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RESEARCH MEMORANDUM

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EQUILIBRIA WITH RATIONING IN AN ECONOMY WITH INCREASING RETURNS

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1. Introduction.[1)]

In this paper it is shown that the theory on (dis)equilibria with quantity rationing, as developed recently by a.o. Drèze (1975), Benassy (1977) and Barro and Grossman (1976), is appropriate to study economies where production takes place under increasing returns to scale. We consider a temporary equilibrium model with quantity rationing. There are three commodities (a single good, labour and money). Consumers sell labour, buy goods and hold money. The producer(s) use(s) labour as the only input to produce goods. Profits are transferred to government and there is a fixed autonomous demand from government. This model was studied a.o. by Malinvaud (1977), Böhm (1978 and 1979), Dehez and Gabszewicz (1977), Gepts (1977), Hildenbrand and Hildenbrand (1978).

A model with a single firm is introduced in section 2 and in section 3 the (temporary) fixed price equilibria (also called "disequilibria" by some authors) are studied under alternative assumptions on the production function (decreasing returns, constant returns and increasing returns). It has been shown (Malinvaud (1977)) that under decreasing returns to scale there can exist four different types of equilibria (plus intermediate cases): Walrasian, Classical, Keynesian and Repressed Inflation. We show that under constant returns Classical equilibrium disappears and that under increasing returns only Keynesian and Repressed Inflation equilibrium (plus an intermediate case) can occur. This is so, because in this case (the) producer(s) has always to be rationed in order to limit the production. In section 4 we consider the same economy with increasing returns but with different potential producers. A rationing scheme for distributing goods demand or labour supply among firms is introduced by means of market share distributions. It appears that the existence of a fixed price equilibrium and the type of equilibrium (Keynesian or Repressed Inflation) that occurs, now also depend on the set of firms that are active, or, if all firms are identical on the number of active firms. A stability concept of equilibrium w.r.t. the set of active firms is introduced, by which an equilibrium is stable if no

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1) I thank Jaques Drèze, Paul van den Heuvel and Pieter Ruys for helpfull conversations. All remaining errors are mine.

non active firm can profitably become active.

I prefer to interpret the model of this paper as a (very simple) micro economic model (with aggregate consumption however), rather than as a macro economic model, as is often done in the literature (e.g. Malinvaud (1977)). Particularly in the case of increasing returns, it seems not reasonable to aggregate implicitely over producers and a fortiori aggregation over different commodities seems not acceptable.

It seems worthwile to study increasing returns, since there are no firm grounds that decreasing returns are normal even in the short run. Also it seems that some phenomena, frequently occurring in times of depression, like forced mergers and failures of firms, could be better explained if one assumes increasing returns. However, the use of a simple short run model like the present one for the analysis of increasing returns seems somewhat unrealistic, it has the advantage that some of the typical problems related to this phenomenon can be made clear.

Obviously, if decreasing returns start only at a very high level of production, a model with increasing returns is appropriate if this high level is not attained by any producer.

2. The model.

Following Böhm (1978), Dehez and Gabszewicz (1977) a.o., we consider an economy with a set I of consumers, a single producer and an autonomous consumer (government). There are three commodities: a consumption good, labour and money. P is the price of the good. W the nominal wage rate and M are the initial money holdings of consumers. In the short period considered, consumers can spend M plus their labour income $W\ell$ on goods and final money holdings M^1. The model will be formulated in terms of real wages $w = W/P$, real initial money holding $m = M/P$ and real final money holdings $m^1 = M^1/P$.

The producer maximizes profit; the profit is transferred to the autonomous sector (and not distributed among consumers). We consider fixed-price temporary equilibria for different values of w and m.

2.1. Consumers

Each consumer makes a plan for present and future trades of goods and labour and consequently for money holdings. Only the first period variables occur in the temporary equilibrium. The plan has to be chosen from a budget set, determined by initial money holdings, the present price and wage rate, present individual constraints on goods and labour, expected future prices and wages and expected individual constraints. Expectations on future parameters may depend on the present values of these parameters and also on the aggregate constraints. This results in individual constrained demand and supply functions for each consumer i, and particularly in constrained functions of present goods and labour:

$$x_i = \xi_i(P,W,M_i;\underline{x}_i,\underline{\ell}_i,\underline{x},\underline{\ell}) \ ,$$

$$\ell_i = \lambda_i(P,W,M_i;\underline{x}_i,\underline{\ell}_i,\underline{x},\underline{\ell}) \ ,$$

where x_i and ℓ_i are constrained demand and supply of goods and labour respectively, $\underline{x}_i$ and $\underline{\ell}_i$ are the individual constraints on goods and labour, i.e. the maximum quantities that could be bought and sold, whereas $\underline{x}$ and $\underline{\ell}$ are the aggregate constraints. Given a rationing scheme, by which individual constraints are determined from aggregate constraints, aggregate demand and supply functions of all consumers can be defined.

This whole process (see Grandmont (1977) for a general treatment and Böhm (1979) and Hildenbrand and Hildenbrand (1978) in relation to a model with rationing) is left implicit in the present paper. We shall introduce demand and supply by aggregate functions. We shall assume that these functions are homogenous of degree zero in P, W and M, so that they can be expressed in terms of real wages $w = W/P \geqq 0$ and real initial money holding $m = W/P \geqq 0$, hence it is also assumed that aggregate demand and supply only depend on total real money holdings $m = \Sigma m_i$ and not on the distribution among consumers.

Total demand x for the good and total supply ℓ of labour by consumers are given by the <u>constrained demand and supply functions</u>:

$$(1) \qquad x = \xi(w,m;\underline{x},\underline{\ell}) \geqq 0 \quad \text{and} \quad L \geqq \ell = \lambda(w,m;\underline{x},\underline{\ell}) \geqq 0 \ ,$$

where $0 \leqq \underline{x} \leqq \infty$ is the total constraint on the good, indicating that consumers can buy at most $\underline{x}$ of the good, and $0 \leqq \underline{\ell} \leqq \infty$ is the total constraint on labour, indicating that at most $\underline{\ell}$ of labour can be sold. L is the maximum quantity of labour available. The constrained demand for money m^1 is

(2) $$m^1 = \mu(w,m;\underline{x},\underline{\ell}) = m + w\lambda(w,m;\underline{x},\underline{\ell}) - \xi(w,m,\underline{x},\underline{\ell}) \geqq 0 .$$

Net savings are defined by

(3) $$s = \sigma(w,m;\underline{x},\underline{\ell}) = \mu(w,m;\underline{x},\underline{\ell}) - m = w\lambda(w,m;\underline{x},\underline{\ell}) - \xi(w,m;\underline{x},\underline{\ell}) .$$

$\underline{x} = \infty$ and $\underline{\ell} = \infty$ indicate that there is no constraint on goods or labour. We call x and ℓ <u>notional</u> (or unconstrained) demand and supply if

(4) $$x = \xi(w,m;\infty,\infty) = \sup_{\substack{\underline{x}\to\infty \\ \underline{\ell}\to\infty}} \xi(w,m;\underline{x},\underline{\ell}) ,$$
$$\ell = \lambda(w,m;\infty,\infty) = \sup_{\substack{\underline{x}\to\infty \\ \underline{\ell}\to\infty}} \lambda(w,m;\underline{x},\underline{\ell}) .$$

Since $m^1 \geqq 0$ and $\lambda(w,m;\underline{x},\underline{\ell}) \leqq L$, also $\xi(w,m;\underline{x},\underline{\ell}) \leqq m + wL$.
So both limits exist and are finite. For ease of notation we write $\xi(w,m)$ and $\lambda(w,m)$ for $\xi(w,m;\infty,\infty)$ and $\lambda(w,m;\infty,\infty)$. We call x and ℓ <u>effective</u> demand and supply if

(5) $$x = \xi(w,m;\infty,\underline{\ell}) = \sup_{\underline{x}\to\infty} \xi(w,m;\underline{x},\underline{\ell}) ,$$
$$\ell = \lambda(w,m;\underline{x},\infty) = \sup_{\underline{\ell}\to\infty} \lambda(w,m;\underline{x},\underline{\ell}) .$$

Effective demand and supply are the quantities that consumers intend to buy or to sell, given a constraint on the other market, but not considering the own constraint. Clearly effective demand equals notional demand if there is no constraint on the other market. A constraint $\underline{x}$ is called <u>binding</u> at $w,m,\underline{\ell}$, <u>if</u> $\xi(w,m;\underline{x},\underline{\ell}) < \xi(w,m;\infty,\underline{\ell})$. Similarly for $\underline{\ell}$. The set <u>of acceptable trades</u> (see Gepts 1977) of consumers is defined by

(6) $$A(w,m) = \{x,\ell \mid \exists\ \underline{x},\underline{\ell} : x = \xi(w,m;\underline{x},\underline{\ell}),\ \ell = \lambda(w,m;\underline{x},\underline{\ell})\}$$

We make the following assumptions on constrained demand functions; for all w and m:

A1 ξ and λ are continuous in all variables; ξ and λ are twice differentiable in w and m and in $\underline{x}$ and $\underline{\ell}$ for $0 < \underline{x} < \xi(w,m)$, $0 < \underline{\ell} < \lambda(w,m)$;

A2 $\xi(w,m;\infty,\underline{\ell}) > 0 \Leftrightarrow m+w\underline{\ell} > 0$;
$\lambda(w,m;\underline{x},\infty) > 0$ if $w > 0$;

A3 $\xi(w,m;\underline{x},\underline{\ell}) = \min\ \{\underline{x},\xi(w,m;\infty,\underline{\ell})\}$ for all $\underline{\ell}$;
$\lambda(w,m;\underline{x},\underline{\ell}) = \min\ \{\underline{\ell},\lambda(w,m;\underline{x},\infty)\}$ for all $\underline{x}$;

A4 $\dfrac{\partial\xi(w,m;\infty,\underline{\ell})}{\partial\underline{\ell}} \geqq 0$ for $0 \leqq \underline{\ell} < \lambda(w,m)$ and $\xi(w,m;\infty,\underline{\ell}) = \xi(w,m)$ for $\underline{\ell} \geqq \lambda(w,m)$;
$\dfrac{\partial\lambda(w,m;\underline{x},\infty)}{\partial\underline{x}} \geqq 0$ for $0 \leqq \underline{x} < \xi(w,m)$ and $\lambda(w,m;\underline{x},\infty) = \lambda(w,m)$ for $\underline{x} \geqq \xi(w,m)$;

A5 $\dfrac{\partial\,\sigma(w,m;\underline{x},\infty)}{\partial\underline{x}} < 0$ for $0 \leqq \underline{x} < \xi(w,m)$;
$\dfrac{\partial\,\sigma(w,m;\infty,\underline{\ell}}{\partial\underline{\ell}} > 0$ for $0 \leqq \underline{\ell} < \lambda(w,m)$;

A6 $\dfrac{\partial\xi(w,m;\infty,\underline{\ell})}{\partial m} > 0$ and $\dfrac{\partial\lambda(w,m;\underline{x},\infty)}{\partial m} < 0$ for all $\underline{x},\underline{\ell}$;

A7 if $w' > w$, then $\xi(w',m;\infty,\underline{\ell}) > \xi(w,m;\infty,\underline{\ell})$ and
$\sigma(w',m;\infty,\underline{\ell}) > \sigma(w\ ,m;\infty,\underline{\ell})$, for all $\underline{\ell}$.

