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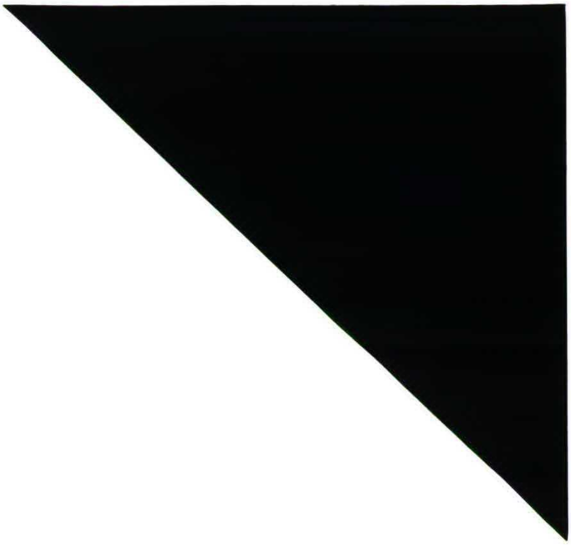

On the interaction between forecasting and inventory control

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ON THE INTERACTION BETWEEN FORECASTING AND INVENTORY CONTROL

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Abstract

The choice of the reorder point in inventory control is often based on the P_1-service criterion. This criterion states that the probability of stockout at the end of an arbitrary period is (at most) $1-P_1$. Standard procedure is to choose for the reorder point a quantile of the distribution of demand during lead time or review plus lead time. If this distribution is assumed normal, only corresponding mean and standard deviation have to be estimated, e.g. by simple exponential smoothing.

This paper shows that this standard procedure does not guarantee stockout probabilities smaller than $1-P_1$. Even if the demand is normally distributed indeed as well as stationary, stockout probabilities can be exceeded by 20% up to 800% (in extreme cases). A correction method, based on simulations, is suggested, which reduces this bias considerably.

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1. Introduction.

In practical inventory management it is inevitable that future demand is forecasted somehow. The decision parameters in the inventory model are then based on an estimated demand distribution, obtained from the results of the forecast procedure. It is very important to understand the interaction between the demand forecasting and the inventory control, since the performance of the inventory system is not only determined by the two components separately, but also by the interaction. The paper reveals an aspect of this interaction which to our best knowledge has not been described before.

Most papers on this interaction focus on the relative performance of several forecasting methods on attained service levels or inventory investments. The most important instrument of analysis is Monte Carlo experimentation. Jacobs and Wagner (1989) investigate the impact on total system cost of using the sample mean and standard deviation as compared to robust parameter estimates, like exponentially smoothed average and mean absolute deviation (MAD). An important outcome of their research is that, in general, the scaling factor of the MAD measure should be larger than the commonly used 1.25. Gardner (1990) compares the influence of several forecasting techniques on the relation between customer service and inventory investment. By means of simulation Watson (1987) showed that for lumpy (stationary stuttering Poisson) demand patterns, demand-forecast fluctuation (using Simple Moving Average) can cause either positive or negative shifts in the customer service level achieved.

Karmarkar (1994) presents an alternative method for estimating the service level, instead of the conventional approach of using the normal model. The method is especially developed to meet situations where the type of the demand distribution is unknown. Silver (1978) provides a heuristic approach to control the inventory of an item having non-stationary demand. Eppen and Martin (1988) very clearly describe some consequences of incorrectly assuming normality of the distribution of forecast errors over lead-time.

Among these studies -and numerous others- we found only one that attempted to investigate analytically the interaction between forecasting and inventory control, using the standard procedure. This one exception is the paper by Silver and Rahnama (1987), where it is argued that underestimating the so-called safety factor k leads to higher cost penalty than overestimating k; therefore, the authors suggest to bias k upwards.

In this paper, we return to basics and show that even if all assumptions are met, the standard procedure does not guarantee the prescribed P_1-service level. To show this we describe and analyse a very simple situation: a periodic review (R,S)-control policy, a stationary demand process assuming normality of demand during review plus lead time, a P_1-service criterion, and single exponential smoothing (SES) as a forecasting model. In Section 2 the standard procedure to obtain the reorder level in practice is described. In Section 3 both a simple and a more refined argument is given why stockout probabilities exceed the prescribed level. Section 4 gives some further analytical results. Section 5 describes a simulation experiment which quantifies the difference between the realised and the prescribed levels of stockout probability. A possible correction leading to stockout probabilities closer to the prescribed level is presented. Finally Section 6 gives the conclusion and proposes further research.

2. Standard procedure.

Assume that inventory is controlled according to an (R,S)-policy. Than it follows from standard inventory theory (cf. Silver and Peterson (1985)) that the unique S that guarantees a non-stockout probability of P_1 is determined by the equation

$$F(S) = P_1,$$

where F denotes the distribution function of demand during review plus lead time. Assume a constant review plus lead time and take this as unit period. Consider a stationary demand process where X_i denotes the stochastic demand in period i ; all X_i are assumed independent. Let F denote the common distribution function of the X_i, so that

$$F(x) = P(X_i \leq x).$$

The $(1-\gamma)$-quantile $\theta_{1-\gamma}$ of F is defined by

$$F(\theta_{1-\gamma}) = 1-\gamma. \tag{1}$$

Now, the P_1-service criterion demands that the probability of stockout is (at most) $\gamma = 1-P_1$. Under this criterion, $\theta_{1-\gamma}$ is the reorder level S. If F were fully known, $\theta_{1-\gamma}$ could be solved from (1), e.g. for $X_i \sim N(\mu,\sigma^2)$,

$$\theta_{1-\gamma} = \mu + u_{1-\gamma}\,\sigma \tag{2}$$

where $u_{1-\gamma}$ is defined by $\Phi(u_{1-\gamma}) = 1-\gamma$ with Φ the standard normal distribution function. However, in practice F is unknown and has to be estimated. A generally accepted standard procedure is Simple Exponential Smoothing (SES). It is in fact a forecasting method for X_{t+1}, based on the observed demands X_i up to period t. The SES updating formulae for the forecast S_t and the variance of the forecast error V_t use the last known forecast error E_t, defined by

$$E_t = X_t - S_{t-1}. \tag{3}$$

These formulae read

$$\left.\begin{aligned} S_t &= \alpha E_t + S_{t-1} \\ V_t &= \omega E_t^2 + (1-\omega) V_{t-1} \end{aligned}\right\} \tag{4}$$

where α and ω are the smoothing constants. Substituting (4) into (2) gives the estimated $(1-\gamma)$-quantile

$$Z = Z_{1-\gamma}(t) = S_t + u_{1-\gamma}\sqrt{V_t}. \tag{5}$$

This estimate is proposed even if F is not known to be normal; see Hax and Candea (1984), or Silver and Peterson (1985) for all this, e.g.

The central question of course is whether this standard procedure guarantees that the stockout probability does not exceed γ, in other words whether $P(X_{t+1} > Z)$ is not larger than γ. The paper investigates this question in the ideal case $F = N(\mu, \sigma^2)$. It is shown that the actual stockout probability may greatly exceed γ even in this simple case. First of all, Section 3 gives simple arguments why even an unbiased estimator Z for $\theta_{1-\gamma}$ generally leads to stockout probabilities exceeding γ.

