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DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

THE GEWIRTZ GRAPH - AN EXERCISE IN THE THEORY OF GRAPH SPECTRA

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FEW 486

The Gewirtz graph - an exercise in the theory of graph spectra

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ABSTRACT

We prove that there is a unique graph (on 56 vertices) with spectrum $10^1\ 2^{35}\ (-4)^{20}$ and examine its structure. It turns out that e.g. the Coxeter graph (on 28 vertices) and the Sylvester graph (on 36 vertices) are induced subgraphs. We give several descriptions of this graph.

1. Goal

Let $\Gamma = (X,E)$ be a strongly regular graph with parameters $(v,k,\lambda,\mu) = (56,10,0,2)$. Then Γ (that is, its 0-1 adjacency matrix A) has spectrum $10^1\ 2^{35}\ (-4)^{20}$, where the exponents denote multiplicities. We will show that up to isomorphism there is a unique such graph Γ, a fact that was first proved by GEWIRTZ [8]. However, Gewirtz used (20 pages of) tedious combinatorial arguments where we use more powerful spectral techniques. In addition we find several new descriptions of the Gewirtz graph and rather precise information about its subgraphs. We show uniqueness in two ways, namely directly and by embedding Γ in the Higman-Sims graph. For the direct proof, we want to show the following:

(i) Γ contains an induced subgraph $6C_4$ (the disjoint union of six quadrangles).
(ii) If T is a 24-subset of X inducing $6C_4$, then $X \setminus T$ is the bipartite point-block incidence graph of $AG(2,4)$ from which a parallel class of lines has been removed.
(iii) The adjacencies between T and $X \setminus T$ are uniquely determined.

Parts (i) and (ii) have short and elegant proofs, but part (iii) is too detailed. This detailed analysis can be avoided by going up instead of down. For the embedding proof, we want to show the following:

(iv) Γ has 42 16-cocliques, and the graph Δ with these 42 16-cocliques as vertices and where two 16-cocliques are adjacent when they are disjoint, is the point-line incidence graph of the projective plane $PG(2,4)$.
(v) Γ can be embedded as the subgraph of points nonadjacent to a given edge in the Higman-Sims graph.

2. Tools

We use the following spectral tools:
(A) *A positive semidefinite symmetric matrix M of order n and rank f is the Gram matrix (or inner product matrix) of a set of n vectors in $\mathbf{R}^f$.*

(B) Let A and B be real symmetric matrices of orders n and m (where $m \le n$) and with eigenvalues $\theta_1 \ge \cdots \ge \theta_n$ and $\eta_1 \ge \cdots \ge \eta_m$, respectively. We say that the eigenvalues of B *interlace* those of A when $\theta_j \ge \eta_j \ge \theta_{n-m+j}$ for all j $(1 \le j \le m)$. We say that the interlacing is *tight* when for some integer l we have $\eta_j = \theta_j$ for $1 \le j \le l$ and $\eta_j = \theta_{n-m+j}$ for $l+1 \le j \le m$. If B is a principal submatrix of A then the eigenvalues of B interlace those of A. Another case of interlacing is given by the following theorem.

Theorem. *Given a symmetric partition of the rows and columns of a symmetric matrix A, let B be the matrix with as entries the average row sums of the parts of A. Then the eigenvalues of B interlace those of A, and when the interlacing is tight, the parts of A have constant row sums. Conversely, if the parts of A have constant row sums, then each eigenvalue of B is an eigenvalue of A.*

(C) Given a symmetric partition of a symmetric matrix A with two eigenvalues into four submatrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

the eigenvalues of A_{22} can be computed from those of A_{11}: *If A has eigenvalues α and β (where $\alpha > \beta$) with multiplicities f and $n-f$, respectively, and A_{11} (of order m) has eigenvalues $\theta_1 \geq \cdots \geq \theta_m$, then A_{22} (of order $n-m$) has eigenvalues $\eta_1 \geq \cdots \geq \eta_{n-m}$, where*

$$\eta_i = \begin{cases} \alpha & \text{if } 1 \leq i \leq f-m, \\ \beta & \text{if } f+1 \leq i \leq n-m, \\ \alpha+\beta-\theta_{f-i+1} & \text{otherwise.} \end{cases}$$

For proofs of the above statements, see, e.g., Chapter 1 of HAEMERS [9].

Notation: we shall call an eigenvector of a matrix with eigenvalue θ a θ-eigenvector of that matrix.

