum_{m=1}^{i-1}(\prod_{k=1}^{m-1}(I-C_kF_k(t)))$$
$$(-C_mF_m^{(\ell)}(t)))\}, \; \ell = 1, 2, .., i-1 \tag{5.7}$$

$$F_i^{(i)}(t) = M_i^{-1}(t)C_i'(\prod_{k=1}^{i-1}(I-C_kF_k(t)))' \tag{5.8}$$

where

$$M_i(t) = R_{ii} + C_i'\{\sum_{j=1}^{i-1}(\prod_{k=j+1}^{i-1}(I-C_kF_k(t)))'F_j'(t)R_{ij}F_j(t)$$

$$(\prod_{k=j+1}^{i-1}(I-C_kF_k(t))) + (\prod_{k=1}^{i-1}(I-C_kF_k(t)))'K_i(t+1)$$

$$(\prod_{k=1}^{i-1}(I-C_kF_k(t)))\}C_i \tag{5.9}$$

for $i = 2, 3, .., N$, $t = t_0, t_0+1, .., t_f$

and $K_i(.)$, $i = 1, 2, .., N$, is the solution of the backward recursive matrix "Riccati" equations

$$K_i(t) = D'Q_iD + A'\{\sum_{j=1}^{N}(\prod_{k=j+1}^{N}(I-C_kF_k(t)))'F_j'(t)R_{ij}$$

$$F_j(t)(\prod_{k=j+1}^{N}(I-C_kF_k(t))) + (\prod_{k=1}^{N}(I-C_kF_k(t)))'K_i(t+1)$$

$$(\prod_{k=1}^{N}(I-C_kF_k(t)))\}A,\ t = t_f, t_f-1, .., t_0+1, \tag{5.10}$$

$$K_i(t_f+1) = D'Q_iD \tag{5.11}$$

and $g_i(.)$, $i = 1, 2, .., N$, is the solution of the backward recursive "tracking" equations

$$g_i(t) = D'Q_i[b_2(t-1)-\hat{z}_i(t-1)] + A'\{\sum_{j=1}^{N}(\prod_{k=j+1}^{N}(I-C_kF_k(t)))'$$

$$F_j(t)R_{ij}[F_j(t)(\prod_{k=j+1}^{N}(I-C_kF_k(t)))[\sum_{j=1}^{N}C_j\hat{x}_j(t)+b_1(t)] +$$

$$+ \sum_{\ell=1}^{N}(F_j^{(\ell)}(t)-F_j(t)\sum_{m=j+1}^{N}(\prod_{k=j+1}^{m-1}(I-C_kF_k(t)))C_mF_m^{(\ell)}(t))$$

$$g_\ell(t+1)] + (\prod_{k=1}^{N}(I-C_kF_k(t)))'(K_i(t+1)\ [(\prod_{k=1}^{N}(I-C_kF_k(t)))$$

$$[\sum_{j=1}^{N} C_j \hat{x}_j(t)+b_1(t)] + \sum_{\ell=1}^{N} \sum_{m=1}^{N} (\prod_{k=1}^{m-1} (I-C_k F_k(t)))$$

$$(-C_m F_m^{(\ell)}(t) g_\ell(t+1)] + g_i(t+1))\},$$

$$t= t_f,\ t_f-1,\ ..,\ t_0+1, \tag{5.12}$$

$$g_i(t_f+1) = D'Q_i[b_2(t_f)-\hat{z}_i(t_f)]. \tag{5.13}$$

The structure of the solution is comparable to Gardner and Cruz [4]. An extensive treatment of the proof can be found in de Zeeuw [10]. Three lemmas are essential. They will be stated here, also as a link to the next section.