By A3, consumers buy and sell as much as they can, if the constraint is binding. By A4, relaxation of a constraint on one market will not lead to a decrease of demand or supply on the other market. By assumption A5 only a part of extra labour income, due to a relaxation of the labour constraint, is used for consumption, whereas only a part of extra consumption possibilities is financed by extra labour supply. A4 and A5 imply:

(7) $$\frac{\partial\xi(w,m;\infty,\underline{\ell})}{\partial\underline{\ell}} < w \text{ and } \frac{\partial\lambda(w,m;\underline{x},\infty)}{\partial\underline{x}} < \frac{1}{w} \quad .$$

From A6, which requires that effective demand of goods and effective supply of labour increase and decrease respectively, after an increase of m, it follows that

(8) $$\frac{\partial\sigma(w,m;\infty,\underline{\ell})}{\partial m} > 0 \text{ and } \frac{\partial\sigma(w,m;\underline{x},\infty)}{\partial m} < 0 \quad .$$

By A7 an increase of the wage rate will always lead to both an increase of consumption and of savings (if $\underline{\ell}$ is binding or not); it implies that wage income will also increase, (but not necessarily labour supply).

However, these assumptions seem reasonable, they do not straightforwardly follow from utility maximizing behaviour of consumers (see Hildenbrand and Hildenbrand (1978) and also Van den Heuvel (1979)).

LEMMA 2.1.: Under assumptions A and if $\xi(w,m;\infty,\ell) \leqq x \leqq \xi(w,m)$ and $\lambda(w,m;x,\infty) \leqq \ell \leqq \lambda(w,m)$, then $x = \xi(w,m)$ and $\ell = \lambda(w,m)$).

Proof: Suppose $x < \xi(w,m) \equiv x^* = \xi(w,m;\infty,\ell^*)$, for $\ell^* = \lambda(w,m)$, by assumption A4, then by assumption A5: $(x^*-x) < w(\ell^*-\ell)$. If $\ell^* \equiv \lambda(w,m) = \ell$, then $x^*-x < 0$, a contradiction; so let $(\ell^*-\ell) > 0$; again by assumption A5: $(\ell^*-\ell) < \frac{1}{w}(x^*-x)$. Combining the two inequalities gives $(x^*-x) < (x^*-x)$, a contradiction. □

PROPOSITION 2.2.: Under assumptions A:

$$A(w,m) = \{x,\ell \mid 0 \leqq x \leqq \xi(w,m;\infty,\ell) \text{ and } 0 \leqq \ell \leqq \lambda(w,m;x,\infty)\}$$

Proof: Let $x = \xi(w,m;\underline{x},\underline{\ell})$ and $\ell = \lambda(w,m;\underline{x},\underline{\ell})$. We prove $x \leqq \xi(w,m;\infty,\ell)$, the proof for ℓ being similar.
If $\ell = \underline{\ell}$, then by A3:

$$x = \xi(w,m;\underline{x},\ell) = \min\{\underline{x},\xi(w,m;\infty,\ell)\} \leqq \xi(w,m;\infty,\ell) \ .$$

So let $\ell < \underline{\ell}$. Since $\ell = \min\{\underline{\ell},\lambda(w,m;\underline{x},\infty)\}$, we have

(i) $\quad \ell = \lambda(w,m;\underline{x},\infty) < \underline{\ell}$

First assume $x = \underline{x}$ and suppose $x > \xi(w,m;\infty,\ell)$. Considering (i) and lemma 2.1. this implies $\ell = \lambda(w,m)$ and $x = \xi(w,m)$, a contradiction since now $x > \xi(w,m;\infty,\ell) = \xi(w,m)$, by A4.

For $x < \underline{x}$, we get, similarly to (i):

(ii) $\quad x = \xi(w,m;\infty,\ell) < \underline{x}$.

Now from (i) and (ii), applying lemma 2.1., it follows $\underline{x} = \xi(w,m)$ and $\ell = \lambda(w,m)$ and that is a contradiction since $x = \xi(w,m;\infty,\underline{\ell}) = \xi(w,m) = \underline{x}$, by A4. □

Any trade $(x,\ell) \in A(w,m)$ may be chosen by consumers under at most two constraints; they will choose: (1) $(\xi(w,m), \lambda(w,m))$ if there is no constraint; (2) $(\xi(w,m;\infty,\underline{\ell}),\underline{\ell})$ if $\underline{\ell} < \lambda(w,m)$ is the only binding constraint; (3) $(\underline{x},\lambda(w,m;\underline{x},\infty))$, if $\underline{x} < \xi(w,m)$ is the only binding constraint, and (4) $(\underline{x},\underline{\ell})$ in all other cases (with both $\underline{x}$ and $\underline{\ell}$ binding constraints).

PROPOSITION 2.3.: Under the assumptions A, the correspondence A is continuous and compact valued; $(0,0) \in A(w,m)$ for all (w,m) and if $\xi(w,m) > 0$ and $\lambda(w,m) > 0$, then Int $A(w,m) \neq \emptyset$.

<u>Proof</u>: Continuity follows from the continuity of ξ and λ; since for all $(x,\ell) \in A(w,m)$, $(x,\ell) \leqq (wL+m,L)$, $A(w,m)$ is compact. By A5 any trade (x,ℓ) such that $x < \xi(w,m)$, $\ell < \lambda(w,m)$ and $w\ell - x = w\lambda(w,m) - \xi(w,m)$, is in the interior of $A(w,m)$. □

2.2. <u>Government</u>

The autonomous demand is $g \geqq 0$ and will be assumed fixed troughout this paper. The government is served by the producer by priority.

2.3. <u>The producer</u>

The firm uses labour input $z \geqq 0$ to produce output $v \geqq 0$. It does not hold money and does not invest in stock or assets, so there is no relation between present and future decisions. The technology is given by a production function

(9) $v = f(z)$.

We shall assume f to be increasing and differentiable; f' and f" denote the first and second derivatives of f. Define $y = v-g$, so y is the output remaining for consumers after fulfilling government demand g. Clearly $-g \leqq y \leqq \infty$; negative values of y mean that production is not sufficient to fulfil autonomous demand. Let $\underline{y}$ and $\underline{z}$ be constraints on sales of goods and on the purchase of labour. π is the constrained maximum profit function:

(10) $\pi(w;\underline{y},\underline{z}) = \sup\{(y+g) - wz \mid (y+g) = f(z),\ y \leqq \underline{y},\ z \leqq \underline{z}\}$.

Notional profit equals $\pi(w;\infty,\infty) = \pi(w)$. η and γ are the constrained supply and demand corresponces for goods and labour:

(11) $\eta(w;\underline{y},\underline{z}) = \{y \mid (y+g) - wf^{-1}(y+g) = \pi(w;\underline{y},\underline{z})$ and $y \leqq \underline{y}, y \leqq f^{-1}(\underline{z})\}$,

$\gamma(w;\underline{y},\underline{z}) = f^{-1}(\eta(w;\underline{y},\underline{z})+g)$.

Notional supply and demand are $\eta(w;\infty,\infty) = \eta(w)$ and $\gamma(w;\infty,\infty) = \gamma(w)$. Effective supply and demand are $\eta(w;\infty,\underline{z})$ and $\gamma(w;\underline{y},\infty)$. The set of acceptable trades of the producer is defined by:

(12) $B(w) = \{(y,z) \mid \exists\, \underline{y},\underline{z} : y = \eta(w;\underline{y},\underline{z})$ and $z = f^{-1}(y+g)\}$.

Any acceptable trade $(y,z) \in B(w)$ is an optimum under at most one constraint: a point (y^*,z^*), such that $y^* \in \eta(w)$ and $z^* = f^{-1}(y^*+g)$ is optimal without constraints; any other point is a constrained optimum under either a constraint on the good or on labour: if $\underline{y}+g = f(\underline{z})$, then $\eta(w;\underline{y},\underline{z}) = \eta(w,\underline{y},\infty) = \eta(w;\infty,\underline{z})$. Clearly B(w) contains all pairs (y,z), such that by a decrease of the level of activity, profit cannot be increased, i.e.: if $(y,z) \in B(w)$, then for all (y',z') such that $y'+g = f(z') < y+g$, we have $y+g-wf(z) >$ $> y'+g-wf(z')$.

DEFINITION 2.4.: A production function is called regular if for all w:

(i) $B(w)$ is closed;

(ii) The projection of $B(w)\setminus\{-g,0\}$ on the z-axis is an interval.

An example of a production function which is not regular is an increasing function f, such that its second derivative f" is first positive, then negative and then positive again, when z increases.