Extremal subgraphs. As an example of the use of (B) above, let us derive bounds for the number of vertices of a regular subgraph of a regular graph. Let $\Gamma = (X,E)$ be regular of valency k on v vertices, and let G be a subset of size u of X, inducing a regular graph of valency g. Let A be the adjacency matrix of Γ, and B the matrix of average row sums of A for the partition $\{G, X \setminus G\}$. Then

$$B = \begin{bmatrix} g & k-g \\ \frac{u(k-g)}{v-u} & k - \frac{u(k-g)}{v-u} \end{bmatrix} \text{ with eigenvalues } k \text{ and } g - \frac{u(k-g)}{v-u}.$$

Thus, if Γ has eigenvalues $k = \theta_1 \geq \theta_2 \geq \cdots \geq \theta_v$, then

$$\theta_2 \geq g - \frac{u(k-g)}{v-u} \geq \theta_v,$$

and equality holds on one side if and only if each point outside G has $u(k-g)/(v-u)$ neighbours in G.

In particular, for our strongly regular graph Γ with parameters $(v,k,\lambda,\mu) = (56,10,0,2)$ (and $\theta_2 = 2$, $\theta_{56} = -4$) we find

(D) *If G is a subgraph of Γ of size u and regular of valency g, then $7(g-2) \leq u \leq 4(g+4)$. Equality holds in the right (resp. left) hand inequality if and only if each point outside G is adjacent to $g+4$ (resp. $g-2$) vertices of G.*

There are numerous examples of equality, and we shall discuss them in the second part of this paper. For the uniqueness proof only the cases $(u,g) = (16,0)$ or $(24,2)$ will play a rôle.

Distance-regularity from the spectrum.

Proposition. *A graph Γ with the spectrum of a distance-regular graph with diameter d and girth at least $2d-1$, is such a graph.*

Proof. Let Γ have adjacency matrix A, and distance-i matrix A_i, so that $A_0 = I$ and $A_1 = A$. As is well-known (cf. [2], 3.2.2) one can see from the spectrum of a graph whether it is regular and connected. The number of nontrivial g-cycles can be found from $\operatorname{tr} A^g$, so we can find the girth from the spectrum. If $\theta_0, \cdots, \theta_d$ are the distinct eigenvalues of Γ, with $\theta_0 \geq 0$ of multiplicity 1, then

$$\sum_{i \geq 0} A_i = J,$$

$$A_2 = A^2 - kI \quad (\text{if } d > 2),$$

$$A_i = AA_{i-1} - (k-1)A_{i-2} \quad (\text{for } 2 < i < d),$$

$$\prod_{i=1}^{d}(A - \theta_i I) = \frac{1}{v}\prod_{i=1}^{d}(\theta_0 - \theta_i)J.$$

Since Γ has diameter at most d, this means that we can express each A_i as a polynomial of degree i in A, and hence Γ is distance-regular. ■

In particular, any graph with spectrum $5^1\, 2^{16}\, (-1)^{10}\, (-3)^9$ is distance-regular with intersection array $\{5,4,2;\, 1,1,4\}$, and hence isomorphic to the Sylvester graph (cf. [2], 13.1.2). Note that one cannot see from the spectrum of a graph whether it is distance-regular (cf. [2], 9.2.6). It is unknown whether 'distance-regularity of diameter 3' can be recognised.

3. First uniqueness proof

(i). Let $\mathbf{T}$ be the collection of subgraphs of Γ on 24 vertices that are regular of degree 2. [Then $\mathbf{T}$ is the collection of subgraphs of Γ induced by the *special sets* of CAMERON, GOETHALS & SEIDEL [4], p. 279. We shall see eventually that $|\mathbf{T}| = 105$, and that $T \cong 6C_4$ for all $T \in \mathbf{T}$.] We show first that each quadrangle is contained in a unique $T \in \mathbf{T}$. Next, we show that at least one $T \in \mathbf{T}$ is isomorphic to $6C_4$. Let Q be a quadrangle in Γ. Then we find a partition $\{Q,R,S\}$ of the vertex set X of Γ, where S is the 32-set containing neighbours of points in Q, and R is the 20-set containing the remaining points. (Note that since $\lambda = 0$ and $\mu = 2$, each point of S has a unique neighbour in Q.) Now since each $r \in R$ has two common neighbours with each $q \in Q$, the vertex r has 8 neighbours in S, so R is regular of valency 2, and so is $T := Q \cup R$. Clearly, T is the unique member of $\mathbf{T}$ containing Q.

Now let T be an arbitrary element of $\mathbf{T}$, with adjacency matrix A_T. We first show that T is a union of polygons, each of which has a length divisible by four. Let Q' be a connected component of T, and put $R' = T \setminus Q'$, and $S = X \setminus T$. Consider the matrix B' corresponding to the partition $\{Q',R',S\}$ of X. If $|Q'| = c$, then

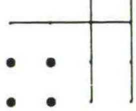
$$B' = \begin{bmatrix} 2 & 0 & 8 \\ 0 & 2 & 8 \\ c/4 & 6-c/4 & 4 \end{bmatrix} \text{ with eigenvalues 10, 2 and } -4.$$

Interlacing is tight, so the row sums of the submatrices of A corresponding to this partition are constant, and since these row sums are clearly integral, we find that c is a multiple of 4.