Lemma 1:

a) $$I - \sum_{i=j}^{n} P_i (\prod_{k=i+1}^{n} (I-P_k)) = \prod_{k=j}^{n} (I-P_k)$$

b) $$\sum_{i=j}^{n} \{\tilde{P}_i - P_i \sum_{k=i+1}^{n} (\prod_{\ell=i+1}^{k-1} (I-P_\ell))\tilde{P}_k = \sum_{k=j}^{n} (\prod_{\ell=j}^{k-1} (I-P_\ell))\tilde{P}_k$$

Lemma 2:

If (5.3) is correct for $i = 1, 2, .., j-1$, then

$$\frac{\partial x_i^*(t)}{\partial x_j^*(t)} = -C_j'(\prod_{k=i+1}^{j-1} (I-C_k F_k(t)))' F_i'(t),\ i = 1, 2, .., j-1,$$

for $j = 2, 3, .., N$, $t = t_0, t_0+1, .., t_f$.

Lemma 3:

If (5.3) is correct for $i = 1, 2, .., j-1$, then

$$x_i^*(t) - \hat{x}_i(t) = -F_i(t)(\prod_{k=i+1}^{j-1} (I-C_k F_k(t)))(Ay(t-1) + \sum_{m=j}^{N} C_m[x_m^*(t)-\hat{x}_m(t)] +$$

$$+ \sum_{m=1}^{N} C_m \hat{x}_m(t) + b_1(t)) + \sum_{\ell=1}^{N} (-F_i^{(\ell)}(t) + F_i(t)$$

$$\sum_{m=i+1}^{j-1} (\prod_{k=i+1}^{m-1} (I-C_k F_k(t)))C_m F_m^{(\ell)}(t)) g_\ell(t+1),\ i = 1, 2, .., j-1,$$

for $j = 2, 3, .., N$, $t = t_0, t_0+1, .., t_f$.

Remarks:

(1) In order to find the costs of the game, the constant terms $c_i(.)$, $i = 1, 2, .., N$, in the value functions should also be evaluated. If the variance $\Sigma(t) \neq 0$, $t = t_0, t_0+1, .., t_f$, an extra term enters the recursive equations.

(2) By inductive reasoning it can be shown, that the Hessian matrices $(R_{11}+C_1'K_1(.)C_1)$ in (5.4) and $M_i(.)$, $i = 2, 3, .., N$, in (5.9) are positive definite, hence non-singular, so that minimum costs, $F_i(.)$ and $K_i(.)$, $i = 1, 2, .., N$, exist (de Zeeuw [10]).

(3) When the N-level hierarchical solution is substituted, the model can be written as follows:

$$y(t) = \left(\prod_{k=1}^{N}(I-C_kF_k(t))\right)\left(Ay(t-1) + \sum_{j=1}^{N} C_j\hat{x}_j(t) + b_1(t)\right),$$

$$t = t_0, t_0+1, .., t_f,$$

$$y(t_0-1) = y_0.$$

(4) The algorithm for the hierarchical solution has a loop backward in time and upward in level, consisting of the equations (5.4) up to (5.11), and a loop forward in time, consisting of the equations (5.3) and the model equations.

6. More than one player at the same decision level.

The decision hierarchy is changed in the sense, that now $q (q > 1)$ players act at decision level p. So, there are N-q+1 decision levels left. At level p a Nash (or Cournot) equilibrium is assumed. Three cases are distinguished: $p = 1$, $1 < p < N-q+1$ and $p = N-q+1$.

(a) $p = 1$.

The set of equations (5.1) and (5.2) changes in the sense, that now (5.1) is effective for $i = 1, 2, .., q$ (instead of only for $i = 1$) and (5.2) is effective for $i = q+1, q+2, .., N$. The solution of (5.1) for $i = 1, 2, .., q$ is given by the Nash solution for such a problem. Here it is preferred to state this Nash solution recursively with an artificial hierarchy of players. In this way the structure of the solution (5.3) does not change, so that the solution (5.2) for i = q+1, q+2, .., N and the "Riccati" and "tracking" equations (5.10) up to (5.13) do not change. So, this decision hierarchy only gives rise to some adjustments in (5.4) up to (5.9):