2.4. Equilibrium

We consider the economy

$$(13) \qquad E = \{f;\xi,\lambda\}$$

with a single production function f and the functions ξ and λ as defined in (1). The set $C(w,m) = A(w,m) \cap B(w)$ contains all trades in E which are acceptable both for consumers and producers. Each $(a,b) \in C(w,m)$ requires a suitable rationing scheme, consisting of at most two constraints on consumers and at most one constraint on the producer. Some $(a,b) \in C(w,m)$ require a constraint on the same market for both consumers and the producer. In an equilibrium, however, only one of the two can be rationed on the same market.

DEFINITION 2.5.: An equilibrium at (w,m) in E is a quadruple of trades (x,ℓ,y,z) and a rationing scheme $(\underline{x},\underline{y},\underline{\ell},\underline{z})$, such that

(i) $\quad x = \xi(w,m;\underline{x},\underline{\ell}) = y \in \eta(w;\underline{y},\underline{z})$ and

$\quad \ell = \lambda(w,m;\underline{x},\underline{\ell}) = f^{-1}(y+g) = z$

(ii) $\quad \underline{y} = \infty$ or $\underline{x} = \infty$; $\underline{z} = \infty$ or $\underline{\ell} = \infty$;

(iii) $\quad \underline{z} = \infty$ or $\underline{y} = \infty$.

If η and γ are functions, then (i) becomes

$$x = \xi(w,m;\underline{x},\underline{\ell}) = \eta(w;\underline{y},\underline{z}) = y$$

$$\ell = \lambda(w,m;\underline{x},\underline{\ell}) = \gamma(w;\underline{y},\underline{z}) = z$$

The set of equilibria at (w,m) is denoted by e(w,m). So an element of e(w,m) is the 8-tuple $(x,\ell,y,z;\underline{x},\underline{\ell},\underline{y},\underline{z})$ satisfying the conditions of definition 2.5. The set of equilibrium trades t(w,m) consists of all pairs $(a,b) \in \mathbb{R}^2$, such that for some rationing scheme, $(a,b,a,b;\underline{x},\underline{\ell},\underline{y},\underline{z}) \in e(w,m)$. Note that an equilibrium is defined in such a way that no equilibrium is possible, where government demand g is not completely satisfied.

By definition 2.5., only the following eight combinations of binding constraints are permitted:

$$(\underline{y},\underline{\ell}),\ (\underline{x},\underline{z}),\ (\underline{x},\underline{\ell}),\ (\underline{y}),\ (\underline{z}),\ (\underline{\ell}),\ (\underline{x}),\ \text{(no constraint)}\ .$$

Each of these combinations corresponds to a particular type of equilibrium:

(K): Keynesian equilibrium: excess supply on both markets; consumers are rationed on the labour market, producers on the goods market; consumer demand for goods is insufficient to employ their own labour supply; see fig. 1.

(I): Repressed inflation equilibrium: excess demand on both markets; consumers are rationed on the goods market, producers on the labour market; consumers do not supply enough labour to produce their own demand for goods.

(C): Classical equilibrium: consumers are rationed on both markets; producers realize their notional supply and demand.

(KI): Intermediate cases between Keynesian and Repressed inflation equilibrium; consumers are not rationed and for the producer there is either: a constraint $\underline{z}$ on labour or a constraint $\underline{y}$ on goods. These two cases are equivalent: a constraint $\underline{z}$ may be replaced by a constraint $\underline{y} = f(\underline{z})-g$ and vice versa.

(KC): Intermediate case between Keynesian and classical equilibrium; consumers rationed on the labour market.

(IC): Intermediate case between Repressed Inflation and Classical equilibrium; consumers are rationed on the goods market.

(W): Walrasian equilibrium: no constraints; intermediate case between all other cases.

These equilibria are summarized in table I. They will be called in the rest of this paper: K-equilibrium, I-equilibrium, etc.

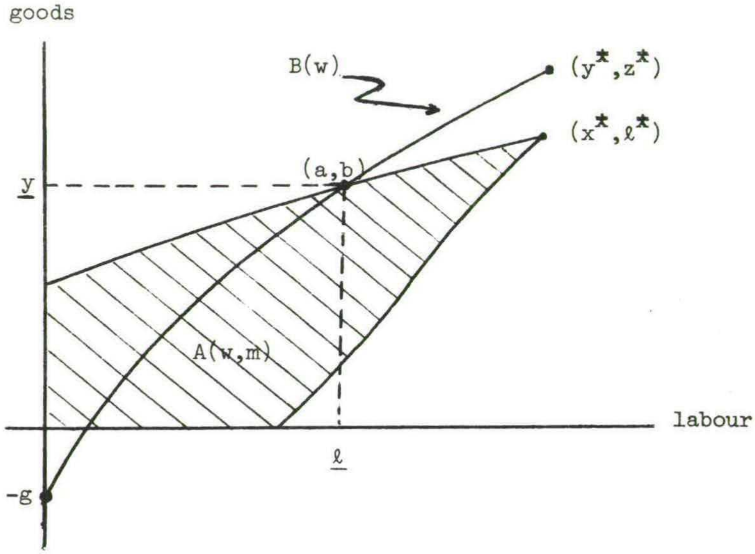

fig. 1

In fig. 1 the shaded area is the set of acceptable trades A(w,m) of consumers and $(x^*,\ell^*) = (\xi(w,m),\lambda(w,m))$; the curve connecting (-g,0) and $(y^*,z^*) = (\eta(w),\gamma(w))$ is the set of acceptable trades B(w) of producers, for a concave production function. The figure depicts a K-equilibrium where $t(w,m) = (a,b) = (\underline{y},\underline{\ell})$. For each type of equilibrium such a picture can be drawn; particularly, in a C-equilibrium (y^*,z^*) is in the interior of A(w,m) and in a KI-equilibrium B(w) contains (x^*,ℓ^*).

Table I. Equilibria

	$\underline{\ell} = b$		$\underline{z} = b$
$\underline{y} = a$	(K) KEYNESIAN $\bar{x} = a < \bar{y}$ $\bar{\ell} > b = \bar{z}$	(KI)$_1$ $x^* = \bar{a} = a < \bar{y}$ $\ell^* = \bar{\ell} = b = \bar{z}$	
	D,C,I	D,C,I	
	(KC) $\bar{x} = a = y^*$ $\bar{\ell} > b = z^*$	(W) WALRASIAN $x^* = \bar{x} = a = y^*$ $\ell^* = \ell = b = z^*$	(KI)$_2$ $x^* = \bar{x} = a = \bar{y}$ $\ell^* = \bar{\ell} = b < \bar{z}$
	D,C	D,C	D,C,I
$\underline{x} = a$	(C) CLASSICAL $\bar{x} > a = y^*$ $\bar{\ell} > b = z^*$	(IC) $\bar{x} = a = y^*$ $\bar{\ell} = b = z^*$	(I) INFLATIONARY $\bar{x} > a = \bar{y}$ $\bar{\ell} = b < \bar{z}$
	D.	D,C	D,C,I

where

$$x^* = \xi(w,m);\ \ell^* = \lambda(w,m);\ y^* \in \eta(w);\ z^* = f^{-1}(y^*+g) \in \gamma(w);$$

$$\bar{x} = \xi(w,m;\infty,\underline{\ell});\ \bar{\ell} = \lambda(w,m;\underline{x},\infty);\ \bar{y} \in \eta(w;\infty,\underline{z});\ \bar{z} = f^{-1}(\bar{y}+g) \in \gamma(w;\infty,\underline{z});$$

$\underline{x}, \underline{\ell}, \underline{y}, \underline{z}$ are binding constraints.

D, C and I indicate that the equilibria are possible under decreasing, constant and increasing returns respectively (sections 3.1., 3.2. and 3.3.).

Let Q be the set of all pairs $(w,m) \in \mathbb{R}_+^2$, such that an equilibrium exists at (w,m). We assume (see Böhm (1978)):

B1 $\quad z_g < \lambda(w,m;0,\infty)$ for all $(w,m) \geqq 0$ and $z_g = f^{-1}(g)$;

B2 $\quad \dfrac{\partial\lambda(w,m;\underline{x},\infty)}{\partial\underline{x}} < \dfrac{d(f^{-1}(\underline{x}+g))}{\partial\underline{x}}$ for $\underline{x} < \xi(w,m)$.

LEMMA 2.6.: Under assumptions A and B and the regularity of f: if $(a,b) \in C(w,m)$ and $(\tilde{a},\tilde{b}) \in B(w,m)$ with $(\tilde{a},\tilde{b}) < (a,b)$, then $(\tilde{a},\tilde{b}) \in \text{Int } A(w,m)$.

<u>Proof</u>: By A5:

$$wa - \xi(w,m;\infty,b) > w\tilde{a} - \xi(w,m;\infty,\tilde{b}) .$$

Since $(a,b) \in C(w,m)$

$$\xi(w,m;\infty,b) \geqq f(b) - g = a .$$

Since $\tilde{b} < b$, and $(f(\tilde{b}) - g,\tilde{b}) \in B(w)$,

$$\pi(w;\infty,b) = f(b) - wb \geqq f(\tilde{b}) - w\tilde{b} = \pi(w;\infty,\tilde{b}) \geqq 0 ,$$

hence

$$\xi(w,m;\infty,\tilde{b}) > \xi(w,m;\infty,b) - wb + w\tilde{b} \geqq$$

$$\xi(w,m;\infty,b) - f(b) + f(\tilde{b}) \geqq f(\tilde{b}) - g ,$$

and

(a) $\quad \xi(w,m;\infty,\tilde{b}) > f(\tilde{b}) - g$.

On the other hand, by B1: $\lambda(w,m;0,\infty) > z_g$; since $(a,b) \in A(w,m)$:

$$\lambda(w,m;a,\infty) \geqq b .$$

By B2, for $0 < \tilde{a} < a$,

(b) $$\lambda(w,m;\tilde{a},\infty) > f^{-1}(\tilde{a}+g) .$$

By (a) and (b), $(\tilde{a},\tilde{b}) \in \text{Int } A(w,m)$. □

THEOREM 2.7.: Under assumptions A and B and the regularity of f:

(i) $$Q = \{(w,m)|C(w,m) \neq \emptyset\}$$

(ii) at $(w,m) \in Q$ there exists a single equilibrium trade $t(w,m)$.

