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RESEARCH MEMORANDUM

OPTIMAL DYNAMIC INVESTMENT POLICIES UNDER CONCAVE-CONVEX ADJUSTMENT COSTS

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A B S T R A C T

This paper considers a dynamic investment model of a monopolistic firm, facing adjustment costs that are concave-convex in the rate of investment. Assuming that the firm is managed by the shareholders, the objective is to maximize the discounted stream of dividends over a finite plannning horizon, plus the terminal value of the capital stock. The problem of finding an optimal investment path is solved by combining results from a model with concave adjustment costs with results from a model with convex adjustment costs. The results are presented in phase diagrams and are economically interpreted.

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1. INTRODUCTION

This paper develops optimal investment rules in a dynamic model of a monopolistic firm, facing a U-shaped average cost of capital stock adjustment. Consequently, the total adjustment cost becomes a concave-convex function of the investment rate. Our results are derived by using optimal control theory for systems in continuous time, with a fixed and finite planning horizon.

The influence of adjustment costs on the firm's optimal behavior has been noted by, for instance, Gould (1968), Rothschild (1971), Nickell (1978), Davidson and Harris (1981). For a survey, see Söderström (1976). However, there does not seem to be general agreement on what should be the "proper" shape of the adjustment cost function. Models with convex adjustment cost functions (i.e. increasing marginal costs of adjusting the capital stock) have been studied by e.g. Gould (1968), Nickell (1978). The use of such cost functions can be realistic, for example, in a situation where a firm, operating in a monopsonistic environment, wishes to acquire capital. Then the firm must face increasing prices because of the increased demand for capital goods. On the other hand, concave adjustment cost functions (implying decreasing marginal costs) have also been proposed; see Rothschild (1971), Nickell (1978). Such cost functions could emerge when fixed ordering costs and quantity discounts are present, but also because of indivisibilities, economies of information in training and so forth (cf. Rothschild (1971, pp. 608-609)).

Davidson and Harris (1981) argue that the adjustment cost function be concave-convex. In a continuous-time model there could be some initial economies to scale when adjusting the capital stock, but as the investment rate becomes larger, in a firm of a given size, average adjustment costs eventually increase. Put in another way: installing capital at a (too) high rate ultimately leads to increasing costs. Nickell (1978) notes that strictly convex adjustment costs imply that it is more expensive to do adjustments quickly than slowly. If we consider internal adjustment costs (which are costs related to the adjustment of capital stocks and/or labor force within the firm) this may be true for sufficiently large amounts of investment expenditure. But internal adjustment costs being convex for low investment rates (for example, installing a new production line over a period of fifty years) make little or no sense. Therefore, when monopsonistic elements in the capital goods market are not predominant it may be better to suppose that adjustment costs are concave for small rates of investment. (Note that the presence of a monopsonistic market structure always lead to a stricly convex adjustment cost function).

The dynamic investment model to be studied in this paper is a modification of the one proposed by van Loon (1983) (see also Kort (1989)) and the reader is referred to these works for further details. The organization of the paper is as follows. Section 2 presents the dynamic optimization problem of the firm and Section 3 contains the mathematical analysis of the optimal control problem as well as economic interpretations of the results. Section 4 concludes the paper.

2. A DYNAMIC INVESTMENT MODEL

Consider a monopolistic firm behaving so as to maximize the shareholders' value of the firm, consisting of the discounted stream of dividends over the planning period plus the discounted value of the amount of equity at the end of the planning period. Hence the firm seeks to maximize the objective functional

$$\int_0^T \exp(-rt)D(t)dt + \exp(-rT)X(T) \tag{1}$$

where $D = D(t)$ is the rate of dividend payout at time t, $X = X(t)$ the stock of equity by time t, $r = \text{constant} \bullet 0$ the shareholders' time preference rate, and T the finite length of the firm's fixed planning period. Suppose that depreciation is proportional to the capital stock and let $I = I(t)$ be the rate of gross investment at time t, $K = K(t)$ the stock of capital by time t and $a = \text{constant} > 0$ the depreciation rate. The equation for the evolution of the stock of capital becomes

$$\dot{K} = I - aK, \qquad K(0) = K_o = \text{constant} > 0. \tag{2}$$

Furthermore, suppose that the firm's only asset is its stock of capital goods which, in turn, can only be financed by equity. After fixing the unit value of the capital stock at one unit of money, it holds that $K = X$ in the firm's balance sheet.

Let the gross earnings of the firm be given by a function $S = S(K)$, such that $S(K) > 0$ for $K > 0$, $S'(K) > 0$, $S''(K) < 0$, $S(0) = 0$. Let $A = A(I)$ denote the rate of adjustment cost incurred when investing at rate I and assume that A(I) satisfies the following conditions: $A(0) = 0$, $A'(I) > 0$ for $I > 0$, and $A''(I) < 0$ for $I < I_o$; $A''(I) > 0$ for $I > I_o$.

Earnings, after deduction of depreciation and adjustment costs, can be used to pay out dividends and/or increase retained earnings (i.e., equity). Hence

$$\dot{X} = S(K) - aK - A(I) - D. \tag{3}$$

Using (2) and (3), and noticing that $\dot{X} = \dot{K}$, yields

$$D = S(K) - I - A(I) \tag{4}$$

and substitution of (4) into (1) gives the final form of the objective functional

$$\int_0^T \exp(-rt)(S(K) - I - A(I))dt + \exp(-rT)K(T). \tag{1a}$$

Introducing lower bounds on dividends as well as investment yields the mixed state-control constraint

$$D = S(K) - I - A(I) \geq 0 \tag{5}$$

and the pure control constraint

$$I \geq 0. \tag{6}$$

Inequality (5) states that total (investment) costs must not exceed total current revenue whereas inequality (6) states that investment is irreversible and thus bounded below by zero.

The following assumption is made for technical reasons.

$$S(K) - aK - A(aK) > 0 \tag{7}$$

and has the interpretation that profits (after depreciation and adjustment costs) are strictly positive for every admissible K, when investment is just at the replacement level. It is possible to introduce (7) explicitly as a pure state constraint (replacing strict inequality by weak inequality). However, wishing to avoid this we have assumed that such a constraint never becomes binding. (This is likely to be the case if the depreciation rate, a, is relatively small).

The optimal control problem of the firm has now been formulated: determine an investment path, $I(t)$, over a fixed and finite planning period $[0,T]$, such that the objective functional in (1a) is maximal, subject to the constraints (2), (5), and (6).

Models related to this control problem can be found in Appelbaum and Harris (1978), Skiba (1978), Davidson and Harris (1981), Hartl and Mehlmann (1983) and Feichtinger and Hartl (1986, Ch. 13). However, all these analyses deal with the case of an infinite planning period where, from obvious reasons, there is no salvage value term in the objective function. In the model at hand, the planning period is finite and we include a salvage value term being the terminal value of the capital stock. The set-up in Davidson and Harris (1981) is close to ours but these authors do not introduce an upper bound on the investment rate. Hence, the paper at hand changes the Davidson and Harris model to deal with a finite planning horizon and specifies an endogenous upper bound on investment given by $S(K) \geq I + A(I)$. An upper bound on investment was considered in Appelbaum and Harris (1978) who, however, disregarded the adjustment costs. Thus the upper bound on investment in Appelbaum and Harris was given by $S(K) \geq I$.

