subfaculteit der econometrie

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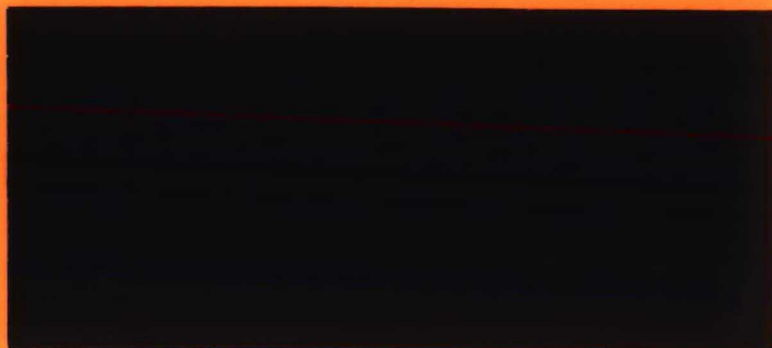
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The 2-ray algorithm for solving
equilibrium problems on the unit
simplex[*]

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330.115

May 1986

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THE 2-RAY ALGORITHM FOR SOLVING EQUILIBRIUM PROBLEMS ON THE UNIT SIMPLEX

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ABSTRACT

In this paper we present a simplicial variable dimension restart algorithm to find economic equilibria on the n-dimensional unit simplex S^n with just two rays to leave the arbitrarily chosen starting point. From this point the algorithm generates a sequence of adjacent simplices of varying dimension in some simplicial subdivision of S^n until a simplex is found which yields an approximate solution. In this way the algorithm traces a piecewise linear path of points which can be considered as an approximation of the piecewise smooth path followed by some price adjustment process. This process solves a sequence of subproblems of varying dimension.

1. Introduction

In this paper a new simplicial variable dimension restart algorithm is presented to solve the equilibrium problem

$$z(x) = 0 \quad x \in S^n \tag{1.1}$$

where z is a continuous function from the n-dimensional unit simplex $S^n = \{x \in R_+^{n+1} \mid \Sigma_{i=1}^{n+1} x_i = 1\}$ into R^{n+1} satisfying the conditions $x^T z(x) = 0$ for all x in S^n and, for all $i=1,\ldots,n+1$, $z_i(x) > 0$ if $x_i = 0$. Simplicial algorithms differ from each other in the number of rays along which the algorithm can leave the arbitrarily chosen starting point. From this point such an algorithm generates a sequence of

adjacent simplices of varying dimension of a simplicial sub-
division of S^n until an approximating simplex is found. An
algorithm with n+1 rays was developed in van der Laan and
Talman (4) (see also Doup and Talman (1)) and with $2^{n+1}-2$
rays in Doup, van der Laan and Talman (2).

Simplicial algorithms to solve the zero point pro-
blem on R^n were developed with n+1 and 2n rays in van der
Laan and Talman (5), with 2^n rays in Wright (8) and with
3^n-1 rays in Kojima and Yamamoto (3). These algorithms on R^n
are closely related to the algorithms on S^n mentioned above.
Moreover, a simplicial algorithm on R^n exists with only two
rays. This algorithm, introduced in Saigal (7) and indepen-
dently in Yamamoto (9), solves a sequence of subproblems of
varying dimension.

We now propose a 2-ray algorithm on S^n which solves
for varying t, $1 < t < n$, a sequence of subproblems consis-
ting of the first t equations of (1.1). The dimension t,
however, needs not to increase monotonically from 1 to n.

In section 2 we describe the piecewise smooth path
of points in S^n of the adjustment process which is approxi-
mately followed by the 2-ray algorithm. In section 3 the
steps of the algorithm to follow the approximating piecewise
linear path are presented. This path is traced by the algo-
rithm by alternating linear programming pivot steps and re-
placement steps in some underlying simplicial subdivision of
S^n. In section 4 some concluding remarks are given.

2. The 2-ray process

The equilibrium problem (EP) can be stated as fol-
lows. Let z be a continuous function from S^n into R^{n+1} such
that $x^T z(x) = 0$ for all x in S^n and for all i in the index
set $I_{n+1} = \{1,\ldots,n+1\}$, $x_i = 0$ implies $z_i(x) > 0$, then the
problem is to find an x^* in S^n such that $z(x^*) = 0$. The
first condition is the so-called Walras' law and the second
condition gives us that x^* lies in the interior of S^n.