Proof:

(i) Since $A(w,m)$ and $B(w)$ are closed sets by proposition 2.2. and by regularity, $C(w,m)$ is also closed.
Clearly, $Q \subset \{(w,m)|C(w,m) \neq \emptyset\}$. So let $C(w,m) \neq \emptyset$. Choose $a = \max \{y|\exists z: (y,z) \in C(w,m)\}$ and $b = f^{-1}(a+g)$.
(α) If $(a,b) \in \text{Int } A(w,m)$, then $(a,b) = t(w,m)$ and $e(w,m) = (a,b,a,b;a,b,\infty,\infty)$ is a C-equilibrium.
(β) If $(a,b) \in \text{Bnd } A(w,m)$, then either:

$$a = \xi(w,m;\infty,b) \quad \text{and} \quad b \leqq \lambda(w,m;a,\infty)$$

and $t(w,m) = (a,b)$ corresponds to a K-equilibrium (when $e(w,m) = (a,b,a,b;\infty,b,a,\infty)$), or to a KC-equilibrium, or to a KI-equilibrium, or to a W-equilibrium, or:

$$a < \xi(w,m;\infty,b) \quad \text{and} \quad b = \lambda(w,m;a,\infty)$$

and we have an I-equilibrium with $e(w,m) = (a,b,a,b;a,\infty,\infty,b)$ or an IC-equilibrium.

(ii) Let $(a,b) \in t(w,m)$. Suppose $(\tilde{a},\tilde{b}) \in t(w,m)$ with $(\tilde{a},\tilde{b}) < (a,b)$. Then by Lemma 2.6 , $(\tilde{a},\tilde{b}) \in \text{Int } A(w,m)$. This implies that consumers are rationed on both markets and the producer on one market. That contradicts the definition of an equilibrium.

□

Without A5 and the (ad hoc) assumptions B, different equilibria with different equilibrium trades could occur at (w,m). Note that $(0,z_g) \in B(w)$, ensures that $Q \neq \emptyset$. If the equilibrium trade is unique, still different equilibria could exist differing only in rationing scheme. However, we shall treat such equilibria as a single one, and speak about the equilibrium e(w,m), if t(w,m) is unique, in order to simplify terminology.

3. Equilibria under different assumptions on production

In this section we shall consider the equilibria that may occur
(1) under decreasing returns to scale,
(2) under constant returns to scale,
(3) under increasing returns to scale.

3.1. Decreasing returns

This is the case considered in most papers (Böhm (1976), Dehez and Gabsewicz (1977), Malinvaud (1977)). We assume:

D1 f is continuous and, for $z > 0$, twice differentiable;

D2 $f(0) = 0$ and for $z > 0$, $f'(z) > 0$;

D3 $f''(z) < 0$, for $z > 0$.

Decreasing returns are defined by D3; by D2 and D3; f is a concave function.

Under assumptions D, notional supply and demand are decreasing functions of w. For $z_g = f^{-1}(g)$, $\alpha = f^{-1}(z_g)$ and $\beta = \lim_{z\to\infty} f'(z) = \sup\{w | \forall z > 0: f(z) - wz > 0\}$, we have: $-g \leqq \eta(w) \leqq 0$ if $w \geqq \alpha$, $0 < \eta(w) < \infty$ if $\alpha < w < \beta$ and $\eta(w) = \infty$ if $w \geqq \beta$. The set of acceptable trades is:

$$B(w) = \{y,z | y+g = f(z) \text{ and } y \leqq \eta(w)\} .$$

Clearly f is regular (def. 2.4). Under assumptions D:

$$\eta(w;\underline{y},\infty) = \min\{\eta(w),\underline{y}\},$$
$$\eta(w;\infty,\underline{z}) = \min\{\eta(w),f(\underline{z})-g\}\ .$$

No equilibrium can occur at $w > \alpha$ if $g > 0$, for then $\eta(w,m;\underline{y},\underline{z}) = -g$, for all $(\underline{y},\underline{z})$. (if $g = 0$, then a trivial equilibrium exists with $(a,b) = 0$).

For $w \leqq \alpha$, non trivial equilibria may occur under the eight possible rationing schemes, mentioned in section 2.4. All equilibria summarized in table I, are possible. Under assumptions A, B and D, a unique equilibrium (trade) exists, by theorem 2.7, for all $(w,m) \in Q$. By B1, $Q = \{(w,m)|w \leqq \alpha\}$ if $g > 0$, and $Q = \mathbb{R}^2_+$ if $g = 0$, but in the latter case only trivial C-equilibria occur at $w > \alpha$. It was shown by Böhm (1978) that under assumptions A, B and D a single pair (w,m) exists, for which $e(w,m)$ is a W-equilibrium, and that Q can be decomposed into seven regions, corresponding to the seven types of equilibrium defined above, each region containing all $(w,m) \in Q$ such that $e(w,m)$ is of that type (see fig. 2). The curves separating the Classical, Keynesian and Repressed Inflation region correspond to the intermediate equilibria (KC, KI, IC). The slope of the curve separating the Keynesian and the Classical region is decreasing; the slope of the curve separating the Repressed Inflation and the Classical region is <u>increasing</u>, provided that it is also assumed:

$$\frac{\partial(w,m;\underline{x},\infty)}{\partial w} > 0\ .$$

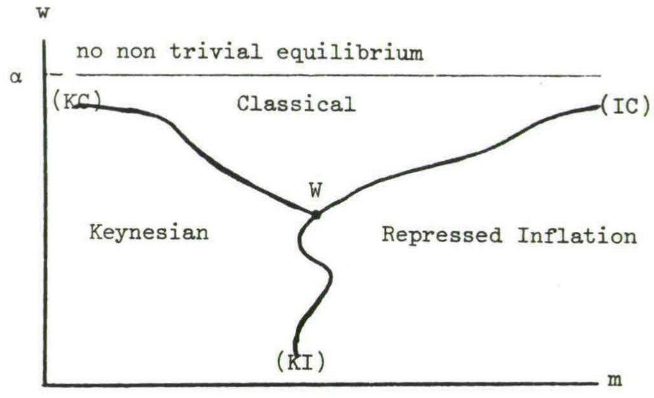

fig. 2

3.2. Constant returns

As an intermediate case we consider an economy with constant returns, which are defined by the assumption:

C There exists $\alpha > 0$, such that $f(z) = \alpha z$, for $z \geqq 0$.

Clearly $f'(z_g) = \alpha = f'(z)$, for all z. Now $\pi(w) = 0$, if $w \geqq \alpha$ and $\pi(w) = \infty$, if $w < \alpha$ and we have

$$\eta(w) = \begin{cases} -g & \text{if } w > \alpha \\ \{y \mid -g \leqq y \leqq \infty\} & \text{if } w = \alpha \\ \infty & \text{if } w < \alpha \end{cases}$$

$$\eta(w;\underline{y},\infty) = \begin{cases} -g & \text{if } w > \alpha \\ \{y \mid -g \leqq y \leqq \underline{y}\} & \text{if } w = \alpha \\ \underline{y} & \text{if } w < \alpha \end{cases}$$

$$\eta(w;\infty,\underline{z}) = \begin{cases} -g & \text{if } w > \alpha \\ \{y \mid -g \leqq y \leqq \alpha\underline{z} - g\} & \text{if } w = \alpha \\ \alpha\underline{z} - g & \text{if } w < \alpha \end{cases}$$

whereas $\gamma(w) = \frac{1}{\alpha}(\eta(w)+g)$ etc.

$$B(w) = \begin{cases} (-g,0) & \text{if } w > \alpha \\ \{y,z \mid -g \leqq y \leqq \eta(w) \ , \ z = \frac{1}{\alpha}(y+g)\} & \text{if } w \leqq \alpha \end{cases}$$

So f is regular (def. 2.4).

At $w > \alpha$, there exists no equilibrium if $g > 0$, since $\eta(w) = -g$. Only if $g = 0$, the equilibrium trade $t(w,m) = (0,0)$ corresponds to a trivial C-equilibrium.

At $w = \alpha$, W-, KC- and IC-equilibria are possible. The set $\{w,m \mid w = \alpha\}$ is the boundary of Q, if $g > 0$ (see fig. 3). Producers are not constrained; note, however, that it is assumed that no rationing scheme is necessary for producers to select the correct points from the sets $\eta(w)$ and $\gamma(w)$, (as is usual in equilibrium theory).

At $w < \alpha$ producers are always constrained, since $\eta(w) = \gamma(w) = \infty$. So only K-, I- and KI-equilibria can occur.

The possible equilibria are summarized in table I (indicated by "C").

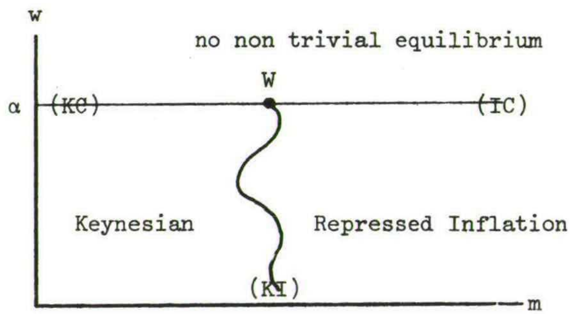

fig. 3

Under assumptions A, B and C, $Q = \{(w,m) \mid w \leqq \alpha\}$ if $g > 0$. The decomposition of Q is depicted in fig. 3, which may be compared with fig. 2. The point W, corresponding to a W-equilibrium is unique and by theorem 2.7 equilibria are unique for all $(w,m) \in Q$.

3.3. Increasing returns

In the case of increasing returns, at given prices and wages, there is a minimum amount of sales necessary to make production profitable. Above this minimum, profit increases with sales. This implies, that production is either not profitable at all, or the producer tries to maximize sales. So, the producer has always to be rationed, if prices are such that production is at all profitable.