3. Failure of standard procedure.

This section starts with a simple example showing that estimating $\theta_{1-\gamma}$ unbiasedly does not mean that the system performance satisfies the P_1-criterion. Assume

$$X_i \sim N(\mu,1)$$

so that (2) reduces to $\theta_{1-\gamma} = \mu + u_{1-\gamma}$. In this case, the only unknown μ can be estimated best by $\overline{X}_t = \sum_{i=1}^{t} X_i/t$, leading to the estimator

$$Z = \overline{X}_t + u_{1-\gamma} \sim N\left(\mu + u_{1-\gamma}, 1/t\right).$$

Figure 1 shows the distributions of both X_{t+1} and Z.

Although Z is unbiased for $\theta_{1-\gamma}$, the asymmetric behavior of $N(\mu,1)$ near $\mu + u_{1-\gamma}$ causes $P\left(X_{t+1} > Z\right) > \gamma = 1 - P_1$. This can be seen directly by calculating

$$q_{\varepsilon,\gamma} = \frac{P\left(X_{t+1} > \mu + u_{1-\gamma} - \varepsilon\right) + P\left(X_{t+1} > \mu + u_{1-\gamma} + \varepsilon\right)}{2\gamma} = \frac{\Phi\left(-u_{1-\gamma} + \varepsilon\right) + \Phi\left(-u_{1-\gamma} - \varepsilon\right)}{2\gamma}.$$

Table 1 illustrates the fact that $q_{\varepsilon,\gamma}$ is increasing in both ε and γ. Furthermore, since all values exceed 1, γ is exceeded as well.

Next assume

$$X_i \sim N(\mu, \sigma^2).$$

For an infinite past, formulae (4) can be rewritten as

$$S_t = \alpha \sum_{i=0}^{\infty} (1-\alpha)^i X_{t-i}$$

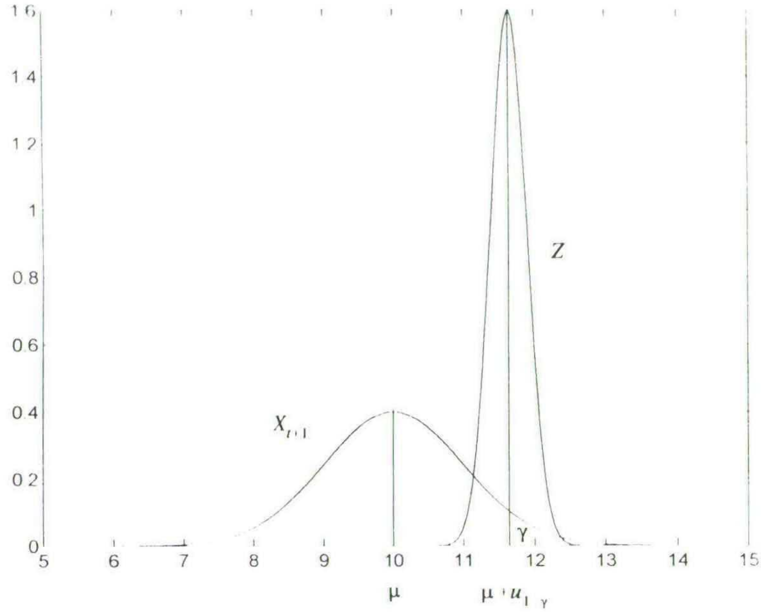

Figure 1: The distribution of both X_{t+1} and Z with $\mu=10$, $\gamma=0.05$ and $t=4$.

$q_{\varepsilon,\gamma}$		γ		
		0.05	0.025	0.01
ε	0.05	1.0042	1.0057	1.0078
	0.1	1.0170	1.0229	1.0311
	0.15	1.0381	1.0516	1.0701

Table 1: Some values of $q_{\varepsilon,\gamma}$.

$$V_t = \omega \sum_{i=0}^{\infty} (1-\omega)^i \left(X_{t-i} - S_{t-i-1}\right)^2 .$$

From

$$\frac{S_t - \mu}{\sigma} = \alpha \sum_{i=0}^{\infty} (1-\alpha)^i \frac{X_{t-i} - \mu}{\sigma}$$

and

$$\frac{Z-\mu}{\sigma} = \frac{S_t - \mu}{\sigma} + u_{1-\gamma} \sqrt{\omega \sum_{i=0}^{\infty} (1-\omega)^i \left(\frac{X_{t-i} - \mu}{\sigma} - \frac{S_{t-i-1} - \mu}{\sigma} \right)^2}$$

plus the fact that the distribution of $(X_{t-i} - \mu)/\sigma$ is independent of μ and σ it follows that the distributions of $(S_t - \mu)/\sigma$ and $(Z-\mu)/\sigma$ do not depend on μ and σ either. Thus the distributions of X_t, S_t and Z belong to a location-scale family and the same holds for the distribution of $X_{t+1} - Z$. If for example $\frac{Z-\mu}{\sigma} \sim N(u_{1-\gamma} + c, k^2 - 1)$ with $k > 1$ and c and k constants which do not depend on μ and σ, we may conclude that

$$Z \sim N\left(\theta_{1-\gamma} + c\sigma, (k^2 - 1)\sigma^2\right)$$

and

$$P\left\{X_{t+1} > Z\right\} = 1 - \Phi\left(\frac{u_{1-\gamma} + c}{k} \right)$$

which is $\neq \gamma$, in general. Even when the distribution of Z is symmetric around $\theta_{1-\gamma}$, i.e.

Z is unbiased, the fact that the probability density function of X_{t+1} is not symmetric around Z ($\gamma<0.5$) implies that $P\{X_{t+1}>Z\}\neq\gamma$. Note that the equality $1-\Phi\left(\frac{u_{1-\gamma}+c}{k}\right)=\gamma$ only holds for $c=(k-1)u_{1-\gamma}$. Thus, when the percentile estimator is positively biased ($c>0$) this may compensate for the effect resulting from $k>1$. This phenomenon was pointed out earlier by Silver and Rahnama (1987), based on other arguments. It is not clear to which location-scale family the distribution of $X_{t+1}-Z$ really belongs. Knowledge of that distribution could lead to a corrective procedure using the particular values of c and k which are constants for given values of α, ω and γ.

4. Exact expression for stockout probability.

To calculate the exact values of the stockout probability $P(X_{t+1}>Z)$ under the standard procedure, the probability distribution of Z in (5) must be known. We report some analytical results on the (co)variances (C and V, respectively) of S_t and V_t in (4), assuming normality and an infinite past:

$$E(S_{t-i})=\mu;\ \text{all } i\geq 0$$

$$C(S_{t-i},S_{t-j})=\frac{\alpha(1-\alpha)^{|i-j|}}{2-\alpha}\sigma^2;\ i\geq 0,\ j\geq 0 \tag{6}$$

$$\sigma_E^2=E(V_t)=V(E_t)=\frac{2}{2-\alpha}\sigma^2 \tag{7}$$

$$\sigma_V^2 = V(V_t) = \frac{\omega\sigma_E^4}{2-\omega}\left\{2 + \frac{\alpha^2(1-\omega)}{1-(1-\omega)(1-\alpha)^2}\right\} \tag{8}$$

$$C(S_t, V_t) = 0$$

Note that (6) implies

$$V(S_{t-i}) = \frac{\alpha}{2-\alpha}\sigma^2; \text{ all } i$$

while (7) is well-known; e.g. Hax and Candea (1984). Further,

$$\frac{\alpha^2(1-\omega)}{1-(1-\omega)(1-\alpha)^2} \le \frac{1}{7} \text{ for } \alpha \le 0.25, \text{ all } \omega$$

so that (8) can be approximated as

$$\sigma_V^2 \approx \frac{2\omega\sigma_E^4}{2-\omega}$$

in most practical cases.