$$F_i(t) = M_i^{-1}(t)C_i'K_i(t+1)\left(\prod_{k=1}^{i-1}(I-C_kF_k(t))\right)$$

$$F_i^{(\ell)}(t) = M_i^{-1}(t)C_i'K_i(t+1)\left(\sum_{m=1}^{i-1}\left(\prod_{k=1}^{m-1}(I-C_kF_k(t))\right)\right.$$

$$\left.(-C_mF_m^{(\ell)}(t))\right), \quad \ell = 1, 2, .., i-1$$

$$F_i^{(i)}(t) = M_i^{-1}(t)C_i'$$

where

$$M_i(t) = R_{ii} + C_i'K_i(t+1)\left(\prod_{k=1}^{i-1}(I-C_kF_k(t))\right)C_i$$

for $i = 1, 2, .., q$, $t = t_0, t_0+1, .., t_f$.

(b) $1 < p < N-q+1$.

The set of equations (5.1) and (5.2) changes in the sense, that in (5.2)

$$\sum_{j=1}^{i-1} \text{ becomes } \sum_{j=1}^{p-1} \text{ for } i = p+1, p+2, .., p+q-1.$$

Again, the solution for i = p+1, p+2, .., p+q-1 is stated in a recursive way with an artificial hierarchy of players, so that the solution for i = p+q, p+q+1, .., N and the "Riccati" and "tracking" equations do not change. Of course, the solution for i = 1, 2, .., p does not change either. So, this decision hierarchy gives rise to the following adjustments: in (5.6) up to (5.9) for i = p+1, p+2, .., p+q-1

$$\left(\prod_{k=1}^{i-1}(I-C_kF_k(t))\right)' \text{ is replaced by}$$

$$I - \left(\prod_{k=p}^{i-1}(I-C_kF_k(t))\right)'\left(I-\left(\prod_{k=1}^{p-1}(I-C_kF_k(t))\right)'\right)$$

and $\sum_{j=1}^{i-1}$ is replaced by $\sum_{j=1}^{p-1}$.

(c) $p = N-q+1$.

This case is not different from case (b), except for the fact, that now the sequence $p+q$, $p+q+1$, .., N is empty.

An extensive treatment of this section can be found in de Zeeuw [11]. Crucial in the proofs is, that lemma 2 and lemma 3 still hold, because (5.3) is kept valid. Other changes in the decision hierarchy can be treated in the same way.

7. Example

A small numerical example is given. Applications to a macroeconometric policy model for the Common Market can be found in Plasmans and de Zeeuw [6].

Cost functionals: $J_1 = 2y^2(0) + 2y^2(1) + x_1^2(0) + x_1^2(1)$

$$J_2 = y^2(0) + y^2(1) + x_2^2(0) + x_2^2(1)$$

Model: $y(i) = y(i-1) + x_1(i) + x_2(i),\ i = 0, 1,$

$y(-1) = 1$

(5.12),(5.13): $g_1(2) = g_2(2) = g_1(1) = g_2(1) = 0;$

(5.11): $K_1(2) = 4$, $K_2(2) = 2;$

(5.4),(5.5): $F_1(1) = \frac{2}{3}$; (5.9): $M_2(1) = \frac{20}{9}$; (5.6): $F_2(1) = \frac{1}{10}$;

(5.10): $K_1(1) = \frac{127}{25}$, $K_2(1) = \frac{11}{5};$

(5.4),(5.5): $F_1(0) = \frac{127}{177};$ (5.9): $M_2(0) = \frac{68158}{31329};$ (5.6): $F_2(0) = \frac{2750}{34079};$

(5.3) and model:
hierarchical solution in values:

$$x_2(0) = -\frac{2750}{34079}\ ;\ x_1(0) = -\frac{22479}{34079}\ ;$$

$$x_2(1) = -\frac{885}{34079}\ ;\ x_1(1) = -\frac{5310}{34079}\ ;$$

with simulation 1, $\frac{8850}{34079}$, $\frac{2655}{34079}$.

