TILBURG UNIVERSITY
KATHOLIEKE UNIVERSITEIT
BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

ON M-FUNCTIONS AND THEIR APPLICATION TO INPUT-OUTPUT MODELS

B. Kaper

FEW 505

On M-Functions and their Application to Input-Output Models

B.Kaper*

September 12, 1991

Abstract

A mathematical model is a description of an Economic system if it reflects the characteristics of the system. This determines the form of the mathematical model as well as the posed conditions. Specific economic systems and special classes of mathematical equations or functions will therefore be joined together.
In this article we will study a class of M-type functions and some related topics in connection with input-output models. One of the referred characteristics will be the existence of a non-negative solution of the mathematical model. Moreover, we will show that the mathematical model is feasible in the sense that it has some properties of comparative static nature.

Keywords: input-output models, M-functions.

*Department of Econometrics, Kath.Univ.Brabant, postbus 90.153, 5000LE, Tilburg, the Netherlands.

1 Introduction

The mathematical model that describes a linear input-output model is a (square) system of linear equations. The descriptive matrix of the model is a matrix with positive diagonal elements and non-positive off-diagonal elements. It is feasible with respect to an input-output model if its inverse is at least a non-negative matrix.
The appropriate conditions that will be imposed on a matrix of the above sign-structure are mainly stated in terms of rowsums or columnsums or, equivalently, the input-output matrix should be a strictly diagonally dominant matrix. The results of a linear input-output model could be made more stringent if the non-negative matrix of input-output coefficients is irreducible. In that case, if the matrix is irreducibly diagonal dominant the solutions of the model are even positive.
The condition on the rowsums of the matrix of input-output coefficients has been given the economic interpretation of availability for a socalled final demand and the condition on the columnsums of a non-negative value added (c.f. [3]).The assumption of irreducibility of the matrix means economically that all sectors do depend in some way on all other sectors in the model (c.f. [1],[3] or [8]).
When proving that the appropriate conditions give rise to a non-negative (or even positive) inverse matrix it is obvious to account on the theory of Perron-Frobenius on non-negative matrices.

One way to study non-linear models is via the matrix of derivatives of the function describing the non-linear model. One transfers the characteristics of the matrix of the linear model to the matrix of derivatives and subsequently tries to translate things to the function itself [2],[7]. In this paper we choose for a more direct way, direct in the sense that no differentiability condition has to be imposed on the function. To that end we will derive first some results on linear models without using the Perron- Frobenius theory. This will lead us to the class of the so-called M-matrices. At the same time , with respect to the feasibility condition on an input-output matrix the irreducibility-condition as well as the strict diagonal-dominancy will be weakened to a new condition: the matrix should be weakly

irreducibly diagonally dominant. In economic terms, there should be so-called "surplus-sectors" and any non- surplus is an intermediate supplier of a surplus-sector, directly or indirectly.
These results on linear models enable us to generalize the feasibility conditions directly to a non-linear function describing an input-output model. Starting point for the study of non-linear input-output models will be a function whose diagonal-subfunctions are isotone and whose off-diagonal subfunctions are antitone. Provided again with the condition that there exists a growth-direction and that the sectors are connected in some way we will prove that a so-called M- function will be feasible with respect to an input-output model in the sense that the model has a non-negative solution for all non-negative demands and that the model answers similar features of comparative static nature as the linear model, at least as much as possible. As distinct from the linear problem, where injectivity implies surjectivity we now also get the problem of surjectivity. In a separate section we will link an M-function and its M-matrix of partial derivatives.

In section 4 we introduce the class of socalled M-functions as a generalization of the input-output matrices. In the main theorem of that section we formulate conditions to be imposed on an off-diagonally antitone function in order to become an M-function: the function should be weakly irreducible and diagonally isotone.
In section 5 we use the concept of order-coerciveness to show the surjectivity of the referred M-type function. In section 6 we enter the comparative static features of a model described by an M-function. In section ?? we formulate conditions on the matrix of partial derivatives of a Gateaux-differentiable function to become an M-function.

2 An Input-Output Model

An input-output model describes an economy which is devided into n (industrial-)sectors, each producing one commodity, and one non-producing sector. The last sector is called the open sector. The output of each industrial sector is supplied as inputs to other industrial-sectors, inclusive the sector itself or goes to meet the final demand of the open sector. The model contains only goods which cease to exist once they are used up in production. There is no production-lag.

Let x_i be the total output of sector i, and $f_{ij}(\mathbf{x})$ the amount of output of industry i absorbed by industry j.The function f_{ij} might be a function of the one variable x_j only. The net output of each sector, i. e. the excess of x_i over $\sum_{j=1}^{n} f_{ij}(\mathbf{x})$, is available for outside use and will meet the final demand. Then the overall input-output balance of the whole economy can be expressed in terms of n equations:

$$x_i = \sum_{j=1}^{n} f_{ij}(\mathbf{x}) + c_i \qquad i = 1, \ldots, n$$

where c_i represents the final demand for output i.

Let $\mathbf{f}$ represent the vectorvalued function on $\mathbf{R}_+^n$ with components f_i, $f_i(\mathbf{x}) = \sum_{j=1}^{n} f_{ij}(\mathbf{x})$. Then the set of n equations can be written as

$$\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{c}$$

or simply as

$$\mathbf{F}(\mathbf{x}) = \mathbf{c}$$

where $\mathbf{F}(\mathbf{x}) = \mathbf{x} - \mathbf{f}(\mathbf{x})$. A generally ackowledged specification on $\mathbf{F}$ is that $\mathbf{F}(\mathbf{0}) = \mathbf{0}$.

We will call $\mathbf{x}$ the (gross) production and $\mathbf{c}$ the final demand vector of the input-output model.

If the model is linear it is assumed that $f_{ij}(\mathbf{x}) = t_{ij}x_j$. Let T denote the $n \times n$ matrix of the non-negative technology coefficients t_{ij}, then the model is described by

$$(I - T)\mathbf{x} = \mathbf{c}$$

An input-output system has a variety of properties of comparative static nature. For example an increase of the level of production of any sector will correspond with a decrease of the net-output of any other sector. An increase in the final demand of a sector will correspond with an increase of the intermediate delivaries. A mathematical model will be called feasible with respect to an input-output model if some of these properties could be derived.

3 Preliminaries

Throughout this paper $\mathbf{R}^n$ is the $n-$dimensional real linear space of columnvectors $\mathbf{x}$ with components $x_1, \ldots, x_n$. The set $\{1, \ldots, n\}$ will be denoted by N. The vectors

$\mathbf{e}_i \in \mathbf{R}^n$, $i \in N$ are the unit basisvectors with i-th component one and all others zero. The vector $\mathbf{e} \in \mathbf{R}^n$ is the one-vector with all components one.
By $L(\mathbf{R}^n)$ we denote the linear space of real $n \times n$ matrices $A = \left[a_{ij} \right]$.
On $\mathbf{R}^n$ and $L(\mathbf{R}^n)$ we use the coordinatewise partial orderings; that is, if $\mathbf{x}$ and $\mathbf{y} \in \mathbf{R}^n$, then

$$\mathbf{x} \leq \mathbf{y} \text{ if } \forall i \in N \;\; x_i \leq y_i$$
$$\mathbf{x} < \mathbf{y} \text{ if } \mathbf{x} \leq \mathbf{y} \text{ and } \{i \mid x_i < y_i\} \neq \emptyset$$
$$\mathbf{x} \ll \mathbf{y} \text{ if } \forall i \in N \;\; x_i < y_i.$$

A vector $\mathbf{x} \gg \mathbf{0}$ is called a positive vector and a vector $\mathbf{x} \geq \mathbf{0}$ as well as a vector $\mathbf{x} \geq \mathbf{0}$ is a non-negative vector. If necessary, in the case of non-negative vectors distinction will be made by adding the mathematical symbol. Similar definitions and agreements hold for the inequalities $\geq$, $>$, and $\gg$ and for matrices A and $B \in L(\mathbf{R}^n)$. The non-negative orthant of $\mathbf{R}^n$ is denoted by $\mathbf{R}^n_+$.
In $\mathbf{R}^n$ we will use the l_∞-norm, $\| \mathbf{x} \|_\infty = \max\{| x_i | \mid i \in N\}$ and in $L(\mathbf{R}^n)$ the corresponding induced operatornorm, $\| A \|_\infty = \max\{\sum_{j=1}^n | a_{ij} | \mid i \in N\}$.

Definition (Principle Subfunction) *Consider* $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ *and let* $\omega = (i_1, \ldots, i_k)$, $1 \leq k \leq n$. *For a fixed* $\mathbf{y} \in \mathbf{R}^n_+$ *we define*

$$D^{[\omega]} = \{\mathbf{u} = (u_1, \ldots, u_k)^T \mid \sum_{j=1}^{k} u_j \mathbf{e}_{i_j} + \sum_{j \notin \omega} y_j \mathbf{e}_j \in \mathbf{R}^n_+\} \subset \mathbf{R}^k.$$

Then $F^{[\omega]} : D^{[\omega]} \subset \mathbf{R}^k \longrightarrow \mathbf{R}^n$ *is a principle subfunction of* F *at* $\mathbf{y} \in \mathbf{R}^n_+$ *if*

$$F_j^{[\omega]}(\mathbf{u}) = F_{i_j}(\sum_{j=1}^{k} u_j \mathbf{e}_{i_j} + \sum_{j \notin \omega} y_j \mathbf{e}_j), \;\; j = 1, \ldots, k.$$

If $\omega = \{1, \ldots, k\}$ we will write $F^{[k]}$ in stead of $F^{[1,\ldots,k]}$. If $\omega = \{i\}$ the principle subfunction $F^{[\omega]}$ is just the i-th diagonal subfunction ϕ_{ii} of F at $\mathbf{y}$ (see definition).