Finally, V_t can be written as a quadratic form in $X^T = (X_t, X_{t-1}, \ldots)^T$:

$$V_t = X^T Q X$$

where the elements of Q are given by

$$Q_{ii} = \omega(1-\omega)^i + \alpha^2\omega\sum_{k=1}^{i}(1-\omega)^{k-1}(1-\alpha)^{2i-2k},$$

$$Q_{ij} = -2\alpha\omega(1-\omega)^i(1-\alpha)^{j-i-1} + 2\alpha^2\omega\sum_{k=1}^{i}(1-\omega)^{k-1}(1-\alpha)^{i+j-2k} \quad (0 \le i < j).$$

By consequence, the distribution of V_t is a linear combination of χ_1^2-distributions with the eigenvalues of Q as weights. See for the proofs Appendix D.

However, the distribution of Z is complicated. Therefore, we used simulation to approximate the actual stockout probabilities.

5. A heuristic for determining the reorder level.

In order to validate the procedure in Section 2 we will conduct a simulation experiment. To be more precise, we want to investigate the following questions:

1 What role play the values of the smoothing constants α and ω?

2 Keeping in mind that high service levels are common in inventory management, is it possible to attain the standard P_1-service levels 0.999, 0,995, 0.99, 0.98, 0.95, 0.90 and 0.80 in the long run?

The precise setup of the simulation experiment is described below. To show the failure of the standard procedure most clearly, we considered precisely those circumstances, where the standard procedure should perform best.

A normal distribution function is used to generate the stationary demand process $\{X_i\}$. As is explained in Section 3 the results of the simulation are not sensitive for the values of μ and σ. Note that possible negative values for X_i, although impossible in practice, do not disturb the investigation of the described issue. In order to get rid of unwanted startup effects a run-in time of 5000 periods is used each time updating the smoothed values of S and V according to (4). Then, during a horizon of n periods $\theta_{1-\gamma}$ is estimated by formula (5). Then the relative frequency $\overline{\gamma}(n)$ of the events $\{X_{t+1} \geq Z,\ t=1,..,n\}$ is determined, which we hope to be γ in the long run. The results are expected to be dependent on the values of α and ω. Thus, α and ω are both varied on different levels viz.

$$0.001,\ 0.005,\ 0.01,\ 0.05,\ 0.1,\ 0.2,\ 0.25 \text{ and } 0.3.$$

The levels of γ in the simulation are

$$0.2,\ 0.1,\ 0.05,\ 0.02,\ 0.01,\ 0.005,\ 0.001.$$

Appendix A shows the statistic

$$100*(\overline{\gamma}(n)-\gamma) = 100*\left(\frac{\#\{X_{t+1} > Z \mid t=1,..,n\}}{n} - \gamma\right)$$

for various values of α, ω and γ obtained with $n=30{,}000$ (larger values do not change the figures substantially). The results reveal that the value of α has a minor effect on the system performance. This phenomenon enables us to present Appendix A by means of Table 2 obtained by averaging over α. Figure 2 gives a visual impression. Note that for γ=0.001 and ω=0.3 the prescribed stockout probability is exceeded by 800%.

The results seem to be reasonable only for small values of ω or larger values of γ. This possibly explains why Karmarkar (1994) "sets all smoothing constants to 0.01 as trial and error suggested that this give good results". The general conclusion is that

$100(\overline{\gamma}(n)-\gamma)$		ω								
		0.001	0.005	0.01	0.05	0.1	0.15	0.2	0.25	0.3
γ	0.2	-0.093	-0.085	-0.047	0.200	0.473	0.679	0.928	1.203	1.456
	0.1	0.101	0.126	0.154	0.389	0.816	1.219	1.597	1.994	2.370
	0.05	0.101	0.123	0.163	0.467	0.867	1.212	1.587	2.023	2.442
	0.02	0.038	0.062	0.101	0.416	0.743	1.096	1.429	1.767	2.116
	0.01	-0.045	-0.016	0.012	0.263	0.580	0.885	1.183	1.476	1.791
	0.005	-0.045	-0.040	-0.017	0.164	0.412	0.669	0.948	1.199	1.480
	0.001	-0.006	-0.001	0.010	0.068	0.165	0.289	0.459	0.620	0.807

Table 2: Simulation results on the difference between attained and prescribed service levels.

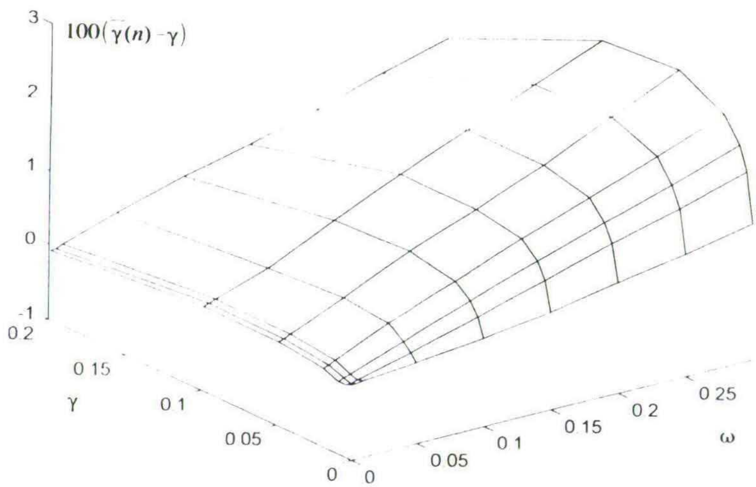

Figure 2: Excess of stockout probability; standard procedure.

the commonly applied procedure of using forecast information to fit a distribution function and base a reorder point on that fitted distribution is biased, even in the ideal case of normality and stationarity of demand.