The matrix $A - \frac{1}{7}J$ has spectrum $2^{36}\,(-4)^{20}$. So, by interlacing, $A_T - \frac{1}{7}J$ has eigenvalue 2 with multiplicity at least $36 - 32 = 4$. In addition, $\mathbf{1}$ is a 2-eigenvector of A_T but not of $A_T - \frac{1}{7}J$. So the multiplicity of the eigenvalue 2 of A_T is at least 5. It follows that T has at least five connected components, so that T is either $6C_4$ or $4C_4 + C_8$. Being in the same member of $\mathbf{T}$ clearly defines an equivalence relation on the quadrangles of Γ. But the total number of quadrangles in Γ is $56 \cdot 45/4 = 630$, which is not a multiple of four, so that $T \cong 6C_4$ occurs at least somewhere in Γ.

(ii) Choose a $T \in \mathbf{T}$ with $\cong 6C_4$. Then T has known spectrum $2^6\, 0^{12}\, (-2)^6$ and $A - \frac{1}{7}J$ has two eigenvalues, so we can compute the spectrum of S by use of tool C, and find $4^1\, 2^{12}\, 0^6\, (-2)^{12}\, (-4)^1$. Since this spectrum is symmetric around 0, S is bipartite, and we may regard S as the point-block incidence graph of a design with 16 points and 16 blocks, block size 4 and 4 blocks on each point. Let D be the point-block incidence matrix of order 16 of this design. (Then the adjacency matrix of S is $\begin{bmatrix} 0 & D \\ D^\top & 0 \end{bmatrix}$.) Then $DD^\top$ has spectrum $16^1\, 4^{12}\, 0^3$ and entries 4 on the diagonal and 0, 1 or 2 off the diagonal. It follows that $E := J + 4I - DD^\top$ has spectrum $4^4\, 0^{12}$ and entries in $\{-1,0,1\}$. In particular, E is a positive semidefinite matrix of rang 4 and we can view E as the Gram matrix of 16 unit vectors in $\mathbb{R}^4$ with inner products in $\{-1,0,1\}$. Thus, these unit vectors can be taken to be $\pm e_i$ $(1 \le i \le 4)$, and since E has row sums 4, each e_i occurs four times, and no -1 entries occur. For our design this means that any two blocks meet in at most one point, and that the blocks come in four parallel classes. Also the points come in four parallel classes, and it follows that our design is $AG(2,4)$ from which one parallel class of lines has been removed (i.e., $GD[4,1,4;16]$).

(iii) Consider the partition $\{T,P,L\}$ of X into T and the points and the lines. Since P and L are 16-cocliques, we find from (D) above that each vertex of T is joined to four points and four lines. For $t \in T$, consider its 4 neighbours x_i in P and its 4 neighbours l_j in L. The set $P \setminus \{x_1,x_2,x_3,x_4\}$ is covered by the l_j, while never $x_i \in l_j$, no three x_i on a line and no three l_j on a point, so the picture of these points and lines is



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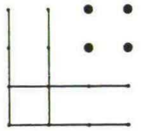
that is, two pairs of parallel lines and two pairs of parallel points (i.e., pairs of points determining the 'missing' direction). Now consider the point $t' \in T$ opposite to t in the same quadrangle, so that the common neighbours of t and t' lie in T. Then t' is adjacent to two points on each of the lines l_j, and it follows that t' determines the 'complementary' picture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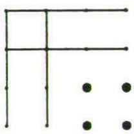
that is, the remaining two lines in the two parallel classes of lines determined by t, and similarly for the points. Let $l_1 /\!/ l_2$. Then l_1 and l_2 have another common neighbour u; let the other two lines adjacent to u be m_1 and m_2. Since we have seen the two common neighbours of t and u already (they are l_1, l_2), u is not adjacent to any of the points x_i, and this forces the 8 neighbours of u to be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The common neighbours s and s' of t and t' determine lines in the other two directions, and so do the common neighbours v and v' of u and u'. This shows that quadrangles come in pairs determining the same partition of 2+2 directions.
Now label the vertices of each of the six quadrangles with 0, 1, 2, 3 (consecutively) and label the 16 points and 16 lines with the 6-tuples of their neighbours in T. Without loss of generality the 16 lines have labels

000000 021111 110213 131320
002222 023333 112031 133102
220022 201133 330231 311302
222200 203311 332013 313120

(where the choice between 131302 and 131320 is forced by considering the point of intersection of 000000 and 021111) and then the 16 points have labels (without loss of generality)

212323 231212 123221 322132
210101 233030 121003 320310
032301 011230 303203 102110
030123 013012 301021 100332.