8. Threat strategies

In this section we will only consider a two person game in a deterministic setting, where the target paths $\hat{z}_i(.)$, the trend paths $\hat{x}_i(.)$ and the exogenous vectors $b_1(.)$ and $b_2(.)$ are all equal to zero. Without loss of generality $D(.) \equiv I$. The matrices $R_{ij}(.)$, $i \neq j$, are also assumed to be positive definite in order to have unique "team solutions" with respect to one cost functional.
So, we have

$$\min_{x_i(.)} J_i(y_0,\ x_1(.),\ x_2(.)) = \frac{1}{2} \sum_{t=t_0}^{t_f} \{\|y(t)\|_{Q_i} + \|x_1(t)\|_{R_{i1}} + \|x_2(t)\|_{R_{i2}}\},$$

$$i = 1,\ 2, \qquad (8.1)$$

subject to

$$y(t) = Ay(t-1) + C_1x_1(t) + C_2x_2(t),\ t = t_0,\ t_0+1,\ ..,\ t_f, \qquad (8.2)$$

$$y(t_0-1) = y_0.$$

Tolwinski [9] consideres the situation, where the second player introduces threats in an attempt to force the first player to a desirable behaviour. This is more in line with the ideas of von Stackelberg about a dominant player solution. In this case the second player needs more information from the past ($y(t-2)$ and $x_2(t-1)$) to be able to judge, whether the other player has behaved according to his wishes. So, the information becomes more "history dependent". In fact, the second player

tries by means of a threat to change the parameters of the game in such a way, that the optimal reaction of the other player coïncides with optimal behaviour with respect to the cost functional of the second player. In case the first player falls for the threat, this concept leads to a solution, which is almost the same as the "team solution" under the second cost functional. We will analyze this concept, in case the second player knows the value of the last action of the first player ($x_1(t-1)$) in addition to $y(t-1)$, $y(t-2)$ and $x_2(t-1)$. Also selection among all the possible threats is considered.
The objective of the second player is the "team solution" $(x_1^T(.),\ x_2^T(.))$ with respect to J_2. He tries to confront the first player with an optimal control problem, of which the solution coïncides with $x_1^T(.)$.
In other words, in case player 1 chooses strategy $x_1^T(.)$, player 2 plays $x_2^T(.)$, but in case player 1 deviates from $x_1^T(.)$, player 2 plays such that player 1 is worse off. Out of all the threats, that will serve his purpose, player 2 chooses one, which does not do too much harm to himself, in case it has to be carried out. Note, that at the last stage player 1 is not vulnerable to any threat of player 2, because the game is over. His rational reaction is given by:

$$x_1(t_f) = -(R_{11}+C_1'Q_1C_1)^{-1}C_1'Q_1(Ay(t_f-1)+C_2x_2(t_f)). \tag{8.3}$$

Substitution of (8.3) in the cost functionals (8.1) and the model equations (8.2) leads to the following change in parameters:

$$Q_1(t_f) := Q_1(I-C_1S) \ ;$$

$$Q_2(t_f) := (I-C_1S)'Q_2(I-C_1S) + S'R_{21}S \ ;$$

$$C_1(t_f) := 0 \ ; \ R_{11}(t_f) := 0 \ ; \ R_{21}(t_f) := 0 \ ,$$

where $\quad S = (R_{11}+C_1'Q_1C_1)^{-1}C_1'Q_1$

and $y(t_f)$ is redefined as $Ay(t_f-1) + C_2x_2(t_f)$.

