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RESEARCH MEMORANDUM

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CENTRAL LIMIT THEOREMS FOR SEQUENCES WITH M(N)-DEPENDENT MAIN PART

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Summary. Let $(X_i^{(n)};\ n \in \mathbb{N},\ 1 \le i \le h(n))$ be a double sequence of random variables with $h(n) \to \infty$ as $n \to \infty$. Suppose that the sequence can be split into two parts: an m(n)-dependent sequence $(X_{i,m(n)};\ n \in \mathbb{N},\ 1 \le i \le h(n))$ of main terms and a sequence $(\bar{X}_{i,m(n)};\ n \in \mathbb{N},\ 1 \le i \le h(n))$ of residual terms. Here $(m(n))$ may be unbounded in $\mathbb{N}$. Adding some conditions, especially on the residual terms, we consider central limit theorems for $(X_i^{(n)})$ based on a theorem for m(n)-dependent sequences. The results are of special interest when score functions are involved, for instance in rank-based procedures.

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1. Introduction; statement of the model

A *(triangular) array* $(Y_i^{(n)})$ in $\mathbb{R}$ is a double sequence $(Y_i^{(n)};\ n \in \mathbb{N},\ 1 \le i \le h(n))$ of random variable (rv's) in $\mathbb{R}$ with $h(n) \to \infty$ as $n \to \infty$. It will be assumed that the rv's which belong to the same row of the triangle are defined on a common probability space. The array is called *m(n)-dependent* if for all $n \in \mathbb{N}$ and $k \in \{2,\dots, h(n)-m(n)\}$ the random vectors $(Y_i^{(n)};\ 1 \le i \le k-1)$ and $(Y_j^{(n)};\ m(n)+k \le j \le h(n))$ are independent. Here $(m(n))$ is a sequence in $\mathbb{N}_0$.
In Berk (1973) a central limit theorem for m(n)-dependent arrays (possibly with (m(n)) unbounded) is proved. We are especially interested in the limit behavior of arrays for which only a main part is m(n)-dependent.

A triangular array $(X_i^{(n)})$ is said to have an m(n)-dependent main part $(X_{i,m(n)})$ and a residual part $(\bar{X}_{i,m(n)})$ if

(1.1) $\quad X_i^{(n)} = X_{i,m(n)} + \bar{X}_{i,m(n)}$ for all $n \in \mathbb{N}$ and $i \in \{1,\dots,h(n)\}$

and

(1.2) $\quad (X_{i,m(n)})$ is m(n)-dependent.

If, moreover,

(1.3) $\quad \max\limits_{1\le i\le h(n)} \mathbf{P}[|\bar{X}_{i,m(n)}| \ge \varepsilon] \to 0$ for all $\varepsilon > 0$,

then the array $(X_i^{(n)})$ will be called *decomposable*. In most part of this research it will even be assumed that $(X_i^{(n)})$ satisfies an $(\varepsilon(n), \delta(n))$-condition, i.e. that (m(n)) can be chosen such that (at least for $n \ge n_0$)

(1.4) $\quad \max\limits_{1\le i\le h(n)} \mathbf{P}[|\bar{X}_{i,m(n)}| \ge \varepsilon(n)] \le \delta(n)$

for some sequences $(\varepsilon(n))$ and $(\delta(n))$ in $(0,\infty)$ tending to 0 as $n \to \infty$.
Decomposability was first introduced in Chanda, Puri & Ruymgaart (1989). The concept was defined in this reference under the additional condition (not assumed in the present research) that the $X_i^{(n)}$, as well as the $X_{i,m(n)}$, are identically distributed.

In the above reference linear processes (e.g. ARMA-processes) and (non-linear) processes with Volterra expansions of a given finite order (Priestley (1981)) are mentioned as examples of decomposable processes which (with certain assumptions) satisfy an $(\varepsilon(n), \delta(n))$-condition. In the next example we will consider special decomposable processes generated by a given decomposable process.

Example 1.1. For each $n \in \mathbb{N}$ let $X_1,\dots,X_n$ be n successive observations from a decomposable time series (X_i) satisfying an $(\varepsilon(n), \delta(n))$-condition for some sequences $(\varepsilon(n))$ and $(\delta(n))$. Suppose that the X_i are identically distributed with continuously differentiable distribution function F (independent of n) with bounded derivative f. The rv's $X_{1,m(n)},\dots,X_{n,m(n)}$ have a common d.f. $F_{m(n)}$. Set $\xi_{i,m(n)} := F(X_{i,m(n)})$, $i \in \{1,\dots,n\}$. By the mean value theorem we obtain:

$$\xi_i = \xi_{i,m(n)} + \bar{X}_{i,m(n)} f(X_{i,m(n)} + \vartheta_n \bar{X}_{i,m(n)}).$$

Here $\vartheta_n \in (0,1)$ is random. So, (ξ_i) is also decomposable with residual part given by $\bar{\xi}_{i,m(n)} := \bar{X}_{i,m(n)} f(X_{i,m(n)} + \vartheta_n \bar{X}_{i,m(n)})$. Note that $|\bar{\xi}_{i,m(n)}| \le A|\bar{X}_{i,m(n)}|$ w.p.1. Hence, (ξ_i) satisfies the $(\tilde{\varepsilon}(n), \tilde{\delta}(n))$-condition with $\tilde{\varepsilon}(n) := A\varepsilon(n)$ and $\tilde{\delta}(n) := \delta(n)$.