This shows uniqueness of the graph Γ.

4. Second uniqueness proof

Now, let us embark upon the second uniqueness proof. In seven steps we determine the properties of Δ, the graph defined on the 16-cocliques in Γ, where disjoint cocliques are adjacent. We make use of the results in (i) and (ii).

(0) *Let C be a coclique in Γ of size 16. Then each vertex $x \in X \setminus C$ is adjacent to precisely 4 vertices in C.*

[Indeed, this is a special case of (D) above.]

(1) *If $T \in \mathbf{T}$ then T is disjoint from some 16-coclique if and only if $T \cong 6C_4$.*

[Indeed, $X \setminus T$ is regular of degree 4 on 32 vertices; if $X \setminus T$ contains a 16-coclique, then $X \setminus T$ is bipartite and therefore has an eigenvalue -4. Using tool C we find that T has eigenvalue 2 with multiplicity at least 6, so $T \cong 6C_4$. The converse implication follows from (ii).]

(2) *Each 16-coclique C is disjoint from five $6C_4$'s (three on each point of $X \setminus C$).*

[Indeed, given a 16-coclique C, there are 120 quadrangles meeting it in a pair (15 on each point of C, 6 on each point of $X \setminus C$), and $16 \cdot (45-15) = 480$ quadrangles meeting it in a single point (30 on each point of C, $4 \cdot 6 + 12 = 36$ on each point of $X \setminus C$), which leaves $630 - 120 - 480 = 30$ quadrangles disjoint from C (3 on each point of $X \setminus C$). But if Q is a quadrangle disjoint from C, then each of the four vertices of Q have 4 neighbours in C, and none occurs twice, so each point of C has a (unique) neighbour in Q, and the $6C_4$ containing Q is disjoint from C.]

(3) *There are at most 42 16-cocliques, each disjoint from 5 others. (I.e., Δ is regular of valency 5.)*

[Indeed, if C is a 16-coclique, and T is a $6C_4$ disjoint from C, then $X \setminus (C \cup T)$ is again a 16-coclique, disjoint from C. Conversely, if C and D are disjoint 16-cocliques, then $X \setminus (C \cup D)$ is regular of valency 2 on 24 vertices, hence a $6C_4$. Thus, by (2) it follows that Δ is regular of valency 5 and has at most $630/6 = 105$ edges.]

(4) *If $C \sim D \sim E$ in Δ, and $C \neq E$, then $|C \cap E| = 4$.*

[Indeed, let $T = X \setminus (C \cup D)$ and $T' = X \setminus (D \cup E)$. If $X \setminus (C \cup D \cup E)$ contains a path of length two (on three points), then it is contained in both T and T', and in both completes to a quadrangle. But since $\mu = 2$, this must be the same quadrangle, and $T = T'$, contradiction. Thus, E contains two opposite points from each of the six quadrangles of T and hence meets C in four points.]

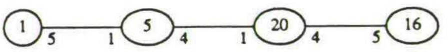
(5) *The union of two disjoint 16-cocliques does not contain a quadrangle.*

[Indeed, this was shown in part (ii): two points are not joined by two lines.]

(6) *If $C \sim D \sim E \sim F$ in Δ, and $C \neq E$, $D \neq F$, then $|C \cap F| = 6$.*

[Indeed, $T := X \setminus (C \cup D)$ is a $6C_4$, and we show that F contains precisely one point of each of its six quadrangles. Since also $|F \cap D| = 4$, it then follows that $|F \cap C| = 6$. If F is disjoint from a quadrangle in T, then F is disjoint from T, so that $F \subseteq C \cup D$, and $|F \cap C| = 12$. But this is ridiculous: we would find (by (0)) a $K_{4,4}$ on $(F \setminus C) \cup (C \setminus F)$, but $\mu = 2$, contradiction. Thus, F contains at least one point from each quadrangle Q of T, and since $|E \cap Q| = 2$ it follows from (5) that F cannot contain more than one point from Q.]