3. ANALYSIS OF THE INVESTMENT PROBLEM

Define the total cost function, $C(I)$, as $C(I) = I + A(I)$ and notice that $C(0) = A(0) = 0$, $C' = 1 + A' > 0$, $C'' = A''$ and $C/I = 1 + A/I$. This implies that C' and A' attain their minimum at $I = I_0$ whereas C/I and A/I attain their minimum at $I = I_1$; see also Figure 1.

[Insert Figure 1 about here]

The current-value Hamiltonian is given by $H(I,K,p) = S(K) - paK + pI - C(I)$ and p is a (current-value) adjoint variable. Using the maximum principle, we should maximize the Hamiltonian with respect to I for any arbitrary but fixed pair (K,p). To determine the sign of p we need the adjoint equation

$$\dot{p} = (r + a)p - S'(K) \quad \text{such that} \qquad p(T) = 1. \tag{8}$$

Integration in (8) shows that the adjoint variable is stricly positive for all $t \in [0,T]$. Recall that p has the usual interpretation as a shadow price (of a unit) of capital stock at date t.

Disregarding for a moment the upper bound on C(I) (i.e., $C(I) \leq S(K)$), an investment policy which maximizes the Hamiltonian is characterized by

$$I = \begin{cases} I(p) \\ 0 \quad \text{or} \quad I_1 \\ 0 \end{cases} \qquad \text{if} \qquad p \begin{matrix} > \\ = \\ < \end{matrix} \bar{p}$$

where $\bar{p}$ is defined by

$$C'(I_1) = C(I_1)/I_1 = \bar{p}. \tag{9}$$

Notice that $I(p) > I_1$ and I(p) is a solution of $C'(I) = p$, that is, $I(p) = (C')^{-1}(p)$.

To take the constraint $C(I) \leq S(K)$ into account we must modify the investment policy as follows. There are two cases to consider.

<u>Case 1</u>: $I_1 \leq C^{-1}(S(K))$.

Here, an optimal investment policy is characterized by

$$I = \begin{cases} \min\{I(p), C^{-1}(S(K))\} \\ 0 \quad \text{or} \quad I_1 \\ 0 \end{cases} \quad \text{if} \quad p \begin{matrix} > \\ = \\ < \end{matrix} \bar{p} \tag{10}$$

<u>Case 2</u>: $I_1 > C^{-1}(S(K))$.

Here, an optimal investment policy is characterized by

$$I = \begin{cases} C^{-1}(S(K)) \\ 0 \quad \text{or} \quad C^{-1}(S(K)) \\ 0 \end{cases} \quad \text{if} \quad p \begin{matrix} > \\ = \\ < \end{matrix} S(K)/C^{-1}(S(K)). \tag{11}$$

The derivation of the optimal policies in Cases 1 and 2 is straightforward and omitted.

Note that the transversality condition in (8) implies that there is always a terminal interval on which the optimal investment rate is zero. On such an interval there is not enough time left to defray the adjustment costs of new investments. Hence, such investments should not be undertaken (cf. also Kort (1989)).

What we have obtained until now is a preliminary characterization of an optimal investment policy. The further analysis will proceed as follows. First we analyze a model with a globally concave adjustment cost function. Second, a model with a globally convex adjustment cost function is studied and, third, we combine the results of these two analyses so as to construct a solution to the problem with a concave-convex adjustment cost function. These derivations are made in Sections 3.1, 3.2 and 3.3, respectively.

3.1 Concave Adjustment Cost Function

A similar model was studied by Kort (1989) but only for the case of a fixed upper bound on the investment rate. (Recall that in the present paper $C^{-1}(S(K))$ provides an upper bound on the investment rate). Let C(I) be a strictly concave (total) investment cost function. Following the standard procedure (see, e.g., Feichtinger and Hartl (1986)) we replace function C(I) by a function, say, $C_1(I,K)$, being linear in I and such that

$$C_1(I,K) = I\ S(K)/C^{-1}(S(K)) := I\ g(K). \tag{12}$$

For any fixed K the function $C_1(I,K)$ coincides with C(I) at I = 0 and at $I = C^{-1}(S(K))$ but $C_1(I,K)$ is below C(I) for $0 < I < C^{-1}(S(K))$.

We proceed with solving the linearized problem. The objective functional for this problem becomes

$$\int_0^T \exp(-rt)[S(K) - I\ g(K)]dt + \exp(-rT)K(T) \tag{1b}$$

and substituting $I = \dot{K} + aK$ into (1b) yields

$$\int_0^T \exp(-rt)[S(K) - g(K)\dot{K} - g(K)aK]dt + \exp(-rT)K(T). \tag{1c}$$

Define the auxiliary functions $M(K) := S(K) - g(K)aK$ and $N(K) := - g(K)$ (which are C^1 functions) and note that a singular solution, say, K*, of the linear problem satisfies (Feichtinger and Hartl (1986, Section 3.3)) $rN(K^*) + M'(K^*) = 0$, that is,

$$- rg(K) + S'(K) - ag(K) - aKg'(K) = 0 \tag{13}$$

where

$$g'(K) = d(S(K)/C^{-1}(S(K)))/dK =$$

$$S'(K)/C^{-1}(S(K)) - S'(K)S(K)/C'(S(K))(C^{-1}(S(K)))^2. \quad (14)$$

Define the set $Q(K) = \{I-aK|\ 0 \le I \le C^{-1}(S(K))\} = [-aK,\ C^{-1}(S(K))-aK]$ and notice that zero is an element in Q because of (7). This implies that the singular solution is sustainable. Inserting (14) into (13) yields

$$S'(K) - (a+r)S(K)/C^{-1}(S(K)) - aK\ S'(K)/C^{-1}(S(K)) +$$

$$aK\ S'(K)S(K)/C'(S(K))(C^{-1}(S(K)))^2 = 0. \quad (14a)$$

It is well known that if no feasible solution(s) exists to (14a) and, in such a case, the left-hand side of (14a) is negative (positive) for all admissible K, then it is optimal to invest so as to approach $K = 0$ ($K = \infty$) as fast as possible, starting from K_0. (Notice that such an investment policy need not exist).

If (14a) has a unique solution, say, K^* (such that $0 \in Q(K^*)$, $K^* > 0$), and the left-hand side of (14a) is positive for $K < K^*$, negative for $K > K^*$, it is optimal to invest so as to approach K^* as fast as possible, starting from K_0. Note that $0 \in Q(K^*)$ holds but the solution of (14a) is not necessarily unique.

In order to secure existence and uniqueness of such a solution to (14a) we impose the following assumptions. Suppose that (i) the derivative $d(rN(K) + M'(K))/dK$ is negative for all $K > 0$ and that (ii) the value $rN(0) + M'(0)$ is positive. Then there exists a $K^* > 0$ being a unique solution to

(14a). In the appendix we present a set of sufficient conditions which will guarantee that hypotheses (i) and (ii) are satisfied.

The optimal investment policy for the linearized problem is given by

$$I^* = \begin{cases} C^{-1}(S(K)) & \text{if } 0 \le K < K^* \quad (15a) \\ aK^* & \text{if } K = K^* \quad (15b) \\ 0 & \text{if } K > K^* \quad (15c) \end{cases}$$

To exclude a pure bang-bang policy we assume that the plannning period, T, is sufficiently long such that the unique optimal solution I* (given by (15)) can reach the steady state value aK* at some instant t < T.