Let $x = v$ be an interior starting point in S^n, then the 2-ray process starts by viewing the sign of only $z_1(v)$. If $z_1(v)$ is positive, then x_1 is increased away from v_1 and all other components of x are kept relatively to v equal to each other and lower than x_1/v_1. If $z_1(v)$ is negative, then x_1 is decreased away from v_1 and all other components of x are kept relatively to v equal to each other and larger than x_1/v_1. The first region is described by

$$A(1) = \{x \in S^n \mid x_2/v_2 = \ldots = x_{n+1}/v_{n+1} < x_1/v_1\}$$

whereas the second region is described by

$$A(-1) = \{x \in S^n \mid x_2/v_2 = \ldots = x_{n+1}/v_{n+1} > x_1/v_1\}.$$

The idea behind the 2-ray process is to solve a sequence of problems of varying dimension, not necessarily monotonic. We start by solving the problem $z_1(x) = 0$ and succesively $z_1(x) = 0,\ldots, z_t(x) = 0$ for $t=2,\ldots,n$. By Walras' law this implies that a solution x for $t=n$ also satisfies $z_{n+1}(x) = 0$, i.e. x is a solution to the EP.

In general, the region $A(k)$, $k=t,-t$, for $t=1,\ldots,n$, is given by

$$A(t) = \{x \in S^n \mid x_i/v_i = b, \ i=t+1,\ldots,n+1, \ x_t/v_t > b, \ b > 0\}$$

and

$$A(-t) = \{x \in S^n \mid x_i/v_i = b, \ i=t+1,\ldots,n+1, \ x_t/v_t < b, \ b > 0\}.$$

Furthermore let $A(0)$ be equal to $\{v\}$. In $A(t)$ $(A(-t))$ the process will generate points x such that $z_1(x) = 0,\ldots,$ $z_{t-1}(x) = 0$ and $z_t(x) > 0$ $(z_t(x) < 0)$. Therefore, let $C(k)$, $k=t,-t$, $1 < t < n$, be given by

$$C(t) = C\ell(\{x \in S^n \mid z_1(x) = 0,\ldots,z_{t-1}(x) = 0, \ z_t(x) > 0\})$$

and

$$C(-t) = C\ell(\{x \in S^n \mid z_1(x) = 0, \ldots, z_{t-1}(x) = 0, z_t(x) < 0\})$$

and $C(n+1) = C(-(n+1)) = \{x \in S^n \mid z(x) = 0\}$. Furthermore let, for $k=t,-t$, $1 < t < n$, $B(k)$ be the intersection of $A(k)$ and $C(k)$. In general, if z is continuously differentiable, each $B(k)$, $k=t,-t$, consists of smooth loops and curves (see e.g. (6)) each curve having two end points in bd $B(k)$, i.e. in either (bd $A(k)$) $\cap$ $C(k)$ or in $A(k)$ $\cap$ (bd $C(k)$). More precisely, the boundary of $B(t)$, $1 < t < n$, is equal to

$$\text{bd } B(t) = (\bigcup_{k=-(t-1),(t-1)} (A(k) \cap C(t))) \cup$$

$$(\bigcup_{k=-(t+1),(t+1)} (A(t) \cap C(k)))$$

and the boundary of $B(-t)$, $1 < t < n$, is equal to

$$\text{bd } B(-t) = (\bigcup_{k=-(t-1),(t-1)} (A(k) \cap C(-t))) \cup$$

$$(\bigcup_{k=-(t+1),(t+1)} (A(-t) \cap C(k))).$$

We observe that no points in $B(k)$, $k=t,-t$, lie in the boundary of S^n. This is caused by the fact that $x_i = 0$ implies $z_i(x) > 0$ and for all x in S^n Walras' law holds.

Therefore, if an end point x of a curve in $B(t)$ lies in (bd $A(t)$) $\cap$ $C(t)$, then x is equal to v if $t=1$ and in the case $t > 1$ x is an end point of a curve in $B(-(t-1))$ if $x_{t-1}/v_{t-1} < b$ and in $B(t-1)$ if $x_{t-1}/v_{t-1} > b$. If the point x lies in $A(t) \cap$ bd $C(t)$, then x is a solution of the EP if $t=n$ and in the case $t < n$ the point x is an end point of a curve in $B(-(t+1))$ if $z_{t+1}(x) < 0$ and in $B(t+1)$ if $z_{t+1}(x) > 0$. Similarly, if an end point x of a curve in

B(-t) lies in (bd A(-t)) ∩ C(-t) then x is equal to v if t=1 and in the case t > 1, x is an end point of a curve in B(-(t-1)) if $x_{t-1}/v_{t-1} <$ b and in B(t-1) if $x_{t-1}/v_{t-1} >$ b. If the point x lies in A(-t) ∩ bd C(-t) then x is a solution of the EP if t=n, and in the case t < n the point x is an end point of a curve in B(-(t+1)) if $z_{t+1}(x) <$ 0 and in B(t+1) if $z_{t+1}(x) >$ 0. In this way we can link all the curves in B(k), k=t,-t, 1 < t < n, together.