From (4) and (6) it immediately follows that Δ has girth at least 6. With (3) this implies that Δ is distance-regular with distance-distribution diagram


$v = 42$

with relations: $d_\Delta(B,C) = 0, 1, 2, 3$ if and only if $|B \cap C| = 16, 0, 4, 6$, respectively. [Since equality holds in the estimates of (3), we now see that $T \equiv 6C_4$ for all $T \in \mathbf{T}$.] In particular, Δ is bipartite, say with bipartite halves Y and Z. Now we can embed Γ and Δ in a graph H on 100 vertices with vertex set $X \cup Y \cup Z \cup \{a,b\}$ where X induces Γ, $Y \cup Z$ induces Δ, $\{a,b\}$ induces an edge K_2, a is adjacent to the points of Y, and b to the points of Z, and finally $x \in X$ is adjacent to $C \in Y \cup Z$ when $x \in C$. We show that H is strongly regular with parameters $(v',k',\lambda',\mu') = (100,22,0,6)$. It is immediately clear that $v' = 100$ and $\lambda' = 0$. And $k' = 22$ and $\mu' = 6$ follow immediately from the following observations (7)-(9).

(7) *Each $x \in X$ lies in 12 16-cocliques, 6 from each bipartite half of Δ.*

[Indeed, each $x \in X$ lies in 45 quadrangles, hence in 45 $6C_4$'s, hence (by (2)) misses $90/3 = 30$ 16-cocliques. Again by (2), Δ induces on these 30 16-cocliques a bipartite graph of valency 3, so there are equal numbers from each bipartite half.]

(8) *If $x \notin B$, where $x \in X$ and B is a 16-coclique, then there are two 16-cocliques containing x and disjoint from B.*

[Indeed, this is immediate from (2).]

(9) *Every two nonadjacent points x, $y \in X$ lie in 4 16-cocliques, two from each bipartite half of Δ.*

[Indeed, for $x \in X$, let Y_x be the set of elements of Y containing x, and for $x \neq u \in X$, let $a_u := |Y_x \cap Y_u|$. Since any two elements of Y_x meet in three elements different from x, we find $\sum_u 1 = 45$, $\sum_u a_u = 6 \cdot 15 = 90$ and $\sum_u a_u(a_u - 1) = 6 \cdot 5 \cdot 3 = 90$, so that $\sum_u (a_u - 2)^2 = 0$.]

Now it is well-known and very easy to prove (see GEWIRTZ [7], Theorem 6.4) that there is a unique strongly regular graph with parameters (100,22,0,6), namely the Higman-Sims graph, and this graph has an automorphism group acting transitively on the edges, so it follows that Γ is uniquely determined.

[The uniqueness of the Higman-Sims graph is derived from the uniqueness of the Steiner system $S(3,6,22)$ by a short and simple argument (cf. BROUWER [1]), while the uniqueness of $S(3,6,22)$ follows by a simple coding-theoretic argument (maybe due to E.S. Lander): the binary code spanned by the lines of $PG(2,4)$ has dimension 10, and so has the extended code C. Now C is selforthogonal with word length 22, so $C^{\perp}$ is a code of dimension 12 containing C, and the words of minimum weight in the three codes of dimension 11 between C and $C^{\perp}$ give just the three ways of extending $PG(2,4)$ to $S(3,6,22)$. Many other proofs can be found in the literature.]

5. Descriptions of the Gewirtz graph

We give several descriptions of Γ, useful in exhibiting (parts of) its automorphism group and showing the existence of various subgraphs.

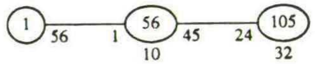
(a) In the Steiner system $S(3,6,22)$, take all 56 blocks missing a fixed symbol. Join two blocks when they are disjoint. This shows the presence of a group of automorphisms $M_{21}.2 = L_3(4).2_2$, a subgroup of index 2 in the full automorphism group of Γ.

(b) In the Higman-Sims graph H (on 100 vertices), take the graph induced on the 56 vertices at distance 2 from both endpoints of a fixed edge. This shows the presence of the full group $L_3(4).2^2$. The Higman-Sims graph can be split into two Hoffman-Singleton graphs (on 50 points). Taking the edge inside one of them, we find that the Gewirtz graph is split into a Sylvester graph (on 36 points) and a $10K_2$. Taking an edge meeting both parts, we find that the Gewirtz graph is split into two Coxeter graphs. (For more details, cf. BROUWER, COHEN & NEUMAIER [2], §13.1.)

(c) The latter partition leads to a description of Γ in terms of the Fano plane. The Coxeter graph can be defined on the 28 antiflags of the Fano plane, where two antiflags (P_1,l_1) and (P_2,l_2) are adjacent whenever $P_1 \neq P_2$, $l_1 \neq l_2$, $P_1 \notin l_2$ and $P_2 \notin l_1$ (see [2], §12.3). Now the Gewirtz graph is defined on two copies of this set of antiflags. Adjacency within one set is the same as for the Coxeter graph. An antiflag $(P_1,l_1)'$ from the first set is adjacent to an antiflag $(P_2,l_2)''$ from the second set if $(P_1,l_1) = (P_2,l_2)$ or ($P_1 \in l_2$ and $P_2 \notin l_1$).

