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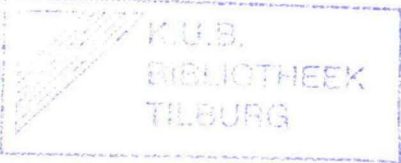
A QUADRATICALLY CONVERGENT PARALLEL  
JACOBI PROCESS FOR ALMOST DIAGONAL  
MATRICES WITH DISTINCT EIGENVALUES

M.H.C. Paardekooper

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A QUADRATICALLY CONVERGENT PARALLEL JACOBI PROCESS

FOR ALMOST DIAGONAL MATRICES WITH

DISTINCT EIGENVALUES

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A Quadratically Convergent Parallel Jacobi Process for Almost Diagonal Matrices with Distinct Eigenvalues.

Abstract

This paper discusses a generalization of the Jacobi process (1846). In each step  $\frac{1}{2}n$  pairs of non-diagonal elements are annihilated in an arbitrary almost diagonal matrix. We prove that the recursively constructed sequence of matrices converges to a diagonal matrix. The quadratically convergent method is adapted to parallel computation on an array processor.

## 1. INTRODUCTION

In the wellknown Jacobi algorithm [5] for the diagonalization of a real symmetric matrix each of the successive orthogonal similarity transformations is designed to annihilate a symmetric pair of off-diagonal elements. The ultimately quadratic convergence of this algorithm has been investigated by several authors [6,7,9,12].

In this paper we describe a natural, despite new generalization of that Jacobi process for the diagonalization of a non-normal almost diagonal matrix  $A \in \mathbb{C}^{n \times n}$  with distinct eigenvalues. We assume  $n$  to be even. In each iteration

$$A^{(k+1)} = S_k^{-1} A^{(k)} S_k, \quad k \in \mathbb{Z}, \quad (1.1)$$

with  $A^{(0)} := A$ ,  $\frac{1}{2}n$  pairs of symmetrically placed off-diagonal elements are annihilated. The transformation matrices  $S_k$  are not unitary. Consequently monotonic decrease of the Frobenius norms of the non-diagonal parts of  $A^{(k)}$  can not be guaranteed despite the annihilation of elements.

In [9] the same lack of monotonicity caused genuine difficulties in the proof of the quadratic convergence of the Eberlein algorithm [3,8] for the algebraic eigenproblem.

Each  $S_k$  is a direct sum of  $\frac{1}{2}n$  unimodular shears  $T_{i,k}$ ,  $i = 1, \dots, n/2$ ;

$$T_{i,k} = \begin{pmatrix} p_{i,k} & q_{i,k} \\ t_{i,k} & s_{i,k} \end{pmatrix} \begin{matrix} \leftarrow \ell(i,k) \\ \leftarrow m(i,k) \end{matrix} \quad (1.2)$$

$$\begin{matrix} \uparrow & \uparrow \\ \ell(i,k) & m(i,k) \end{matrix}$$

They annihilate  $\frac{1}{2}n$  pairs of the transformed matrix. In case of hermitian  $A$  each  $T_{i,k}$  is a unitary  $2 \times 2$  block. Our strategy for the pivots  $(\ell(i,k), m(i,k))$  aims to annihilate each off diagonal element exactly once in  $n-1$  successive steps:

$$\{(\ell(i,k), m(i,k)) \mid i=1, \dots, n/2, k=0, \dots, n-2\} = \{(i,j) \mid 1 \leq i < j \leq n\}. \quad (1.3)$$

If the difference of  $A^{(k)} = D^{(k)} + E^{(k)}$  and its diagonal part  $D^{(k)} = \text{diag}(a_{jj}^{(k)})$  is small relatively the separation of the eigenvalues, the transformation matrix  $S_k$  almost equals the unit matrix. Then the previously annihilated elements can increase only slowly. That phenomenon gives rise to the quadratic convergence:

$$\|E^{(n-1)}\|_F \leq \text{constant} \|E^{(0)}\|_F^2. \quad (1.4)$$

The ideas of Brent and Luk [2] for the solution of the symmetric eigenproblem on a systolic array can be carried over to our method. Further may be remarked that our procedure can be used as final stage in Sameh's (slowly convergent) parallel normreducing eigenvalue algorithm [10], in order to obtain a quadratically convergent process.

The pivots of  $T_{ik}$  are  $(\ell(i,k), m(i,k))$ ,  $i = 1, \dots, n/2$ , with  $\ell(i,k) m(i,k)$  and  $\ell(i,k) < \ell(j,k)$  for  $i < j$ . (1.5)

The caterpillar permutation  $K_n$  [2] generates a pivot strategy that annihilates each off-diagonal element exactly once in a sweep of  $\frac{1}{2} n(n-1)$  shear transformations. These are performed in  $n-1$  steps. Permutation  $K_n$  can be read off from figure 1.



fig. 1 Caterpillar permutation  $K_n$ .