Recall (cf. (12) and (1a),(1b)) the relationship between the problem with concave cost function C(I) and the linearized problem with cost function $C_1(I,K)$ and notice that both problems have the same set of admissible solutions. For any admissible policy I(t) it holds that

$$0 \le J := \int_0^T \exp(-rt)(S(K) - C(I))dt + \exp(-rT)K(T) \le$$

$$\int_0^T \exp(-rt)(S(K) - C_1(I,K))dt + \exp(-rT)K(T) \le \quad (16)$$

$$\int_0^T \exp(-rt)(S(K) - C_1(I^*,K))dt + \exp(-rt)K(T) := J_1$$

where the first inequality follows from the constraint $C(I) \le S(K)$ and the second one from the definition of function $C_1(I,K)$. The third inequality follows from the optimality of I* as defined in (15). Hence, the optimal value of the objective functional of the linearized problem, J_1, provides an upper bound on the value of the objective functional of the concave problem, J.

If the linearized problem does not admit a singular solution, the solutions of the concave problem and the linear problem are the same. (This is intuitively clear but see also, for example, Feichtinger and Hartl (1986, Th. 3.3)). When the linearized problem does have a singular solution, it can be seen from (16) that no solution of the concave problem will give a value of J which actually reaches the upper bound J_1. However, it can be shown (Feichtinger and Hartl (1986, Section 3.5)) that J_1 can be approximated arbitrarily close by using a chattering control. Such a control consists of switching the investment rate, in principle infinitely fast, between the bounds $I = 0$ and $I = C^{-1}(S(K))$, in order to keep $K(t)$ as close as possible to K^* on the time interval where K^* (the singular solution) is optimal. A solution containing a phase of chattering control will be denoted as ε-optimal (Davidson and Harris (1981)).

The ε-optimal solution of the concave adjustment cost problem is depicted in Figure 2 for the case of a small initial stock of capital ($K_o < K^*$); the chattering control is represented by a dot.

[Insert Figure 2 about here]

The results obtained for a concave adjustment cost function can be summarized as follows. For the case of a small initial stock of capital, $K_o < K^*$ (K^* given by (14a)), an ε-optimal investment policy consists of an initial phase of maximal investment, $I = C^{-1}(S(K))$, followed by an intermediate interval of chattering between $I = 0$ and $I = C^{-1}(S(K))$. This interval is the one where K^* is optimal in the linearized problem. The purpose of the chattering control is to keep K as close as possible to K^*. On a terminal interval investment is zero (due to the transversality condition). Some economic interpretations will be offered later, in connection with the construction of the full solution of the concave-convex problem.

3.2 Convex Adjustment Cost Function

This situation has been studied extensively in the literature and we only give a brief account of the main results. In this section, let $C(I)$ be a strictly convex (total) cost function.

In the convex case it is easily shown that an optimal policy has the same structure as (15) (cf. Kort (1989, Section 3.3) but K^* is no longer constant. The optimal investment policy is characterized by

$I = C^{-1}(S(K))$ (Path 1)

$I = I(p)$ (Path 2)

$I = 0$ (Path 3).

Recall that $I(p) = (C')^{-1}(p)$. $I(p)$ is also determined through the relation

$$\dot{I} = [(r+a)C'(I) - S'(K)]/C''(I). \qquad (17)$$

An analysis of the synthesis problem yields the following trajectories as candidates for optimality, depending on the specific set of model parameters.

I. Path 1 → Path 2 → Path 3

II. Path 2 → Path 3

III. Path 3 → Path 2 → Path 3

IV. Path 3.

The investment policy I(p) on Path 2 is non-increasing over time when it is a part of trajectory I (cf. Kort (1989), Lemma 4, p. 149). It can be shown that

$$\text{Trajectory} \begin{cases} \text{I} \\ \text{II} \\ \text{III, IV} \end{cases} \quad \text{occurs if} \quad \text{NPVMI} \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{at } t = 0$$

where NPVMI means "net present value of marginal investment" The expressions for NPVMI on the various paths can be found in Equations (3.20-22) in Kort (1989).

An equilibrium point $(\hat{K},\hat{I})$ must satisfy $\dot{K} = \dot{I} = 0$. Equations (2) and (17) show that such a point is characterized by

$$S'(\hat{K}) = (r+a)C'(a\hat{K}) \tag{18}$$

and it is a saddle point in the K-I plane.

In Figure 3 the optimal solution is depicted for a case where NPVMI > 0 at t=0. In such a case it is optimal to start with maximal investment at the rate $I = C^{-1}(S(K))$, followed by the interior policy I(p) (where investment is non-increasing over time). The final phase of zero investment also occurs here.

[Insert Figure 3 about here]

Notice that the equilibrium point, $\hat{K}$, may never be reached (for a detailed discussion; see Kort (1989, pp. 149-50). This fact is due to the presence of the flexible accelerator mechanism which shows up in problems with convex adjustment costs; see also Gould (1968).

3.3 Concave-convex Adjustment Cost Function

In this section we combine the solution of the problem having a concave adjustment cost function with the solution of the problem having a convex adjustment cost function. This will yield a complete solution of the problem posed in Section 2.

Consider the K-I plane and notice that the phase diagram (Figure 3) for the convex case is valid when $I > I_1$. For $I \leq I_1$ the results for the concave case (Figure 2) can be used, with a slight modification (to which we shall return). Assumption (7) yields $a\hat{K} < C^{-1}(S(\hat{K}))$ and, depending on the functional forms of S(K) and C(I) (and the parameters), one of the following inequalities must always be true.

Case (i): $I_1 < \hat{I} < C^{-1}(S(\hat{K}))$

Case (ii): $\hat{I} < I_1 < C^{-1}(S(\hat{K}))$

Case (iii): $\hat{I} < C^{-1}(S(\hat{K})) < I_1$.

Here, $\hat{I} = a\hat{K}$ and $\hat{K}$ is given by (18). The solution (depicted by a solid curve) in each of the three cases (i), (ii) and (iii), respectively, is graphed in Figures 4, 5 and 6. If we assume that K_0 is sufficiently small, an optimal investment policy always contains an initial phase of maximal investment. Furthermore, there is always a terminal phase of zero investment. That is, $I = C^{-1}(S(K))$ on an initial interval, say, $[0,t_1]$ and $I = 0$ on a final interval, say, $[t_2,T]$.

REMARK

In what follows we shall use the generic symbols t_1, t_2, and t_3 to indicate instants where the investment policy switches from one type to another. In order not to complicate the notation unnecessarily we have chosen to employ the symbols t_1, t_2, and t_3 in all three cases (i)-(iii). However, it is obvious that a particular switching instant will depend on the characteristics of the case under consideration.

Now we proceed with characterizing the investment policy to be employed in Cases (i)-(iii).

[Insert Figure 4 about here]

Case (i): Refer to Figure 4.

For $I > I_1$ we use the results of Section 3.2 (where the model with a convex adjustment cost function was analyzed). The solution depicted in Figure 3 shows that, at some moment, say, t_1, the initially optimal policy of maximal investment, $I = C^{-1}(S(K))$, must be replaced by the (non-increasing) policy $I = I(p) = (C')^{-1}(p)$. We shall need the following lemma.