4 On M-Matrices and M-Functions

As has been derived in section 2 a matrix that describes a linear input-output model is of the form $A = I - T$, where T denotes the non-negative technology-matrix. In literature the technology-matrix T is often required to be an irreducible matrix, or even a positive matrix.

Definition (Irreducible Matrix) *A non-negative matrix T is irreducible if, for any two indices $i, j \in N$ there is a sequence of positive elements of T of the form $\{t_{ii_1}, t_{i_1 i_2}, \ldots, t_{i_m j}\}$.*

Irreducibility is a kind of *"connectivity condition"* : in the graph associated with the non-negative matrix there is a path from every vertex to any other vertex. If T is a technology-matrix it has the economic interpretation that each sector is connected with any other sector of the economy in the sense of intermediate delivaries.
Since the technologymatrix T is a non-negative matrix it is obvious to study input-output models with the results of the Perron-Frobenius theory on non-negative matrices. A non-negative matrix has a non-negative eigenvalue that is at least as large as the absolute value of any eigenvalue of the matrix and a non-negative eigenvector corresponding to that eigenvalue. The eigenvalue is called the *maximal eigenvalue* of the matrix. If the matrix is an irreducible (non-negative) matrix the maximal eigenvalue is positive and the corresponding eigenvector is a positive eigenvector, c. f. [4].
A matrix of the form $A = I - T$, with $T > 0$ is feasible with respect to an input-output if its inverse A^{-1} is a non-negative matrix.

Lemma 4.1 *Let $T \in L(\mathbf{R}^n)$ be a (/ irreducible) non-negative matrix.*
$I - T$ is non-singular and $(I - T)^{-1} > 0$ (/ $\gg 0$) if and only if $r(T) < 1$, where $r(T)$ denotes the maximal eigenvalue.

The condition on the maximal eigenvalue, $r(T) < 1$, could be related to a condition on the row (or column-) sums of the technology-matrix T.

Lemma 4.2 *Let $T \in L(\mathbf{R}^n)$ be a non-negative matrix.*

a If

$$\sum_{j=1}^{n} t_{ij} < 1 \text{ , for each } i \in N \qquad \text{(, all row sums are less than one,)}$$

then $r(T) < 1$
(the input-output matrix $I - T$ satisfies the conditions of a *strictly diagonally dominant matrix).*

b If T is irreducible and

$$\sum_{j=1}^{n} t_{ij} \leq 1 \text{ , for each } i \in N \text{ and for at least one } i \in N \text{ strict inequality holds,}$$

then $r(T) < 1$

(the input-output matrix $I - T$ is an *irreducibly diagonally dominant matrix).*

The conditions on the row sums have been given an economic interpretation by several authors (c. f. [3] or [8]).
The proof of a lemma of this kind is mainly based on the existence of a non-negative (c. q. positive) maximal eigenvector of the technology-matrix. However, the eigenvalue theory of Perron-Frobenius is not very appropriate for an extension to nonlinear input-output models. These deserve a more direct treatment. Moreover, the direct treatment gives us the opportunity to replace the diagonal dominancy of the technology matrix by the weaker condition of what will be called weakly irreducible diagonal dominancy.
We will first give the definition of a class of matrices that contains matrices describing a linear input-output model.

Definition (M-Matrix) *A matrix* $A \in L(\mathbf{R}^n)$ *is an M-matrix if* $a_{ij} \leq 0,\ i, j \in N,\ i \neq j$ *and* A^{-1} *exists and is non-negative.*

Unlike the usual definition (c. f. [1] or [4]) the existence of the inverse is included in the definition of an M-matrix, anticipating the theory of non-linear input-output functions. The 'almost' equivalence of the class of M-matrices with input-output matrices is given by the following lemma.

Lemma 4.3 *For any M-matrix A there exists a non-negative matrix B with maximal eigenvalue* $r(B)$ *such that*

$$A = sI - B, \quad \text{where } r(B) < s.$$

Proof:

Define $s = \max_i\{a_{ii}\}$. Then $B = sI - A$ is a non-negative matrix. Let $r(B)$ be its maximal eigenvalue and let $\mathbf{x}$ be a non-negative eigenvector. We have

$$A\mathbf{x} = (sI - B)\mathbf{x} = (s - r(B))\mathbf{x}$$

and therefore

$$\mathbf{x} = (s - r(B))A^{-1}\mathbf{x}.$$

But $\mathbf{x}$ and $A^{-1}\mathbf{x}$ are non-negative and $s - r(B) \neq 0$, since A is non-singular. Hence $s - r(B) > 0$, and $A = sI - B$.

It is clear that, if A is an M-matrix, for any $\mathbf{x}$ and $\mathbf{y} \in \mathbf{R}^n_+$, $A\mathbf{x} < A\mathbf{y}$ implies $\mathbf{x} < \mathbf{y}$ (having the following economic interpretation in the case A is an input-output matrix: an increase in the final demand of any sector leads to an increase of the level of production of at least one sector).

Moreover, the diagonal elements of an M-matrix as well as the diagonal elements of its inverse A^{-1} are positive (c. f. lemma A.1) (having the following economic interpretation: an increase of the final demand of a sector will correspond with an increase of the level of production of that sector and v. v.).

In the first theorem of this section we will formulate two conditions for a matrix to be an M-matrix when the off diagonal elements of the matrix are non positive. The proof of the theorem is straightforward and not based on the theory of Perron-Frobenius.

Note that a matrix A is non-singular and its inverse is non-negative, if for any $\mathbf{x} \in \mathbf{R}^n$: $A\mathbf{x} \geq \mathbf{0}$ implies $\mathbf{x} \geq \mathbf{0}$.

(a linear function that is injective is bijective , c. f. lemma A.2)

Theorem 4.4 *Let* $A = \left[\, a_{ij} \,\right] \in L(\mathbf{R}^n)$ *with* $a_{ij} \leq 0, i, j \in N, i \neq j$.

Assume that

- *there exists a vector* $\mathbf{u} \gg \mathbf{0}$ *with* $A\mathbf{u} = \mathbf{v} > \mathbf{0}$

- *for any* $i \notin J_+(\mathbf{v}) = \{j \in N \mid v_j > 0\}$ *there is a sequence of negative (non-diagonal) elements* $\{a_{ii_1}, \ldots, a_{i_m l}\}$ *with* $l = l(i) \in J_+(\mathbf{v})$.

Then A *is an M-Matrix.*

Proof:

Let $A\mathbf{x} \geq \mathbf{0}$ for some $\mathbf{x} \in \mathbf{R}^n$ and suppose there exists an index $i \in N$ for which $x_i < 0$. Define the diagonal matrix $U = \text{diag}(u_1, \ldots, u_n)$ and the index set $J_{min}(U^{-1}\mathbf{x}) = \{j \in N \mid u_j^{-1}x_j = \min_{k \in N} u_k^{-1}x_k\}$.
If $i \in J_{min}(U^{-1}\mathbf{x})$ then $i \in J_+(\mathbf{v})$ or there exists an index $j \in N$ with $a_{ij} < 0$. In the last case we will prove that $j \in J_{min}(U^{-1}\mathbf{x})$.
Suppose $j \notin J_{min}(U^{-1}\mathbf{x})$, then $u_i^{-1}x_i < u_j^{-1}x_j$ and hence $a_{ij}u_i^{-1}x_i > a_{ij}u_j^{-1}x_j$. Using the inequalities $u_i^{-1}x_i \leq u_k^{-1}x_k$ for any $k \in N, k \neq i$ and $k \neq j$ and thus $a_{ik}u_i^{-1}x_i \geq a_{ik}u_k^{-1}x_k$,we find that

$$\sum_{k=1}^{n} a_{ik}x_k = \sum_{k=1}^{n} a_{ik}u_k(u_k^{-1}x_k)$$
$$< \sum_{k \neq j} a_{ik}u_k(u_k^{-1}x_k) + a_{ij}u_j(u_i^{-1}x_i) \leq \sum_{k \neq j} a_{ik}u_k(u_i^{-1}x_i) + a_{ij}u_j(u_i^{-1}x_i)$$
$$= (\sum_{k=1}^{n} a_{ik}u_k)(u_i^{-1}x_i) = v_i(u_i^{-1}x_i) = 0$$

which is a contradicion.
By the second assumption of the theorem there exists at least one index $i \in J_{min}(U^{-1}\mathbf{x})$ such that $i \in J_+(\mathbf{v})$. But then $x_i < 0$, $v_i > 0$ and, for any $k \in N$, $u_i^{-1}x_i \leq u_k^{-1}x_k$ lead to the contradiction

$$\sum_{k=1}^{n} a_{ik}x_k = \sum_{k=1}^{n} a_{ik}u_k(u_k^{-1}x_k)$$
$$< \sum_{k=1}^{n} a_{ik}u_k(u_k^{-1}x_k) - \sum_{k=1}^{n} a_{ik}u_k(u_i^{-1}x_i) = \sum_{k=1}^{n} a_{ik}u_k(u_k^{-1}x_k - u_i^{-1}x_i) \leq 0.$$

Thus we must have $x_i \geq 0$ and hence $\mathbf{x} \geq \mathbf{0}$.

If in the first assumption of theorem 4.4 $\mathbf{u} = \mathbf{e}$, the matrix A is a *diagonally dominant matrix* (c. f. [6]). Since $\mathbf{u}$ is a strictly positive vector a matrix that satisfies the first assumption of theorem 4.4 will still be considered a diagonally dominant matrix.
The second assumption is a weaker form of irreducibility: in the graph that could be associated with the matrix A there should exist a path from any vertex not in $J_+(\mathbf{v})$ to a vertex in $J_+(\mathbf{v})$. We will call a matrix that satisfies an assumption of this kind a *weakly irreducible matrix (with respect to a set)* .