Section 3 suggests a heuristic way to improve upon these results by using

$$Z^* = S_t + \left(u_{1-\gamma} - c(\alpha,\omega,\gamma)\right) k(\alpha,\omega,\gamma)\sqrt{V_t}$$

instead of (5) as an estimator of $\theta_{1-\gamma}$, where the correction terms $c(\alpha,\omega,\gamma)$ and $k(\alpha,\omega,\gamma)$ are estimated from the simulated sample mean and sample standard deviation of $Z = Z_{1-\gamma}(t)$, $t=1,\ldots,n$. A relative frequency $\overline{\gamma}^*(n)$ is defined in analogy to $\overline{\gamma}(n)$. Appendices B_1 and B_2 give the values of $c(\alpha,\omega,\gamma)$ and $k(\alpha,\omega,\gamma)$, respectively. Appendix C shows the statistic $100\left(\overline{\gamma}^*(n)-\gamma\right)$. Table 3 and Figure 3 are again obtained by averaging over α. Comparison with Table 2 shows that a significant improvement has been obtained. E.g. in the extreme case $\gamma=0.001$ and $\omega=0.3$, the relative difference of 800% is reduced to 77%. The corrective procedure appears to be less effective when using larger values of α and ω than 0.3 (based on simulation results not given here), but these are very seldom met in practice. By means of multiple regression analysis (with c and $k-1$ as dependent variables and without intercept) the following relations can be established:

$$c(\alpha,\omega,\gamma) = 0.63\alpha - 0.39\omega - 0.08\gamma - 2.26\alpha\gamma + 1.66\omega\gamma$$

$$k(\alpha,\omega,\gamma) = 1 + 0.29\alpha + 0.75\omega - 4.07\omega\gamma$$

The R-square values as recorded by the statistical package used (SAS) are 0.96 and 0.94 for c and $k-1$, respectively. Including regression variables up to γ^4 and $\omega*\gamma^4$

yields relations for $c(\alpha,\omega,\gamma)$ and $k(\alpha,\omega,\gamma)$ with R-square values of 0.98 and 0.99 respectively. Using these more complex relations yields corrections which are nearly as good as the one based on Appendices B_1 and B_2.

$100(\bar{\gamma}(n)-\gamma)$		ω								
		0.001	0.005	0.01	0.05	0.1	0.15	0.2	0.25	0.3
γ	0.2	-0.306	-0.313	-0.307	-0.280	-0.248	-0.286	-0.301	-0.283	-0.229
	0.1	-0.071	-0.060	-0.084	-0.141	-0.112	-0.059	0.031	0.056	0.104
	0.05	0.022	0.008	0.006	-0.016	0.002	-0.030	-0.017	0.036	0.137
	0.02	0.002	-0.009	-0.000	-0.002	0.059	0.097	0.135	0.196	0.249
	0.01	-0.056	-0.044	-0.043	-0.028	0.016	0.079	0.120	0.165	0.206
	0.005	-0.069	-0.067	-0.053	-0.019	0.014	0.045	0.084	0.111	0.144
	0.001	-0.008	-0.011	-0.010	0.002	0.008	0.018	0.030	0.056	0.077

Table 3: Simulation results on the difference between attained and prescribed service levels using the corrective procedure.

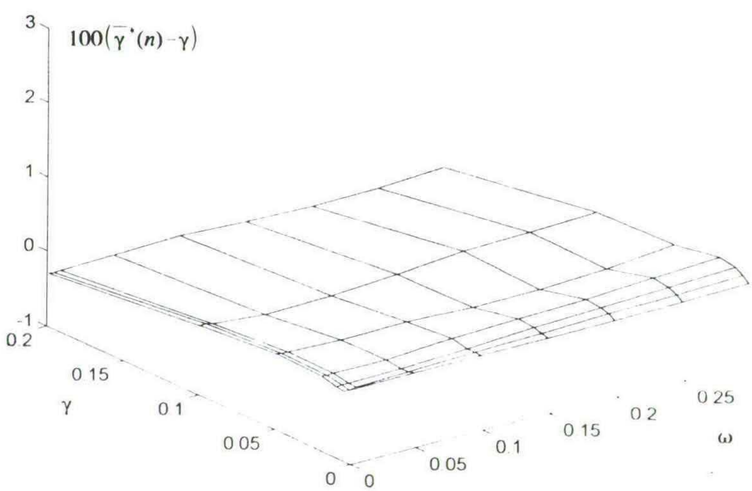

Figure 3: Excess of stockout probability; corrective procedure.

6. Conclusions and further research.

In inventory control one has to make use of estimated parameters of the distribution of demand during lead time or review plus lead time. Generally these estimates are obtained with forecasting procedures. Since demand is usually non-stationary in practice, exponential smoothing procedures are mostly applied. First of all, we demonstrated in the simplest possible setting that the standard procedures from literature do not guarantee the desired service levels. The same conclusions can be drawn with any other control policy than an (R,S)-policy. Additional simulations (not presented here) show that these conclusions even hold if other distributions than the normal are assumed. It is plausible that for non-stationary data the performance of the standard procedure is even worse. To our best knowledge this serious drawback of the standard procedure has remained unnoticed up to now. This is especially important, as the standard procedure is described and advocated in leading handbooks, like Silver & Peterson (1985), Fogarty et al. (1991).

Next, a correction method was presented, which in this simple setting greatly reduced the gap between attained and prescribed service level. We plan to investigate how this correction behaves if not all assumptions are satisfied, especially if the process is non-stationary. A second line of future research concerns the behavior of the standard procedure with regards to the P_2-service criterion.

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APPENDIX A: the statistic $100(\overline{\gamma}(n)-\gamma)$ as function of α, ω and γ.

$\gamma=0.2$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.063	-0.073	-0.027	0.167	0.420	0.633	0.903	1.173	1.480
0.005	-0.107	-0.097	-0.080	0.200	0.493	0.707	0.977	1.253	1.467
0.010	-0.123	-0.120	-0.060	0.193	0.463	0.650	0.943	1.220	1.417
0.050	-0.093	-0.093	-0.013	0.213	0.427	0.560	0.820	1.173	1.463
0.100	-0.163	-0.097	-0.007	0.193	0.413	0.693	0.900	1.160	1.423
0.150	-0.087	-0.083	-0.043	0.290	0.517	0.693	0.877	1.080	1.403
0.200	-0.030	-0.010	-0.013	0.243	0.450	0.717	0.917	1.177	1.377
0.250	-0.063	-0.057	-0.043	0.133	0.583	0.740	0.977	1.253	1.477
0.300	-0.110	-0.137	-0.140	0.167	0.493	0.720	1.040	1.340	1.593

$\gamma=0.1$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	0.080	0.150	0.200	0.430	0.873	1.307	1.637	2.093	2.497
0.005	0.167	0.207	0.247	0.497	0.933	1.280	1.653	2.100	2.513
0.010	0.137	0.210	0.213	0.500	0.913	1.270	1.737	2.120	2.497
0.050	0.180	0.193	0.230	0.430	0.903	1.350	1.650	2.067	2.417
0.100	0.163	0.140	0.163	0.367	0.813	1.233	1.633	2.017	2.353
0.150	0.070	0.083	0.110	0.333	0.760	1.137	1.547	1.987	2.323
0.200	0.113	0.080	0.047	0.353	0.690	1.147	1.503	1.827	2.280
0.250	-0.020	0.037	0.087	0.297	0.717	1.167	1.493	1.867	2.227
0.300	0.020	0.033	0.093	0.293	0.743	1.077	1.523	1.873	2.223

$\gamma=0.05$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	0.113	0.107	0.087	0.463	0.927	1.257	1.607	1.993	2.413
0.005	0.100	0.053	0.127	0.467	0.907	1.307	1.610	2.050	2.440
0.010	0.080	0.067	0.120	0.483	0.927	1.273	1.620	2.053	2.477
0.050	0.067	0.143	0.160	0.520	0.923	1.303	1.697	2.087	2.427
0.100	0.097	0.150	0.217	0.550	0.887	1.260	1.600	2.067	2.487
0.150	0.110	0.150	0.140	0.487	0.867	1.197	1.573	2.043	2.480
0.200	0.127	0.160	0.223	0.427	0.857	1.153	1.590	1.993	2.480
0.250	0.123	0.190	0.247	0.397	0.747	1.093	1.517	1.997	2.420
0.300	0.093	0.087	0.147	0.407	0.760	1.063	1.467	1.920	2.357