If  $b^{(0)} = (1, 2, \dots, n)^T$ , then

$$(K_n b^{(0)})_i = \begin{cases} i+2 & , & 2 \leq i = 2k \leq n-2 \\ i-2 & , & 5 \leq i = 2k-1 \leq n-1 \\ i-1 & , & i = 3, n \\ i & , & i = 1. \end{cases} \quad (1.6)$$

With  $b^{(k)} = K_n^k b^{(0)}$  we have

$$\{\ell(i,k), m(i,k)\} = \{b_{2i-1}^{(k)}, b_{2i}^{(k)}\}, \quad i = 1, \dots, \frac{1}{2}n. \quad (1.7)$$

Figure 2 gives the pivot pairs  $(\ell(i,k), m(i,k))$  in the caterpillar order for  $n = 6$

$k \backslash i$	1	2	3
0	(1,2)	(3,4)	(5,6)
1	(1,4)	(2,6)	(3,5)
2	(1,6)	(2,3)	(4,5)
3	(1,5)	(2,4)	(3,6)
4	(1,3)	(2,5)	(4,6)

fig. 2

For the proof of the quadratic convergence of the cyclic generalized Jacobi-process the rule (1.3) is essential.

For the description of the process we divide  $A^{(k)}$  into a diagonal and a nondiagonal part  $D^{(k)}$  and  $E^{(k)}$  respectively.

$$E^{(k)} = A^{(k)} - D^{(k)}, \quad d_{ij}^{(k)} = \begin{cases} a_{ij}^{(k)}, & i = j \\ 0 & , i \neq j. \end{cases} \quad (1.8)$$

We set

$$\epsilon_k = \| E^{(k)} \|_F \quad (1.9)$$

and

$$\vartheta_k = \min_{i \neq j} |a_{ii}^{(k)} - a_{jj}^{(k)}|. \quad (1.10)$$

The initial matrix  $A$  is almost diagonal namely

$$\epsilon_0 \leq \eta/10, \quad (1.11)$$



where  $\eta$  denotes the separation of the spectrum of A:

$$\eta = \min_{i \neq j} |\lambda_i - \lambda_j|.$$

Condition (1.11) implies [1] that each eigenvalue  $\lambda_i$  is affiliated to exactly one diagonal element, say  $a_{ii}$

$$|\lambda_i - a_{ii}| \leq \epsilon_0 \quad (1.12)$$

Consequently  $|a_{ii} - a_{jj}| \geq \eta - 2\epsilon_0 \geq 8\epsilon_0$ . Thus

$$\epsilon_0 \leq \eta/8. \quad (1.13)$$

The organisation of our paper is as follows. In section 2 we present the parallel annihilators together with their properties and effects during the first step. In section 3 we describe the consequences of the annihilations in a complete sweep. In that section the main result is attained:

if  $n \geq 7$  and  $\epsilon_0/\eta \leq (10n)^{-1}$  then  $\epsilon_{n-1}/\eta \leq 2n(\epsilon_0/\eta)^2 \leq \frac{1}{5} \epsilon_0/n$ .

Section 4 discusses conclusions and some unsolved problems.

## 2. PARALLEL ANNIHILATORS. THE FIRST STEP

Without loss of generality we may assume that

$$(\ell(i,0), m(i,0)) = (2i-1, 2i), \quad i = 1, \dots, \frac{1}{2}n$$

as in the caterpillar strategy. Now we consider the annihilation in the first step,  $k = 0$ , with the annihilators  $T_{i,0}$ . For reasons of simplicity the subscript  $k = 0$ , referring to the first step will be omitted in the formulae of this section.

The shear transformation  $T_i$  only affects the elements in the  $(2i-1)$ -th and the  $2i$ -th row and column.

The Jacobi-parameters of  $T_i$  are  $p_i, q_i, r_i, s_i$ :

$$T_i = \begin{bmatrix} p_i & q_i \\ r_i & s_i \end{bmatrix}, \quad p_i s_i - q_i r_i = 1, \quad i = 1, \dots, \frac{1}{2}n. \quad (2.1)$$

Let matrix  $A$  be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1, \frac{1}{2}n} \\ A_{21} & A_{22} & \cdots & A_{2, \frac{1}{2}n} \\ \vdots & \vdots & & \vdots \\ \cdot & \cdot & & \cdot \\ A_{\frac{1}{2}n, 1} & A_{\frac{1}{2}n, 2} & \cdots & A_{\frac{1}{2}n, \frac{1}{2}n} \end{bmatrix} = (A_{ij}), \quad (2.2)$$

where each  $2 \times 2$  submatrix  $A_{ij}$  is given by

$$A_{ij} = \begin{bmatrix} a_{2i-1, 2j-1} & a_{2i-1, 2j} \\ a_{2i, 2j-1} & a_{2i, 2j} \end{bmatrix}, \quad i, j = 1, \dots, \frac{1}{2}n. \quad (2.3)$$

For the  $2 \times 2$  blocks  $A_{ij}^{(1)}$  of  $A^{(1)} = S^{-1}AS$ , where  $S = \text{diag}(T_i)$ , we find

$$A_{ij}^{(1)} = T_i^{-1} \otimes T_j^T A_{ij}. \quad (2.4)$$

Especially for the diagonal block  $A_{ii}^{(1)}$ :

$$\begin{pmatrix} a_{2i-1,2i-1}^{(1)} \\ a_{2i-1,2i}^{(1)} \\ a_{2i,2i-1}^{(1)} \\ a_{2i,2i}^{(1)} \end{pmatrix} = \begin{pmatrix} s_i p_i & s_i r_i & -q_i p_i & -q_i r_i \\ s_i q_i & s_i^2 & -q_i^2 & -q_i s_i \\ -r_i p_i & -r_i^2 & p_i^2 & p_i r_i \\ -r_i q_i & -r_i s_i & p_i q_i & p_i s_i \end{pmatrix} \begin{pmatrix} a_{2i-1,2i-1} \\ a_{2i-1,2i} \\ a_{2i,2i-1} \\ a_{2i,2i} \end{pmatrix}. \quad (2.5)$$

In order to simplify the formulae we set

$$\sigma_i = a_{2i,2i-1}, \mu_i = a_{2i-1,2i}, \nu_i = a_{2i-1,2i-1} - a_{2i,2i}, \quad i=1, \dots, \frac{1}{2}n. \quad (2.6)$$

The annihilation of both  $\sigma_i$  and  $\mu_i$  requires the solution of a quadratic equation with discriminant  $\nu_i^2 + 4\sigma_i\mu_i$ .