Lemma 1. Let t_2 be the instant at which $I(p)$ becomes equal to I_1 and let K denote the corresponding value of K. For $\varepsilon > 0$ and sufficiently small it holds that

$$I = I(p) \geq I_1 \quad \text{for} \quad t_2 - \varepsilon < t \leq t_2$$

$$I = 0 \quad \text{for} \quad t_2 < t < t_2 + \varepsilon.$$

Proof. See the appendix.

The lemma implies that from the instant t_2 and till the end of the planning period, the investment rate is set equal to zero.

To summarize, in Case (i) an optimal investment policy is given by

$$I = \begin{cases} C^{-1}(S(K)) & \\ I(p) & \text{for} \\ 0 & \end{cases} \qquad \begin{matrix} 0 \le t < t_1 \\ t_1 \le t \le t_2 \\ t_2 < t \le T. \end{matrix} \tag{19}$$

The economic implications of such a policy are as follows. Endowed with a small initial amount of capital goods, K_o, the firm starts out by investing at the maximal rate on the interval $[0,t_1)$. This is a growth phase where no dividends are paid out since all revenues are used for investment (recall that $C(I) = S(K)$ during this phase). The investment rate and the stock of capital are both increasing over time (due to assumption (7); see also Kort (1989, p. 142)). At the instant t_1 this growth phase stops because of the fact that marginal earnings have become too small to finance the rising marginal investment costs. (Note that $S'(K)$ decreases when K increases and that $C'(S(K))$ increases when K increases and $S(K) > I_1$; see also Figure 1).

At $t = t_1$ the firm switches from a maximal to an "interior" investment policy, $I = I(p)$. The latter is characterized by $C'(I) = p$, that is, marginal investment costs equal the shadow price of capital stock. During the interval $[t_1,t_2]$ the investment rate is continuously non-increasing with respect to time. On this interval the capital stock first increases but then starts to decrease as soon as the investment rate falls below the replacement level. Since investment is not at its upper bound, a positive amount of dividends is paid out.

The instant t_2 is determined as the moment where the investment rate becomes equal to I_1. On the final interval $(t_2,T]$ contraction occurs in the sense that no investment is undertaken. All profits are distributed as dividends and the stock of capital goods continues to decrease on that interval.

In Case (i) we have seen that the investment rate drops from $I(p) = I_1$ to zero just after $t = t_2$. In the problem with a globally convex adjustment cost function (Figure 3) there is an interval, preceding the final phase of zero investment, on which the investment rate is continuously decreasing towards zero. If the policy of Case (i) included such an interval, on which I is lower than I_1, then such investment rates would be unprofitable because of the higher average costs (C/I) (see also Figure 1).

[Insert Figure 5 about here]

Case (ii): Refer to Figure 5.

We start by proving the following lemma which, as Lemma 1, characterizes the investment policy on an intermediate time interval.

Lemma 2. Define a level of capital, say, $\hat{K}$, by $S'(\hat{K}) = (a+r)C'(I_1)$ and let t_3 be an instant of time such that $t_1 < t_3 < t_2$. There exists an interval $[t_1,t_3]$ on which investment is given by $I = I(p)$ and investment decreases on this interval until $I = I_1$, at which point it holds that $K = \hat{K}$. The instant of time when this happens is t_3. On the interval $[t_3,t_2]$ investment chatters between 0 and I_1 in order to keep K as close as possible to $\hat{K}$.

Proof: See the appendix.

The implication of the lemma is as follows. As previously, on an initial interval $[0,t_1]$, investment is maximal, $I = C^{-1}(S(K))$, and both capital stock and investments are growing over time. Hence, on this interval marginal earnings decrease whereas (for $I > I_1$) marginal adjustment costs increase. At the instant t_1 the firm should start to let the investment rate decrease. On the interval $[t_1,t_3]$ investment is kept at its "interior" level $I = I(p)$, implying a continuously non-increasing investment rate (as in Case (i)). But, in contrast to Case (i), the investment rate reaches at time $t = t_3$ the line $I = I_1$ before falling below the replacement level. Therefore, the capital stock increases on the whole interval $[t_1,t_3]$.

At time $t=t_3$ the capital stock reaches its equilibrium value $\hat{K}$ where it holds that the marginal earnings rate equals the marginal cost rate. The latter consists of the sum of the shareholders' time preference rate (r) and the depreciation rate (a), corrected for the fact that adjustment costs must be paid in order to increase the capital stock by one dollar.

Notice that one might argue that the firm should employ an investment policy which maintains the capital stock at the level $\hat{K}$. In the problem at hand such a replacement policy would, however, be suboptimal since it would induce a higher average investment cost (see Figure 1 and note that $a\hat{K} < I_1$). The better policy is to keep K as close as possible to $\hat{K}$ by using a chattering control, that is, the investment rate switches rapidly between 0 and I_1. This is what happens on the interval (t_3,t_2). From the instant t_2 and onwards investment is zero.

To summarize, in Case (ii) an ε-optimal investment policy is given by

$$I = \begin{cases} C^{-1}(S(K)) & 0 \le t < t_1 \\ I(p) & t_1 \le t \le t_3 \\ \text{Chattering between 0 and } I_1 & t_3 < t < t_2 \\ 0 & t_2 \le t \le T \end{cases}$$

Notice, in contrast to Case (i), that the policy I(p) is not extended all over the intermediate interval $[t_1,t_2]$ but a period of chattering investment on (t_3,t_2) precedes the final interval of zero investment.

[Insert Figure 6 about here]

Case (iii): Refer to Figure 6.

This case has been treated in some detail in Section 3.1 (cf. Figure 2). It suffices to state the following characteristics of the investment policy. On the initial interval $[0,t_1)$ it is optimal to invest maximally, that is, $I = C^{-1}(S(K))$, due to the high marginal earnings (K_o is relatively low and hence $S'(K)$ is high). This policy is followed by chattering such that I switches as fast as possible between 0 and $C^{-1}(S(K))$ on the interval $[t_1,t_2)$, in order to keep K as close as possible to the singular level K* (cf. (14b)). As in Case (ii), a chattering policy is better than replacement investment because the chattering policy carries a lower adjustment cost. On the final interval, $[t_2,t_3]$, no investment is undertaken.

An ε-optimal investment policy is characterized by

$$I = \begin{cases} C^{-1}(S(K)) & 0 \le t < t_1 \\ \text{Chattering between 0 and } C^{-1}(S(K)) & t_1 \le t < t_2 \\ 0 & t_2 \le t \le T. \end{cases}$$

Notice that only in the convex adjustment cost model (Section 3.2) the investment rate is continuous with respect to time throughout the interval $[0,T]$. Of the three alternative solutions of the concave-convex adjustment cost model we see that the solution in Case (i) is the one which comes closest to having this property. Recall that the policy in Case (i) only has one discontinuity (at $t = t_2$).

The solution in Case (iii) has the same structure as the solution of the concave adjustment cost model, which is due to the relatively high value of I_1, below which the average cost function is decreasing. In Case

(ii) the intermediate phase consists of a region where the investment rate is continuous in time and a chattering control region. Hence, the influence of the convex and the concave parts of the adjustment cost function is perhaps most clearly reflected here. This is due to the fact that the value of I_1 is neither particularly high nor low.