$\gamma=0.02$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	0.023	0.067	0.100	0.390	0.720	1.023	1.410	1.783	2.103
0.005	0.033	0.047	0.100	0.393	0.757	1.070	1.417	1.850	2.160
0.010	0.007	0.057	0.087	0.413	0.730	1.070	1.410	1.833	2.190
0.050	0.070	0.090	0.090	0.417	0.737	1.083	1.453	1.757	2.173
0.100	0.103	0.113	0.140	0.357	0.753	1.123	1.420	1.730	2.110
0.150	0.053	0.083	0.110	0.443	0.757	1.143	1.443	1.757	2.093
0.200	0.003	0.040	0.107	0.477	0.750	1.153	1.433	1.763	2.083
0.250	0.053	0.067	0.110	0.443	0.750	1.137	1.447	1.730	2.073
0.300	-0.007	-0.003	0.067	0.413	0.730	1.063	1.430	1.703	2.053

$\gamma=0.01$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.033	-0.003	0.020	0.210	0.513	0.820	1.133	1.453	1.797
0.005	-0.033	0.010	0.020	0.233	0.537	0.833	1.140	1.473	1.810
0.010	-0.030	-0.007	0.007	0.243	0.553	0.863	1.137	1.470	1.813
0.050	-0.053	-0.020	0.010	0.270	0.557	0.867	1.167	1.463	1.790
0.100	-0.073	-0.043	0.017	0.257	0.587	0.883	1.190	1.483	1.803
0.150	-0.047	-0.010	0.033	0.293	0.600	0.920	1.213	1.507	1.773
0.200	-0.053	-0.017	0.007	0.307	0.637	0.920	1.220	1.493	1.803
0.250	-0.040	-0.030	-0.007	0.290	0.633	0.930	1.227	1.483	1.770
0.300	-0.043	-0.023	0.000	0.263	0.607	0.930	1.223	1.453	1.757

$\gamma=0.005$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.057	-0.057	-0.030	0.140	0.387	0.613	0.880	1.163	1.487
0.005	-0.057	-0.047	-0.023	0.150	0.377	0.620	0.920	1.160	1.490
0.010	-0.060	-0.057	-0.020	0.157	0.383	0.637	0.923	1.190	1.500
0.050	-0.047	-0.043	-0.017	0.170	0.400	0.643	0.907	1.180	1.517
0.100	-0.040	-0.040	-0.010	0.163	0.430	0.670	0.950	1.217	1.497
0.150	-0.040	-0.040	-0.030	0.153	0.427	0.690	0.937	1.207	1.490
0.200	-0.053	-0.023	-0.020	0.187	0.427	0.733	0.987	1.223	1.453
0.250	-0.040	-0.027	0.007	0.180	0.433	0.700	1.007	1.227	1.453
0.300	-0.013	-0.030	-0.010	0.173	0.447	0.717	1.023	1.223	1.437

$\gamma=0.001$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.013	-0.000	0.010	0.047	0.143	0.280	0.430	0.590	0.783
0.005	-0.010	0.007	0.013	0.047	0.147	0.280	0.443	0.590	0.767
0.010	-0.007	0.003	0.020	0.047	0.157	0.290	0.443	0.593	0.770
0.050	-0.000	-0.007	0.013	0.073	0.183	0.323	0.497	0.607	0.760
0.100	-0.000	-0.007	0.020	0.080	0.180	0.310	0.463	0.643	0.807
0.150	-0.010	-0.007	0.010	0.070	0.173	0.283	0.443	0.650	0.843
0.200	-0.007	0.003	0.010	0.077	0.170	0.277	0.463	0.650	0.857
0.250	-0.000	-0.003	-0.000	0.083	0.163	0.287	0.473	0.637	0.853
0.300	-0.007	-0.003	-0.003	0.087	0.167	0.273	0.473	0.620	0.823

APPENDIX B1: the correction term $c(\alpha, \omega, \gamma)$ as function of α, ω and γ.

$\gamma=0.2$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.008	-0.009	-0.009	-0.014	-0.019	-0.025	-0.030	-0.036	-0.041
0.005	-0.008	-0.009	-0.009	-0.014	-0.019	-0.025	-0.030	-0.036	-0.042
0.010	-0.008	-0.008	-0.009	-0.013	-0.019	-0.024	-0.029	-0.035	-0.041
0.050	0.000	-0.000	-0.001	-0.005	-0.011	-0.016	-0.022	-0.028	-0.033
0.100	0.011	0.011	0.010	0.006	0.000	-0.006	-0.011	-0.017	-0.023
0.150	0.023	0.023	0.022	0.017	0.011	0.006	-0.000	-0.006	-0.012
0.200	0.035	0.035	0.034	0.029	0.023	0.017	0.011	0.005	-0.001
0.250	0.048	0.047	0.047	0.042	0.036	0.030	0.023	0.017	0.011
0.300	0.061	0.060	0.060	0.055	0.048	0.042	0.036	0.030	0.023

$\gamma=0.1$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.010	-0.011	-0.012	-0.018	-0.026	-0.035	-0.043	-0.052	-0.061
0.005	-0.010	-0.010	-0.011	-0.018	-0.026	-0.034	-0.043	-0.051	-0.060
0.010	-0.008	-0.009	-0.010	-0.017	-0.025	-0.033	-0.041	-0.050	-0.059
0.050	0.004	0.003	0.003	-0.004	-0.013	-0.021	-0.029	-0.038	-0.047
0.100	0.021	0.020	0.019	0.012	0.004	-0.005	-0.013	-0.022	-0.031
0.150	0.039	0.038	0.037	0.030	0.021	0.012	0.004	-0.005	-0.015
0.200	0.057	0.057	0.056	0.048	0.039	0.030	0.021	0.012	0.003
0.250	0.076	0.076	0.075	0.067	0.058	0.049	0.040	0.030	0.021
0.300	0.096	0.096	0.095	0.087	0.077	0.068	0.059	0.049	0.039

$\gamma=0.05$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.012	-0.012	-0.014	-0.022	-0.033	-0.043	-0.054	-0.065	-0.076
0.005	-0.011	-0.011	-0.013	-0.021	-0.032	-0.042	-0.053	-0.064	-0.075
0.010	-0.009	-0.010	-0.011	-0.019	-0.030	-0.041	-0.051	-0.062	-0.074
0.050	0.007	0.006	0.005	-0.003	-0.014	-0.025	-0.036	-0.047	-0.059
0.100	0.029	0.028	0.027	0.018	0.007	-0.004	-0.015	-0.026	-0.038
0.150	0.052	0.051	0.050	0.040	0.029	0.018	0.007	-0.005	-0.017
0.200	0.075	0.075	0.073	0.064	0.052	0.041	0.029	0.018	0.005
0.250	0.100	0.099	0.098	0.088	0.076	0.065	0.053	0.041	0.029
0.300	0.126	0.125	0.124	0.114	0.101	0.089	0.077	0.065	0.053