Let B be given by

$$B = \bigcup_{k \in I_{\pm n}} B(k)$$

where $I_{\pm n} = \{-n,-n+1,\ldots,-1,1,\ldots,n\}$, then the set B is in general the union of piecewise smooth curves and loops and contains one curve, say P, connecting v and a solution point x^* of the EP. All other curves connect two solution points. This is illustrated in figure 1 for n=2.
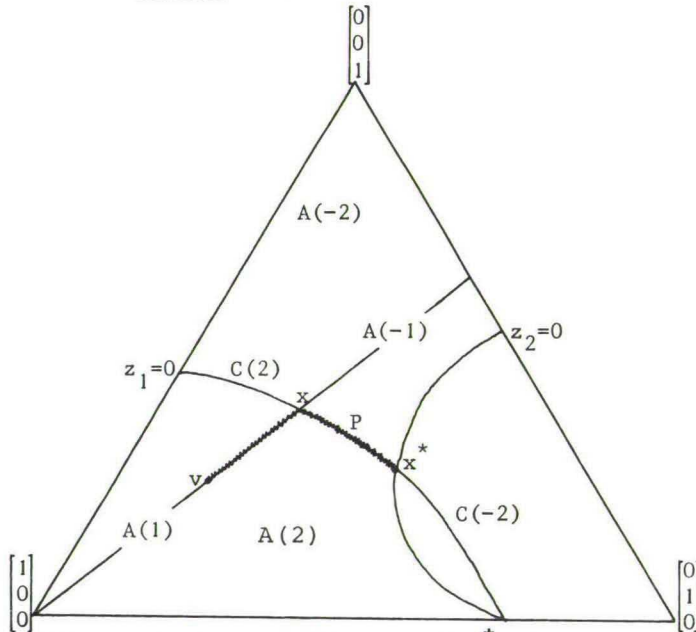

Figure 1. The curve P connects v and x^*. The sign of $z_1(v)$ is negative and in the point x we have $z_1(x) = 0$ and $z_2(x) > 0$

The curve P will be followed approximately by the 2-ray algorithm which we will present in the following section.

3. The 2-ray algorithm on S^n

To follow the piecewise smooth curve P of the 2-ray process, this curve is approximated by the piecewise linear path generated by the steps of a simplicial variable dimension restart algorithm. We call this algorithm the 2-ray algorithm on S^n. The piecewise linear path traced by the algorithm is followed by generating a sequence of adjacent simplices of varying dimension of some simplicial subdivision of S^n containing this p.l. path. This sequence of simplices is generated by alternating pivot steps in a system of linear equations and replacement steps in the simplicial subdivision. The underlying simplicial subdivision is the so-called V-triangulation, developed in (1). Each t-dimensional region A(t) and A(-t) is triangulated in t-simplices. More precisely, each A(k), k=t,-t, is subdivided in $\frac{1}{2}(t+1)!$ subsets A(γ) for permutations $\gamma = (\gamma_1,\ldots,\gamma_t)$ of t elements of the set $I_{\pm t} = \{-t,-t+1,\ldots,-1,1,\ldots,t\}$ such that for all i, $1 < i < t-1$, either i or -i is an element of $\{\gamma_1,\ldots,\gamma_t\}$, k is an element of this set and such that for some s, $0 < s < t$,

$$\gamma_1 < 0,\ldots,\gamma_s < 0, \ \gamma_{s+1} > 0,\ldots,\gamma_t > 0.$$

We call such a permutation a feasible permutation of t elements in $I_{\pm t}$.

Definition 3.1. Let $\gamma = (\gamma_1,\ldots,\gamma_t)$ be a feasible permutation of t elements in $I_{\pm t}$. The region A(γ) is given by

$$A(\gamma) = \{x \in S^n \mid x = v + \Sigma_{h=1}^t \alpha(\gamma_h)q(\gamma_h),$$

$$0 < \alpha(\gamma_t) < \ldots < \alpha(\gamma_1) < 1 \} \quad (3.1)$$

where the vectors $q(\gamma_h)$, $h=1,\ldots,t$ are given by

$$q(\gamma_h) = p(\{\gamma_1,\ldots,\gamma_h\}) - p(\{\gamma_1,\ldots,\gamma_{h-1}\})$$

with for $K \subset I_{-n} = \{-n,\ldots,-1\}$, $K \neq \emptyset$, $p(K)$ given by

$$p_i(K) = \begin{cases} v_i/\Sigma_{k \notin -K}v_k & i \notin -K \\ \\ 0 & i \in -K \end{cases} \quad , \ i \in I_{n+1}$$

and for $K \subset I_{\pm n}$, $K \not\subset I_{-n}$, $K \neq \emptyset$, $p(K)$ given by

$$p_i(K) = \begin{cases} v_i/\Sigma_{k \in K^+}v_k & i \in K^+ \\ \\ 0 & i \notin K^+ \end{cases} \quad , \ i \in I_{n+1}$$

where $K^+ = \{k \in K \mid k > 0\}$. In the case $K = \emptyset$, $p(K)$ is given by $p(K) = v$.