By solving a standard optimal control problem, the "team solution" with respect to J_2 can be found. Suppose this solution is given by (after the change in parameters):

$$x_1^T(t) = -L_1(t)y(t-1), \quad t = t_0, t_0+1, .., t_f-1,$$

$$x_2^T(t) = -L_2(t)y(t-1), \quad t = t_0, t_0+1, .., t_f \qquad (8.4)$$

When we assume, that player 2 will be successfull in realizing this team solution during the time to come, again we can analyze the problem stagewise, backward in time. So, player 2 introduces a threat at stage t, $t = t_f, t_f-1, .., t_0+1$, in order to influence the action of player 1 at stage t-1, as a function of the information $\{y(t-1), y(t-2), x_1(t-1), x_2(t-1)\}$. At stage t_0 player 2 plays $-L_2(t_0)y_0$. The threat strategy must fullfill three conditions:

(a) if $x_1(t-1) = -L_1(t-1)y(t-2)$, then $x_2(t) = -L_2(t)y(t-1)$.

(b) $-L_1(t-1)y(t-2)$ is the optimal control for player 1 at stage t-1.

(c) if $x_1(t-1) \neq -L_1(t-1)y(t-2)$ (in which case the threat should be carried out), then the costs of player 2 are minimized (a selection from all the possible threats).

We consider the following class of controls, which fullfill condition (a):

$$x_2(t) = -L_2(t)y(t-1) + v,$$

where $v = 0$, if $x_1(t-1) = -L_1(t-1)y(t-2)$ $\qquad$ (8.5)

The cost functionals become:

$$J_i = \sum_{s=t_0}^{t-2} \tfrac{1}{2}\{\|y(s)\|_{Q_i} + \|x_1(s)\|_{R_{i1}} + \|x_2(s)\|_{R_{i2}}\} +$$

$$+ \tfrac{1}{2}\|x_1(t-1)\|_{R_{i1}} + \tfrac{1}{2}\|x_2(t-1)\|_{R_{i2}} + \tfrac{1}{2}\|y(t-1)\|_{P_i(t)} + w_i(t),$$

where

$$P_i(t_f) = T'(t_f)Q_iT(t_f) + L_2'(t_f)R_{i2}L_2(t_f) + Q_i$$

$$P_i(t) = T'(t)P_i(t+1)T(t) + L_1'(t)R_{i1}L_1(t) + L_2'(t)R_{i2}L_2(t) + Q_i$$

$$T(t_f) = A - C_2L_2(t_f)$$

$$T(t) = A - C_1L_1(t) - C_2L_2(t)$$

$$w_i(t) = \frac{1}{2}\|v\|_{N_i(t)} + d_i'(t)\ v \tag{8.6}$$

$$d_i'(t_f) = y'(t_f-1)(T'(t_f)Q_iC_2 - L_2(t_f)R_{i2})$$

$$d_i'(t) = y'(t-1)(T'(t)P_i(t+1)C_2 - L_2(t)R_{i2})$$

$$N_i(t_f) = R_{i2} + C_2'Q_iC_2$$

$$N_i(t) = R_{i2} + C_2'P_i(t+1)C_2$$

for $t = t_f-1,\ t_f-2,\ ..,\ t_0+1\ ,\ i = 1,2.$

The terms $w_i(.)$ are the "extra costs", in case the threat is carried out.

If $w_1(t)$ is a differentiable function of $x_1(t-1)$, the conditions

(i) $$\frac{\partial J_1}{\partial x_1(t-1)}\ (-L_1(t-1)y(t-2)) = 0$$

(ii) $$\frac{\partial^2 J_1}{\partial x_1^2(t-1)} > 0$$

(iii) $$w_1(-L_1(t-1)y(t-2)) \geqq 0$$

guarantee condition (b), for:

(i) + (ii) $\Rightarrow J_1(-L_1(t-1)y(t-2),\ -L_2(t)y(t-1) + v) \leqq$

$$J_1(x_1(t-1),\ -L_2(t)y(t-1) + v),\ \forall x_i(t-1)$$

and

$$\text{(iii)} \Rightarrow\ J_1(-L_1(t-1)y(t-2),\ -L_2(t)y(t-1)) \leqq$$

$$J_1(-L_1(t-1)y(t-2),\ -L_2(t)y(t-1) + v). \tag{8.7}$$

Condition (c) implies, that player 2 minimizes $w_2(t)$.