$\gamma=0.02$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.013	-0.014	-0.016	-0.026	-0.039	-0.053	-0.066	-0.080	-0.094
0.005	-0.012	-0.013	-0.014	-0.025	-0.038	-0.051	-0.065	-0.079	-0.093
0.010	-0.010	-0.010	-0.012	-0.023	-0.036	-0.049	-0.063	-0.076	-0.090
0.050	0.011	0.010	0.008	-0.003	-0.016	-0.029	-0.043	-0.057	-0.071
0.100	0.038	0.037	0.035	0.024	0.011	-0.003	-0.017	-0.031	-0.046
0.150	0.066	0.065	0.064	0.052	0.038	0.024	0.010	-0.004	-0.019
0.200	0.096	0.095	0.093	0.081	0.067	0.053	0.038	0.024	0.009
0.250	0.127	0.126	0.124	0.112	0.097	0.083	0.068	0.053	0.037
0.300	0.159	0.158	0.156	0.144	0.128	0.113	0.098	0.083	0.067

$\gamma=0.01$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.015	-0.015	-0.017	-0.029	-0.044	-0.059	-0.074	-0.090	-0.106
0.005	-0.013	-0.014	-0.015	-0.027	-0.042	-0.057	-0.073	-0.088	-0.104
0.010	-0.010	-0.011	-0.013	-0.025	-0.040	-0.055	-0.070	-0.086	-0.102
0.050	0.013	0.012	0.010	-0.002	-0.017	-0.032	-0.048	-0.064	-0.080
0.100	0.044	0.043	0.041	0.028	0.013	-0.003	-0.018	-0.034	-0.051
0.150	0.076	0.075	0.073	0.060	0.044	0.028	0.012	-0.004	-0.021
0.200	0.109	0.108	0.107	0.093	0.077	0.061	0.044	0.028	0.011
0.250	0.144	0.143	0.142	0.128	0.111	0.094	0.078	0.061	0.043
0.300	0.181	0.180	0.178	0.164	0.146	0.129	0.112	0.095	0.077

$\gamma=0.005$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.016	-0.017	-0.018	-0.032	-0.048	-0.065	-0.082	-0.099	-0.117
0.005	-0.013	-0.015	-0.016	-0.030	-0.046	-0.063	-0.080	-0.097	-0.115
0.010	-0.010	-0.011	-0.013	-0.027	-0.043	-0.060	-0.077	-0.094	-0.112
0.050	0.015	0.014	0.012	-0.001	-0.018	-0.035	-0.052	-0.070	-0.088
0.100	0.049	0.048	0.046	0.032	0.015	-0.002	-0.020	-0.037	-0.056
0.150	0.085	0.084	0.082	0.067	0.050	0.032	0.015	-0.004	-0.022
0.200	0.122	0.121	0.119	0.104	0.086	0.068	0.050	0.032	0.013
0.250	0.161	0.160	0.157	0.142	0.124	0.105	0.087	0.068	0.049
0.300	0.201	0.200	0.198	0.182	0.163	0.144	0.125	0.106	0.086

$\gamma=0.001$

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.018	-0.019	-0.021	-0.037	-0.057	-0.077	-0.097	-0.118	-0.139
0.005	-0.015	-0.016	-0.018	-0.034	-0.054	-0.074	-0.094	-0.115	-0.136
0.010	-0.011	-0.012	-0.015	-0.031	-0.051	-0.070	-0.091	-0.112	-0.133
0.050	0.020	0.018	0.016	-0.000	-0.020	-0.041	-0.061	-0.082	-0.104
0.100	0.060	0.059	0.057	0.040	0.019	-0.001	-0.022	-0.043	-0.065
0.150	0.103	0.102	0.100	0.082	0.061	0.040	0.019	-0.003	-0.025
0.200	0.148	0.146	0.144	0.126	0.105	0.083	0.061	0.039	0.017
0.250	0.194	0.193	0.190	0.172	0.150	0.128	0.106	0.083	0.060
0.300	0.242	0.241	0.239	0.220	0.197	0.174	0.152	0.129	0.105

APPENDIX B2: the correction term $k(\alpha, \omega, \gamma)$ as function of α, ω and γ.

γ=0.2

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.000	1.001	1.001	1.005	1.009	1.013	1.018	1.022	1.027
0.005	1.001	1.002	1.002	1.006	1.010	1.014	1.019	1.023	1.028
0.010	1.002	1.003	1.003	1.007	1.011	1.016	1.020	1.025	1.029
0.050	1.012	1.013	1.013	1.017	1.021	1.026	1.031	1.035	1.040
0.100	1.026	1.026	1.026	1.030	1.035	1.039	1.044	1.049	1.054
0.150	1.039	1.039	1.040	1.043	1.048	1.053	1.058	1.063	1.068
0.200	1.053	1.054	1.054	1.058	1.063	1.068	1.073	1.078	1.083
0.250	1.068	1.068	1.069	1.073	1.078	1.083	1.088	1.093	1.098
0.300	1.083	1.084	1.084	1.088	1.093	1.099	1.104	1.109	1.114

γ=0.1

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.000	1.001	1.002	1.010	1.021	1.031	1.041	1.051	1.061
0.005	1.001	1.002	1.003	1.011	1.022	1.032	1.042	1.052	1.062
0.010	1.002	1.003	1.004	1.013	1.023	1.033	1.043	1.054	1.064
0.050	1.013	1.013	1.014	1.023	1.033	1.044	1.054	1.065	1.075
0.100	1.026	1.026	1.027	1.036	1.047	1.057	1.068	1.079	1.089
0.150	1.039	1.040	1.041	1.050	1.061	1.071	1.082	1.093	1.104
0.200	1.053	1.054	1.055	1.064	1.075	1.086	1.098	1.109	1.120
0.250	1.068	1.069	1.070	1.079	1.091	1.102	1.113	1.125	1.136
0.300	1.084	1.084	1.085	1.095	1.107	1.118	1.130	1.142	1.153

γ=0.05

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.001	1.002	1.003	1.017	1.034	1.050	1.066	1.083	1.099
0.005	1.001	1.003	1.004	1.018	1.035	1.051	1.068	1.084	1.100
0.010	1.003	1.004	1.006	1.019	1.036	1.053	1.069	1.085	1.102
0.050	1.013	1.014	1.016	1.029	1.047	1.063	1.080	1.097	1.114
0.100	1.026	1.027	1.029	1.043	1.060	1.077	1.094	1.112	1.129
0.150	1.039	1.041	1.042	1.056	1.074	1.092	1.110	1.127	1.144
0.200	1.054	1.055	1.056	1.071	1.089	1.107	1.125	1.143	1.161
0.250	1.068	1.070	1.071	1.086	1.105	1.124	1.142	1.160	1.178
0.300	1.084	1.085	1.087	1.102	1.121	1.141	1.159	1.178	1.196

γ=0.02

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.001	1.003	1.005	1.026	1.052	1.077	1.101	1.126	1.150
0.005	1.002	1.004	1.006	1.027	1.053	1.078	1.103	1.127	1.152
0.010	1.003	1.005	1.007	1.029	1.054	1.080	1.104	1.129	1.154
0.050	1.013	1.015	1.017	1.039	1.065	1.091	1.116	1.141	1.166
0.100	1.026	1.028	1.031	1.052	1.079	1.105	1.131	1.157	1.182
0.150	1.040	1.042	1.044	1.066	1.094	1.121	1.147	1.173	1.199
0.200	1.054	1.056	1.058	1.081	1.109	1.137	1.164	1.190	1.217
0.250	1.068	1.071	1.073	1.097	1.125	1.154	1.181	1.208	1.235
0.300	1.084	1.086	1.089	1.113	1.142	1.171	1.200	1.227	1.255