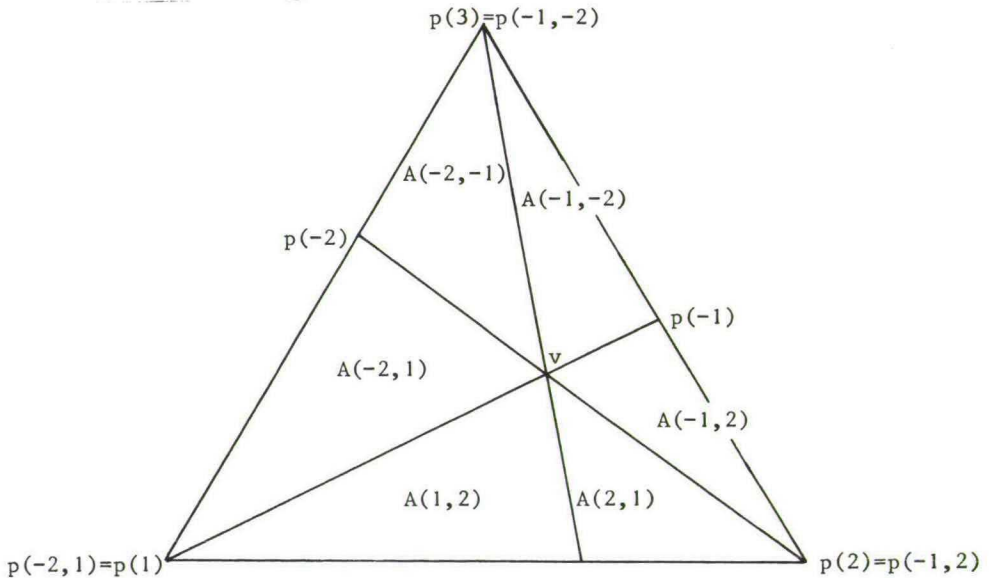Some regions $A(\gamma)$ are given in figure 2 for $n=2$.

Figure 2. The 2-dimensional unit simplex subdivided into 2-dimensional regions $A(\gamma)$

The region $A(\gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_t)$, is t-dimensional for feasible γ. The set $A(k)$, $k = t, -t$, is equal to the union of the $A(\gamma)$'s over all $\gamma = (\gamma_1, \ldots, \gamma_t)$ containing k.

The boundary of $A(\gamma)$ consists of a number of $(t-1)$-dimensional subsets. These are obtained by setting one inequality in (3.1) to an equality. The following four cases can occur: i) $\alpha(\gamma_t) = 0$, ii) $\alpha(\gamma_i) = \alpha(\gamma_{i-1})$, $i \neq s+1$, $2 \leqslant i \leqslant t$, iii) $\alpha(\gamma_{s+1}) = \alpha(\gamma_s)$ and iv) $\alpha(\gamma_1) = 1$. Let x be an interior point in bd $A(\gamma)$. In case i, x lies also in $A(\bar{\gamma})$, with $\bar{\gamma}$ given by

$$\bar{\gamma} = \begin{cases} (\gamma_1, \ldots, \gamma_{t-1}) & \text{if } \gamma_t = t \text{ or } \gamma_t = -t \\ (\gamma_1, \ldots, \gamma_{t-1}, -\gamma_t) & \text{if } \gamma_t < 0 \text{ and } \gamma_t \neq -t \\ (\gamma_1, \ldots, \gamma_s, -\gamma_t, \gamma_{s+1}, \ldots, \gamma_{t-1}) & \text{if } \gamma_t > 0 \text{ and } \gamma_t \neq t. \end{cases}$$

In case ii, x lies also in $A(\bar{\gamma})$, with $\bar{\gamma}$ given by $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{i-2}, \gamma_i, \gamma_{i-1}, \ldots, \gamma_t)$. In case iii, which can only occur if $0 < s < t$, x lies also in $A(\bar{\gamma})$, with $\bar{\gamma}$ given by

$$\bar{\gamma} = \begin{cases} (\gamma_1, \ldots, \gamma_{s-1}, \gamma_{s+1}, \ldots, \gamma_t) & \text{if } \gamma_s = -t \\ (\gamma_1, \ldots, \gamma_{s-1}, \gamma_{s+1}, \ldots, \gamma_t, -\gamma_s) & \text{if } \gamma_s \neq -t. \end{cases}$$

Finally, if case iv holds, x lies in the boundary of S^n. More precisely, if γ_1 is negative, then x lies in the boundary face $S_h^n = \{x \in S^n \mid x_h = 0\}$, $h = -\gamma_1$, and if γ_1 is positive, then x lies in $S^n(t) = \{x \in S^n \mid x_h = 0, h = t+1, \ldots, n+1\}$. So, if γ contains $-t$, then x_h can be equal to zero for some $h = 1, \ldots, t$, and if γ contains t, then x_h can be equal to zero for some $h = 1, \ldots, t-1$ or x_h is equal to zero for all h, $h = t+1, \ldots, n+1$. This concludes the description of the boundary of $A(\gamma)$.