(d) It is possible to split the collinearity graph of the unique generalized quadrangle $GQ(3,9)$ (on 112 vertices) into two copies of the Gewirtz graph: In the Steiner system $S(5,8,24)$, take all 112 blocks starting 110... or 101...; join two blocks of the same kind when they have only two symbols in common, and join two blocks starting differently when they have four symbols in common. This produces the collinearity graph Ξ of $GQ(3,9)$, and clearly shows two copies of Γ. (The presence of subgraphs $10K_2$ in Γ is also visible here: the ten lines on a fixed point in one half of a split of Ξ into two Gewirtz graphs hit the other half in $10K_2$.) Adding the 56 blocks starting 011... we find a partial linear space on 168 points and with 280 lines of size 6, with a partition into three Gewirtz graphs, and such that the union of any two Gewirtz graphs induces a $GQ(3,9)$. (See also BROUWER & VAN LINT [3], p. 113.)

(e) The McLaughlin graph Λ (on 275 vertices) is locally $GQ(3,9)$. If we fix two nonadjacent vertices x and y, then both $\Lambda(x) \cap \Lambda(y)$ and $\Lambda(x) \setminus \Lambda(y)$ are isomorphic to the Gewirtz graph. In particular, the second subconstituent Σ of Λ (on 162 vertices) is locally Gewirtz. It is strongly regular with distance distribution diagram



$v = 162$

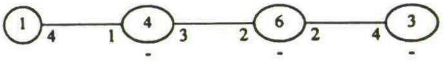
and its μ-graphs are $6C_4$'s. (CAMERON, GOETHALS & SEIDEL [4] proved uniqueness of this graph (given its parameters) using the uniqueness of the Gewirtz graph. The fact that the μ-graphs are disconnected leads to the existence of a distance-regular antipodal 3-cover of this graph, found by L. Soicher.) Thus, the incidence graph of the symmetric $GD[4,1,4;16]$ (on 32 vertices) is found as the graph on the common neighbours of three pairwise nonadjacent vertices of the McLaughlin graph.

(f) For a construction in terms of the ternary Golay code, see below.

6. Subgraphs

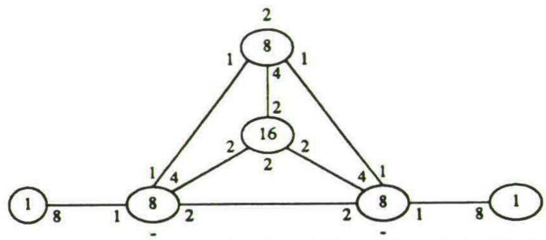
It is of interest to determine what subgraphs have equality in (D) above. Thus, we want to find the subgraphs of Γ that are regular of degree k on $u = 7k-14$ or $u = 4k+16$ vertices. If $u = 7k-14$, then k is even (since no regular graph with odd valency has an odd number of vertices), and the nontrivial possibilities are $k \in \{4,6,8\}$.

(1) $k = 4$ or $k = 8$. This is realised by the 120 subgraphs isomorphic to the Heawood graph on 14 vertices, and their complements on 42 vertices (with spectra $(\pm 4)^1 (\pm\sqrt{3})^6$ and $8^1\, 2^{22}\, (-2\pm\sqrt{3})^6\, (-4)^7$). Their distance distribution diagrams are



$v = 14$

and



$v = 42$

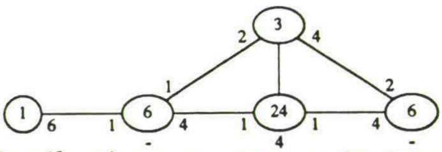
Their presence can be seen from construction (a): if Γ consists of the blocks of $S(5,8,24)$ starting with 110, then fix a block B starting with 001. There are 7, 42, 7 blocks in Γ that have 4, 2, 0 symbols in common with B. Each Heawood graph is fixed pointwise by an involution from M_{24} fixing B pointwise. That there are no other embeddings can be seen as follows: In the Heawood graph, if one starts with a 3-claw and repeatedly completes a path of length two (on 3 vertices) to a quadrangle, one finds the entire graph. Now $\mathrm{Aut}\,\Gamma$ is transitive on 3-claws, so each 3-claw in Γ is contained in a unique Heawood subgraph, and there are precisely 120 Heawood subgraphs. The presence of the Heawood graph also follows from construction (c): take as incidence matrix of the Fano plane the circulant with top row (0110100). Then the 7+7 antiflags corresponding to the zeros on the diagonal induce a Heawood graph.