γ=0.01

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.001	1.003	1.007	1.033	1.066	1.097	1.128	1.159	1.190
0.005	1.002	1.005	1.008	1.035	1.067	1.099	1.130	1.161	1.191
0.010	1.003	1.006	1.009	1.036	1.069	1.100	1.132	1.162	1.193
0.050	1.013	1.016	1.019	1.046	1.080	1.112	1.144	1.175	1.206
0.100	1.026	1.029	1.032	1.060	1.094	1.127	1.159	1.191	1.223
0.150	1.040	1.042	1.046	1.074	1.109	1.143	1.176	1.208	1.240
0.200	1.054	1.057	1.060	1.089	1.125	1.159	1.193	1.226	1.259
0.250	1.069	1.071	1.075	1.105	1.141	1.177	1.211	1.245	1.278
0.300	1.084	1.087	1.091	1.121	1.159	1.195	1.230	1.265	1.299

γ=0.005

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.001	1.004	1.008	1.041	1.080	1.118	1.155	1.192	1.228
0.005	1.002	1.005	1.009	1.042	1.082	1.120	1.157	1.194	1.230
0.010	1.003	1.006	1.010	1.043	1.083	1.121	1.159	1.195	1.232
0.050	1.013	1.016	1.020	1.054	1.094	1.133	1.171	1.209	1.246
0.100	1.026	1.030	1.033	1.067	1.108	1.148	1.187	1.225	1.263
0.150	1.040	1.043	1.047	1.082	1.124	1.165	1.204	1.243	1.281
0.200	1.054	1.057	1.061	1.097	1.140	1.182	1.222	1.262	1.301
0.250	1.069	1.072	1.076	1.113	1.157	1.200	1.241	1.282	1.321
0.300	1.084	1.088	1.092	1.130	1.175	1.219	1.261	1.302	1.342

γ=0.001

α\ω	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	1.001	1.006	1.012	1.058	1.113	1.166	1.217	1.267	1.316
0.005	1.002	1.007	1.013	1.059	1.115	1.168	1.219	1.269	1.318
0.010	1.003	1.008	1.014	1.061	1.116	1.170	1.221	1.271	1.320
0.050	1.014	1.018	1.024	1.071	1.128	1.182	1.234	1.285	1.335
0.100	1.027	1.031	1.037	1.085	1.143	1.198	1.252	1.303	1.354
0.150	1.040	1.045	1.051	1.100	1.159	1.216	1.270	1.323	1.374
0.200	1.054	1.059	1.065	1.116	1.176	1.234	1.290	1.343	1.395
0.250	1.069	1.074	1.080	1.132	1.194	1.253	1.310	1.365	1.418
0.300	1.085	1.090	1.096	1.150	1.213	1.274	1.332	1.387	1.441

APPENDIX C: the statistic $100(\overline{\gamma}^*(n)-\gamma)$ as function of α, ω and γ.

$\gamma=0.2$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.337	-0.270	-0.323	-0.270	-0.247	-0.310	-0.240	-0.217	-0.123
0.005	-0.353	-0.377	-0.373	-0.303	-0.220	-0.253	-0.150	-0.233	-0.137
0.010	-0.390	-0.380	-0.380	-0.313	-0.207	-0.243	-0.247	-0.240	-0.160
0.050	-0.403	-0.393	-0.347	-0.290	-0.277	-0.330	-0.450	-0.370	-0.237
0.100	-0.430	-0.413	-0.353	-0.343	-0.333	-0.310	-0.313	-0.287	-0.270
0.150	-0.363	-0.333	-0.303	-0.163	-0.230	-0.347	-0.380	-0.340	-0.310
0.200	-0.340	-0.357	-0.297	-0.347	-0.283	-0.313	-0.290	-0.333	-0.280
0.250	-0.330	-0.330	-0.340	-0.430	-0.407	-0.380	-0.417	-0.323	-0.273
0.300	-0.377	-0.407	-0.430	-0.403	-0.340	-0.327	-0.347	-0.340	-0.347

$\gamma=0.1$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.143	-0.077	-0.067	-0.123	0.070	0.120	0.223	0.220	0.297
0.005	-0.090	-0.037	0.003	-0.070	0.063	0.133	0.183	0.213	0.263
0.010	-0.073	-0.003	0.010	-0.007	0.063	0.087	0.100	0.140	0.287
0.050	-0.030	-0.050	-0.093	-0.077	0.003	0.017	0.153	0.233	0.187
0.100	-0.043	-0.077	-0.150	-0.123	-0.087	-0.017	0.083	0.077	0.150
0.150	-0.127	-0.127	-0.137	-0.217	-0.197	-0.083	-0.033	-0.033	0.057
0.200	-0.107	-0.127	-0.177	-0.250	-0.277	-0.213	-0.087	0.033	0.007
0.250	-0.180	-0.197	-0.190	-0.283	-0.377	-0.337	-0.110	-0.070	-0.037
0.300	-0.157	-0.157	-0.157	-0.283	-0.340	-0.333	-0.227	-0.183	-0.080

$\gamma=0.05$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.017	-0.067	-0.083	-0.043	0.040	0.090	0.080	0.177	0.273
0.005	-0.057	-0.087	-0.087	-0.010	0.027	0.053	0.077	0.187	0.257
0.010	-0.057	-0.097	-0.100	-0.037	0.047	0.033	0.090	0.150	0.240
0.050	-0.060	-0.063	-0.067	0.047	0.030	0.057	0.083	0.170	0.327
0.100	-0.057	-0.013	-0.003	-0.027	0.010	0.007	0.013	0.090	0.177
0.150	-0.023	-0.017	-0.027	0.003	-0.003	-0.077	-0.030	0.080	0.157
0.200	-0.000	-0.000	0.010	-0.120	-0.047	-0.150	-0.087	-0.067	0.113
0.250	-0.017	0.040	-0.003	-0.113	-0.073	-0.170	-0.110	-0.077	0.010
0.300	-0.007	-0.060	-0.080	-0.143	-0.133	-0.167	-0.143	-0.080	-0.013

$\gamma=0.02$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.063	-0.033	-0.057	-0.037	0.003	0.067	0.127	0.230	0.333
0.005	-0.040	-0.047	-0.043	-0.033	0.030	0.070	0.140	0.230	0.353
0.010	-0.073	-0.040	-0.030	-0.003	0.057	0.047	0.127	0.230	0.353
0.050	-0.010	-0.030	-0.027	-0.020	0.010	0.110	0.133	0.217	0.320
0.100	0.030	-0.003	0.000	-0.033	0.047	0.110	0.130	0.213	0.283
0.150	-0.017	-0.020	0.013	0.013	0.040	0.093	0.147	0.203	0.250
0.200	-0.073	-0.023	-0.017	0.010	0.077	0.113	0.127	0.167	0.217
0.250	-0.027	-0.040	-0.057	-0.027	0.037	0.080	0.100	0.160	0.180
0.300	-0.063	-0.097	-0.050	-0.030	0.010	0.033	0.097	0.150	0.137