Definition (Weakly Irreducible) *Let J be a subset of the indexset N.*
A matrix $A \in L(\mathbf{R}^n)$ is weakly irreducible if there exists a non-empty set J such that for any

$i \notin J$ there is a sequence of non-zero (non-diagonal) elements of A, $\{a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_m l}\}$ with $l = l(i) \in J$.

A matrix that satisfies both the condition of diagonal dominancy and the condition of weak irreducibility is a *weakly irreducibly diagonally dominant matrix*.

If a matrix T satisfies the conditions of lemma 4.2, the input-output matrix $A = I - T$ satisfies both assumptions of theorem 4.4 with $\mathbf{u} = \mathbf{e}$:

if $A = I - T$ is strictly diagonally dominant, $A\mathbf{e} \gg \mathbf{0}$ and hence $J_+(A\mathbf{e}) = N$,

if $A = I - T$ is irreducibly diagonally dominant, $A\mathbf{e} > \mathbf{0}$ and any sector is connected with any other sector and the required sequence of negative non diagonal elements will surely exist.

The inverse of a weakly irreducible matrix need not be positive, however, as can be seen by the following numerical example.

Example *Consider the matrix T,*

$$T = \begin{bmatrix} 0 & .5 & .5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $A = I - T$, where I denotes the 3×3 unit-matrix, then A satisfies the assumptions of theorem 4.4 with $\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and $J_+(A\mathbf{u}) = \{2, 3\}$. Its inverse is the non-negative matrix

$$A^{-1} = \begin{bmatrix} 1 & .5 & .5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

If A represents an input-output matrix, weakly irreducibly diagonally dominancy (, where $J = J_+(A\mathbf{u})$) has the following economic interpretation :

since the vector $\mathbf{u}$ is determined apart from a constant it reflects the ratio's of the level of production of the different sectors. If any sector $i \in J_+(A\mathbf{u})$ is called a surplus sector then the first assumption asserts that at least one sector is a surplus sector. The second assumption asserts that any non-surplus sector should be connected with a surplus sector in the sense of intermediate delivaries. In other words, a matrix is feasible with respect to an input-output model if there are surplus sectors and any non-surplus sector is a (probably indirect) intermediate supplier of a surplus sector.

The conclusion of theorem 4.4 also holds if in the diagonally dominancy assumption of the theorem the matrix A is replaced by the transponed matrix A^T. The assumptions could then be given an economic interpretation in terms of the financial profitability of the sectors.

In order to generalize further to non-linear functions we replace the different signs of the elements of an M-matrix A and its inverse A^{-1} by concepts of monotonicity . The conditions of the different concepts can easily be checked in case of the linear function $F(\mathbf{x}) = A\mathbf{x}$, where A is an M-matrix.

Definition (monotonicity) *Consider a function $F : D \subset \mathbf{R}^n \longrightarrow \mathbf{R}^n$.*

a *F is isotone (/ antitone) on D if $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x}, \mathbf{y} \in D$ implies that $F(\mathbf{x}) \leq F(\mathbf{y})$ (/ $F(\mathbf{x}) \geq F(\mathbf{y})$) and strictly isotone (/ strictly antitone) if, in addition, it follows from $\mathbf{x} \ll \mathbf{y}, \mathbf{x}, \mathbf{y} \in D$ that also $F(\mathbf{x}) \ll F(\mathbf{y})$ (/ $F(\mathbf{x}) \gg F(\mathbf{y})$)*

b *F is off-diagonally antitone on D if for any $\mathbf{x} \in D$ and for any $i, j \in N, i \neq j$, $\phi_{ij} : \{t \in \mathbf{R} \mid \mathbf{x} + t\mathbf{e}_j \in D\} \longrightarrow \mathbf{R}$, $\phi_{ij}(t) = F_i(\mathbf{x} + t\mathbf{e}_j)$ is antitone (ϕ_{ij} is sometimes called an off-diagonal subfunction of F)*

c *F is diagonally isotone (/ strictly diagonally isotone) if for any $\mathbf{x} \in D$ and for any $i \in N$ $\phi_{ii} : \{t \in \mathbf{R} \mid \mathbf{x} + t\mathbf{e}_i \in D\} \longrightarrow \mathbf{R}$, $\phi_{ii}(t) = F_i(\mathbf{x} + t\mathbf{e}_i)$ are isotone (/ strictly isotone).(ϕ_{ii} is called the i-th diagonal subfunction of F.*

d *F is inverse isotone on D if $F(\mathbf{x}) \leq F(\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in D$,implies that $\mathbf{x} \leq \mathbf{y}$.*

The class of functions that contains the set of off-diagonally antitone functions feasible with respect an input-output model is the class of M-functions.

Definition (M-function) *A function $F : D \subset \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is an M-function on D if F is off-diagonally antitone and inverse isotone on D*

The next lemma shows that the definition of an M-function represents a generalization of the M-matrix.

Lemma 4.5 *A matrix $A \in L(\mathbf{R}^n)$ is an M-matrix if and only if the induced mapping $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n, F(\mathbf{x}) = A\mathbf{x}$ is an M-function.*

With respect to the diagonal subfunctions of F the following result hold.

Lemma 4.6 *Let $F : D \subset \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be an M-function.*
Then F and $F^{-1} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ are strictly diagonally isotone.
Proof:
c. f. Appendix, lemma A.3

Lemma 4.6 has the following economic interpretation: an increase in the level of production of one industrial sector causes an increase in the netoutput of that sector and, conversely, an increase in the netoutput of one industrial sector causes an increase in the level of production of that sector. Because of properties of this kind an M- function will be considered feasible with respect to an input-output model.
The next theorem is a generalization of theorem 4.4.

Theorem 4.7 *Let $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ be an off-diagonally antitone function.*
Assume that

- *there exists a positive vector $\mathbf{u} \in \mathbf{R}^n, \mathbf{u} \gg \mathbf{0}$ such that, for any $\mathbf{x} \in \mathbf{R}^n_+$ the function $P : \mathbf{R}_+ \longrightarrow \mathbf{R}^n, P_i(t) = F_i(\mathbf{x} + t\mathbf{u}), \quad i \in N$, is isotone,*
- *$J_+ = \{j \in N \mid$ for any $\mathbf{x} \in \mathbf{R}^n_+, P_j$ is strictly isotone $\}$ is not empty*
- *for any $i \notin J_+$ there exists a chain of strictly antitone (off-diagonally sub-)functons $\{\phi_{ii_1}, \ldots, \phi_{i_m l}\}$ where $l = l(i) \in J_+$.*

Then F is inverse isotone and hence an M-function.
Proof:

Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^n$ with $F\mathbf{x} \leq F\mathbf{y}$.

Define the diagonal matrix $H = \text{diag}(u_1, \ldots, u_n)$ and $\sigma = max\{u_i^{-1}(x_i - y_i) \mid i \in N\}$. Then $U^{-1}(\mathbf{x} - \mathbf{y}) \leq \sigma\mathbf{e}$ or $\mathbf{x} \leq \mathbf{y} + \sigma\mathbf{u}$. Notice that $\sigma \leq 0$ if and only if $\mathbf{x} \leq \mathbf{y}$.

Suppose $\sigma > 0$. Define $J_{max}(\mathbf{x}, \mathbf{y}) = \{i \in N \mid x_i = y_i + \sigma u_i\}$ and, hence, $x_i < y_i + \sigma u_i$ if $i \notin J_{max}(\mathbf{x}, \mathbf{y})$.

For $i \in J_{max}(\mathbf{x}, \mathbf{y})$ the componentfunction P_i is strictly isotone or there exists an (off-diagonal sub-) function ϕ_{ij} that is strictly antitone at $\mathbf{x}$. In the last case we will prove that $j \in J_{max}(\mathbf{x}, \mathbf{y})$.

Suppose $j \notin J_{max}(\mathbf{x}, \mathbf{y})$ in which case $x_j < y_j + \sigma u_j$, whereas $x_i = y_i + \sigma u_i$. Because of the isotony of P_i and the strict antitonicity of ϕ_{ij} we have

$$\begin{aligned} F_i(\mathbf{x}) &\leq F_i(\mathbf{y}) \leq F_i(\mathbf{y} + \sigma\mathbf{u}) \\ &< F_i(y_1 + \sigma u_1, \ldots, y_{j-1} + \sigma u_{j-1}, x_j, y_{j+1} + \sigma u_{j+1}, \ldots, y_n + \sigma u_n) \\ &\leq F_i(x_1, \ldots, x_{i-1}, y_i + \sigma u_i, \ldots, x_n) = F_i(\mathbf{x}). \end{aligned}$$

Hence $j \in J_{max}(\mathbf{x}, \mathbf{y})$.

Because of the connectivity condition of the theorem there exists an index $i \in J_{max}(\mathbf{x}, \mathbf{y})$ for which the component function P_i is strictly isotone. Together with the off-diagonally antitonicity of F and the assumption $\sigma > 0$ we have

$$\begin{aligned} F_i(\mathbf{x}) &\leq F_i(\mathbf{y}) < F_i(\mathbf{y} + \sigma\mathbf{u}) \\ &= F_i(y_1 + \sigma u_1, \ldots, y_{i-1} + \sigma u_{i-1}, x_i, y_{i+1} + \sigma u_{i+1}, \ldots, y_n + \sigma u_n) \leq F_i(\mathbf{x}). \end{aligned}$$

Hence, $\sigma \leq 0$ and $\mathbf{x} \leq \mathbf{y}$.