With (1.13) we get  $|2\sigma_i\mu_i| \leq \epsilon^2 \leq \theta^2/64 \leq |\nu_i|^2/64$ . Thus

$$|1 + 4\sigma_i\mu_i \nu_i^{-2}| \geq 31/32. \quad (2.7)$$

Let be

$$F_i := (1 + 4\sigma_i\mu_i\nu_i^{-2})^{\frac{1}{2}}, \quad \text{Re}(F_i) > 0, \quad 1 \leq i \leq \frac{1}{2}n.$$

Lowerbound (2.7) enables the formulation of

THEOREM 2.1 The shears

$$T_i = \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} \begin{pmatrix} (\frac{1}{2} + \frac{1}{2F_i})^{\frac{1}{2}} & \frac{-\mu_i}{\nu_i(F_i(1+F_i)/2)^{\frac{1}{2}}} \\ \frac{\sigma_i}{\nu_i(F_i(1+F_i)/2)^{\frac{1}{2}}} & (\frac{1}{2} + \frac{1}{2F_i})^{\frac{1}{2}} \end{pmatrix}, \quad i=1, \dots, n/2 \quad (2.8)$$

annihilate both  $\sigma_i$  and  $\mu_i$ . Such a shear  $T_i$  is called the annihilator, related to the pivots in question.

PROOF. With (2.11) we derive that  $a_{2i,2i-1}^{(1)} = 0$  and  $a_{2i-1,2i}^{(1)} = 0$  if

$$\begin{bmatrix} p_i \\ q_i \end{bmatrix} = \alpha_i \begin{bmatrix} (1+F_i)/2 \\ \sigma_i \nu_i^{-1} \end{bmatrix}, \quad \begin{bmatrix} r_i \\ s_i \end{bmatrix} = \beta_i \begin{bmatrix} -\mu_i \nu_i^{-1} \\ (1+F_i)/2 \end{bmatrix}, \quad \alpha_i, \beta_i \in \mathbb{C} \setminus \{0\}.$$

The condition  $p_i s_i - q_i r_i = 1$  implies  $\alpha_i \beta_i = 2(F_i + F_i^2)^{-1}$ . With  $\alpha_i = \beta_i = (2/(F_i + F_i^2))^{\frac{1}{2}}$  we get (2.15).  $\square$

In the sequel we need the estimates in

THEOREM 2.2. If  $\epsilon \leq \eta/10$  then the following inequalities hold for  $T_i$ ,  $i = 1, \dots, \frac{1}{2}n$ .

$$(a) \quad |p_i|^2 \leq 1 + (\epsilon/\eta)^2 \leq 1 + (\epsilon/\vartheta)^2; \quad (2.9)$$

$$(b) \quad |F_i + F_i^2|^{-1} \leq \frac{1}{2} + \frac{5}{4} \epsilon^2/\eta^2; \quad (2.10)$$

$$(c) \quad \|T_i\|_2^2 \leq 1 + 2\epsilon/\eta \leq \frac{6}{5}. \quad (2.11)$$

$$(d) \quad |(F_i + F_i^2)/2|^{-\frac{1}{2}} \leq 1 + \frac{6}{5} \epsilon^2/\eta^2. \quad (2.12)$$

$$\begin{aligned} \text{PROOF (a). } |p_i|^2 &= \left| \frac{1}{2} + \frac{1}{2} (1+4\sigma_i \nu_i^{-2})^{-\frac{1}{2}} \right| \leq \frac{1}{2} + \frac{1}{2} (1-2\epsilon^2/\vartheta^2)^{-\frac{1}{2}}/\eta^{-2} \\ &\leq \frac{1}{2} + \frac{1}{2} (1-2(\epsilon/\eta)^2(1-2\epsilon/\eta)^{-2})^{-\frac{1}{2}} \end{aligned}$$

since  $\vartheta \geq \eta - 2\epsilon$ . If  $|x| < 1/10$  then  $(1-2x^2(1-2x)^{-2})^{-\frac{1}{2}} \leq 1 + 2x^2$ .

$$\text{Hence } |p_i|^2 \leq 1 + \epsilon^2/\eta^2 \leq 1 + \epsilon^2/\vartheta^2.$$

$$\begin{aligned} (b) \quad |F_i + F_i^2|^{-1} &= \left| 1+4\sigma_i \mu_i \nu_i^{-2} + (1+4\sigma_i \mu_i \nu_i^{-2})^{\frac{1}{2}} \right|^{-1} \\ &\leq (1-2x^2(1-2x)^{-2} + (1-2x^2(1-2x)^{-2})^{\frac{1}{2}})^{-1} \end{aligned}$$

where as above  $x = \epsilon/\eta$ . Simple but tedious calculations affirm

$$(1-2x^2(1-2x)^{-2} + (1-2x^2)(1-2x)^{-2})^{\frac{1}{2}}^{-1} \leq \frac{1}{2} + \frac{5}{4} x^2, \quad |x| \leq 1/10.$$

(c) Let be  $x_i = |p_i|^2 + |q_i|^2$ ,  $y_i = |r_i|^2 + |s_i|^2$ ,  $z = p\bar{r}_i + q\bar{s}_i$ . Then

$$T_i T_i^* = \begin{bmatrix} x_i & z_i \\ \bar{z}_i & y_i \end{bmatrix}$$

and  $x_i y_i - |z_i|^2 = 1$  as follows from the unimodularity of annihilator  $T_i$ .