The V-triangulation of S^n triangulates each region $A(\gamma)$ into t-simplices. Let $G(\gamma)$ denote the collection of t-simplices which triangulates $A(\gamma)$ into t-simplices. Let m be a positive integer, m^{-1} denoting the grid size of the V-triangulation, then the collection is defined as follows.

Definition 3.2. The collection $G(\gamma)$ of t-dimensional sim-
plices $\sigma(y^1, \pi(t))$ with vertices $y^1, \ldots, y^{t+1}$ is given by

1) $y^1 = v + \sum_{i=1}^{t} a(\gamma_i) m^{-1} q(\gamma_i)$ for integers $a(\gamma_i)$,
   $i=1,\ldots,t$ such that $0 < a(\gamma_t) < \ldots < a(\gamma_1) < m-1$

2) $\pi(t) = (\pi_1, \ldots, \pi_t)$ is a permutation of the t elements
   in $\{\gamma_1, \ldots, \gamma_t\}$ such that $p < p'$ if $\pi_p = \gamma_{i-1}, \pi_{p'} = \gamma_i$
   for certain i and $a(\pi_p) = a(\pi_{p'})$

and

3) $y^{i+1} = y^i + m^{-1} q(\pi_i)$, $i=1,\ldots,t$.

The triangulation of the regions $A(\gamma)$ is illustrated in fi-
gure 3 for n=2 and m=2. The arrows give the directions of the
vectors $q(.)$.

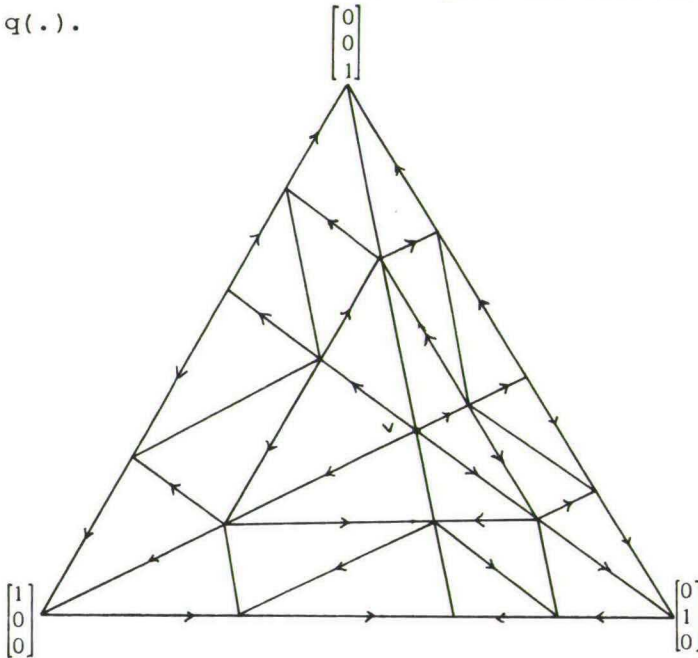Figure 3. Triangulation of the regions $A(\gamma)$ in S^n, n=2, with
grid size $m^{-1} = \frac{1}{2}$

We call the union of $G(\gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_t)$, over all γ's such that γ contains k, the triangulation $G(k)$ of $A(k)$, $k=t,-t$, $1 < t < n$.

The algorithm in general moves from one t-simplex $\sigma(y^1, \pi(t))$ in $A(k)$, $k=t,-t$, to an adjacent t-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(t))$ in $A(k)$. If both simplices lie in $G(\gamma)$, the parameters $\bar{y}^1, \bar{\pi}(t)$ and $\bar{a}$ are obtained from table 1 where p is the index of the vertex of σ such that σ and $\bar{\sigma}$ share the common facet $\tau(y^1, \ldots, y^{p-1}, y^{p+1}, \ldots, y^{t+1})$, where $a_j = a(\gamma_i)$, $j = |\gamma_i|$, $i=1,\ldots,t$, $a_j = 0$ for $j = t+1,\ldots,n+1$, and where $e(h)$ is the h-th unit vector in R^{n+1}.

	$\bar{y}^1$	$\bar{\pi}(t)$	$\bar{a}$
p=1	$y^1 + m^{-1}q(\pi_1)$	$(\pi_2, \ldots, \pi_t, \pi_1)$	$a + e(|\pi_1|)$
1<p<t+1	y^1	$(\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \ldots, \pi_t)$	a
p=t+1	$y^1 - m^{-1}q(\pi_t)$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$	$a - e(|\pi_t|)$

Table 1. p is the index of the vertex to be replaced

We now consider the case that an adjacent simplex of $\sigma(y^1, \pi(t))$ does not lie in $G(\gamma)$. The following three lemmas describe this case.

Lemma 3.3. Let $\sigma(y^1, \pi(t))$ be a t-simplex in $G(\gamma)$. The facet τ opposite vertex y^1 lies in the boundary of $A(\gamma)$ iff

$$\pi_1 = \gamma_1 \text{ and } a(\gamma_1) = m-1.$$

In the case $\gamma_1 < 0$, τ is a (t-1)-simplex in $S^n_{-\gamma_1}$, i.e. all points x in τ satisfy $x_{-\gamma_1} = 0$. In the case $\gamma_1 > 0$, τ is a (t-1)-simplex in $S^n(t)$.