After the matrix-terminology a function that satisfies the three conditions of theorem 4.7 will be called a *weakly irreducibly diagonally isotone function.*

5 On Surjective M-Functions

In this section we examine the surjectivity of an M-function on $\mathbf{R}_+^n$. Preceding the actual theorem on a surjective M-function we start with a lemma that replaces surjectivity by the equivalent concept of order-coerciveness. The function in question is a continuous M-function that satisfies the (natural input-output) condition $F\mathbf{0} = \mathbf{0}$.

Definition (Order-coerciveness)

a For any sequence $\{\mathbf{x}^k\} \subset \mathbf{R}^n$ *we write*

$$\lim_{k\to\infty} \mathbf{x}^k = \infty \quad \textit{if} \quad \lim_{k\to\infty} x_i^k = \infty$$

for at least one index i.

b The function $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ *is order-coercive if for any sequence* $\mathbf{x}^k \subset \mathbf{R}^n$

$$\mathbf{x}^k \leq \mathbf{x}^{k+1}, \quad k = 0, 1, \ldots, \quad \lim_{k\to\infty} \mathbf{x}^k = \infty$$

implies that $\lim_{k\to\infty} F(\mathbf{x}^k) = \infty$

Lemma 5.1 *Let* $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ *be a continuous M-function,* $F(\mathbf{0}) = \mathbf{0}$. *Then* F *is surjective if and only if* F *is order-coercive.*

Proof:

We will start with the 'only if' part.

An M-function is injective and hence in this case bijective. Any sequence $\{\mathbf{x}^k\}$ in $\mathbf{R}^n_+$ for which $F\mathbf{x}^k \leq \mathbf{a}$ is bounded since $\mathbf{x}^k \leq F^{-1}\mathbf{a}$ for $k = 0, 1, \ldots$. Hence for a sequence $\{\mathbf{x}^k\}$ for which $\mathbf{x}^k \leq \mathbf{x}^{k+1}$ and $\lim_{k\to\infty} \mathbf{x}^k = \infty$ must hold $\lim_{k\to\infty} F\mathbf{x}^k = \infty$. A surjective function is therefore order-coercive.

In order to prove that order-coerciveness implies surjectivity we will show first that for any $\mathbf{z} \in \mathbf{R}^n_+$ there exists an $\mathbf{y}^0 \in \mathbf{R}^n_+$ such that $\mathbf{z} \leq F\mathbf{y}^0$ (then $F\mathbf{0} \leq \mathbf{z} \leq F\mathbf{y}^0$).

From lemma 4.6 we allready know that F is strictly diagonally isotone. We will show that for any vector $\mathbf{x} \geq \mathbf{0}$ and for any $i \in N$ $\lim_{t\to\infty} F_i(\mathbf{x} + t\mathbf{e}_i) = \infty$ so that F is surjectively diagonally isotone. Because of the continuity of F it is equivalent to show that for any vector $\mathbf{x} \in \mathbf{R}^n_+$, for any sequence $\{t^k\}$, $\lim_{k\to\infty} t^k = \infty$ holds $\lim_{k\to\infty} F_i(\mathbf{x} + t^k\mathbf{e}_i) = \infty$. Suppose there exists an $\mathbf{x} \in \mathbf{R}^n_+$, an index $i \in N$ and a sequence $\{t^k\} \subset [0, \infty)$ with $\lim_{k\to\infty} t^k = \infty$ for which $F_i(\mathbf{x} + t^k\mathbf{e}_i) \leq a_i < \infty$. Suppose $t^k \leq t^{k+1}$. The off-diagonal antitonicity of F implies

$$F_j(\mathbf{x} + t^k\mathbf{e}_i) \leq F_j(\mathbf{x}) = a_j < \infty, \quad j \neq i, \; j \in N, \; k = 0, 1, 2, \ldots$$

and hence

$$F(\mathbf{x} + t^k \mathbf{e}_i) \le \mathbf{a}.$$

The order-coerciveness of F leads to the contradiction that the sequence $\{t^k\}$ is bounded. Therefore F is surjectively diagonally isotone.

Let $\mathbf{u}^0 \in \mathbf{R}^n_+$ be an arbitrary point. Define $\mathbf{z}' \in \mathbf{R}^n_+$ by $z'_i = \max\{F_i(\mathbf{u}^0), z_i\}$, $i \in N$. Because F is surjectively diagonally isotone we can solve successively the following set of equations

$$F_i(u_1^k, \ldots, u_{i-1}^k, u_i, u_{i+1}^k, \ldots, u_n^k) = z'_i, \qquad i \in N,\ k = 0, 1, \ldots .$$

The solution u_i^{k+1} is unique and satisfies the following inequalities

$$\mathbf{u}^k \le \mathbf{u}^{k+1} \text{ and } F(\mathbf{u}^k) \le \mathbf{z}', \qquad k = 0, 1, \ldots .$$

Clearly $F(\mathbf{u}^0) \le \mathbf{z}'$. Assume $F(\mathbf{u}^k) \le \mathbf{z}'$ for some $k \ge 0$, then

$$F_i(u_1^k, \ldots, u_{i-1}^k, u_i^{k+1}, u_{i+1}^k, \ldots, u_n^k) = z'_i \ge F_i(\mathbf{u}^k),$$

and, because F is strictly diagonally isotone we have $u_i^{k+1} \ge u_i^k$, $i \in N$. Because F is off-diagonally antitone we have

$$z'_i = F_i(u_1^k, \ldots, u_{i-1}^k, u_i^{k+1}, u_{i+1}^k, \ldots, u_n^k),\ i \in N.$$

From the ordercoercivity of F it follows that the increasing sequence $\{\mathbf{u}^k\}$ is bounded above and hence convergent. Thus $\lim_{k\to\infty} \mathbf{u}^k = \mathbf{y}^0$ and by the continuity of F we have $F(\mathbf{y}^0) = \mathbf{z}' \ge \mathbf{z}$.

Now, since there exists a vector $\mathbf{0} \le \mathbf{y}^0$ for which $F\mathbf{0} \le \mathbf{z} \le F\mathbf{y}^0$ we can prove the existence of a vector $\mathbf{x} \in \mathbf{R}^n_+$ such that $F\mathbf{x} = \mathbf{z}$.

Consider the Jacobi processes ($\omega = 1$)

$$F_i(x_1^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_n^k) = z_i,\ i \in N,\ k = 0, 1, \ldots,$$

with $\mathbf{x}^0 = \mathbf{0}$ and

$$F_i(y_1^k, \ldots, y_{i-1}^k, y_i, y_{i+1}^k, \ldots, y_n^k) = z_i,\ i \in N,\ k = 0, 1, \ldots .$$

Each of these equations has a unique solution, x_i^{k+1} respectively y_i^{k+1}. In a simsilar way as above we prove the following set of inequalities

$$\mathbf{x}^0 \le \mathbf{x}^k \le \mathbf{x}^{k+1} \le \mathbf{y}^{k+1} \le \mathbf{y}^k \le \mathbf{y}^0$$

and

$$F\mathbf{x}^k \le \mathbf{z} \le F\mathbf{y}^k.$$

Assume for some $k \ge 0$

$$F\mathbf{x}^k \le \mathbf{z} \le F\mathbf{y}^k \text{ and } \mathbf{x}^k \le \mathbf{y}^k$$

then $F_i(x_1^k, \dots x_{i-1}^k, x_i^{k+1}, x_{i+1}^k, \dots, x_n^k) = z_i \geq F_i(\mathbf{x}^k)$
and $F_i(y_1^k, \dots, y_{i-1}^k, y_i^{k+1}, y_{i+1}^k, \dots y_n^k) = z_i \leq F_i(\mathbf{y}^k)$, $i \in N$. By the strictly diagonal isotonicity we have

$$x_i^{k+1} \geq x_i^k \text{ and } y_i^{k+1} \leq y_i^k,\ i \in N.$$

By the off-diagonal antitonicity we have

$$F_i(x_1^k, \dots, x_{i-1}^k, t, x_{i+1}^k, \dots, x_n^k) \geq F_i(y_1^k, \dots, y_{i-1}^k, t, y_{i+1}^k, \dots, y_n^k) \text{ for all } t \in [x_i^0, y_i^0],$$

and hence $x_i^{k+1} \leq y_i^{k+1}$, $i \in N$.

Moreover,

$$z_i = F_i(x_1^k, \dots, x_{i-1}^k, x_i^{k+1}, x_{i+1}^k, \dots, x_n^k) \geq F_i(\mathbf{x}^{k+1}),\ i \in N.$$

The monotonic sequences $\{\mathbf{x}^k\}$ and $\{\mathbf{y}^k\}$ are bounded and hence convergent. Thus $\lim_{k\to\infty} \mathbf{x}^k = \mathbf{x}^*$ and $\lim_{k\to\infty} \mathbf{y}^k = \mathbf{y}^*$, $\mathbf{x}^* \leq \mathbf{y}^*$. By the continuity of F we have $F(\mathbf{x}^*) = F(\mathbf{y}^*) = \mathbf{z}$.

We now state the main theorem of this section.

Theorem 5.2 *Let $F : \mathbf{R}_+^n \longrightarrow \mathbf{R}^n$ be a continuous , off-diagonally antitone function . We assume that*

- *there exists a positive vector $\mathbf{u} \in \mathbf{R}^n, \mathbf{u} \gg \mathbf{0}$ such that, for any $\mathbf{x} \in \mathbf{R}_+^n$ the function $P : \mathbf{R}_+ \longrightarrow \mathbf{R}^n, P_i(t) = F_i(\mathbf{x} + t\mathbf{u}),\ \ i \in N$, is isotone,*
- *$J_+ = \{j \in N \mid$ for any $\mathbf{x} \in \mathbf{R}_+^n, P_j$ is strictly isotone $\}$ is not empty ,*
- *for any $i \notin J_+$ there exists a chain of (off-diagonally sub-) functions $\{\phi_{ii_1}, \dots, \phi_{i_m l}\}$ which , for any $\mathbf{x} \in \mathbf{R}_+^n$,are surjective , strictly isotone and such that P_l is strictly isotone and surjective.*

Then F is a surjective M-function.