So  $x_i + y_i = ((x_i - y_i)^2 + 4|z_i|^2 + 4)^{\frac{1}{2}} \geq 2$  and

$$\|T_i\|_2^2 = (x_i + y_i + ((x_i - y_i)^2 - 4)^{\frac{1}{2}}) / 2 \quad (2.13)$$

With (2.8) we get

$$\begin{aligned} x_i + y_i - 2 &= 2\left(\left|\frac{1}{2} + \frac{1}{2} F_i^{-1}\right| - 1\right) + 2(|\sigma_i|^2 + |\mu_i|^2) / |\nu_i^2(F_i + F_i^2)| \\ &\leq |(1 - F_i) / F_i| + 2\epsilon^2 / |\nu_i^2(F_i + F_i^2)| \\ &\leq (4|\sigma_i \mu_i| + 2\epsilon^2) / |\nu_i^2(F_i + F_i^2)| \leq (2 + 5\epsilon^2 / \eta^2) \epsilon^2 / \theta^2 \end{aligned}$$

as follows from (2.10). Hence

$$\|T_i\|_F^2 = x_i + y_i \leq 2 + 2.05 \epsilon^2 / \theta^2 \quad (2.14)$$

From (2.13) and (2.14) it follows with easy calculations that

$$\|T_i\|_2^2 \leq 1 + \frac{8}{5} \epsilon / \theta \leq 1 + \frac{8}{5} \epsilon \eta^{-1} (1 - 2\epsilon / \eta)^{-1} \leq 1 + 2 \epsilon / \eta \leq \frac{6}{5}$$

for  $\epsilon \leq \eta / 10$ .

(d) The derivation of (2.12) is similar to that of (2.10).  $\square$

An upperbound for the growth of norm of the nondiagonal part after the first step is given in

THEOREM 2.3. If  $\epsilon < \eta/10$ , then

$$\epsilon_1 \leq (1+2\epsilon/\eta) \epsilon \leq \frac{6}{5} \epsilon. \quad (2.15)$$

PROOF. The  $2 \times 2$  blocks in the partitioned  $A^{(1)}$  are

$$A_{ij}^{(1)} = T_i^{-1} A_{ij} T_j, \quad i, j = 1, \dots, \frac{n}{2}.$$

Since  $a_{2i-1, 2i}^{(1)} = a_{2i, 2i-1}^{(1)} = 0$ ,  $i = 1, \dots, \frac{n}{2}$ , with (2.11) we find

$$\begin{aligned} \epsilon_1^2 &= \sum_{i \neq j} \|A_{ij}^{(1)}\|_F^2 \leq \sum_{i \neq j} \|T_i^{-1}\|_2^2 \|T_j\|_2^2 \|A_{ij}\|_F^2 \\ &\leq \sum_{i \neq j} \|T_i\|_2^2 \|T_j\|_2^2 \|A_{ij}\|_F^2 \leq (1+2\epsilon/\eta)^2 \epsilon^2. \quad \square \end{aligned}$$

According to (1.12)  $a_{ii}^{(0)}$  is affiliated with  $\lambda_i$  and

$$|a_{ii}^{(0)} - \lambda_i| \leq \epsilon_0.$$

This affiliation is not changed by the annihilating transformation. This will be proved in

THEOREM 2.4. If  $\epsilon < \eta/10$  then the affiliation of the eigenvalues remains unchanged by the annihilating transformation

$$A^{(1)} = S^{-1} A S$$

where  $S = \text{diag}(T_i)$  with  $T_i$  given by (2.8)

PROOF. From (2.5) and (2.8) one gets

$$\begin{aligned} |a_{2i-1, 2i-1}^{(1)} - a_{2i-1, 2i-1}^{(0)}| &= |2 \sigma_i \mu_i \nu_i^{-1} (1+F_i)^{-1}| \\ &\leq \epsilon^2 |\nu_i|^{-2} (1+(1-2\epsilon^2 |\nu_i|^{-2})^{\frac{1}{2}})^{-1} |\nu_i| \\ &\leq (\epsilon/\theta)^2 (1+(1-2(\epsilon/\theta)^2)^{\frac{1}{2}})^{-1} |\nu_i| \leq |\nu_i|/126, \end{aligned} \quad (2.16)$$

for  $\epsilon/\theta \leq 1/8$ . A change of affiliation only can be an exchange of affiliation between two elements at say the  $i$ -th shear transformation. Assume  $a_{2i-1,2i-1}$  and  $a_{2i-1,2i-1}^{(1)}$  to be affiliated with  $\lambda_{2i-1}$  and  $\lambda_{2i}$  respectively. Then as follows from (2.15)

$$|a_{2i-1,2i-1} - \lambda_{2i-1}| \leq \epsilon \quad , \quad |a_{2i-1,2i-1}^{(1)} - \lambda_{2i}| \leq \frac{6}{5} \epsilon .$$