(2) Subgraphs with $u = 28$, $k = 6$ can be found as follows: Above we found that a $GD\,[4,1,4;\,16]$ ($AG\,(2,4)$ minus a parallel class of lines) is obtained from $PG\,(2,4)$ by fixing a flag (x,l), and throwing away the points and lines incident with l and x (respectively). Now pick a unital U in $PG\,(2,4)$ (i.e., a subplane $AG\,(2,3)$) containing x with tangent l. In the $GD\,[4,1,4;\,16]$ this unital determines 8 points and 8 lines such that each of these points (resp. lines) is on three of these lines (resp. points), i.e., we find a $GD\,[3,1,2;\,8]$. In the $6C_4$ in the complement (in Γ) of the incidence graph of $GD\,[4,1,4;\,16]$, pick three quadrangles. It is straightforward to check that the 12 vertices of these quadrangles and the 8 points and the 8 lines of our $GD\,[3,1,2;\,8]$ induce a graph on 28 vertices, regular of degree 6.

If $u = 4k+16$, then $k \in \{0,1,2,3,4,5,6,10\}$. We shall see that all possibilities do in fact occur.

(0) $k = 10$. This is the entire graph.

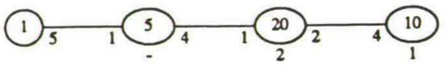
(1) $k = 0$ or $k = 6$. These are the 42 16-cocliques and their complements on 40 vertices. The distance distribution diagram of a complement is



$v = 40$

and its spectrum is $6^1\, 2^{20}\, (-2)^{15}\, (-4)^4$. It is a four-fold cover of the Petersen graph.

(2) $k = 1$ or $k = 5$. These are the 112 subgraphs $10K_2$ and their complements on 36 vertices. These complements on 36 vertices have spectrum $5^1\, 2^{16}\, (-1)^{10}\, (-3)^9$ by tool C, and hence are distance-regular with distance-distribution diagram

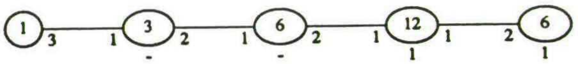


$v = 36$

that is, are Sylvester graphs. The presence of subgraphs $10K_2$ allows a construction of Γ in terms of the perfect ternary Golay code C: First of all, the Sylvester graph is obtained as the graph on the 36 words of weight 6 starting with 1 in C, where two such words are adjacent when their Hamming distance is 9. And then the Gewirtz graph is obtained by adding the 20 words of weight 2 starting with 1, joining one of them with any other vertex when they have inner product 0.

(3) $k = 2$ or $k = 4$. These are the 105 subgraphs $6C_4$ and their complements, the incidence graphs of the symmetric 4-nets as discussed above.

(4) $k = 3$. This is realised by the 240 subgraphs isomorphic to the Coxeter graph (Coxeter [6]), the unique distance-regular graph with distance-distribution diagram



$v = 28.$

(Its spectrum is $3^1\, 2^8\, (-1)^7\, (-1 \pm \sqrt{2})^6$.) Thus, we have found one more example of a 'remarkable split' into two isomorphic halves: $GQ\,(3,9)$ can be split into two Gewirtz graphs, the Higman-Sims graph can be split into two Hoffman-Singleton graphs, the Gewirtz graph can be split into two Coxeter graphs, and the Hoffman-Singleton graph can be split into two $5C_5$'s. (See also Brouwer & van Lint [3], §10.)

Completely regular codes. A subset C of the vertex set X of a graph Γ is called *completely regular* if for all $x \in X$ and all integers j the number of vertices in C at distance j from x does not depend on x but only on j and the distance $d(x,C)$ from x to C. When Γ is distance-regular, an equivalent condition is that the distance-distribution diagram around C is linear. The *covering radius* of C is the maximum value of $d(x,C)$ for $x \in X$. A completely regular subset C of covering radius 1 is an extremal subgraph as studied above. Remains to examine the case of completely regular subsets C of covering radius 2. Let $C_i := \{x \in X \mid d(x,C) = i\}$, where each vertex of C_i has c_i neighbours in C_{i-1}, a_i neighbours in C_i and b_i neighbours in C_{i+1}. Then the tridiagonal matrix

$$\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix}$$

has 3 distinct eigenvalues occurring among the eigenvalues of Γ. In our case this means that this matrix has eigenvalues 10, 2, -4, and by computing determinant and trace we find

$$a_0 a_2 - a_0 b_1 - a_2 c_1 = -8$$

$$a_0 + a_1 + a_2 = 8$$

$$a_1 + b_1 + c_1 = 10.$$

Combining these yields $(a_0+1)^2 - 9 = c_1(a_0 - a_2)$. Moreover, since also C_2 is completely regular, and $C = (C_2)_2$, we may assume that $a_0 \le a_2$. We find solutions:

a_0	b_0	c_1	a_1	b_1	c_2	a_2	partition	C
0	10	1	0	9	2	8	1+10+45	vertex
0	10	2	4	4	6	4	6+30+20	Desargues subgraph
1	9	1	1	8	4	6	2+18+36	edge
2	8	j	4	$6-j$	8	2	$4j$+32+$4(6-j)$	jC_4, $1 \le j \le 5$

In each case except for the second it is immediately clear what the graphs are, and that there are no other examples than those given. For the second type, see the next section.