Proof:

According to theorem 4.7 an off-diagonally antitone function that satisfies the conditions of theorem 5.2 is an M-function on $\mathbf{R}_+^n$. According to lemma 5.1 it suffices to show that F is an order-coercive function on $\mathbf{R}_+^n$.

Consider the sequence $\{\mathbf{x}^k\} \subset \mathbf{R}_+^n$, $\mathbf{x}^k \leq \mathbf{x}^{k+1}$ and $\lim_{k\to\infty} \mathbf{x}^k = \infty$. Suppose there exists a vector $\mathbf{a} \in \mathbf{R}_+^n$ such that $F(\mathbf{x}^k) \leq \mathbf{a},\ k = 1,2,\ldots$.. There exists a subsequence of $\{\mathbf{x}^k\}$, that will be indicated again by $\{\mathbf{x}^k\}$, such that $x_{i_0}^k = \max_i x_i^k,\ k = 1,2,\ldots$ for some fixed index i_0. The index set $J_\infty = \{i \in N \mid \lim_{k\to\infty} x_i^k = \infty\}$ is not empty.

Define the diagonal matrix $U = \text{diag}(u_1,\ldots,u_n)$ and consider the subset $J'_\infty \subset J_\infty$,

$$J'_\infty = \{i \in N \mid \exists \beta_i \in \mathbf{R} such that \forall k \in \mathbb{N}\ \ U^{-1}\mathbf{x}^k \leq u_i^{-1}(x_i^k + \beta_i)\mathbf{e}.$$

For $i = i_0 \in J'_\infty$ we have $\beta_{i_0} = 0$, else $\beta_i \geq 0$.

With respect to the index set J'_∞ we will prove that there exists an index $i \in J'_\infty$ for which P_i is strictly isotone and surjective.

Take an $i \in J'_\infty$. If P_i is strictly isotone and surjective we are ready. Else P_i is isotone and there exists an off-diagonal subfunction ϕ_{ij} that is ,for any $\mathbf{x} \in \mathbf{R}_+^n$, surjective and strictly antitone.

We will show that there exists a constant β_j such that $u_i^{-1}(x_i^k + \beta_i) \leq u_j^{-1}(x_j^k + \beta_j)$ and hence $j \in J'_\infty$.

Suppose for any $n \in \mathbb{N}$ there exists a number k_n with $u_i^{-1}(x_i^{k_n} + \beta_i) > u_j^{-1}(x_j^{k_n} + n)$. A subsequence will be created, that will be indicated by $\{\mathbf{x}^k\}$ with the property that

$$\mathbf{x}^k \leq \mathbf{x}^{k+1}, \mathbf{x}^k \leq u_i^{-1}(x_i^k + \beta_i)\mathbf{u} \quad \text{and} \quad x_j^k < u_j u_i^{-1}(x_i^k + \beta_i) - k.$$

Consider the following sequence $\{\mathbf{y}^k\} \subset \mathbf{R}_+^n$,

$$\mathbf{y}^k = u_i^{-1}(x_i^k + \beta_i)\mathbf{u} - \beta_i\mathbf{e}_i - k\mathbf{e}_j.$$

Then, because ϕ_{ij} is strictly antitone,

$$\begin{aligned} F_i(\mathbf{y}^k) &\leq F_i(u_i^{-1}(x_i^{k+1} + \beta_i)\mathbf{u} - \beta_i\mathbf{e}_i - k\mathbf{e}_j) \\ &< F_i(u_i^{-1}(x_i^{k+1} + \beta_i)\mathbf{u} - \beta_i\mathbf{e}_i - (k+1)\mathbf{e}_j) = F_i(\mathbf{y}^{k+1}) \end{aligned}$$

and , because ϕ_{ij} is surjective

$$\lim_{k\to\infty} F_i(\mathbf{y}^{k+1}) = \infty.$$

Furthermore, $F_i(\mathbf{x}^k) \geq F_i(\mathbf{y}^k)$ and hence $\lim_{k\to\infty} F_i(\mathbf{x}^k) = \infty$

This contradicts the assumption $F_i(\mathbf{x}^k) \leq a_i$ thus $j \in J'_\infty$.

By vitue of the connectivity assumption of the theorem we know that there exists an index $i \in J'_\infty$for wich P_i is strictly isotone and surjective. Moreover,

$$a_i \geq F_i(\mathbf{x}^k) \geq F_i(u_i^{-1}(x_i^k + \beta_i)\mathbf{u} - \beta_i\mathbf{e}_i).$$

Because P_i is (strictly) isotone and surjective

$$\sup\{t \mid t \geq 0, F_i((t + u_i^{-1}\beta_i)\mathbf{u} - \beta\mathbf{e}_i) \leq a_i\} = \gamma < \infty.$$

Hence $x_i^k \leq u_i\gamma$, $k = 1, 2, \ldots$ which contradicts $i \in J'_\infty \subset J_\infty$. The assumption that there exists a vector $\mathbf{a}$ such that $F\mathbf{x}^k \leq \mathbf{a}$ is false and the conclusion is now that F is order-coercive and hence a surjective M-function.

6 Comparative Statics

For any off-diagonally antitone function that is weakly irreducibly diagonally dominant we can prove the following theorem.

Theorem 6.1 *Let $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ be an off-diagonally antitone function. Assume that*

- *for any $\mathbf{x} \in \mathbf{R}^n_+$, the function $P : \mathbf{R}_+ \longrightarrow \mathbf{R}^n$, $P_i(t) = F_i(\mathbf{x} + t\mathbf{e})$, $i \in N$, is isotone*
- *$J_+ = \{j \in N \mid$, for any $\mathbf{x} \in \mathbf{R}^n_+, P_j$ is strictly isotone $\}$ is not empty*
- *for any $j \notin J_+$ there exists a chain of (off-diagonally sub-) functions $\{\phi_{ii_1}, \ldots, \phi_{i_m l}\}$ which, for any $\mathbf{x} \in \mathbf{R}^n_+$, are strictly antitone and such that $l \in J_+$.*

For each $\mathbf{x} \in \mathbf{R}^n_+$ and $F(\mathbf{x}) \leq F(\mathbf{y})$ for some $\mathbf{y} \in \mathbf{R}^n_+$ (hence $\mathbf{x} \leq \mathbf{y}$) there holds

a *for any $i \in J_+$ $F_i\mathbf{x} = F_i\mathbf{y}$ implies $y_i - x_i <\| \mathbf{y} - \mathbf{x} \|_\infty$*

b *for any $i \in N$ $F_i\mathbf{x} = F_i\mathbf{y}$ implies $y_i - x_i <\| \mathbf{y} - \mathbf{x} \|_\infty$
or $y_i - x_i = \| \mathbf{y} - \mathbf{x} \|_\infty = y_j - x_j$ if $j \in N$ and ϕ_{ij} is strictly antitone.*

c *there exists $i \in N$ for which $y_i - x_i = \| \mathbf{y} - \mathbf{x} \|_\infty$ and $F_i\mathbf{x} < F_i\mathbf{y}$ (and hence $(y_i - x_i)(F_i\mathbf{y} - F_i\mathbf{x}) > 0$.*

Proof:

a Suppose $y_i - x_i = \| \mathbf{y} - \mathbf{x} \|_\infty$. Because $i \in J_+$ we have

$$F_i(\mathbf{x}) < F_i(\mathbf{x} + \| \mathbf{y} - \mathbf{x} \|_\infty \mathbf{e}) \leq F_i(\mathbf{y})$$

which is a contradiction.

b If $y_i - x_i <\| \mathbf{y}-\mathbf{x} \|_\infty$ there is nothing to prove. Assume $y_i - x_i =\| \mathbf{y}-\mathbf{x} \|_\infty$ and suppose $y_i - x_i > y_j - x_j$ for $j \in N$ where ϕ_{ij} is strictly antitone. Because $y_i - x_i \geq y_k - x_k$ for $k \neq j$ and ϕ_{ij} is strictly antitone we have

$$F_i(\mathbf{x}) \leq F_i(\mathbf{x}+ \| \mathbf{y}-\mathbf{x} \|_\infty \mathbf{e}) < F_i(\mathbf{y})$$

which is a contradiction.

c Assume there exists an index $i \in N$ for which $y_i - x_i =\| \mathbf{y}-\mathbf{x} \|_\infty$ and $F_i\mathbf{x} = F_i\mathbf{y}$. Then $i \notin J_+$ and there exists an index $j \in N$ for which ϕ_{ij} is strictly antitone and $y_i - x_i =\| \mathbf{y}-\mathbf{x} \|_\infty = y_j - x_j$. If $F_j(\mathbf{x}) < F_j(\mathbf{y})$ we are ready, else $j \notin J_+$. According to the connectivity assumptions of the theorem eventually there should exist an index $l \in N$ for which $y_l - x_l =\| \mathbf{y}-\mathbf{x} \|_\infty$ and P_l is strictly isotone. In that case $F_l(\mathbf{x}) < F_l(\mathbf{y})$.

If F describes an input-output model and $\mathbf{y}$ is the level of production after the demand $F(\mathbf{x})$ has been increased (until $F(\mathbf{y})$) then the conclusions of theorem 6.1 have the following economic interpretation:

- There is no sector with a decreasing level of production.
- The increase in the level of production of a productive sector without a change in the demand will be less than the maximum of all increments (part a). In the case $J = N$ the sector with the greatest change (in the absolute sense) in the level of production has a nonzero increase in the demand.
- If the increase in the level of production of a non-productive sector without a change in the demand is the maximum of all increments then there should be another sector to which production that sector is actively contributing. Moreover, the increment of the other sector also equals the maximum of all increments.
- There is a sector with an increasing demand whose increment in the level of production is the maximum of all increments.