By (2.16)

$$\begin{aligned} |a_{2i-1,2i-1}^{(1)} - a_{2i-1,2i-1}| &\leq \frac{1}{126} |\nu_i| \\ &\leq \frac{1}{126} (|\lambda_{2i-1} - \lambda_{2i}| + 2\epsilon) . \end{aligned} \tag{2.17}$$

The change of affiliation implies

$$|a_{2i-1,2i-1}^{(1)} - a_{2i-1,2i-1}| \geq |\lambda_{2i-1} - \lambda_{2i}| - \frac{11}{5} \epsilon . \tag{2.18}$$

Consequently we get with (2.17) and (2.18)

$$|\lambda_{2i-1} - \lambda_{2i}| - \frac{11}{5} \epsilon \leq \frac{1}{126} |\lambda_{2i-1} - \lambda_{2i}| + \frac{1}{63} \epsilon$$

Thus  $\epsilon > \frac{625}{1396} |\lambda_{2i-1} - \lambda_i| \geq \frac{2}{5} \eta$ . This contradicts (1.11). □

## 3. THE EFFECT OF A COMPLETE SWEEP

For the estimates of the norm  $\epsilon_k$  of the nondiagonal part  $E^{(k)}$  of  $A^{(k)}$ ,  $k = 0, \dots, n-1$ , we make use of the conclusion based on theorem 2.3: if  $\epsilon_k/\eta \leq \frac{1}{10}$  then

$$\epsilon_{k+1} \leq (1+2\epsilon_k/\eta)\epsilon_k. \quad (3.1)$$

In the first theorem we proof that a sufficient small  $\epsilon_0$  guarantees a slow growth of  $\epsilon_k$ .

THEOREM 3.1. If  $\epsilon_0 \leq \frac{1}{10} \eta n^{-1}$  then

$$\epsilon_k \leq \frac{1}{10} \eta(n-k)^{-1}, \quad k = 0, \dots, n-1. \quad (3.2)$$

PROOF. Assume  $\epsilon_k/\eta \leq \frac{1}{10} (\eta-k)^{-1}$ . Then, as follows from (3.1)

$$\epsilon_{k+1} \eta^{-1} \leq (1+2\epsilon_k \eta^{-1}) \epsilon_k \eta^{-1} \leq \frac{1}{10} (n-k)^{-1} (1 + \frac{1}{5}(n-k)^{-1}).$$

It is easy to verify that

$$\frac{1}{10} (n-k)^{-1} (1 + \frac{1}{5} (n-k)^{-1}) \leq \frac{1}{10} (n-k-1)^{-1}. \quad \square$$

The transform  $A^{(k+1)}$  is obtained from  $A^{(k)}$  in two stages:

$$\begin{cases} A^{(k+\frac{1}{2})} = S_k^{-1} A^{(k)} = H^{(k)} A^{(k)} + B^{(k)}, & (3.3) \\ A^{(k+1)} = A^{(k+\frac{1}{2})} S_k = A^{(k+\frac{1}{2})} H^{(k)} + C^{(k)} & (3.4) \end{cases}$$

where  $H^{(k)}$  denotes the diagonal part of both  $S_k$  and  $S_k^{-1}$ .

Thus by (1.2)

$$\begin{cases} b_{\ell(i,k),j}^{(k)} = -q_{i,k} a_{m(i,k),j}^{(k)}, \quad b_{m(i,k),j}^{(k)} = -r_{i,k} a_{\ell(i,k),j}^{(k)}, \quad i = 1, \dots, n/2, & (3.5) \end{cases}$$

$$\begin{cases} c_{j,\ell(i,k)}^{(k)} = r_{i,k} a_{j,m(i,k)}^{(k+\frac{1}{2})}, \quad c_{j,m(i,k)}^{(k)} = q_{i,k} = q_{i,k} a_{j,\ell(i,k)}^{(k+\frac{1}{2})}, \quad j = 1, \dots, n. & (3.6) \end{cases}$$



Let be

$$G^{(k)} = B^{(k)} H^{(k)} + C^{(k)} . \quad (3.7)$$

Then

$$A^{(k+1)} = H^{(k)} A^{(k)} H^{(k)} + G^{(k)} . \quad (3.8)$$

According to (3.8) the updating  $A^{(k)} \rightarrow A^{(k+1)}$  comes about by two actions: a multiplication by two diagonal elements of  $S_k$  followed by the addition of  $g_{i,j}^{(k)}$ . The multipliers  $H^{(k)}$  are rather tame as can be seen in

THEOREM 3.2. If  $\epsilon_0 \leq \frac{1}{10} \eta/n$  then

$$\prod_{k=2}^{n-2} \|H^{(k)}\|_2^2 \leq 155/154 .$$

PROOF. With (2.9) and (3.2), we get

$$\begin{aligned} \prod_{k=2}^{n-2} \|H^{(k)}\|_2^2 &\leq \prod_{k=2}^{n-2} (1 + \epsilon_k^2 / \eta^2) \leq \prod_{k=2}^{n-2} (1 + (n-k)^{-2} / 100) \\ &\leq \prod_{k=2}^{\infty} (1 + k^{-2} / 100) \leq 155/154 \end{aligned}$$

for  $\frac{1}{100} \sum_{k=2}^{\infty} k^{-2} = (\pi^2/6-1)/100$  and  $\exp(\pi^2/6-1)/100 \leq 155/154$ .  $\square$