$\gamma=0.01$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.080	-0.047	-0.060	-0.047	-0.017	0.060	0.120	0.163	0.223
0.005	-0.070	-0.063	-0.063	-0.060	-0.013	0.057	0.100	0.160	0.230
0.010	-0.090	-0.067	-0.053	-0.067	-0.017	0.050	0.097	0.153	0.227
0.050	-0.100	-0.107	-0.087	-0.047	0.020	0.050	0.087	0.163	0.223
0.100	-0.127	-0.083	-0.077	-0.030	0.027	0.070	0.130	0.177	0.193
0.150	-0.097	-0.077	-0.090	-0.050	0.013	0.103	0.150	0.183	0.207
0.200	-0.080	-0.083	-0.080	-0.043	-0.010	0.093	0.137	0.187	0.217
0.250	-0.077	-0.077	-0.107	-0.060	-0.003	0.077	0.080	0.160	0.193
0.300	-0.077	-0.093	-0.070	-0.073	-0.007	0.033	0.053	0.143	0.200

$\gamma=0.005$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.113	-0.093	-0.080	-0.040	0.007	0.023	0.060	0.090	0.123
0.005	-0.107	-0.100	-0.080	-0.043	0.003	0.040	0.087	0.090	0.130
0.010	-0.087	-0.100	-0.077	-0.037	0.013	0.050	0.073	0.103	0.127
0.050	-0.080	-0.090	-0.070	-0.047	-0.003	0.070	0.137	0.147	0.157
0.100	-0.083	-0.083	-0.063	-0.050	0.003	0.053	0.073	0.127	0.153
0.150	-0.077	-0.057	-0.060	-0.027	-0.017	0.037	0.050	0.100	0.140
0.200	-0.077	-0.073	-0.060	-0.057	-0.010	0.007	0.053	0.080	0.133
0.250	-0.057	-0.063	-0.057	-0.057	-0.050	-0.010	0.053	0.073	0.127
0.300	-0.043	-0.060	-0.060	-0.063	-0.063	-0.030	0.037	0.073	0.103

$\gamma=0.001$

$\alpha\backslash\omega$	0.001	0.005	0.010	0.050	0.100	0.150	0.200	0.250	0.300
0.001	-0.017	-0.007	-0.003	-0.013	0.010	0.010	0.027	0.060	0.077
0.005	-0.013	-0.010	-0.007	-0.007	0.007	0.013	0.023	0.050	0.073
0.010	-0.010	-0.010	-0.017	-0.007	0.003	0.013	0.010	0.047	0.073
0.050	-0.007	-0.020	-0.010	-0.000	0.007	0.017	0.023	0.067	0.090
0.100	-0.007	-0.010	-0.010	-0.003	0.003	0.013	0.033	0.057	0.093
0.150	-0.013	-0.023	-0.013	-0.010	-0.000	0.013	0.027	0.050	0.077
0.200	-0.013	-0.017	-0.023	0.007	0.003	0.003	0.023	0.050	0.063
0.250	-0.010	-0.013	-0.023	-0.010	-0.000	0.010	0.023	0.030	0.050
0.300	-0.013	-0.017	-0.023	-0.013	-0.007	-0.000	0.010	0.027	0.053

APPENDIX D: Derivation of $V(V_t)$.

The forecast update formula for the variance of the forecast error is given by

$$V_t = \omega E_t^2 + (1-\omega) V_{t-1} = \omega \sum_{i=0}^{\infty} (1-\omega)^i E_{t-i}^2 .$$

Because of stationarity, V_t and V_{t-1} are identically distributed. Using this update formula and $\sigma_V^2 = V(V_t) = V(V_{t-1})$ we may write:

$$\sigma_V^2 = 2\omega^2 \sigma_E^4 + (1-\omega)^2 \sigma_V^2 + 2\omega(1-\omega) C(E_t^2, V_{t-1}),$$

or

$$(2-\omega)\sigma_V^2 = 2\omega\sigma_E^4 + 2\omega \sum_{j=1}^{\infty} (1-\omega)^j C(E_t^2, E_{t-j}^2). \qquad \text{(a)}$$

Using the well-known result

$$E(Y_1 Y_2 Y_3 Y_4) = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}$$

for a multivariate normal distributed variable $Y = (Y_1, Y_2, Y_3, Y_4)$ with $E(Y) = 0$ and variance-covariance matrix $\Sigma = \{\sigma_{ij}\}_{i=1,..,4;\, j=1,..4}$, we may establish

$$C(E_t^2, E_{t-j}^2) = 2C^2(E_t, E_{t-j}). \qquad \text{(b)}$$

An expression for $C(E_t, E_{t-j})$ can be found in a few consecutive steps.

$$C(S_{t-i}, X_{t-j}) = \begin{cases} \alpha(1-\alpha)^{j-i}\sigma^2 & , j \geq i \\ 0 & , j < i \end{cases}$$

$$C(S_{t-i},S_{t-j})=\frac{\alpha}{2-\alpha}(1-\alpha)^{|i-j|}\sigma^2$$

$$C(S_{t-i},E_{t-j})=\begin{cases}\dfrac{\alpha}{2-\alpha}(1-\alpha)^{j-i}\sigma^2\ , & j\geq i\\ \dfrac{-\alpha}{2-\alpha}(1-\alpha)^{i-j-1}\sigma^2\ , & j<i\end{cases}$$

$$C(E_{t-i},X_{t-j})=\begin{cases}0\ , & j<i\\ \sigma^2\ , & j=i\\ -\alpha(1-\alpha)^{j-i-1}\sigma^2\ , & j>i\end{cases}$$

$$C(E_{t-i},E_{t-j})=\begin{cases}\dfrac{-\alpha}{2-\alpha}(1-\alpha)^{|i-j|-1}\sigma^2\ , & j\neq i\\ \dfrac{2}{2-\alpha}\sigma^2=\sigma_E^2\ , & j=i\end{cases}$$

Thus,

$$C(E_t,E_{t-j})=\frac{-\alpha}{2-\alpha}(1-\alpha)^{j-1}\sigma^2,\ \text{for } j\geq 1$$

and with (a) and (b) this gives

$$\sigma_V^2=\frac{\omega\sigma_E^4}{2-\omega}\left\{2+\frac{\alpha^2(1-\omega)}{1-(1-\omega)(1-\alpha)^2}\right\}.$$

Note that $\dfrac{\alpha^2(1-\omega)}{1-(1-\omega)(1-\alpha)^2}\leq\dfrac{1}{7}$ for $\alpha\leq 0.25$ so that σ_V^2 may be approximated by $\dfrac{2\omega\sigma_E^4}{2-\omega}$ in most practical cases.

Assuming $\mu = 0$ for simplicity, another result can easily be obtained:

$$C(X_t,\ X_s X_u) = 0\ (\forall_{s,t,u})\ \Rightarrow\ C(S_t,\ E_{t-i}^2) = 0\ (\forall_{i \geq 0})\ \Rightarrow\ C(S_t, V_t) = 0.$$

In other words, there is no correlation between the forecast and the variance of the forecast error.

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