The following corollary is a direct consequence of property c) of theorem 6.1.

Corollary 6.2 *Let $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ be an off-diagonally antitone function, that is J-irreducibly diagonally isotone where J is the (non-empty) set of indices of strictly isotone diagonal subfunctions.*

If

$$F(\mathbf{y}) = F(\mathbf{x}) + \gamma \mathbf{e}_k \text{ where } \gamma > 0 \text{ then } y_k - x_k = \| \mathbf{y} - \mathbf{x} \|_\infty$$

Proof:

Index k is the unique index that suits the index i in property c) of theorem 6.1.

The comparative statics results of this section have been derived for the special case that $\mathbf{u} = \mathbf{e}$. Similar results could easily be obtained in the general case of a positive vector $\mathbf{u}$ if we apply the transformation $U^{-1}(\mathbf{y} - \mathbf{x}$, where $U = \text{diag}(u_1, \ldots, u_n)$.

8 Diagonal Dominancy

In section 4 we imposed conditions on an off-diagonally antitone function in order to be an M-function. In the main theorem of that section no differentiability condition was required. In this section we examine the inverse isotony of an off-diagonally antitone function from the matrix of derivatives.

In the first part of this section we replace the conditions of lemma 4.2 on the rowsums of the technology-matrix T by an equivalent statement that enables us to generalize the concept of diagonal dominancy to non-linear functions.

Theorem 8.1 *Consider the matrix $A = I - T \in L(\mathbf{R}^n)$, where T is a non-negative matrix. Then the following statements are equivalent:*

a $\forall k \in N \;\; \sum_{j=1}^n t_{kj} < 1$ *(hence A is a strictly diagonally dominant matrix)*

b $\forall k \in N$ *and* $\forall \mathbf{x} \in \mathbf{R}^n_+, \mathbf{x} \neq \mathbf{0}, \;\; x_k - \sum_{j=1}^n t_{kj} x_j = 0$ *implies* $x_k < \| \mathbf{x} \|_\infty$.

Proof:
If a) holds and for some $k \in N \quad x_k - \sum_{j=1}^{n} t_{kj} x_j = 0,\ \mathbf{x} \in \mathbf{R}_+^n, \mathbf{x} \neq \mathbf{0}$, then

$$\sum_{j \neq k} t_{kj} x_k < (1 - t_{kk}) x_k = \sum_{j \neq k} t_{kj} x_j \leq \| \mathbf{x} \|_\infty \sum_{j \neq k} t_{kj}.$$

Hence $x_k < \| \mathbf{x} \|_\infty$.

Conversely, if b) holds, assume that for some $k \in N \quad 1 - t_{kk} \leq \sum_{j \neq k} t_{kj}$. Then $1 - t_{kk} = \lambda \sum_{j \neq k} t_{kj}$ with $0 < \lambda \leq 1$. Define $\mathbf{x} \in \mathbf{R}_+^n$ with $x_k = 1$ and $x_j = \lambda$, $j \neq k$. Then $\| \mathbf{x} \|_\infty = x_k = 1$ and $x_k - \sum_{j=1}^{n} t_{kj} x_j = 0$. This contradicts b) since $\mathbf{x} \neq \mathbf{0}$. Hence, a) holds.

Note that the statement in part b) of theorem 8.1 is equivalent to property a) of theorem 6.1 (in the case of a strictly diagonally dominant matrix the set J of strictly isotone diagonal functions is just the whole set N).
Because of the equivalence of the two statements in the linear case the statement in part b) of theorem 8.1 will therefore be used to define a strictly diagonally dominant function.

Definition (Strictly Diagonally Dominant Function) *A function $F : \mathbf{R}_+^n \longrightarrow \mathbf{R}^n$ is strictly diagonally dominant if for each $k \in N$ the k-th component function of F, F_k, is strictly dominant with respect to the k-th variable, that is, for every $\mathbf{x}$ and $\mathbf{y} \in \mathbf{R}_+^n$, $\mathbf{x} \neq \mathbf{y}$*

$$F_k(\mathbf{x}) = F_k(\mathbf{y}) \text{ implies that } | y_k - x_k | < \| \mathbf{y} - \mathbf{x} \|_\infty .$$

Next we state three lemma's preparing a theorem on a differentiable function that is off-diagonally antitone , whose derivative is a strictly diagonally dominant matrix (see [5] for a detailed treatment on diagonally dominant functions).

Lemma 8.2 *Let $F : \mathbf{R}_+^n \longrightarrow \mathbf{R}^n$, be an off-diagonally antitone and differentiable function whose derivative $DF(\mathbf{x})$ is a strictly diagonally dominant matrix on $\mathbf{R}_+^n$. Then F is a strictly diagonally dominant function on $\mathbf{R}_+^n$.*

Proof:
Let $k \in N$ and $F_k(\mathbf{x}) = F_k(\mathbf{y})$ with $\mathbf{x} \neq \mathbf{y} \in \mathbf{R}_+^n$. The function $\psi : [0, 1] \longrightarrow \mathbf{R}$,

$$\psi(t) = F_k(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

is a differentiable function on $[0, 1]$ with the property that $\psi(0) = \psi(1)$. According to Rolle's theorem there exists a $t_0 \in (0, 1)$ such that

$$\psi'(t_0) = \sum_{j=1}^{n} F'_{kj}(\mathbf{x} + t_0(\mathbf{y} - \mathbf{x}))(y_j - x_j) = 0$$

or, equivalently,

$$F'_{kk}(\mathbf{x} + t_0(\mathbf{y} - \mathbf{x}))(y_k - x_k) = -\sum_{j\neq k} F'_{kj}(\mathbf{x} + t_0(\mathbf{y} - \mathbf{x}))(y_j - x_j).$$

Because $DF(\mathbf{x} + t_0(\mathbf{y} - \mathbf{x}))$ is a strictly diagonally dominant matrix there holds

$$\begin{aligned} F'_{kk}(\xi_0) \mid y_k - x_k \mid &\leq -\sum_{j\neq k} F'_{kj}(\xi_0) \mid y_j - x_j \mid \\ &\leq -\sum_{j\neq k} F'_{kj}(\xi_0) \parallel \mathbf{y} - \mathbf{x} \parallel_\infty \\ &< F'_{kk}(\xi_0) \parallel \mathbf{y} - \mathbf{x} \parallel_\infty, \end{aligned}$$

where $\xi_0 = \mathbf{x} + t_0(\mathbf{y} - \mathbf{x})$. From wich it follows that

$$\mid y_k - x_k \mid < \parallel \mathbf{y} - \mathbf{x} \parallel_\infty .$$

The next lemma is a generalization of the property that a principal submatrix of a strictly diagonally dominant matrix is a strictly diagonally dominant matrix.

Lemma 8.3 *A principle subfunction of a strictly diagonally dominant function that is off-diagonally antitone on $\mathbf{R}^n_+$ is a strictly diagonally dominant function.*

The proof of lemma 8.3 is an immediate consequence of lemma 8.2 and the remark that a principle submatrix of a strictly diagonally dominant matrix is a strictly diagonally dominant matrix.

Lemma 8.4 *Let $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ be a continuous function.*
If F is a strictly diagonally dominant and strictly diagonally isotone function on $\mathbf{R}^n_+$ then

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+,\ \mathbf{x} \neq \mathbf{y},\ \exists k = k\{\mathbf{x}, \mathbf{y}\}\ :\ (y_k - x_k)(F_k(\mathbf{y}) - F_k(\mathbf{x})) > 0.$$

Proof:

Let $k \in N$ and $\mid x_k - y_k \mid = \parallel \mathbf{x} - \mathbf{y} \parallel_\infty$. Because F is strictly diagonally dominant $F_k(\mathbf{x}) \neq F_k(\mathbf{y})$.

Assume that $y_k - x_k > 0$. Consider the (convex) set

$$K = \{\mathbf{z} \in \mathbf{R}^n_+ \mid \mathbf{z} \neq \mathbf{x},\ z_k - x_k = \parallel \mathbf{z} - \mathbf{x} \parallel_\infty\}$$

and the function $H_k : K \times [0,1] \longrightarrow \mathbf{R}$,

$H_k(\mathbf{z},t) = F_k(\mathbf{x}+t(\mathbf{z}-\mathbf{x}))$.

Then

- $\forall t \in [0,1]$ the function $H_k(.,t)$ is a continuous function on K ,

- $\forall \mathbf{z} \in K$ the function $H_k(\mathbf{z},.)$ is a continuous and injective function on $[0,1]$:

let $s < t, \mathbf{p} = \mathbf{x}+t(\mathbf{z}-\mathbf{x})$ and $\mathbf{q} = \mathbf{x}+s(\mathbf{z}-\mathbf{x})$; then

$$\begin{aligned} p_k - q_k &= x_k + t(z_k - x_k) - x_k - s(z_k - x_k) = (t-s)(z_k - x_k) \\ &= (t-s) \parallel \mathbf{z}-\mathbf{x} \parallel_\infty = \parallel (t-s)(\mathbf{z}-\mathbf{x}) \parallel_\infty \\ &= \parallel \mathbf{x}+t(\mathbf{z}-\mathbf{x}) - \mathbf{x} - (t-s)(\mathbf{z}-\mathbf{x}) \parallel_\infty = \parallel \mathbf{p}-\mathbf{q} \parallel_\infty . \end{aligned}$$

in which case $F_k(\mathbf{p}) \neq F_k(\mathbf{q})$ and hence $H_k(\mathbf{y},t) \neq H_k(\mathbf{y},s)$.