The twofold action (3.8) for the construction of  $A^{(k+1)}$  is the starting point for the description on the history of the element  $a_{i,j}^{(n-1)}$ ,  $i \neq j$ , after its annihilation in some step. Let the element in the position  $(i,j)$  be annihilated in step  $N(i,j) - 1 = N-1$ . Then

$$a_{i,j}^{(N)} = 0 \quad , \quad a_{i,j}^{(N+1)} = g_{i,j}^{(N)} \quad , \quad i \neq j. \quad (3.9)$$

and

$$a_{i,j}^{(k+1)} = h_{i,i}^{(k)} h_{j,j}^{(k)} a_{i,j}^{(k)} + g_{i,j}^{(k)} \quad , \quad N+1 \leq k \leq n-2. \quad (3.10)$$

So

$$|a_{i,j}^{(k+1)}| \leq \|H^{(k)}\|_2^2 |a_{i,j}^{(k)}| + |g_{i,j}^{(k)}|$$

and consequently

$$\begin{aligned} |a_{i,j}^{(n-1)}| &\leq \sum_{k=N}^{n-2} \prod_{m=k+1}^{n-2} \|H^{(m)}\|_2^2 |g_{i,j}^{(k)}| \\ &\leq \frac{155}{154} \sum_{k=N}^{n-2} |g_{i,j}^{(k)}| \quad , i \neq j , \end{aligned} \quad (3.11)$$

as follows from theorem 3.1.

The notational problems in the analysis of the parallel cyclic Jacobi-process are reduced with

DEFINITION 3.3. Let be  $V = (v_{i,j}) \in \mathbb{C}^{n \times n}$ . Then  $\hat{V}^{(k)} = (\hat{v}_{i,j}^{(k)})$  and  $\tilde{V}^{(k)} = (\tilde{v}_{i,j}^{(k)})$ ,  $k = 0, \dots, n-2$ , are defined by

$$\hat{v}_{i,j}^{(k)} = \begin{cases} 0 & , i = j \text{ or } \{i,j\} \in \{\{\ell(i,k), m(i,k)\} \mid 0 \leq i \leq n/2\} \\ v_{i,j} & , (i,j) \text{ otherwise,} \end{cases} \quad (3.12)$$

$$\tilde{v}_{i,j}^{(k)} = \begin{cases} v_{i,j} & , i \neq j \wedge k > N(i,j) \\ 0 & , \text{ otherwise,} \end{cases} \quad (3.13)$$

$$\delta_k(V) := \|\hat{V}^{(k)}\|_F . \quad (3.14)$$

□

Figure 3 illustrates the pattern of zeros in the successive matrices  $\tilde{G}^{(k)}$ ,  $k=1, \dots, n-2=4$ . The □ indicates the position of an element annihilated in a preceding step. The corresponding elements in  $\tilde{G}^{(k)}$  contribute according (3.8) to  $a_{ij}^{(n-1)}$ .

$$\tilde{G}^{(1)} = \begin{pmatrix} 0 & \square & 0 & 0 & 0 & 0 \\ \square & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \square & 0 & 0 \\ 0 & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & 0 & \square & 0 \end{pmatrix}; \tilde{G}^{(2)} = \begin{pmatrix} 0 & \square & 0 & \square & 0 & 0 \\ \square & 0 & 0 & 0 & 0 & \square \\ 0 & 0 & 0 & \square & \square & 0 \\ \square & 0 & \square & 0 & 0 & 0 \\ 0 & 0 & \square & 0 & 0 & \square \\ 0 & \square & 0 & 0 & \square & 0 \end{pmatrix};$$

$$\tilde{G}^{(3)} = \begin{pmatrix} 0 & \square & 0 & \square & 0 & \square \\ \square & 0 & \square & 0 & 0 & \square \\ 0 & \square & 0 & \square & \square & 0 \\ \square & 0 & \square & 0 & \square & 0 \\ 0 & 0 & \square & \square & 0 & \square \\ \square & \square & 0 & \square & 0 & 0 \end{pmatrix}; \tilde{G}^{(4)} = \begin{pmatrix} 0 & \square & \square & \square & 0 & \square \\ \square & 0 & \square & 0 & \square & \square \\ \square & \square & 0 & \square & \square & 0 \\ 0 & \square & \square & \square & 0 & \square \\ 0 & \square & \square & \square & 0 & \square \\ \square & \square & 0 & \square & \square & 0 \end{pmatrix}$$

Fig. 3. The matrices  $\tilde{G}^{(k)}$ ,  $k=1, \dots, n-2=4$ , in case of the caterpillar cyclic order.

Evidently  $|\tilde{G}^{(k)}| \leq |\hat{G}^{(k)}|$ . Our notation allows to reformulate (3.11) in matricial form:

$$|E^{(n-1)}| \leq \frac{155}{154} \sum_{k=1}^{n-2} |\tilde{G}^{(k)}| \leq \frac{155}{154} \sum_{k=1}^{n-2} |\hat{G}^{(k)}|.$$

Thus

$$\epsilon_{n-1} \leq \frac{155}{154} \sum_{k=1}^{n-2} \|\hat{G}^{(k)}\|_F. \quad (3.15)$$

In the proof of the quadratic convergence an essential role has been reserved for

THEOREM 3.4. Let be  $G^{(k)}$  as defined in (3.7). If  $\epsilon_0/\eta \leq (10n)^{-1}$ , then

$$\|\hat{G}^{(k)}\|_F \leq \frac{84}{83} \left(1 + \frac{7}{10} (n-k)^{-1}\right) \epsilon_k^2/\eta \quad (3.16)$$