Increasing returns are defined by the increase of the mean output per unit of labour input, or, equivalently, by the decrease of mean labour input per unit of output. Let $\bar{f}(z) = f(z)/z$ be the mean output function. $\bar{f}'(z)$ and $\bar{f}''(z)$ denote the first and second derivatives of $\bar{f}$. We assume:

I1 $\quad$ f is continuous and $\bar{f}$ is twice differentiable at z such that $f(z) > 0$;

I2 $\quad$ $f(0) = 0$ and $f(z) \geqq 0$ for all $z \geqq 0$;

I3 $\quad$ $f(z) > 0$ implies $\bar{f}'(z) > 0$.

Bij I1 it is allowed that $f(z) = 0$ for some $z > 0$. (See remark below). Assumption I3 defines increasing returns. Clearly I3 implies that $f'(z) > 0$ for $f(z) > 0$.

Let $\delta = \lim_{z\to\infty} \bar{f}(z)$; δ may be infinite. We define two functions $\hat{\gamma}$ and $\hat{\eta}$ of the interval $[0,\delta[$ into $\mathbb{R}$: the minimum demand function $\hat{\gamma}$ for labour is defined by:

(14) $\quad$ $\hat{\gamma}(w) = \min \{z \mid z > 0 \text{ and } f(z)-wz \geqq 0\}$ for $0 \leqq w \leqq \delta$;

the minimum supply function of goods is $\hat{\eta}(w) = f(\hat{\gamma}(w)) - g$. (See fig. 4). $\hat{\gamma}$ is the inverse of the mean output function; since

(15) $\quad$ $z \in \hat{\gamma}(w) \Leftrightarrow w = \bar{f}(z)$,

$\hat{\gamma}(w) = \bar{f}^{-1}(w)$ and $\hat{\eta}(w) = f(\bar{f}^{-1}(w)) - g = w\hat{\gamma}(w) - g$.

Production is profitable only if sales of goods and available labour are above $\hat{\eta}(w)$ and $\hat{\gamma}(w)$. At $w \geqq \delta$ production is not profitable at any level. Clearly

$$\frac{\partial \hat{\gamma}(w)}{\partial w} > 0 \quad \text{and} \quad \frac{\partial \hat{\eta}(w)}{\partial w} = w \frac{\partial \hat{\gamma}(w)}{\partial w} + \hat{\gamma}(w) > 0 \ .$$

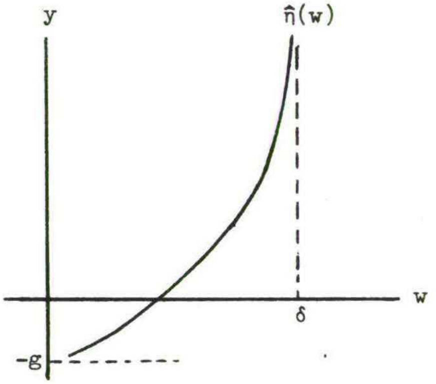

fig. 4

REMARK: Increasing returns are consistent with the existence of a fixed minimum labour input (fixed costs). Let z_0 be the minimum input, then there exists an increasing function h, such that

$$f(z) = \begin{cases} 0 & \text{if } z \leqq z_0 \\ h(z-z_0) > 0 & \text{if } z > z_0 \end{cases}$$

Also $\bar{f}(z) > 0$ if and only if $z > z_0$. Now $\hat{\gamma}(0) = z_0$ and $\hat{\gamma}(w) > z_0$, if $w > 0$. If there are no fixed costs, then $f(z) > 0$ for $z > 0$. Now, for $\varphi = \lim_{z \to \infty} \bar{f}(z) \geqq 0$, $\hat{\gamma}(\varphi) = 0$ and for $0 \leqq w \leqq \varphi$, $\hat{\gamma}(w) = 0$. Note that in the non-fixed costs-case, $\bar{f}'(z) > 0$ is equivalent to $f''(z) > 0$, i.e. to the convexity of the production function.

For profits we have: $\pi(w) = 0$ if $w \geqq \delta$ and $\pi(w) = \infty$ if $w < \delta$; the supply functions are

$$\eta(w) = \begin{cases} -g & \text{if } w \geqq \delta \\ \infty & \text{if } w < \delta \end{cases}$$

(16) $$\eta(w;\underline{y},\infty) = \begin{cases} -g & \text{if } w \geqq \delta \quad \text{or} \quad \underline{y} < \hat{\eta}(w) \\ \underline{y} & \text{if } w < \delta \quad \text{and} \quad \underline{y} \geqq \hat{\eta}(w) \end{cases}$$

$$\eta(w;\infty,\underline{z}) = \begin{cases} -g & \text{if } w \geqq \delta \quad \text{or} \quad \underline{z} < \hat{\gamma}(w) \\ f(\underline{z})-g & \text{if } w < \delta \quad \text{and} \quad \underline{z} \geqq \hat{\gamma}(w) \end{cases}$$

and $\gamma(w) = f^{-1}(\eta(w)+g)$, etc. The set of acceptable trades is

(17) $$B(w) = \begin{cases} (-g,0) & \text{if } w \geqq \delta \\ \{(y,z)\,|\,y+g = f(z) \text{ and } z \geqq \hat{\gamma}(w)\} & \text{if } w < \delta \end{cases}$$

So f is <u>regular</u> (def. 2.4); hence by theorem 2.7, under assumptions A, B and I, an equilibrium is unique for all $(w,m) \in Q$.

PROPOSITION 3.1.: Under the assumptions A, B and I:

(i) $(w,m) \in Q \Leftrightarrow (\hat{\eta}(w),\hat{\gamma}(w)) \in A(w,m)$,

(ii) Q is a closed set,

(iii) if $(w,m) \in \text{Bnd } Q$, then $t(w,m) = (\hat{\eta}(w),\hat{\gamma}(w))$.

<u>Proof</u>: Let $(\hat{y},\hat{z}) = (\hat{\eta}(w),\hat{\gamma}(w))$.

(i) If $(\hat{y},\hat{z}) \in A(w,m)$, then $C(w,m) \neq \emptyset$, hence by theorem 2.7, $(w,m) \in Q$. Suppose $(\hat{y},\hat{z}) \notin C(w,m)$ but $(y,z) \in C(w,m)$. Then $(y,z) > (\hat{y},\hat{z})$ and this implies by lemma 2.6, that $(\hat{y},\hat{z}) \in \text{Int } A(w,m)$, a contradiction.

(ii) By proposition 2.3, A is a continuous correspondence, whereas by assumption I, the function $w \to (\hat{\eta}(w),\hat{\gamma}(w))$ is continuous. So if $(w^t,m^t) \to$

$\to (w,m)$ for $t \to \infty$, and $(w^t, m^t) \in Q$ for all t, then $(\hat{\eta}(w^t), \hat{\gamma}(w^t)) \in$ $\in A(w^t, m^t)$, for all t and therefore $(\hat{\eta}(w), \hat{\gamma}(w)) \in A(w,m)$, hence $(w,m) \in Q$.

(iii) As a consequence of (ii), if $(w,m) \in \text{Bnd } Q$, then $(\hat{y}, \hat{z}) \in \text{Bnd } A(w,m)$ and hence $(\hat{y}, \hat{z}) \in t(w,m)$. □

Since ξ and λ are non decreasing in $\underline{\ell}$ and $\underline{x}$ respectively, by assumptions A4, it follows from (i) of proposition 3.1, that an equilibrium exists at (w,m), if and only if simultaneously:

$$(18) \qquad \xi(w,m;\infty,\hat{\gamma}(w)) \geqq \hat{\eta}(w) \quad \text{and} \quad \lambda(w,m;\hat{\eta}(w),\infty) \geqq \hat{\gamma}(w) .$$

At $w \geqq \delta$, no non trivial equilibrium can occur. Only if $g = 0$, a trivial C-equilibrium exists with $t(w,m) = 0$ and $\underline{\ell} = \underline{x} = 0$.

So non trivial equilibria only occur at $w < \delta$. Since at $w < \delta$, $\eta(w) = \gamma(w) = \infty$, the producer has always to be rationed. Hence W-, C-, KC-, and IC-equilibria are impossible. This leads to the following theorem:

THEOREM 3.2.: Under assumption A and I all equilibria with non zero production are K-, I-, or KI-equilibria.

At values $w < \delta$ no equilibrium occurs, if demand for goods is smaller than minimum profitable supply, or supply of labour is smaller than minimum labour demand. In that case $A(w,m) \cap B(w) = \emptyset$. (Fig. 5, see also fig. 1)

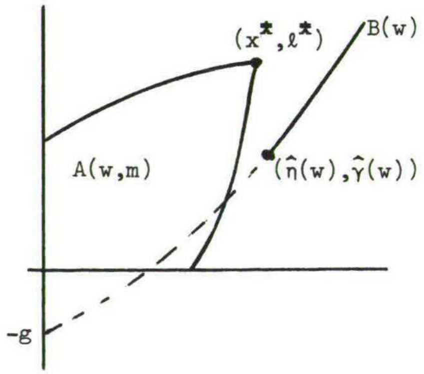

fig. 5

By (iii) of proposition 3.1, the boundary (relative to $\mathbb{R}^2_+$) of the set Q gives equilibria where the firm only realizes its minimum profitable sales and therefore has zero-profits. There are three types of boundary cases:

$$\xi(w,m;\sim,b) = \hat{\eta}(w) = a \quad \text{and} \quad \lambda(w,m) > \hat{\gamma}(w) = b \qquad \text{(K-equilibrium)}$$

$$\text{(19)} \qquad \xi(w,m) > \hat{\eta}(w) = a \quad \text{and} \quad \lambda(w,m;a,\sim) = \hat{\gamma}(w) = b \qquad \text{(I-equilibrium)}$$

$$\xi(w,m) = \hat{\eta}(w) = a \quad \text{and} \quad \lambda(w,m) = \hat{\gamma}(w) = b \qquad \text{(KI-equilibrium)}$$

The last case bears some similarity to Walrasian equilibrium in the case of non increasing returns. It is a boundary case to all other cases and its position in the pictures of Q (see fig. 5), is similar to the one of W-equilibrium in fig. 2 and particularly 3. Therefore we shall call such a KI-equilibrium where consumers are not rationed and producers realize their minimum profitable sales, a pseudo-Walras equilibrium, abreviated PW-equilibrium.