For $\mathbf{z} = \mathbf{x} + (y_k - x_k)\mathbf{e}_k \in K$ the function $H_k(\mathbf{z},.)$ is strictly isotone, hence $\forall \mathbf{z} \in K$ $H_k(\mathbf{z},.)$ is strictly isotone, especially for $\mathbf{z} = \mathbf{y}$:

$F_k(\mathbf{x}) = H_k(\mathbf{y},0) < H_k(\mathbf{y},1) = F_k(\mathbf{y})$.

Hence $(y_k - x_k)(F_k(\mathbf{y}) - F_k(\mathbf{x})) > 0$.

Theorem 8.5 *Let $F : \mathbf{R}^n_+ \longrightarrow \mathbf{R}^n$ be a differentiable function that is off-diagonally antitone.*

If $\forall \mathbf{x} \in \mathbf{R}^n_+$ $DF(\mathbf{x})$ is a strictly diagonally dominant matrix then F is inverse isotone and hence an M-function.

Proof:

Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$, $\mathbf{x} \neq \mathbf{y}$ and $F(\mathbf{x}) \leq F(\mathbf{y})$.

Define the indexset $J_> = \{i \in N \mid x_i > y_i\}$ and suppose that $J_> \neq \emptyset$. Moreover assume that $J_> = 1,\ldots,k,\ 1 \leq k \leq n$.

Consider the principle subfunction F_ω of F at $\mathbf{y}$ with $\omega = (1,\ldots,k)$,

$F_{\omega\, i}(t_1,\ldots,t_k) = F_i(t_1,\ldots,t_k,y_{k+1},\ldots,y_n),\ \ i = 1,\ldots,k.$

Since F is an off-diagonally antitone function we have

$$\begin{aligned} F_{\omega\, i}(\mathbf{y}_\omega) &= F_i(\mathbf{y}) \geq F_i(\mathbf{x}) \\ &\geq F_i(x_1,\ldots,x_k,y_{k+1},x_{k+2},\ldots,x_n) \geq \ldots \geq F_i(x_1,\ldots,x_k,y_{k+1},\ldots,y_n) \\ &= F_{\omega\, i}(\mathbf{x}_\omega). \end{aligned}$$

Hence

$(y_i - x_i)(F_{\omega\, i}(\mathbf{y}_\omega) - F_{\omega\, i}(\mathbf{x}_\omega)) \leq 0, \;\; i = 1, \ldots, k.$

According to lemma 8.2 F is a strictly diagonally dominant function on $\mathbf{R}^n_+$ and hence, according to lemma 8.3 the principle subfunction F_ω is also a strictly diagonally dominant function.

Since F is a strictly diagonally isotone function the principle subfunction F_ω is also strictly diagonally isotone. Following lemma 8.4 there exists an index $k = k(\mathbf{x}_\omega, \mathbf{y}_\omega) \in \{1, \ldots, k\}$such that

$(y_k - x_k)(F_{\omega\, k}(\mathbf{y}_\omega) - F_{\omega\, k}(\mathbf{x}_\omega)) > 0.$

This contradicts the forgoing inequality. Hence $J_> = \emptyset$ and $\mathbf{x} \leq \mathbf{y}$.

æ

A Appendix

Lemma A.1 *Let $A \in L(\mathbf{R}^n)$ be an M-matrix.*

Then the diagonal elements of A and A^{-1} are positive.

Proof:

Define $A^{-1} = \left[\; b_{ij} \;\right]$. Then $\forall i \in N \;\; a_{ii}b_{ii} = 1 - \sum_{j \neq i} a_{ij}b_{ji}$. Since $a_{ij} \leq 0, \; j \in N, \; j \neq i$ and $b_{ij} \leq 0, \; j \in N$ we have $a_{ii}b_{ii} \geq 1$ from which follows a_{ii} and b_{ii} are positive.

Lemma A.2 *Let $A \in L(\mathbf{R}^n)$ and $\mathbf{x} \in \mathbf{R}^n$.*

$A\mathbf{x} \geq \mathbf{0}$ implies $\mathbf{x} \geq \mathbf{0}$ if and only if A is nonsingular and $A^{-1} \geq 0$.

Proof:

Let $A\mathbf{x} = \mathbf{0}$ for some vector $\mathbf{x} \in \mathbf{R}^n$. Then $A\mathbf{x} \geq \mathbf{0}$ and hence $\mathbf{x} \geq \mathbf{0}$. Also $A(-\mathbf{x}) \geq \mathbf{0}$ and $-\mathbf{x} \geq \mathbf{0}$ which means that $\mathbf{x} = \mathbf{0}$ and hence A^{-1} is non-singular. Moreover, since $\mathbf{e}_i = A(A^{-1}\mathbf{e}_i) \geq \mathbf{0}$ we have $A^{-1}\mathbf{e}_i \geq \mathbf{0}, \; i \in N$. Hence $A^{-1} \geq 0$.

Let $A\mathbf{x} \geq \mathbf{0}$ for some $\mathbf{x} \in \mathbf{R}^n$. Since $A^{-1} \geq 0$ we have $\mathbf{x} = A^{-1}(A\mathbf{x}) \geq A^{-1}\mathbf{0} = \mathbf{0}$.

Lemma A.3 *Let $F : D \subset \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be an M-function.*

Then F and $F^{-1} : F(D) \subset \mathbf{R}^n \longrightarrow \mathbf{R}^n$ are strictly diagonally isotone.

Proof:

Take $\mathbf{x} \in D$, $i \in N$ and $s < t$ with $\mathbf{x} + s\mathbf{e}_i, \mathbf{x} + t\mathbf{e}_i \in D$. Suppose $F_i(\mathbf{x} + s\mathbf{e}_i) \geq F_i(\mathbf{x} + t\mathbf{e}_i)$. The off-diagonal antitonicity then implies $F(\mathbf{x} + s\mathbf{e}_i) \geq F(\mathbf{x} + t\mathbf{e}_i$. By the inverse isotonicity this leads to the contradiction $s \geq t$, which shows that F must be strictly diagonally isotone.

Take $\mathbf{y} \in F(D)$, $i \in N$ and $s < t$ with $\mathbf{y} + s\mathbf{e}_i, \mathbf{y} + t\mathbf{e}_i \in F(D)$. The inverse isotonicity of F implies $F^{-1}(\mathbf{y} + s\mathbf{e}_i) \leq F^{-1}(\mathbf{y} + t\mathbf{e}_i)$. Suppose $F_i^{-1}(\mathbf{y} + s\mathbf{e}_i) = F_i^{-1}(\mathbf{y} + t\mathbf{e}_i)$. By the off-diagonal antitonicity this lead to the inequality $F_i(F^{-1}(\mathbf{y} + s\mathbf{e}_i)) \geq F_i(F^{-1}(\mathbf{y} + t\mathbf{e}_i))$ or, equivalently, to the contradiction $s \geq t$. Hence $F_i^{-1}(\mathbf{y} + s\mathbf{e}_i) < F_i^{-1}(\mathbf{y} + t\mathbf{e}_i)$ which shows that F^{-1} must be strictly diagonally isotone.

Lemma A.4 *Let* $\mathbf{a} \in \mathbf{R}^n$ *with* $a_j \leq 0, j \neq i$ *(,* $i \in N$ *is some fixed index,) and* $\mathbf{a}^T\mathbf{e} \geq 0$. *Consider a vector* $\mathbf{v} \in \mathbf{R}^n$ *with* $\mathbf{a}^T\mathbf{v} = 0$.

If $\mathbf{a}^T\mathbf{e} > 0$ *then* $| v_i | < \| \mathbf{v} \|_\infty$,

if $\mathbf{a}^T\mathbf{e} = 0$ *then* $| v_i | < \| \mathbf{v} \|_\infty$ *or* $| v_i | = \| \mathbf{v} \|_\infty = | v_j | \; \forall j \in N$ *with* $a_j < 0$.

Proof:

Let $\mathbf{v} \in \mathbf{R}^n$ be such that $\sum_{j=1}^n a_j v_j = 0$. Assume that $\sum_{j=1}^n a_j > 0$.Then

$$a_i \mid v_i \mid \leq \sum_{j \neq i} (-a_j) \mid v_j \mid \leq \sum_{j \neq i} (-a_j) \parallel \mathbf{v} \parallel_\infty < a_i \parallel \mathbf{v} \parallel_\infty$$

from wich follows $| v_i | < \| \mathbf{v} \|_\infty$.

Let us assume next that $\sum_{j=1}^n a_j = 0$ and that $| v_i | = \| \mathbf{v} \|_\infty$. Suppose that $| v_j | < \| \mathbf{v} \|_\infty$ for any $j \in N$ with $a_j < 0$. Then

$$a_i \parallel \mathbf{v} \parallel_\infty = a_i \mid v_i \mid \leq \sum_{j \neq i} (-a_j) \mid v_j \mid < \sum_{j \neq i} (-a_j) \parallel \mathbf{v} \parallel_\infty$$

from which follows the inequality $a_i < \sum_{j \neq i} (-a_j)$. This contradicts the assumed equality on $\mathbf{a}$.

References

[1] A.Berman and R.J.Plemmons *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York 1979.

[2] P.Chandar *The Nonlinear Input-Output Model.* Journal of Economic Theory, 30(1983) pp. 219-229.

[3] L.Mc Kenzie *Matrices with dominant diagonal and economic theory.* in K.J.Arrow, S.Karlin and P.Suppes (eds) Mathematical Methods in the Social Sciences, 1959 Stanford Un. Press, Chapter 4, pp.47-62 (1960)

[4] H.Minc *Nonnegative Matrices.* John Wiley & Sons, New York 1988.