PROOF. Remark that  $\|\hat{G}^{(k)}\|_F \leq \|P^{(k)}\|_2 \|\hat{B}^{(k)}\|_F + \|\hat{C}^{(k)}\|_F$ . Firstly we estimate  $\|\hat{C}^{(k)}\|_F$ . By (3.6) and (3.12)

$$\begin{aligned} \|\hat{C}^{(k)}\|_F &\leq \sum_{i=1}^{n/2} \left( \left( \sum_{j=1}^{n(i)} |c_{j,\ell(i,k)}^{(k)}|^2 \right)^{\frac{1}{2}} + \left( \sum_{j=1}^{n(i)} |c_{j,m(i,k)}^{(k)}|^2 \right)^{\frac{1}{2}} \right) \\ &\leq \sum_{i=1}^{n/2} \left( |r_{i,k}| \left( \sum_{j=1}^{n(i)} |a_{j,m(i,k)}^{(k+\frac{1}{2})}|^2 \right)^{\frac{1}{2}} + |q_{i,k}| \left( \sum_{j=1}^{n(i)} |a_{j,\ell(i,k)}^{(k+\frac{1}{2})}|^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

where  $\sum_{j=1}^{n(i)}$  denotes summation for all  $j$  except  $j = \ell(i,k), m(i,k)$ .

With (2.8) it follows that

$$\|\hat{C}^{(k)}\|_F \leq \sum_{i=1}^{n/2} \rho_{i,k} \left( |\sigma_{i,k}| \left( \sum_{j=1}^{n(i)} |a_{j,m(i,k)}^{(k+\frac{1}{2})}|^2 \right)^{\frac{1}{2}} + |\mu_{i,k}| \left( \sum_{j=1}^{n(i)} |a_{j,\ell(i,k)}^{(k+\frac{1}{2})}|^2 \right)^{\frac{1}{2}} \right)$$

where  $\sigma_{i,k} = a_{m(i,k),\ell(i,k)}^{(k)}$ ,  $\mu_{i,k} = a_{\ell(i,k),m(i,k)}^{(k)}$  and

$$\rho_{i,k} = |\nu_{i,k}^2 (F_{i,k} + F_{i,k}^2)^2|^{-\frac{1}{2}}.$$

Now  $|\nu_{i,k}|^{-1} \leq \eta^{-1} (1 - 2\epsilon_k/\eta)^{-1} \leq \eta^{-1} (1 - (n-k)^{-1}/5)^{-1} \leq (1 + (n-k)^{-1}/4)/\eta$  for

$n - k \geq 2$ . (2.12) and (3.2) imply  $|(F_{i,k} + F_{i,k}^2)/2|^{-\frac{1}{2}} \leq 1 + \frac{6}{5} \epsilon_k^2/\eta^2 \leq 1.012$ .

Hence

$$\begin{aligned} \|\hat{C}^{(k)}\|_F &\leq 0.506 (1 + (n-k)^{-1}/4) \left( \zeta_k^2 (A^{(k+\frac{1}{2})}) + \sum_{i=2}^{n/2} (|\sigma_{ik}|^2 + |\mu_{ik}|^2)/\eta \right) \\ &\leq \frac{42}{83} (1 + (n-k)^{-1}/4) \|S_k\|_2^2 \epsilon_k^2/\eta \leq \frac{42}{83} (1 + (n-k)^{-1}/4) (1 + (n-k)^{-1}/5) \epsilon_k^2/\eta \end{aligned}$$

as follows from (2.11) and (3.2). Thus

$$\|\hat{C}^{(k)}\|_F \leq \frac{42}{83} (1 + (n-k)^{-1}/2) \epsilon_k^2/\eta.$$

In a similar way one gets, now with (2.9):

$$\begin{aligned} \delta_k(B^{(k)}P^{(k)}) &\leq \|\hat{B}^{(k)}\|_F \|\mathbb{H}^{(k)}\|_2 \leq \frac{42}{83} (1+(n-k)^{-1}/4) (1+(n-k)^{-2}/100) \epsilon_k^2/\eta \\ &\leq \frac{42}{83} (1+3(n-k)^{-1}/8) \epsilon_k^2/\eta . \end{aligned}$$

Hence

$$\|\hat{G}^{(k)}\|_F \leq \frac{84}{83} (1+7(n-k)^{-1}/10) \epsilon_k^2/\eta . \quad \square$$

THEOREM 3.5. If  $\epsilon_0/\eta \leq (10n)^{-1}$  then

$$\epsilon_{n-1} \leq \frac{54}{53} \sum_{k=1}^{n-2} (1 + \frac{7}{10} (n-k)^{-1}) \prod_{j=0}^{k-1} (1+(n-j)^{-1}/5)^2 \epsilon_0^2/\eta . \quad (3.17)$$

PROOF. By (3.15) and (3.16)

$$\epsilon_{n-1} \leq \frac{155}{154} \frac{84}{83} \sum_{k=1}^{n-2} (1 + \frac{7}{10} (n-k)^{-1}) \epsilon_k^2/\eta . \quad (3.18)$$

According to theorem 3.1  $\epsilon_k/\eta \leq 1/10$  for  $k = 0, 1, \dots, n-1$  if  $\epsilon_0/\eta \leq \frac{1}{10} \eta^{-1}$ . Hence by (3.1) and (3.2),

$$\epsilon_{k+1} \leq (1+(n-k)^{-1}/5) \epsilon_k, \quad k = 0, \dots, n-2$$

Thus

$$\epsilon_k \leq \prod_{j=0}^{k-1} (1+(n-j)^{-1}/5) \epsilon_0, \quad k = 0, \dots, n-1 . \quad (3.19)$$