4. CONCLUDING REMARKS

In this paper a dynamic investment model of a firm has been analyzed. A key feature was the assumption of a concave-convex investment adjustment cost function. Depending on the parameter values and the specific functional forms of the earnings and cost functions, three different types of optimal solutions emerged. These solutions were illustrated in phase diagrams for the case of a small initial stock of capital goods.

A common characteristic of all solutions was an initial maximal growth phase and a terminal phase of zero investment. However, on an intermediate interval the solutions were different. In the first case the investment rate decreased continuously until reaching a value at which the average adjustment costs are minimal. In the second case such a phase also appeared but was followed by a chattering investment policy where the investment rate switched rapidly between zero and the value at which average adjustment costs are minimal. In the third case the intermediate interval only contained the chattering policy where the investment rate switches rapidly between zero and its upper bound.

The literature on optimal dynamic investment under adjustment costs has mainly focused on the case of convex adjustment costs. However, as already indicated in the introduction, some authors have argued that a partly concave adjustment cost function could also be economically plausible. In particular, an adjustment cost function being concave for low investment rates and convex for higher investment rates could also be realistic but not much attention has been paid to investment policies subject to adjustment costs of such a shape.

Davidson and Harris (1981) studied the implications of a concave-convex adjustment cost function when the firm maximizes its discounted cash flow over an infinite planning period and where there is no upper bound on the rate of investment. In the paper at hand, the planning period is finite and we impose an endogenous upper bound on the investment rate. This bound arises from the requirement that dividends must remain non-negative.

In all cases we obtained an initial growth phase with maximal investment and a final contraction phase (where investment is zero due to the finite planning period). These phases did not appear in the work by Davidson and Harris. Moreover, the policy to be followed during the intermediate phase seems to have a richer structure compared to that in Davidson and Harris. For instance, these authors obtain a final policy of investment which chatters between zero and an unspecified value (being greater than or equal to I_1). (This policy is to be applied after having reached the equilibrium value of K). Introduction of an upper bound on investment made it possible for us to give a precise characterization of the levels of investment to be employed in the chattering policies in Cases (ii) and (iii). Finally notice that, due to their assumptions, Davidson and Harris did not need to distinguish between the situations treated in our Cases (ii) and (iii).

A major point of critique which can be raised against the optimal investment policies of Section 3 is the occurrence of chattering controls. A chattering investment policy, alternating between zero and "high" levels of investment, can be interpreted to imply in practice, the faster the switching the better. Although this interpretation implies that a "pulsing" policy with the highest possible pulsing frequency may be the most profitable, the chattering itself is not an implementable policy in practice (see, e.g., Feichtinger and Sorger (1986), Hahn and Hyun (1989)). However, the investment "cycles" of continuous time models can be thought of as counterparts of investment in "lumps", encountered in discrete time investment models.

Avoiding the occurence of chattering policies requires a change of the model set-up. A few remarks on this issue seem to be in order. Chattering investment policies can be excluded from optimality if we incorporate a sufficiently large "start-up" cost which is incurred every time the firm raises the investment rate from zero to a positive level (Davidson and Harris (1981, Section 4)). Incidentally, such a cost plays the same role as "re-entry" costs in the theory of optimal extraction of resources (Lewis and Schmalensee (1982)) or "pulsing costs" in models of optimal advertising (Hahn and Hyun (1989)). From studies like these it is known that the introduction of such costs prevents the occurence of chattering. (For instance, Hahn and Hyun shows that periodic pulsing (rather than chattering) is the better advertising policy). Yet another possibility is to introduce adjustment costs being convex in the rate of change of the investment rate, that is, the adjustment cost function is modified to depend on I as well as $\dot{I}$ and are concave-convex in I but convex in $\dot{I}$.

APPENDIX

The purpose of the appendix is first to establish conditions for existence and uniqueness of a solution to Eq.(14a) in Section 3.1. Without loss of generality we can assume a zero discount rate. Differentiation of function $rN(K) + M'(K)$ shows that

$$d(rN(K)+M'(K))/dK = S''(K)\{1-aK/C^{-1}(S(K))\} - 2aS'(K)/C^{-1}(S(K)) + \{2aK[S'(K)]^2 + 2aS(K)S'(K) + aKS(K)S''(K)\}/C'(S(K))[C^{-1}(S(K))]^2 - \{aK[S'(K)]^2S(K)C''(S(K)) + 2aK[S'(K)]^2S(K)\}/[C'(S(K)]^2[C^{-1}(S(K)]^2.$$

The first term on the right-hand side is negative by assumption (7). A sufficient (but not necessary) condition for global negativity of the derivative $d(rN+M')/dK$ is as follows.

$$-2K[S'(K)]^2C'(S(K)) - 2S(K)S'(K)C'(S(K)) - KS(K)S''(K)C'(S(K)) + K[S'(K)]^2S(K)C''(S(K)) + 2K[S'(K)]^2S(K) > 0. \quad (A.1)$$

Denote the left-hand side of inequality (A.1) by $W(K)$. Hence $W(K) > 0$ is a sufficient condition for global negativity of $d(rN+M')/dK$. Suppose that the revenue function $S(K)$ satisfies

$$2[S'(K)]^2 < - S''(K)S(K) \quad (A.2)$$

which is guaranteed if $S(K)$ does not grow "too fast". (For example, (A.2) holds for $S(K) = K^b$ provided that $0 < b < 1/3$). Under assumption (A.2) it holds that

$$W(K) > S(K)S'(K)[-2C'(S(K)) + KS'(K)C''(S(K)) + 2KS'(K)]. \quad (A.3)$$

We wish to establish a condition which guarantees that the right-hand side of inequality (A.3) is positive. Having done so we know that (A.1) is satisfied and hence that $d(rN+M')/dK$ is globally negative. Positivity of the right-hand side of (A.3) is equivalent to requiring

$$C'(S(K))/KS'(K) < 1 + C''(S(K))/2. \tag{A.4}$$

Recall that we consider the case of an adjustment cost function having $C''(K) < 0$. Unfortunately, condition (A.4) does not seem to have a straightforward economic interpretation. However, with a concave cost function C defined by $C(S(K)) = S(K)^d$ such that $0 < d < 1$, and with $S(K) = K^b$ (as above), inequality (A.4) becomes

$$S(K)^{2-d} > [2d + bd(1-d)]/2b \tag{A.4'}$$

Notice that (A.4') holds for almost all $S(K) > 0$ if the parameter "d" is sufficiently small, that is, if the cost function C does not grow "too fast".

To summarize, we have demonstrated that if (i) there is no discounting and (ii) functions S(K) and C(S(K)) do not grow "too fast" (i.e., (A.2) and (A.4) holds) then global negativity of $d(rN+M')/dK$ should be guaranteed.