Lemma 3.4. Let $\sigma(y^1, \pi(t))$ be a t-simplex in $G(\gamma)$. The facet τ opposite vertex y^p, for some p, $1 < p < t+1$, lies in the boundary of $A(\gamma)$ iff

$$\pi_{p-1} = \gamma_{i-1}, \ \pi_p = \gamma_i \text{ for certain } i, \ 1 < i < t, \text{ and}$$
$$a(\gamma_{i-1}) = a(\gamma_i).$$

In this case we consider three subcases:

1) $\text{sgn}(\gamma_{i-1}) = \text{sgn}(\gamma_i)$: the facet τ is also a facet of the t-simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{i-2}, \gamma_i, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_t)$ and $\bar{\pi}(t) = (\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \pi_{p+1}, \ldots, \pi_t)$

2) $\gamma_{i-1} = -t$ and $\gamma_i > 0$: the facet τ lies in bd $A(-t)$ and is the (t-1)-simplex $\bar{\sigma}(y^1, \bar{\pi}(t-1))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{i-2}, \gamma_i, \ldots, \gamma_t)$ and $\bar{\pi}(t-1) = (\pi_1, \ldots, \pi_{p-2}, \pi_p, \ldots, \pi_t)$

3) $\gamma_{i-1} < 0$, $\gamma_{i-1} \neq -t$ and $\gamma_i > 0$: the facet τ is also a facet of the t-simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$ where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{i-2}, \gamma_i, \gamma_{i+1}, \ldots, \gamma_t, -\gamma_{i-1})$, $\bar{a}(-\gamma_{i-1}) = 0$ and $\bar{\pi}(t) = (\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p+1}, \ldots, \pi_t, -\pi_{p-1})$.

Lemma 3.5. Let $\sigma(y^1, \pi(t))$ be a t-simplex in $G(\gamma)$. The facet τ opposite vertex y^{t+1} lies in the boundary of $A(\gamma)$ iff

$$\pi_t = \gamma_t \text{ and } a(\gamma_t) = 0.$$

In this case we again consider three subcases:

1) $\gamma_t = -t$ or $\gamma_t = t$: the facet τ lies in bd $A(\gamma_t)$ and is a (t-1)-simplex $\bar{\sigma}(y^1, \bar{\pi}(t-1))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{t-1})$ and $\bar{\pi}(t-1) = (\pi_1, \ldots, \pi_{t-1})$

2) $\gamma_t < 0$ and $\gamma_t \neq -t$: the facet τ is also a facet of the t-simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{t-1}, -\gamma_t)$, $\bar{a}(-\gamma_t) = 0$ and $\bar{\pi}(t) = (\pi_1, \ldots, \pi_{t-1}, -\pi_t)$

3) $\gamma_t > 0$ and $\gamma_t \neq t$: the facet τ is also a facet of the t-simplex $\bar{\sigma}(y^1, \bar{\pi}(t))$ in $G(\bar{\gamma})$, where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_s, -\gamma_t, \gamma_{s+1}, \ldots, \gamma_{t-1})$, with s such that $\gamma_s < 0$ and $\gamma_{s+1} > 0$, $\bar{a}(-\gamma_t) = a(\gamma_{s+1})$ and $\bar{\pi}(t) = (\pi_1, \ldots, \pi_{p-1}, -\pi_t, \pi_p, \ldots, \pi_{t-1})$ where $\pi_p = \gamma_{s+1}$.

This concludes the case that an adjacent simplex of a simplex $\sigma(y^1, \pi(t))$ in $G(\gamma)$ does not lie in $G(\gamma)$.

The algorithm now generates adjacent t-simplices $\sigma(y^1, \pi(t))$ in $G(k)$, $k=t,-t$, for varying t, $1 < t < n$, with k-complete common facets.

<u>Definition</u> 3.6. A g-simplex $\sigma(y^1, \ldots, y^{g+1})$, $g=t-1,t$, is k-complete, $k=t,-t$, if the linear system

$$\Sigma_{i=1}^{g+1} \lambda_i \binom{z(y^i)}{1} + \Sigma_{h=t}^{n+1} \mu_h \binom{e(h)}{0} = \binom{0}{1} \tag{3.2}$$

has a solution (λ, μ) such that $\lambda_i > 0$, $i=1, \ldots, g+1$, and $\mu_t > 0$ if $k=t$ and $\mu_t < 0$ if $k = -t$.

Observe that the system has n+2 rows and for $g=t-1$, n+2 columns and for $g=t$ one column more. To guarantee convergence we need the following assumption.

<u>Nondegeneracy</u> assumption. The linear system (3.2) has for $g=t-1$ a unique solution (λ, μ) such that $\lambda_i > 0$, $i=1, \ldots, t$, and $\mu_t \neq 0$, whereas for $g=t$ at most one of the variables λ_i, $i=1, \ldots, t+1$, and μ_t is equal to zero.