Now player 2 chooses at stage t a threat v, differentiable with respect to $x_1(t-1)$, such that $w_2(t)$ is minimized under the restrictions (i), (ii) and (iii). This is a rather complex problem. We restrict ourselves to solutions with the properties

$$w_1(-L_1(t-1)y(t-2)) = 0 \text{ and } \frac{\partial w_1}{\partial x_1(t-1)} \text{ constant.} \tag{8.8}$$

Then:

$$\frac{\partial J_1}{\partial x_1(t-1)} = (C_1'P_1(t)C_1 + R_{11})x_1(t-1) + \frac{\partial w_1}{\partial x_1(t-1)} +$$

$$+ C_1'P_1(t)(Ay(t-2)+C_2x_2(t-1)).$$

Condition (i) yields:

$$\frac{\partial w_1}{\partial x_1(t-1)}(-L_1(t-1)y(t-2)) = a(t),$$

where $\quad a(t) = (C_1'P_1(t)C_1+R_{11})L_1(t-1)y(t-2) -$

$$- C_1'P_1(t)(Ay(t-2)+C_2x_2(t-1)).$$

Together with assumptions (8.8) this leads to:

$$w_1(x_1(t-1)) = a'(t)(x_1(t-1)+L_1(t-1)y(t-2)). \tag{8.9}$$

We check condition (ii):

$$\frac{\partial^2 J_1}{\partial x_1^2(t-1)} = (C_1' P_1(t) C_1 + R_{11}) > 0.$$

Now we try to solve for the threat v from (8.6) and (8.9). Still the problem is complex. Again we restrict ourselves severely, We consider only threats v, where all but one component are equal to zero. Furthermore, note that threats v, which imply higher extra costs than $w_1(t)$, serve the purpose of player 2 as well (see (8.7)).

In case $d_1(t) = 0$, solutions are given by

$$\begin{aligned} v_j &= \left[\frac{|a'(t)(x_1(t-1) + L_1(t-1)y(t-2))|}{(N_1(t))_{jj}} \right]^{\frac{1}{2}} &&, j = j_0 \\ & && j_0 \in \{1, 2, .., s_1\}, \\ v_j &= 0 &&, j \neq j_0 \end{aligned}$$

because $|a'(t)(x_1(t-1)+L_1(t-1)y(t-2))| \geqq w_1(t)$.

In case $d_1(t) \neq 0$, solutions are given by

$$\begin{aligned} v_j &= \frac{a'(t)(x_1(t-1)+L_1(t-1)y(t-2))}{(d_1(t))_j} &&, j = j_0 \\ & && j_0 \in \{\{1,2,..,s_1\} | (d_1(t))_{j_0} \neq 0\}, \\ v_j &= 0 &&, j \neq j_0 \end{aligned}$$

because $\frac{1}{2}\| v \|_{N_1(t)} \geqq 0$.

Now minimization of $w_2(t)$ has become very simple:

$$\min_{j_0} \{ (N_2(t))_{j_0 j_0} v_{j_0}^2 + (d_2(t))_{j_0} v_{j_0} \}.$$

We realize, that the problem is by far not generally solved. Besides, the model looks not very realistic. In case player 1 also introduces threats in the same way, the situation occurs, that the threats are actually carried out. At $t = t_0$ player 2 plays $-L_2(t_0)y_0$, which is generally not according to the team solution under J_1. So player 1 executes his threat, which is generally not according to the team solution under J_2. So, player 2 executes his threat at $t = t_0+1$, etc. The question rises, who wants to and who can continue this costly proces. We need a concept for relative strength. The dominant player might not be the one, who decides first!

9. Conclusion

In this paper algorithms are given for optimal control methods in a decentralized (partly) hierarchical decision structure for econometric models in reduced form. These algorithms can be used for policy evaluation in case of more than one policy maker with possibly conflicting objectives. Also a concept for threat strategies is investigated. The last results are not very satisfying.

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