It remains to show that function $rN(K) + M'(K)$ has a positive value at $K = 0$. As above, suppose that $r = 0$ and recall that

$$\begin{aligned} rN(K) + M'(K) &= S'(K) - a[g(K) + Kg'(K)] = \\ & S'(K) - a\{S(K)/C^{-1}(S(K)) + KS'(K)/C^{-1}(S(K)) - \\ & KS'(K)S(K)/C'(S(K))[C^{-1}(S(K))]^2\}. \end{aligned} \tag{A.5}$$

Notice that $S'(0) > 0$, $C'(0) > 0$. By, L'Hôpital's rule the first term in the curly brackets in (A.5) has the limit $C'(0)$ for $K > 0$. The same holds true for the second term. To evaluate the third term, apply L'Hôpital's rule twice. This yields a limit of the third term being equal to $- C'(0)$. Collecting results we obtain

$$\lim_{K \to 0} [S'(K) - a(g(K)+Kg'(K))] = S'(0) - aC'(0) \tag{A.6}$$

which is positive for $S'(0) > aC'(0)$. Recalling that the total cost function $C(I)$ equals $I + A(I))$ we see that $S'(0) > aC'(0)$ is equivalent to $S'(0) > a(1 + A'(0))$. [A similar condition appears in van Schijndel (1988, Eq. (2.31))]. The condition $S'(0) > aC'(0)$ simply states that the marginal profit derived from the first unit of capital stock must be positive.

<u>Remark</u>

It is easy to show that for $r > 0$, condition (A.6) is replaced by $S'(0) - (a+r)C'(0)$. Positivity of this expression means that the marginal profit derived from the first unit of capital stock must exceed what could have been earned by paying the amount $C'(0)$ out as dividend (when the shareholders' time preference rate equals r). If $S'(0)$ were less than $(a+r)C'(0)$, investment in capital goods would be inferior to paying out dividends.

The next thing we need to do is to prove Lemma 1. At $t = t_2$ we have a singular investment policy which satisfies (see also (9))

$$p = C'(I_1) \Rightarrow p = \bar{p} = C'(I_1) = C(I_1)/I_1 \qquad (\text{at } t = t_2) \tag{A.7}$$

Due to (8), (17), (A.7) and $\dot{I} < 0$ at $t = t_2$ we obtain

$$\dot{p} = (r+a)p - S'(K) < 0 \tag{A.8}$$

If we consider the optimal investment policy with $I_1 \leq C^{-1}(S(K))$ (cf. Case 1 at the beginning of Section 3) we conclude from (A.7) and (A.8) that

$$\begin{aligned} I &= I(p) \geq I_1 \quad \text{for} \quad t_2 - \varepsilon < t \leq t_2 \\ I &= 0 \qquad\qquad \text{for} \quad t_2 < t < t + \varepsilon \quad . \end{aligned} \tag{A.9}$$

which completes the proof of Lemma 1.

Finally we prove Lemma 2. Let C(I) be a convex (total) cost function and define a linear-convex cost function, $\tilde{C}(I)$, by

$$\tilde{C}(I) = \bar{p}I \quad \text{for } I \leq I_1 \ ; \quad \tilde{C}(I) = C(I) \text{ for } I > I_1. \tag{A.10}$$

The singular investment policy of the problem with cost given by (A.10) is easily found to satisfy

$$\tilde{C}''(I)\,\dot{I} = (r+a)\tilde{C}'(I) - S'(K). \tag{A.11}$$

If $\tilde{C}(I) = C(I)$ from (A.10) is substituted into (A.11) we obtain (17) and conclude that the singular investment rate of the linear-convex problem equals the singular investment rate of the convex problem for $I > I_1$. When investment I reaches the level $I = I_1$ it must hold that

$$S'(K) = (r+a)\tilde{C}'(I_1) = (r+a)\bar{p}, \tag{A.12}$$

cf. (9), (A.10) and (A.11). Below the level $I = I_1$ we are faced with a linear problem which is easily solved (cf. Feichtinger and Hartl (1986, Th. 3.2)) and admits a singular solution defined by

$$S'(\hat{\hat{K}}) = (r+a)\bar{p}. \qquad (A.13)$$

From (A.12) and (A.13) we see that investment I reaches the level $I = I_1$ for $K = \hat{\hat{K}}$.

The phase diagram of the linear-convex problem looks very much the same as the one depicted in Figure 5 (with C replaced by $\tilde{C}$ and the chattering control replaced by a policy of replacement investment). Having characterized the solution of the linear-convex problem allows us to state the following results for the concave-convex problem.

(a) For $I > I_1$ the optimal solution of the concave-convex problem is the same as the one of the linear-convex problem.

(b) For $I \leq I_1$ it is optimal (in the concave-convex problem) to use a chattering control which consists of switching as fast as possible between 0 and I_1, in order to keep K as close as possible to the singular level $\hat{\hat{K}}$.

In this way the payoff functional of the concave-convex problem can be made arbitrarily close to its upper bound being the optimal value of the payoff functional of the linear-convex problem. This can be seen by comparing the (total) cost function of the concave-convex problem with that of the linear-convex problem).

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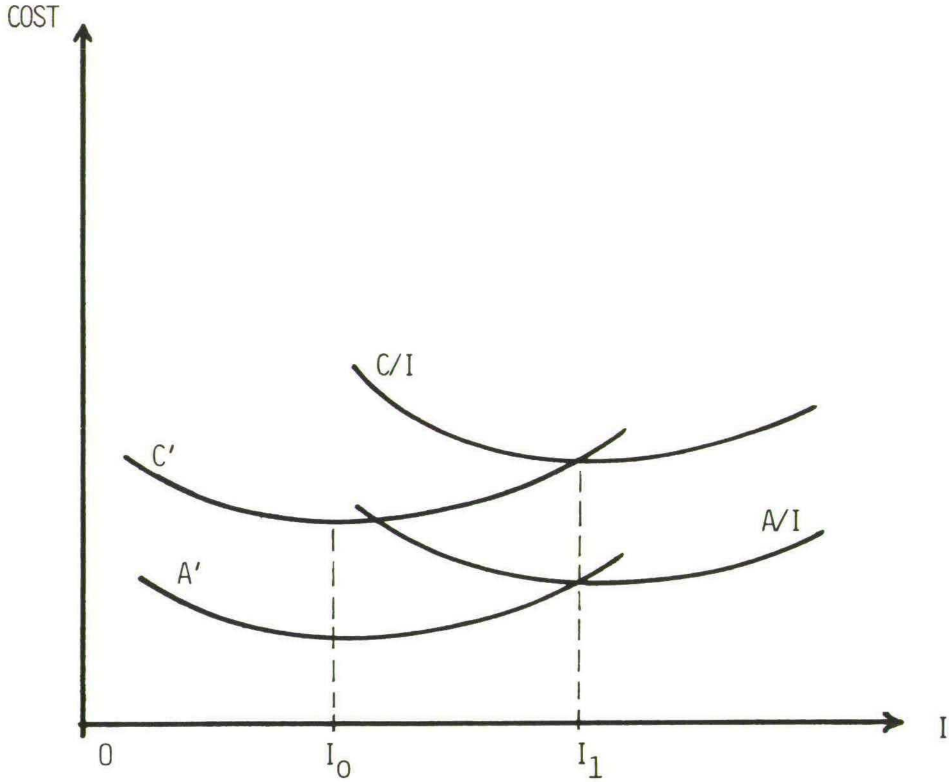