[5] J.J.Moré *Nonlinear Generalizations of Matrix Diagonal Dominance with Applications to Gauss-Seidel Iterations.* Siam J.Numer.Anal.,9(1972) pp.357-378.

[6] J.M.Ortega and W.C.Rheinboldt *Iterative Solution of Nonlinear Equations in Severable Variables.* Academic Press, New York 1970.

[7] I.W.Sandberg *A Nonlinear Input-Output Model of a Multisectored Economy.* Econometrica, 6(1973) pp.1167-1182.

[8] G.Sierksma *Non-negative Matrices: the Open Leontief Model* Linear Algebra and its Applications, 26(1979) pp.175-201.

IN 1990 REEDS VERSCHENEN

419 Bertrand Melenberg, Rob Alessie
A method to construct moments in the multi-good life cycle consumption model

420 J. Kriens
On the differentiability of the set of efficient (μ,σ^2) combinations in the Markowitz portfolio selection method

421 Steffen Jørgensen, Peter M. Kort
Optimal dynamic investment policies under concave-convex adjustment costs

422 J.P.C. Blanc
Cyclic polling systems: limited service versus Bernoulli schedules

423 M.H.C. Paardekooper
Parallel normreducing transformations for the algebraic eigenvalue problem

424 Hans Gremmen
On the political (ir)relevance of classical customs union theory

425 Ed Nijssen
Marketingstrategie in Machtsperspectief

426 Jack P.C. Kleijnen
Regression Metamodels for Simulation with Common Random Numbers: Comparison of Techniques

427 Harry H. Tigelaar
The correlation structure of stationary bilinear processes

428 Drs. C.H. Veld en Drs. A.H.F. Verboven
De waardering van aandelenwarrants en langlopende call-opties

429 Theo van de Klundert en Anton B. van Schaik
Liquidity Constraints and the Keynesian Corridor

430 Gert Nieuwenhuis
Central limit theorems for sequences with m(n)-dependent main part

431 Hans J. Gremmen
Macro-Economic Implications of Profit Optimizing Investment Behaviour

432 J.M. Schumacher
System-Theoretic Trends in Econometrics

433 Peter M. Kort, Paul M.J.J. van Loon, Mikulás Luptacik
Optimal Dynamic Environmental Policies of a Profit Maximizing Firm

434 Raymond Gradus
Optimal Dynamic Profit Taxation: The Derivation of Feedback Stackelberg Equilibria

435 Jack P.C. Kleijnen
Statistics and Deterministic Simulation Models: Why Not?

436 M.J.G. van Eijs, R.J.M. Heuts, J.P.C. Kleijnen
Analysis and comparison of two strategies for multi-item inventory systems with joint replenishment costs

437 Jan A. Weststrate
Waiting times in a two-queue model with exhaustive and Bernoulli service

438 Alfons Daems
Typologie van non-profit organisaties

439 Drs. C.H. Veld en Drs. J. Grazell
Motieven voor de uitgifte van converteerbare obligatieleningen en warrantobligatieleningen

440 Jack P.C. Kleijnen
Sensitivity analysis of simulation experiments: regression analysis and statistical design

441 C.H. Veld en A.H.F. Verboven
De waardering van conversierechten van Nederlandse converteerbare obligaties

442 Drs. C.H. Veld en Drs. P.J.W. Duffhues
Verslaggevingsaspecten van aandelenwarrants

443 Jack P.C. Kleijnen and Ben Annink
Vector computers, Monte Carlo simulation, and regression analysis: an introduction

444 Alfons Daems
"Non-market failures": Imperfecties in de budgetsector

445 J.P.C. Blanc
The power-series algorithm applied to cyclic polling systems

446 L.W.G. Strijbosch and R.M.J. Heuts
Modelling (s,Q) inventory systems: parametric versus non-parametric approximations for the lead time demand distribution

447 Jack P.C. Kleijnen
Supercomputers for Monte Carlo simulation: cross-validation versus Rao's test in multivariate regression

448 Jack P.C. Kleijnen, Greet van Ham and Jan Rotmans
Techniques for sensitivity analysis of simulation models: a case study of the CO_2 greenhouse effect

449 Harrie A.A. Verbon and Marijn J.M. Verhoeven
Decision-making on pension schemes: expectation-formation under demographic change

450 Drs. W. Reijnders en Drs. P. Verstappen
Logistiek management marketinginstrument van de jaren negentig

451 Alfons J. Daems
Budgeting the non-profit organization
An agency theoretic approach

452 W.H. Haemers, D.G. Higman, S.A. Hobart
Strongly regular graphs induced by polarities of symmetric designs

453 M.J.G. van Eijs
Two notes on the joint replenishment problem under constant demand

454 B.B. van der Genugten
Iterated WLS using residuals for improved efficiency in the linear model with completely unknown heteroskedasticity

455 F.A. van der Duyn Schouten and S.G. Vanneste
Two Simple Control Policies for a Multicomponent Maintenance System

456 Geert J. Almekinders and Sylvester C.W. Eijffinger
Objectives and effectiveness of foreign exchange market intervention
A survey of the empirical literature

457 Saskia Oortwijn, Peter Borm, Hans Keiding and Stef Tijs
Extensions of the τ-value to NTU-games

458 Willem H. Haemers, Christopher Parker, Vera Pless and Vladimir D. Tonchev
A design and a code invariant under the simple group Co_3

459 J.P.C. Blanc
Performance evaluation of polling systems by means of the power-series algorithm

460 Leo W.G. Strijbosch, Arno G.M. van Doorne, Willem J. Selen
A simplified MOLP algorithm: The MOLP-S procedure

461 Arie Kapteyn and Aart de Zeeuw
Changing incentives for economic research in The Netherlands

462 W. Spanjers
Equilibrium with co-ordination and exchange institutions: A comment

463 Sylvester Eijffinger and Adrian van Rixtel
The Japanese financial system and monetary policy: A descriptive review

464 Hans Kremers and Dolf Talman
A new algorithm for the linear complementarity problem allowing for an arbitrary starting point

465 René van den Brink, Robert P. Gilles
A social power index for hierarchically structured populations of economic agents

IN 1991 REEDS VERSCHENEN

466 Prof.Dr. Th.C.M.J. van de Klundert - Prof.Dr. A.B.T.M. van Schaik
Economische groei in Nederland in een internationaal perspectief

467 Dr. Sylvester C.W. Eijffinger
The convergence of monetary policy - Germany and France as an example

468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche

469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication

470 Drs. J. Grazell en Drs. C.H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering

471 P.C. van Batenburg, J. Kriens, W.M. Lammerts van Bueren and R.H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling

472 Marcel Kerkhofs
Identification and Estimation of Household Production Models

473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach

474 Jack P.C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design

475 C.P.M. van Hoesel
An $O(n \log n)$ algorithm for the two-machine flow shop problem with controllable machine speeds

476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance

477 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
Coordinated replenishment systems with discount opportunities

478 A. van den Nouweland, J. Potters, S. Tijs and J. Zarzuelo
Cores and related solution concepts for multi-choice games

479 Drs. C.H. Veld
Warrant pricing: a review of theoretical and empirical research

480 E. Nijssen
De Miles and Snow-typologie: Een exploratieve studie in de meubelbranche

481 Harry G. Barkema
Are managers indeed motivated by their bonuses?

482 Jacob C. Engwerda, André C.M. Ran, Arie L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$

483 Peter M. Kort
A dynamic model of the firm with uncertain earnings and adjustment costs

484 Raymond H.J.M. Gradus, Peter M. Kort
Optimal taxation on profit and pollution within a macroeconomic framework

485 René van den Brink, Robert P. Gilles
Axiomatizations of the Conjunctive Permission Value for Games with Permission Structures

486 A.E. Brouwer & W.H. Haemers
The Gewirtz graph - an exercise in the theory of graph spectra

487 Pim Adang, Bertrand Melenberg
Intratemporal uncertainty in the multi-good life cycle consumption model: motivation and application

488 J.H.J. Roemen
The long term elasticity of the milk supply with respect to the milk price in the Netherlands in the period 1969-1984

489 Herbert Hamers
The Shapley-Entrance Game

490 Rezaul Kabir and Theo Vermaelen
Insider trading restrictions and the stock market

491 Piet A. Verheyen
The economic explanation of the jump of the co-state variable

492 Drs. F.L.J.W. Manders en Dr. J.A.C. de Haan
De organisatorische aspecten bij systeemontwikkeling
een beschouwing op besturing en verandering

493 Paul C. van Batenburg and J. Kriens
Applications of statistical methods and techniques to auditing and accounting

494 Ruud T. Frambach
The diffusion of innovations: the influence of supply-side factors

495 J.H.J. Roemen
A decision rule for the (des)investments in the dairy cow stock

496 Hans Kremers and Dolf Talman
An SLSPP-algorithm to compute an equilibrium in an economy with linear production technologies

497 L.W.G. Strijbosch and R.M.J. Heuts
Investigating several alternatives for estimating the compound lead time demand in an (s,Q) inventory model

498 Bert Bettonvil and Jack P.C. Kleijnen
Identifying the important factors in simulation models with many factors

499 Drs. H.C.A. Roest, Drs. F.L. Tijssen
Beheersing van het kwaliteitsperceptieproces bij diensten door middel van keurmerken

500 B.B. van der Genugten
Density of the F-statistic in the linear model with arbitrarily normal distributed errors

501 Harry Barkema and Sytse Douma
The direction, mode and location of corporate expansions

502 Gert Nieuwenhuis
Bridging the gap between a stationary point process and its Palm distribution

503 Chris Veld
Motives for the use of equity-warrants by Dutch companies

504 Pieter K. Jagersma
Een etiologie van horizontale internationale ondernemingsexpansie