In the remainder of this section we consider the shape and the composition of the set Q.

First note that $Q \neq \emptyset$ under assumption A, B and I: let $w_g = \bar{f}(z_g)$, then $\hat{\gamma}(w_g) = z_g$ and $\hat{\eta}(w_g) = 0$. By A2, $\xi(w,m;\infty,z_g) > 0$ for all m and by B2, $\lambda(w,m;0,\infty) > z_g$, for all m; so by theorem 2.7, $(w_g,m) \in Q$ for all m. It also follows that $(w,m) \in Q$ for all m and $w \leqq w_g$.

If $e(\bar{w},\bar{m})$ is a boundary K-equilibrium, and $m < \bar{m}$, then $(\bar{w},m) \notin Q$: by A6:

$$\xi(\bar{w},m;\infty,\hat{\gamma}(\bar{w})) < \xi(\bar{w},\bar{m};\infty,\hat{\gamma}(\bar{w})) = \hat{\eta}(\bar{w}) .$$

If $e(\bar{w},\bar{m})$ is a boundary I-equilibrium and $m > \bar{m}$, then $(\bar{w},m) \notin Q$: again by A6:

$$\lambda(\bar{w},m;\hat{\eta}(\bar{w}),\infty) < \lambda(\bar{w},\bar{m};\hat{\eta}(\bar{w}),\infty) = \hat{\gamma}(\bar{w}).$$

This implies that a PW-equilibrium $e(\bar{w},\bar{m})$ is unique given $\bar{w}$: no $(\bar{w},m)$ with $m \neq \bar{m}$ is in Q.

If (w,m_1) and (w,m_2) are in Q and $m_1 < m < m_2$, then $(w,m) \in Q$:

$$\hat{\eta}(w) \leqq \xi(w,m_1;\infty,\hat{\gamma}(w)) < \xi(w,m;\infty,\hat{\gamma}(w)) < \xi(w,m_2;\infty,\hat{\gamma}(w))$$

$$\lambda(w,m_1;\hat{\eta}(w),\infty) > \lambda(w,m;\hat{\eta}(w),\infty) > \lambda(w,m_2;\hat{\eta}(w),\infty) \geqq \hat{\gamma}(w) .$$

Given $(w,m) \in Q$, then by A6:

(i) either $(w,0) \in Q$, or for some $0 < m' \leqq m$, $e(w,m')$ is a boundary K-equilibrium;

(ii) if $m' > m$ and $(w,m') \notin Q$, then there exists m'', such that $e(w,m'')$ is a boundary I-equilibrium for $m \leqq m'' < m'$.

If $e(w,m_1)$ and $e(w,m_2)$ are a K-equilibrium and an I-equilibrium respectively, then $m_1 < m_2$ and there exists $m_1 < m < m_2$, such that $e(w,m)$ is a KI-equilibrium (with consumers not rationed).

On the boundary of Q w.r.t. $\mathbb{R}^2_+$ equilibrium trades satisfy $t(w,m) = (\hat{\eta}(w),\hat{\gamma}(w))$, by proposition 2.7. Boundary K-equilibria and PW-equilibria lie on a curve, where w is an increasing function of m.

PROPOSITION 3.3: Under assumptions A, B and I: if (w_1,m_1) and (w_2,m_2) are such that

$$\begin{aligned} &\xi(w_i,m_i;\infty,\hat{\gamma}(w_i)) = w_i\hat{\gamma}(w_i)-g \\ &\lambda(w_i,m_i) \geqq \hat{\gamma}(w_i) \end{aligned} \qquad \text{for } i = 1,2$$

and $\quad w_1 < w_2$, then $m_1 < m_2$.

<u>Proof</u>: By A7 and A6 respectively

$$w_2\hat{\gamma}(w_2) - \xi(w_2,m_1;\infty,\hat{\gamma}(w_2)) >$$

$$w_2 \min (\hat{\gamma}(w_2),\lambda(w_1,m_1)) - \xi(w_1,m_1;\infty,\hat{\gamma}(w_2)) \geqq$$

$$w_1\hat{\gamma}(w_1) - \xi(w_1,m_1;\infty,\hat{\gamma}(w_1)) = g$$

hence

$$\xi(w_2,m_1;\infty,\hat{\gamma}(w_2)) < w_2\hat{\gamma}(w_2)-g = \xi(w_2,m_2;\infty,\hat{\gamma}(w_2))$$

and since $\frac{\partial\xi(w,m;\infty,\ell)}{\partial m} > 0$, we have $m_2 > m_1$. □

To say more we need two assumptions:

E1 $\lambda(w,m)$ is concave in w, for all m;

E2 $f(z) > 0 \Rightarrow \bar{f}''(z) < 0$.

E2 requires that returns do not increase too fast. Particularly consider $f(z) = z^{\alpha}$, $\alpha > 1$; then $\bar{f}(z) = z^{\alpha-1}$, $\bar{f}''(z) = (\alpha-1)(\alpha-2)z^{\alpha-3}$; now $\bar{f}''(z) < 0$, if $\alpha < 2$. E2 implies that $\hat{\gamma}(w)$ is <u>convex</u>: by I3 and E2, $\bar{f}$ is concave and since $\hat{\gamma}(w) = \bar{f}^{-1}(w)$, $\hat{\gamma}$ is a convex function.

Let $P = \{w,m | \lambda(w,m) \geqq \hat{\gamma}(w)\}$. Clearly $Q \subset P$. The upper boundary of P is given by a function $h : \mathbb{R}^+ \to \mathbb{R}^+$, where the boundary of P consists of $(h(m),m)$ for all m, and $\lambda(h(m),m) = \hat{\gamma}(h(m)$. Now h is a <u>decreasing</u> function of m, since

(i) h is defined for each $m > 0$: by assumption A2: $\lambda(w_g,m) > \hat{\gamma}(w_g)$ and for $w_L = \bar{f}(L)$ we have $\lambda(w_L,m) < L = \hat{\gamma}(w_L)$. By continuity of λ and $\hat{\gamma}$, there exists w, such that $\lambda(w,m) = \hat{\gamma}(w)$, and this w is unique by the concavity of λ and the convexity of $\hat{\gamma}$.

(ii) h is decreasing in m: let $F(w,m) = \lambda(w,m) - \hat{\gamma}(w))$. Then at (w,m) such that $F(w,m) = 0$, we have

$$\frac{\partial F}{\partial w} = \frac{\partial\lambda(w,m)}{\partial w} - \frac{\partial\hat{\gamma}(w)}{\partial w} < 0 ,$$

by concavity and convexity, and by A5:

$$\frac{\partial F}{\partial m} = \frac{\partial(w,m)}{\partial w} < 0 .$$

Hence $\frac{dw}{dm} < 0$.

If e(w,m) is a PW-equilibrium, then $\lambda(w,m) = \hat{\gamma}(w)$, hence (w,m) is on the curve h, which is decreasing. Now from proposition 3.3. it follows that (w,m) giving a PW-equilibrium is unique. There are four cases:

(a) For no $(w,m) \in Q$, $e(w,m)$ is a PW-equilibrium and for all $(w,m) \in$ Bnd Q, $e(w,m)$ is a boundary K-equilibrium. Now the boundary of Q can be described by a curve where w is an increasing function of m.
(b) For no $(w,m) \in Q$, $e(w,m)$ is a PW-equilibrium and for all $(w,m) \in$ Bnd Q, $e(w,m)$ is a boundary I-equilibrium.
(c) $(h(0),0) \in Q$ and $e(h(0),0)$ is the unique PW-equilibrium; all other points on Bnd Q give boundary I-equilibria.
(d) For some $\bar{m} > 0$, $e(h(\bar{m}),\bar{m})$ is the unique PW-equilibrium. In this case for some $w_0 < h(\bar{m})$, $e(w_0,0)$ is a boundary K-equilibrium and an increasing curve joining $(w_0,0)$ and $(h(\bar{m}),\bar{m})$ consists of boundary points of Q giving boundary K-equilibria. All boundary points of Q where $m > \bar{m}$, give boundary I-equilibria. This part of Bnd Q lies completely below the curve h. In this case the picture of Q looks like the one given in fig. 6.

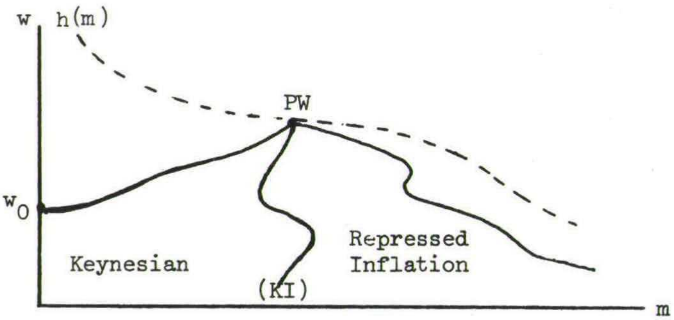

fig. 6

4. Equilibria with many firms and increasing returns

We consider an economy $E^N = \{N, f_j; \xi, \lambda\}$. Consumers are defined by the constrained demand and supply functions ξ and λ (1). $N = \{1,2,\dots,n\}$ is a finite set of potential firms. $\mathfrak{N}$ is the set of all subsets of N. Each $j \in N$ has a production function $f_j(z_j)$. We only consider the increasing returns case so assumptions I are satisfied by all f_j. $\eta_j, \gamma_j, \hat{\eta}_j$ and $\hat{\gamma}_j$ are as defined in (14) and (16). For sake of simplicity we assume $g = 0$ in this section. A (temporary) fixed price equilibrium in E_N is defined as follows (by assumptions I all η and γ are functions):

DEFINITION 4.1.: An equilibrium at (w,m) in E^N, is a 2(n+1)-tuple of trades (x,ℓ,y_j,z_j) and a rationing scheme $(\underline{x},\underline{\ell},\underline{y}_j,\underline{z}_j)$, for $j \in N$, such that:

(i) $x = \xi(w,m;\underline{x},\underline{\ell}) = \Sigma y_j$ and $\ell = \lambda(w,m;\underline{x},\underline{\ell}) = \Sigma \underline{z}_j$;

(ii) $\forall_j: y_j = \eta_j(w;\underline{y}_j,\underline{z}_j)$ and $z_j = \gamma_j(w;\underline{y}_j,\underline{z}_j)$;

(iii) $[\forall_j: \underline{y}_j = \infty]$ or $\underline{x} = \infty$;$[\forall_j: \underline{z}_j = \infty]$ or $\underline{\ell} = \infty$;

(iv) $\forall_j: [\underline{y}_j = \infty$ or $\underline{z}_j = \infty]$.