FIGURE 1. MARGINAL AND UNIT COST FUNCTIONS

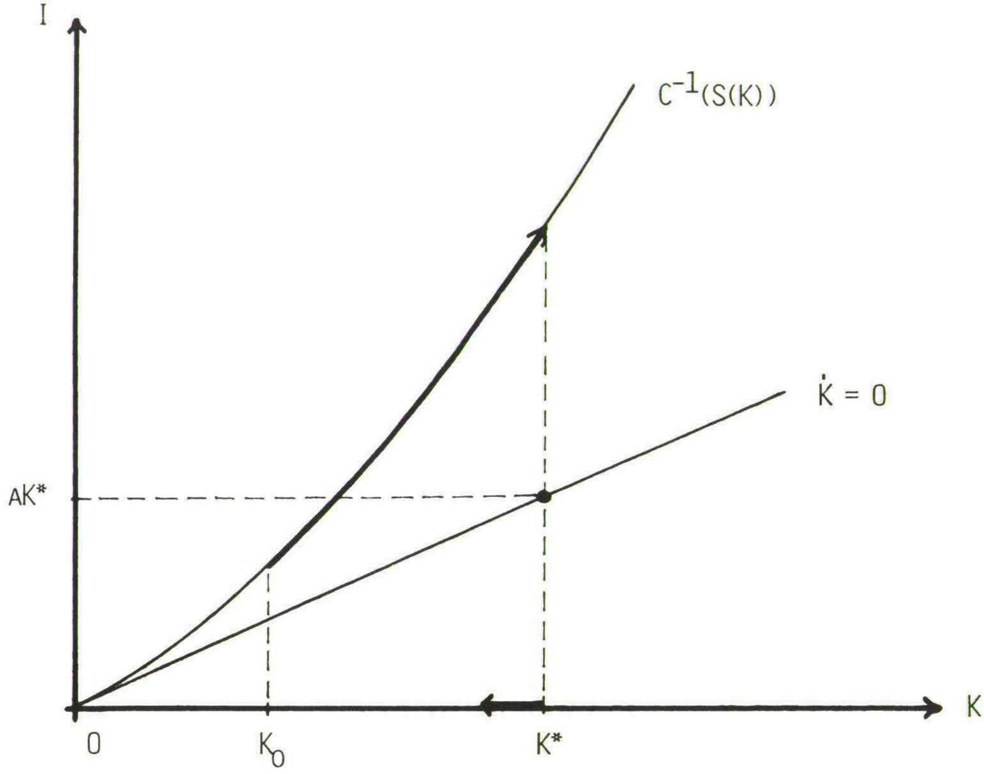

FIGURE 2. OPTIMAL SOLUTION TO THE CONCAVE ADJUSTMENT COST PROBLEM FOR THE CASE $K_0 < K^*$

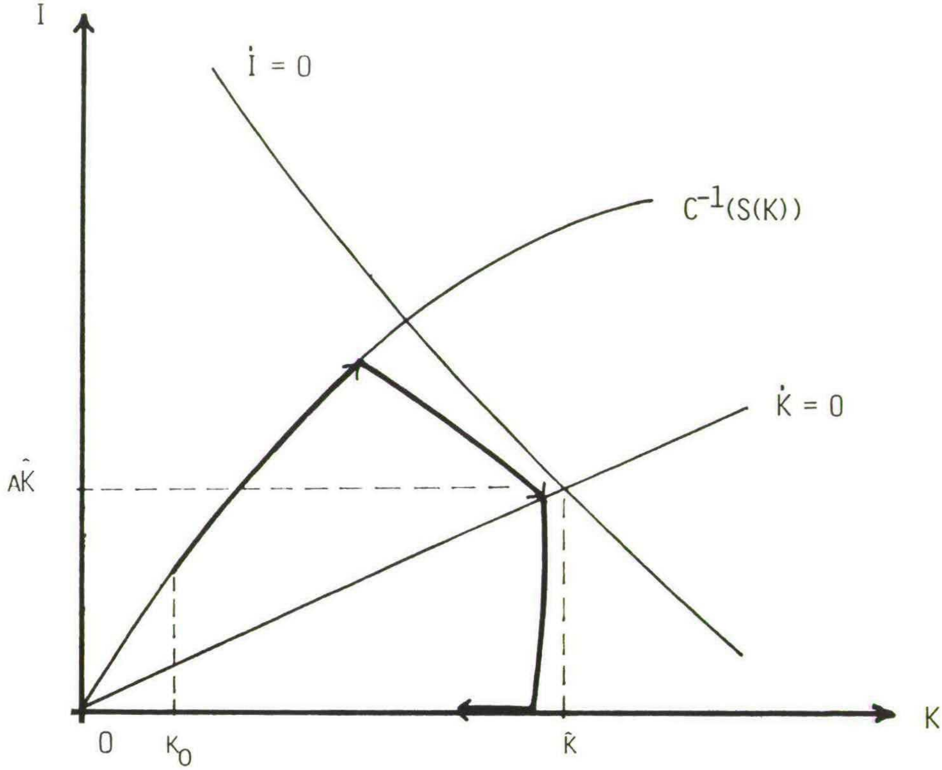

FIGURE 3. OPTIMAL SOLUTION TO THE CONVEX ADJUSTMENT COST PROBLEM FOR THE CASE $K_0 < \hat{K}$

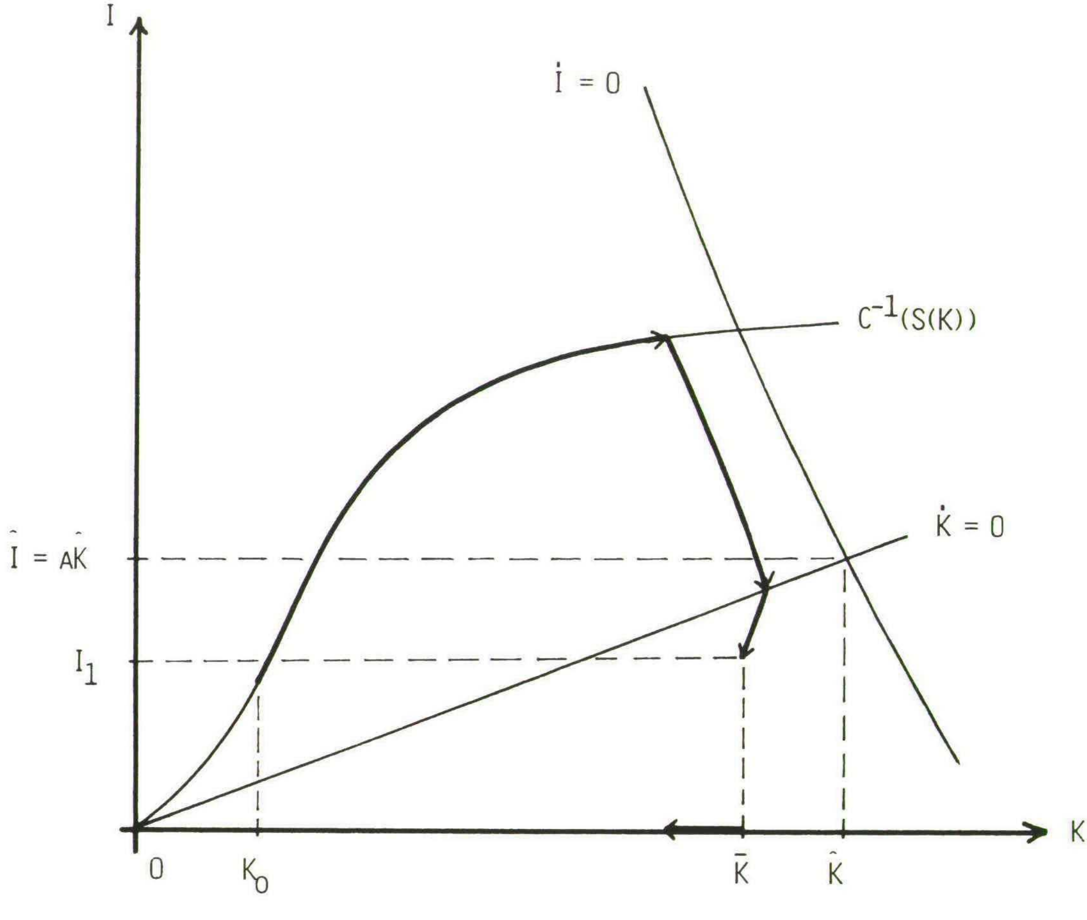

FIGURE 4. OPTIMAL SOLUTION TO THE CONCAVE-CONVEX ADJUSTMENT COST PROBLEM: CASE (I)