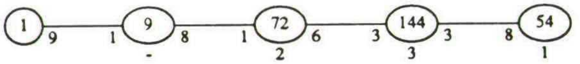
7. Group and subgroups

The automorphism group of Γ is $L_3(4).2^2$, and according to the *Atlas* [5] the maximal subgroups of this group are as given in the following table.

Index	Group	Orbit sizes	Associated subgraph in Γ
42	$2^4{:}S_5$	16+40	16-coclique
56	$A_6{\cdot}2^2$	1+10+45	vertex
105	$2^{2+4}.3.2^2$	24+32	$6C_4$
112	M_{10}	20+36	$10K_2$, Sylvester graph
120	$L_2(7){:}2{\times}2$	7+7+42	Heawood graph
240	$L_2(7){:}2$	28+28	Coxeter graph
280	$3^2Q_8.2{\times}2$	2+18+36	edge
336	$S_5{\times}2$	6+20+30	Desargues graph

We have seen all of these already, except for the last type, which can be found inside the Higman-Sims graph by fixing a path of length three $a \sim b \sim c \sim d$ (with $a \neq d$) and taking all vertices adjacent to either a or d (but not both). This yields a graph on 20 vertices, the extended bipartite double (cf. [2], p. 26) of the Petersen graph, with spectrum $4^1\, 2^5\, 1^4\, (-1)^4\, (-2)^5\, (-4)^1$.

Concerning the last type but one, we may note that the 280 edges may be identified with the 280 unitals (subplanes $AG(2,3)$) in $PG(2,4)$; indeed, the complement of two disjoint hyperovals from the same $L_3(4)$ orbit is such a unital. The graph on the 280 edges, where two edges are adjacent when they are opposite edges of a quadrangle, is distance-transitive with spectrum $9^1\, 4^{64}\, 1^{105}\, (-3)^{90}\, (-5)^{20}$ and distance distribution diagram



$v = 280.$

Its automorphism group $\mathrm{Aut}(L_3(4))$ is three times larger than that of Γ. The uniqueness of this graph (given its parameters) was shown in LAMBECK [10].

8. The p-rank of the adjacency matrix

Proposition. *The adjacency matrix A of the Gewirtz graph Γ has 2-rank 20, 5-rank 55, and p-rank 56 for $p \neq 2, 5$. The matrix $A+I$ has 3-rank 20, 11-rank 55, and p-rank 56 for $p \neq 3, 11$.*

Proof. (i) If A does not have full rank over $\boldsymbol{F}_p$, its determinant equals 0 (mod p). Thus $p = 2$ or $p = 5$, and clearly the 5-rank of A is 55 (since $\det(J-A) \neq 0$ (mod 5)). So, let us consider $p = 2$. Since Γ has a subgraph $10K_2$, the 2-rank of A is at least 20. On the other hand, the matrix $7A - 14I - J$ has rank 20 over $\boldsymbol{R}$ and hence $A - J$ has rank at most 20 over $\boldsymbol{F}_2$. A Coxeter subgraph of Γ shows that $\mathbf{1}$ is in the rowspace of $A - J$, so A has rank at most 20 over $\boldsymbol{F}_2$.
(ii) For $A+I$ we proceed similarly. Its eigenvalues are 11, 3 and -3 with multiplicities 1, 35 and 20, so the only interesting case is $p = 3$. Put $A = \begin{bmatrix} C & N \\ N' & C \end{bmatrix}$, where C represents the Coxeter graph. Then $u = (\mathbf{1}, -\mathbf{1})^T$ is a (-4)-eigenvector of A, hence $28A - 56I + 3uu^T$ has rank 20 over $\boldsymbol{R}$. So $A + I$ has rank at most 20 over $\boldsymbol{F}_3$. For a lower bound, consider a subgraph $6C_4$. The 12 rows of $A+I$ corresponding to $3C_4$ have 3-rank (at least) 10. The same holds for the 12 columns corresponding to the remaining $3C_4$. Since these rows and columns intersect in a zero matrix, we find that $A + I$ has 3-rank at least 20. ■

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