The set of equilibria at (w,m) is denoted by $e^N(w,m)$ and equilibrium trades are now defined by $t^N(w,m) = \{x,\ell,\underline{y}_j,\underline{z}_j \mid \exists(\underline{x},\underline{\ell},y_j,\underline{z}_j): (x,\ell,y_j,z_j;\underline{x},\underline{\ell},\underline{y}_j\underline{z}_j)\} \in$ $\in e^N(w,m)\}$.

Since we assume increasing returns, all firms have to be rationed. If in $e^N(w,m)$, $\underline{y}_j < \infty$ and $\underline{z}_j = \infty$, then $y_j = \underline{y}_j$ if $\underline{y}_j \geqq \hat{\eta}(w)$ and $y_j = 0$ if $\underline{y}_j < \hat{\eta}(w)$, and similarly if labour is constrained. A firm which cannot make a non-negative profit has to leave the market. So in an equilibrium it is also determined which firms have a positive production. Such firms will be called active.

The equilibria that are possible with different firms are the ones considered in section 3.3 for a single firm: (i) K-equilibrium (all firms rationed on the goods market and consumers rationed on the labour market); (ii) I-equilibrium (all firms rationed on the labour market and consumers on the goods market) and (iii) an intermediate case of KI-equilibria (consumers not rationed and all firms rationed on one market, but not necessarily the same). If in the last case some firms are rationed on the goods market and the other ones on the labour market, that rationing scheme could be replaced by one with all firms rationed on the same market, e.g. by replacing constraints $\underline{z}_j$ by $\underline{y}_j = f_j(\underline{z}_j)$, without affecting the equilibrium trades.

4.1. Market share distributions

At any $(w,m) \in Q$ many equilibria can exist, differing not only in rationing schemes, but also in equilibrium trades and in the composition of the set of active firms. Up to now, no restrictions have been put on the rationing schemes and thus on the rules by which the distribution among firms of sales or available labour is determined. A complete description of the economy certainly requires rules of this type. Therefore we introduce market share distributions for goods and labour. The idea behind this concept (applied in a partial equilibrium model in Weddepohl (1978)) is that, if there is excess supply of goods, each consumer has to decide from which firm he will try to buy first (and because of increasing returns he certainly will be served). The decisions of all consumers will lead to a specific distribution of sales among firms. In the case of excess demand for goods this distribution does not matter, for consumers will address themselves to different firms consecutively. Similarly the choice of an employer by consumers will generate a market share distribution of labour. Clearly these distributions also depend on which firms are in the market, i.e. are active.

Let $T = \{\tau_i \in \mathbb{R}^n | \tau_i \geqq 0, \Sigma \tau_i = 1\}$ be the unit simplex. Firstly $\rho : \mathfrak{N} \to T \cup \{0\}$ is the market share distribution of good's sales. It associates to each set of active firms $H \subset N$, the share of each firm in H of the sales of goods, if there is excess (effective) supply, i.e. if $\underline{y}$ is the aggregate constraint on sales, then each (active) firm $j \in H$ has the share $\rho_i(H)$ and therefore his constraint is $\underline{y}_j = \rho_i(H)\underline{y}$ (for $j \notin H$, $\rho_j(H) = 0$).

Secondly $\theta : \mathfrak{N} \to T \cup \{0\}$ is the market share distribution of labour. It associates to a set $H \subset N$ the share of each $j \in H$ of labour supply, if there is excess (effective) demand for labour. Hence, given the aggregate constraint $\underline{z}$, each firm's constraint on labour will be $\theta_j(H)\underline{z} = \underline{z}_j$ (for $j \notin H$, $\theta_j(H) = \emptyset$). On the market share distributions we assume:

M1 $\quad \rho(H) = 0 \Leftrightarrow H = \emptyset;\ \theta(H) = 0 \Leftrightarrow H = \emptyset$

M2 $\quad j \in H \subset H'$ and $H \neq H' \Rightarrow [\rho_j(H) > \rho_j(H')$ and $\theta_j(H) > \theta_j(H')]$

Among the equilibria in E^N (def. 4.1), only a subset will respect given market share distributions and only these equilibria can realize if the market share distributions dictate the distribution of good's sales or labour among firms. We define equilibria of this type for each set $H \subset N$ of active firms.

DEFINITION 4.2.: $e_H^N(w,m) \subset e^N(w,m)$ for $H \subset N$, is the set of equilibria in E^N, such that

(i) if $\forall_j$: $\underline{y}_j < \infty$, then $\forall_j$: $\underline{y}_j = \rho_j(H)x$,

(ii) if $\forall_j$: $\underline{z}_j < \infty$, then $\forall_j$: $\underline{z}_j = \theta_j(H)\ell$.

Note that $\rho_j(H) = \theta_j(H) = 0$, if $j \notin H$. (i) obtains if all firms are constrained on the goods market (so at a K- of KI-equilibrium), (ii) if all firms are constrained on the labour market (so at an I- or KI-equilibrium).

The definition does not allow equilibria where some firms are rationed on the goods market and other firms on the labour market, unless e.g. $\underline{z}_j = \theta_j(H)\ell$ may be replaced by $\underline{y}_j = f_j(\underline{z}_j)$, which requires $\underline{y}_j = \rho_j(H)x$. To include this case would require a refinement of our concept of market share distribution, in order to allow for the distribution of the "unused" part of some firm's share among the other firms, in order to determine $\underline{y}_j$ and $\underline{z}_j$. This problem is considered in a different paper (Weddepohl, 1979). In the present paper we shall not pursue this matter further; in section 4.3. we consider a special case where the problem is ruled out.

Given an equilibrium in $e_H^N(w,m)$ there will, under suitable assumptions, also exists an equilibrium in $e_{H'}^N(w,m)$. $H' \subset H$. There may also exist an equilibrium for $H' \supset H$, and particularly it may be profitable for a non active firm to become active. We define

DEFINITION 4.3.: The equilibria $e_H^N(w,m)$ are stable w.r.t. H, if there exists no $j \notin H$, such that there exists an equilibrium in $e_{H \cup j}^N(w,m)$ where j makes a positive profit.

If an equilibrium is not stable, a non-active firm could make a positive profit, if it realizes its market share, i.e. by becoming active. We shall study stability in section 4.3. for the special case, where all firms are identical.

4.2. Aggregate production functions

Given a set of active firms $H \subset N$ and the market shares $\rho_j = \rho_j(H)$ and $\theta_j = \theta_j(H)$, we can define two aggregate production functions, giving total output as a function of total labour input:

(i) If labour is rationed and distributed among firms in H according to shares θ_j, then

$$(20) \qquad f_{\ell H} = \sum_H f_j(\theta_j z)$$

for z total labour input. Now

$$\bar{f}_{\ell H} = \sum_H \theta_j \bar{f}_j(\theta_j z)$$

If assumptions I hold for all $j \in N$, then $f_{\ell H}$ also satisfies I, particularly (for I3):

$$\bar{f}'_{\ell H} = \sum_H \theta_j^2 \bar{f}'_j(\theta_j z) > 0$$

Similarly E3 holds for $f_{\ell H}$ if it does for all j.

(ii) If goods are rationed for firms and sales are distributed among active firms in $H \subset N$ according to shares $\rho_j = \rho_j(H)$, then f_{gH} is defined by:

$$(21) \qquad f_{gH}^{-1}(y) = \sum_H \rho_j f_j^{-1}(\rho_j y)$$

y being total goods sales. Now also assumptions I and E2 hold for f_{gM} if they hold for all f_j.

Both with the help of $f_{\ell H}$ and f_{gH} the (constrained) demand and supply functions η_{gH} and γ_{gH} (see (16)) can be defined. The minimum profitable sales and labour input functions (see (14)) are defined, however, by

$$(22) \qquad \hat{\eta}_{gH}(w) = \min_H \frac{\hat{\eta}_j(w)}{\rho_j}, \quad \hat{\gamma}_{\ell H}(w) = \min_H \frac{\hat{\gamma}_j(w)}{\theta_j}$$

for then we have for all $j \in H$:

$$\rho_j \hat{n}_{gH}(w) \geq \hat{n}_j(w) .$$

Clearly $f_{\ell H}$ and f_{gH} need not coincide.

We define two "single firm" economies.

$$(23) \qquad E_{gH} = \{f_{gH};\xi,\lambda\} \quad \text{and} \quad E_{\ell H} = \{f_{\ell H};\xi,\lambda\} ,$$

as defined in section 2.3. and studied in section 3.3. for the increasing returns case, with equilibrium sets $e_{gH}(w,m)$ and $e_{\ell H}(w,m)$ (see definition 2.5.). Let $(x,\ell,y_j,z_j;\infty,\underline{\ell},\underline{y}_j,\infty) \in e^N_H(w,m)$ be a K-equilibrium (or a KI-equilibrium with $\ell = \infty$) in E^N, then $(x,\ell,\Sigma y_j,\Sigma z_j;\infty,\underline{\ell},\Sigma\underline{y}_j,\infty) \in e_{gH}(w,m)$ is a K-equilibrium or a KI-equilibrium in E_{gH}. The converse is also true. Similarly an I- or a KI-equilibrium, with all firms rationed on the labour market, from $e^N_H(w,m)$ in E^N, is equivalent to an I- or KI-equilibrium from $e_{\ell H}(w,m)$ in $E_{\ell H}$.