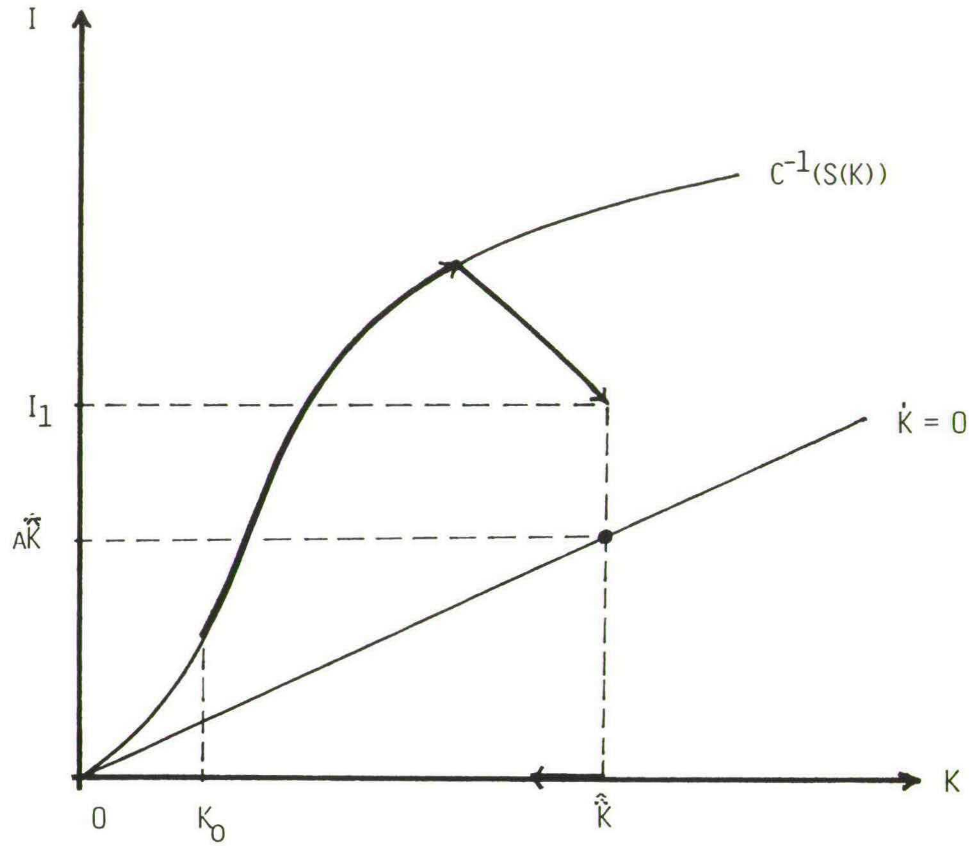

FIGURE 5. OPTIMAL SOLUTION TO THE CONCAVE-CONVEX ADJUSTMENT COST PROBLEM: CASE (II). THE DOT REPRESENTS A CHATTERING CONTROL POLICY

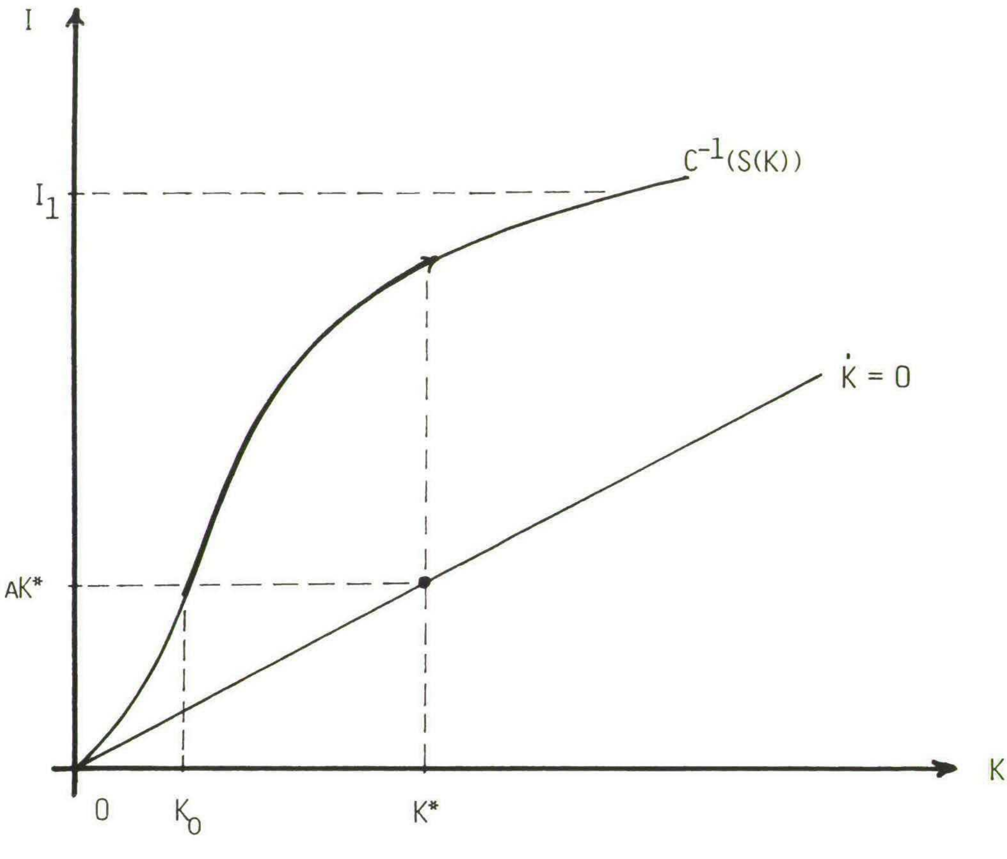

FIGURE 6. OPTIMAL SOLUTION TO THE CONCAVE-CONVEX ADJUSTMENT COST PROBLEM: CASE (III). THE DOT REPRESENTS A CHATTERING CONTROL POLICY

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