So (3.17) is a consequence of (3.18) and (3.19). □

LEMMA 3.6. Let be

$$t_n = \sum_{k=1}^{n-2} (1 + \frac{7}{10} (n-k)^{-1}) \prod_{j=0}^{k-1} (1+(n-j)^{-1}/5)^2$$

Then for  $n \geq 3$

$$t_n \leq 11n/6 + 1$$

PROOF. It is easy to see that  $t_n$  satisfies the recurrence relation

$$t_{n+1} = (1 + \frac{1}{5} (n+1)^{-1})^2 (t_n + 7n^{-1}/5) .$$

With starting value  $t_3 = \frac{27}{20} (16/15)^2$  one verifies that

$t_n \leq \frac{11}{6} n + 1$ ,  $n = 3, 4, \dots, 9$ . Now assume  $t_n \leq 11n/6 + 1$ . Then

$$t_{n+1} \leq (1 + \frac{1}{5} (n+1)^{-1})^2 (\frac{11}{6} n + 2 + 7n^{-1}/5) \leq 11(n+1)/6 + 1$$

for  $n \geq 9$ . □

This lemma together with theorem 3.6 enables us to finish the proof of the quadratic convergence as formulated in

**THEOREM 3.7.** If  $\epsilon_0/\eta \leq (10n)^{-1}$  then

$$\epsilon_{n-1} \leq c_n \epsilon_0^2/\eta \tag{3.20}$$

where  $c_n = \frac{54}{53} (11n/6 + 1)$ . So

$$\epsilon_{k(n-1)} \leq (c_n/\eta)^{2^{k-1}} \epsilon_0^{2^k} .$$

PROOF. (3.17) and (3.18) give

$$\epsilon_{n-1} \leq \frac{54}{53} (11n/6 + 1) \epsilon_0^2/\eta < \frac{1}{5} \epsilon_0$$

since  $\epsilon_0/\eta \leq (10n)^{-1}$ . This guarantees the quadratic convergence. □

## 4. CONCLUSIONS

It goes without saying that our proof shares important features with the proof of the analogous problem for real symmetric matrices [6,7,11,12]. As in the paper of Ruhe [9] the complexity of the proof rises from the non-unitary transformations. These hinder the monotonic decrease of the norm of the non-diagonal parts  $E^{(k)}$  of the successive matrices  $A^{(k)}$ .

The proof is restricted to the case of almost diagonal matrices in the sense that  $\|E^{(0)}\|_F$  is small relatively  $\eta = \min_{i \neq j} |\lambda_i - \lambda_j|$ , the separation of  $A$ 's spectrum.

The method clarifies that in the final stage of a norm reducing eigenvalue algorithm the annihilation procedure suffices for the convergence of the process and can be executed in stead of the complicated and time consuming normreducing shear transforms in both the Eberlein as in the Sameh algorithm [3,8,14].

The long sequence of worst case estimates brings about that despite their stepwise sharpness even the final bound (3.20) underestimates the convergence quite considerably. The uniform lowerbound  $\eta(1-2\epsilon_k/\eta) \leq \vartheta_k$  for all  $|a_{ii}^{(k)} - a_{jj}^{(k)}|$  especially leads to a pessimistic result.

A numerical result illustrates that the condition  $\epsilon_0/\eta \leq (10n)^{-1}$  is far too strong as appeared in many test cases. The matrix is of order 6.  $A = D + E$  with

$$D = \text{diag}(1,36,29,22,8,15), \quad e_{ij} = 2.8(j-i) . \quad (4.1)$$

The eigenvalues of A are 3.416218, 9.792723, 15.704854, 21.295146, 27.207277 and 33.583782. So  $\eta = 5.590292$ .

k	$\epsilon_k$	k	$\epsilon_k$	k	$\epsilon_k$
0	<u>15.336223</u>	7	3.647763	14	$0.973630 \cdot 10^{-3}$
1	22.305149	8	2.061459	15	<u><math>0.624204 \cdot 10^{-4}</math></u>
2	19.654710	9	0.698883	16	$0.631612 \cdot 10^{-5}$
3	15.030077	10	<u>0.257238</u>	17	$0.816325 \cdot 10^{-7}$
4	11.300451	11	$0.907285 \cdot 10^{-1}$	18	$0.544994 \cdot 10^{-8}$
5	<u>5.509465</u>	12	$0.595277 \cdot 10^{-1}$	19	$0.476108 \cdot 10^{-10}$
6	4.941476	13	$0.470223 \cdot 10^{-2}$	20	<u><math>0.473872 \cdot 10^{-15}</math></u>

Fig. 4. The sequence  $\epsilon_k$  with caterpillar cyclic order.

Figure 4 illustrates that the convergence occurs despite the large  $\epsilon_0/\eta$ . There rest interesting researchproblems. Firstly how to modify the algorithm in case of multiple eigenvalues. The papers [4,13] give useful information for the structure of an almost diagonal matrix with multiple eigenvalues. It seems reasonable to skip the shear transform  $T_{ik}$  if the corresponding diagonal elements of  $A^{(k)}$  are affiliated with equal eigenvalues.

The criterion, based on  $\min_{i \neq j} |a_{i,i}^{(k)} - a_{j,j}^{(k)}|$  and  $\epsilon_k$ , for the transfer from the normreduction in the Sameh process to the annihilation procedure is an important question for future research [14].



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