The production function to be used depends on the type of equilibrium obtained. It could occur that at some (w,m), an I-equilibrium exists in $E_{\ell H}$ and that simultaneously in E_{gH} there exists a K-equilibrium. Then clearly $e^N_H(w,m)$ contains both a K- and an I-equilibrium. It could also happen that at (w,m), $E_{\ell H}$ only has a K-equilibrium and that E_{gh} only has an I-equilibrium. In that case $e^N_H(w,m) = \emptyset$. It seems that for the definition of an equilibrium in such a case an extension of the market share concept as considered above is required with firms rationed on different markets.

4.3. Identical firms

Let all firms have the same production function f, satisfying assumptions I, and have identical market shares. For $|H|$ denoting the number of members of $H \subset N$, define $h = |H|$ and $n = |N|$. Now $\forall_j \in N$: $f_j(z_j) = f(z_j)$ and given a set H of active firms, for all $j \in H$: $\theta_j(H) = \frac{1}{h} = \rho_j(H)$ and for $j \notin H$: $\theta_j(H) = 0 = \rho_j(H)$. Clearly ρ and θ satisfy assumptions M1 and M2.

We need only consider the number of active firms, in the set H, its composition being irrelevant. There exists a single aggregate production function for a given number h of active firms, since f_{gh} and $f_{\ell h}$, as defined in (20) and (21), coincide:

$$(24) \qquad f_h(z) = hf(\frac{1}{h}z)$$

Let $\hat{\eta}$ and $\hat{\gamma}$ be the minimum supply and demand functions w.r.t. f (see (14)), then $\hat{\eta}_h$ and $\hat{\gamma}_h$ w.r.t. f_h satisfy:

$$(25) \qquad \hat{\eta}_h(w) = h\,\hat{\eta}(w) \text{ and } \hat{\gamma}_h(w) = h\,\hat{\gamma}(w)\,.$$

For given h, we may limit ourselves to the single firm economy $E_h = \{f_h;\xi,\lambda\}$ (see (13)) with equilibrium sets $e_h(w,m)$ and equilibrium trade $t_h(w,m)$ since the equilibria in E_h are equivalent to the ones in E^N, satisfying definition 4.2. (i.e. equilibria from $e^N_H(w,m)$ for $|H| = h$). Clearly if $(a,b) \in t_h(w,m)$, then an equilibrium in $e_h(w,m)$ will be such that $x = a; \ell = b; y_j = a/h$ and $z_j = b/h$ if $j \in H; y_j = z_j = 0$ if $j \notin H$. We can apply the analysis of section 3.3. for studying h firm equilibria. Let Q_h be the set of (w,m) pairs such that an equilibrium exists in E_h at $(w,m); Q_h$ will have the properties found in section 3.3. if assumptions A, B and I and eventually E hold. By assumptions I2 and I3:

$$(26) \qquad f_h(z) > f_{h+1}(z)$$

since $\quad \frac{1}{z} f_h(z) = \frac{h}{z} f(\frac{1}{h}z) = \bar{f}(\frac{1}{h}z) > \bar{f}(\frac{1}{h+1}z) = \frac{1}{z} f_{h+1}(z)$.

This implies:

$$\hat{\gamma}_h(w) < \hat{\gamma}_{h+1}(w) \text{ and } \hat{\eta}_h(w) < \hat{\eta}_{h+1}(w)\,.$$

We are now able to compare the equilibria for different numbers of active firms. An equilibrium $e_h(w,m)$ is stable, according to definition 4.3. if and only if $(w,m) \in Q_h \setminus \text{Int } Q_{h+1}$ (since boundary points of Q_{h+1} give zero profits for all firms). Under assumptions A, B and I, equilibrium trades are unique, so $t_h(w,m)$ in E_h is a singleton. Theorem 4.4. gives the relations between equilibria at (w,m) for different h. Note that assumptions B hold for all $h \leqq n$, if they hold for f_n, due to (26).

THEOREM 4.4.: Under the assumptions A, B and I:

(i) $Q_h \subset Q_{h-1}$;

(ii) if $(w,m) \in \text{Bnd } Q_h$, then $(w,m) \in \text{Int } Q_{h-1}$;

(iii) if $e_h(w,m)$ is a K- or a KI-equilibrium, then $e_{h-1}(w,m)$ is a K-equilibrium;

(iv) if $e_h^N(w,m)$ is an I- or a KI-equilibrium and $(w,m) \in Q_{h+1}$, then $e_{h+1}(w,m)$ is an I-equilibrium.

Proof:

(i) Since $(w,m) \in Q_h$, $(\hat{\eta}_h(w), \hat{\gamma}_h(w)) \in A(w,m)$, by proposition 3.1, and therefore $\lambda(w,m;\hat{\eta}_h(w),\infty) \geqq \hat{\gamma}(w) = f_h^{-1}(\hat{\eta}(w))$. Since $\hat{\eta}_{h-1}(w) < \hat{\eta}_h(w)$ and by B1, B2 and (26),

(a) $\lambda(w,m;\hat{\eta}_{h-1}(w),\infty) \geqq f_h^{-1}(\hat{\eta}_{h-1}(w)) > f_{h-1}^{-1}(\hat{\eta}_{h-1}(w)) = \hat{\gamma}_{h-1}(w)$.

Also $\xi(w,m;\infty,\hat{\gamma}_h(w)) \geqq \hat{\eta}_h(w)$. By assumption A4, particularly (7), since $\hat{\eta}_h(w) - \hat{\eta}_{h-1}(w) = w(\hat{\gamma}_h(w) - \hat{\gamma}_{h-1}(w))$, it follows

(b) $\xi(w,m;\infty,\hat{\gamma}_{h-1}(w)) > \hat{\eta}_{h-1}(w)$.

So by (a), (b) and proposition 3.1., $(w,m) \in Q_{h-1}$.

(ii) If $(w,m) \in \text{Bnd } Q_h$, then $(\hat{\eta}_h(w), \hat{\gamma}_h(w)) \in \text{Bnd } A(w,m)$. By (a) and (b) of (i), $\hat{\eta}_{h-1}(w), \hat{\gamma}_{h-2}(w)) \in \text{Int } A(w,m)$.

(iii) Let $e_h(w,m)$ be a K- or KI-equilibrium. Then for all $v \leqq \xi(w,m)$, $\lambda(w,m;v,\infty) = f_h^{-1}(v)$, by B1 and B2. Since $f_{h-1}^{-1}(w) < f_h^{-1}(v)$ for all $v > 0$, $\lambda(w,m;v,\infty) > f_{h-1}^{-1}(v)$, for all $v \leqq \xi(w,m)$, hence $e_{h-1}(w)$ is a K-equilibrium.

(iv) if $e_{h+1}(w,m)$ would be a K- or a KI-equilibrium, then by (iii) $e_h(w,m)$ would be a K-equilibrium. □

Theorem 4.4. entails, together with the results of section 3.3., that the set Q_h shifts upwards if h increases, as shown in figure 7. Note that (iii) of theorem 4 implies that the PW-equilibrium pair (w,m) increases in both w and m, and that the right hand boundary of the Keynesian region shifts to the right. So generally a sufficiently large increase of the real wage rate will cause that some firms have to leave the market. Conversely a decrease of real wages will attract new firms into the market. Due to the more efficient use of labour by less firms, an increase of wages with decrease of the number of firms, may cause that an I-equilibrium is replaced by a K-equilibrium.

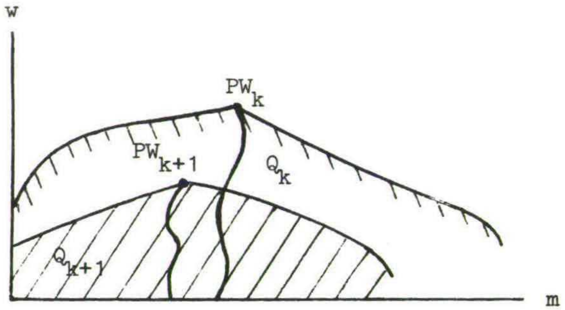

fig. 7

5. Final remarks.

Typically under increasing returns only equilibria can exist where at least the producers are rationed, so that Walrasian and Classical equilibria are excluded. The model considered in this paper is, however, very restrictive, since no investment is possible. It may be argued that fixed capital ought to be included in the model and that it is more likely than that the short run production function shows decreasing returns. In that case increasing returns are rather a long run phenomenon. Now increasing returns come into the picture when investment decisions are to be made, i.e. when it must be decided how much equipment and of what type shall be ordered. For these decisions not only the expected wages and prices ought to be taken into account, but also the expected rationing of goods sales and labour, and it will precisely be these constraints that limit the purchase of new equipment. Therefore it seems that, however with fixed capital and a decreasing returns-short run production function Classical and Walrasian equilibrium may occur, such equilibria cannot persist as soon as capital may be adjusted, if the long run-production function shows increasing returns, (even if prices and wages also vary).

We only considered fixed price equilibria in this paper. The model of section 4 seems also appropriate for the analysis of price making behaviour of firms. The stability concept of definition 4.3 should be extended by introducing the conditions under which no firm is inclined to change its price. (For a partial analysis see Weddepohl (1978, 1979).

It must be noted that the market shares, as introduced in section 4.1, may also be applied if there are decreasing returns and if the producers are rationed. These shares could vary over time, not only as a result of entry and exit of firms, but also as a function of selling activities by the firms.

All these problems need further research. In models where more than a single period is considered, the assumption that profits are transferred to the autonomous sector, is not reasonable. It should either be distributed among firms or have some relation to new investment.

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