If $\bar{z}$ is the piecewise linear approximation of z with respect to the underlying triangulation and $\sigma(y^1, \ldots, y^{t+1})$ is a k-complete simplex, then according to (3.2) the point $x = \Sigma_i \lambda_i y^i$ lies in σ and satisfies

$$\begin{aligned} \bar{z}_h(x) &= 0 & h&=1, \ldots, t-1 \\ \bar{z}_h(x) &= -\mu_h & h&=t, \ldots, n+1. \end{aligned} \tag{3.3}$$

A k-complete t-simplex contains a whole line segment of points satisfying (3.3). Such a line segment can be followed by making an l.p. pivot step in (3.2). At an end point either $\lambda_p = 0$ for some p or $\mu_t = 0$.

Let $\bar{C}(k)$, k=t,-t, be defined as the regions C(k) but with respect to the function $\bar{z}$, then x satisfying (3.3) lies in $\bar{C}(k)$. The algorithm now follows the piecewise linear path $\bar{P}$ in the union of A(k) $\cap$ $\bar{C}(k)$, k $\in$ $I_{\pm n}$, which starts at the point v. The algorithm terminates when an approximate solution to the EP has been found, i.e. when t=n and μ_n becomes equal to zero. We will now describe the case that for a k-complete t-simplex $\sigma(y^1, \pi(t))$ μ_t becomes equal to zero in the linear system (3.2). If t=n the point x = $\Sigma_{i=1}^{n+1} \lambda_i y^i$ is an approximate solution to the EP.

Lemma 3.7. Let $\sigma(y^1, \pi(t))$ be a k-complete t-simplex of G(γ) in A(k), k=t,-t, 1 < t < n, with solution (λ, μ) of the linear system (3.2) such that $\mu_t = 0$. Then, if $\mu_{t+1} > 0$, $\sigma(y^1, \pi(t))$ is a facet of the (t+1)-complete (t+1)-simplex $\bar{\sigma}(y^1, \bar{\pi}(t+1))$ of G($\bar{\gamma}$) in A(t+1) with $\bar{\gamma} = (\gamma_1, \ldots, \gamma_t, t+1)$, $\bar{a}(t+1) = 0$ and $\bar{\pi}(t+1) = (\pi_1, \ldots, \pi_t, t+1)$, and if $\mu_{t+1} < 0$, then $\sigma(y^1, \pi(t))$ is a facet of the -(t+1)-complete (t+1)-simplex $\bar{\sigma}(y^1, \bar{\pi}(t+1))$ of G($\bar{\gamma}$) in A(-(t+1)) with $\bar{\gamma} = (\gamma_1, \ldots, \gamma_s, -(t+1), \gamma_{s+1}, \ldots, \gamma_t)$, $\bar{a}(-(t+1)) = a(\gamma_{s+1})$ and $\bar{\pi}(t+1) = (\pi_1, \ldots, \pi_{p-1}, -(t+1), \pi_p, \ldots, \pi_t)$ where $\pi_p = \gamma_{s+1}$ and s is such that $\gamma_s < 0$ and $\gamma_{s+1} > 0$.

In both cases of lemma 3.7 the algorithm continues by making an l.p. pivot step with $(z^T(\bar{y}), 1)$ in the linear system (3.2) where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet σ.

If in the case that for a k-complete t-simplex $\sigma(y^1,\pi(t))$ of $G(\gamma)$ in $A(k)$, λ_p becomes equal to zero for some p, $1 < p < t+1$, then the facet τ is either a facet of another k-complete t-simplex in $G(\gamma)$ or not. The latter case is described in the lemmas 3.3, 3.4 and 3.5 whereas the former case is described in table 1. If we consider lemma 3.3, then we observe that a k-complete facet τ cannot lie in S_h^n, for certain h, $1 < h < t-1$, since $\bar{z}_h(x)$ is positive for all x in S_h^n, so that according to (3.3) such an x cannot lie in $\bar{C}(k)$. The same holds in this case for h=t and k=-t. If the algorithm generates a vertex y in $A(t) \cap S^n(t)$, then we set $z_t(y)$ equal to $-\varepsilon$, where ε is some arbitrary small positive number. This prevents the algorithm from generating a t-complete facet τ in $A(t) \cap S^n(t)$. Recall from section 2 that the 2-ray process cannot generate points in $A(t) \cap S^n(t)$.

The algorithm starts with the 0-dimensional simplex $\sigma(v)$. If $z_1(v) > 0$, then v is the facet of the 1-dimensional 1-complete simplex $\sigma(y^1,(1))$ in $A(1)$ with $y^1 = v$, and if $z_1(v) < 0$, then v is the facet of the (-1)-complete 1-simplex $\sigma(y^1,(-1))$ in $A(-1)$ with $y^1 = v$. In general the algorithm generates for varying k, k=t,-t, $1 < t < n$, in $A(k)$ a sequence of adjacent t-dimensional k-complete simplices $\sigma(y^1,\pi(t))$ in $G(\gamma)$ for varying γ such that γ contains k. The common facet of two adjacent simplices in $A(k)$ is k-complete and the parameters $\bar{y}^{-1},\bar{\pi}(t)$ and $\bar{a}$ of a t-simplex $\bar{\sigma}(\bar{y}^{-1},\bar{\pi}(t))$ in $A(k)$ adjacent to a t-simplex $\sigma(y^1,\pi(t))$ in $A(\gamma)$ are obtained from $y^1,\pi(t)$ and a as described in table 1 if $\bar{\sigma}$ lies also in $A(\gamma)$, and in lemma 3.4 cases 1) and 3) and lemma 3.5 cases 2) and 3) if $\bar{\sigma}$ lies in $A(\bar{\gamma})$ for some $\bar{\gamma} \neq \gamma$. In these cases the algorithm continues by making an l.p. pivot step in (3.2) with $(z^T(\bar{y}),1)$ where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the common facet τ. If the k-complete facet τ lies in the boundary of $A(k)$, i.e. when case 2) of lemma 3.4 or case 1) of lemma 3.5 occurs, then τ is a (t-1)-simplex $\sigma(y^1,\bar{\pi}(t-1))$ in either $A(t-1)$ or $A(-(t-1))$ and the algorithm continues by

reintroducing $e(t-1)$ in (3.2) by either increasing μ_{t-1} or decreasing μ_{t-1} from zero. If the algorithm generates a solution (λ,μ) of the linear system (3.2) with $\mu_t = 0$, then in the case $t=n$ the point x given by $x = \Sigma_{i=1}^{n+1}\lambda_i y^i$ is an approximate solution of the EP, and in the case $t < n$, σ is a $(t+1)$-complete t-simplex in $A(t+1)$ if $\mu_{t+1} > 0$ and a $-(t+1)$-complete t-simplex in $A(-(t+1))$ if $\mu_{t+1} < 0$. The parameters of the unique $(t+1)$-simplex $\bar\sigma(y^1,\bar\pi(t+1))$ in $A(k)$, $k=t+1$ or $-(t+1)$, containing σ as facet, are obtained as described in lemma 3.7.

By the description given above the algorithm traces a piecewise linear path $\bar P$ in $\bar B = \cup_k A(k) \cap \bar C(k)$ where the union is over all k in $I_{\pm n}$. The path $\bar P$ connects v with an approximate solution x of the EP. Notice that according to (3.3) the path $\bar P$ can be considered as an approximation of the piecewise smooth path P in B. If the accuracy of the approximate solution x is not satisfactory, the algorithm can be restarted with v equal to x and with a smaller grid size of the triangulation.

4. Concluding remarks

As described in section 3 the 2-ray algorithm solves a sequence of subproblems $\bar z_1(x) = 0,\ldots,\bar z_t(x) = 0$ for varying t, $1 < t < n$. However, it is also possible to solve the sequence of subproblems $\bar z_{i_1}(x) = 0,\ldots,\bar z_{i_t}(x) = 0$, where $(i_1,\ldots,i_t)$ is a (fixed) permutation of t elements in I_{n+1}. Once a permutation is chosen it remains fixed during the remaining part of the algorithm. If an index i_{t+1} has to be added, then we can choose i_{t+1} in such a way that $|\mu_{i_{t+1}}| = \max\{|\mu_h| \mid h \in I_{n+1}\backslash\{i_1,\ldots,i_t\}\}$. For the starting point $x=v$ this coincides with choosing i_1 in such a way that $|z_{i_1}(v)| = \max_{h \in I_{n+1}} |z_h(v)|$. The regions $A(k)$, $k=t,-t$, are adapted such that the permutation $(1,\ldots,t)$ is replaced by $(i_1,\ldots,i_t)$, $t=1,\ldots,n$.

The 2-ray algorithm for solving the equilibrium problem can be easily adapted to solve the more general nonlinear complementarity problem (NLCP) on S^n which can be stated as follows. Let z be a continuous function from S^n into R^{n+1} such that $x^T z(x) = 0$ for all x in S^n, then the NLCP is to find an x^* in S^n such that $z(x^*) \leqslant 0$. In this case a starting point can lie on the boundary of S^n and we must allow for movements on the boundary.

Algorithms discussed on S^n can be generalized to the product space S of N, N $>$ 1, unit simplices S^{n_j}, j=1,...,N. A generalization of the 2-ray algorithm would then be the 2^N-ray algorithm on S which can be applied to the EP and the NLCP on S. Again we must take into account that the starting point can lie on the boundary of S and we must allow for movements on the boundary.

The 2^N-ray algorithm on $S = \pi_{j=1}^N S^{n_j}$, N $>$ 1, to solve the NLCP on S will be discussed in a subsequent paper.

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