Let, furthermore, $R_{i,n}$ be the rank of X_i in the sample $X_1,\dots,X_n$. Decompose the array $(R_{i,n}/n)$ as follows:

$$\hat{\Gamma}_n(\xi_i) = \frac{R_{i,n}}{n} = \xi_{i,m(n)} + \left[\frac{R_{i,n}}{n} - \xi_{i,m(n)}\right].$$

Here $\hat{\Gamma}_n$ is the empirical distribution function of $\xi_1,\dots,\xi_n$. Let U_n with $U_n(t) := \sqrt{n}\,(\hat{\Gamma}_n(t) - t)$, $t \in [0,1]$, be the (reduced) empirical process and define

$$\Omega_n := [\max_{1\le i\le n} |\bar{X}_{i,m(n)}| \le \varepsilon(n)].$$

Then $\mathbf{P}(\Omega_n^c) \le n\delta(n)$. If $\varepsilon'(n)$ is chosen such that $\varepsilon(n)/\varepsilon'(n) \to 0$, then we have for $n \ge n_0$:

$$P\left[\left|\frac{R_{i,n}}{n} - \xi_{i,m(n)}\right| \geq \varepsilon'(n)\right] \leq$$

$$\leq P[|\hat{\Gamma}_n(\xi_i) - \xi_i| \geq \frac{1}{2}\varepsilon'(n)] + P[|\bar{\xi}_{i,m(n)}| \geq \frac{1}{2}\varepsilon'(n)]$$

$$\leq P[\sup_{0\leq t\leq 1} |U_n(t)| \geq \frac{1}{2}\sqrt{n}\,\varepsilon'(n)] + \delta(n)$$

$$\leq P(\Omega_n \cap [\sup_{0\leq s<t\leq 1} |U_n(t) - U_n(s)| \geq \frac{1}{2}\sqrt{n}\,\varepsilon'(n)] + (n+1)\delta(n)$$

$$=: \mathcal{P}_n + (n+1)\delta_n.$$

According to Nieuwenhuis & Ruymgaart (1989; Th. 2.1) there exists an exponential upper bound for $\mathcal{P}_n$, provided that $\delta(n)/\varepsilon'(n) \to 0$, $\varepsilon(n)/\varepsilon'(n) \to 0$ and (X_i) is a linear process. The generalization of this theorem to more general decomposable processes is, however, straightforward and will not be proved here. Consequently,

(1.5) $\quad \mathcal{P}_n \leq Cm(n)\exp(-A\sigma_n\psi(B\tau_n))$,

if $\delta(n)/\varepsilon'(n) \to 0$ and $\varepsilon(n)/\varepsilon'(n) \to 0$. Here $\sigma_n := n(\varepsilon'(n))^2/m(n)$, $\tau_n := \varepsilon'(n)$ and ψ is some decreasing and continuous function on $[-1,\infty)$ for which $\psi(x) \downarrow 0$ as $x \uparrow \infty$. (cf. Shorack & Wellner (1986)). Next assume that $\varepsilon(n) = \mathcal{O}(n^{-\eta})$, $\delta(n) = \mathcal{O}(n^{-\eta_1})$, $\varepsilon'(n) = \mathcal{O}(n^{-\eta'})$, $\delta'(n) = \mathcal{O}(n^{-\eta_1'})$ and $m(n) = [cn^{\rho}]$ for some $c \in (0,\infty)$ and $\rho \in (0,1)$. Then the upper bound in (1.5) tends to 0 exponentially fast provided that $0 < \eta' < \frac{1}{2} - \frac{1}{2}\rho$. Consequently, the array $(R_{i,n}/n)$ is decomposable and satisfies the $(\varepsilon'(n), \delta'(n))$-condition if η' and η_1' are such that $0 < \eta' < \min\{\eta, \frac{1}{2} - \frac{1}{2}\rho\}$ and $0 < \eta_1' < \eta_1 - 1$. It should also be assumed here that $0 < \rho < 1$ and $\eta_1 > 1$. This decomposability of $(R_{i,n}/n)$ might be interesting. Unfortunately the condition that η' should be between 0 and $\frac{1}{2} - \frac{1}{2}\rho$ is rather strong. □

According to Chanda, Puri & Ruymgaart (1989) decomposability might be an alternative to the classical mixing concepts. The definition in (1.1) - (1.3) is such that it might provide a useful model to many practical situations, especially when (1.4) is also assumed.

That is why it is worthwhile to consider the asymptotic behavior of decomposable processes. In this monograph conditions will be presented which guarantee asymptotic normality of the partial sums (if suitably standardized) of such processes. Berk's theorem for m(n)-dependent rv's will be the guide to this research.

In Section 2 some classical results will be generalized. Berk's CLT (central limit theorem) for m(n)-dependent rv's and a well-known CLT for linear processes generated by an i.i.d. sequence (cf. e.g. Anderson (1971; Th. 7.7.8)) are generalized to a CLT for arrays with m(n)-dependent main part. It is proved that, when compared to the CLT for linear processes, the resulting theorem is an *almost* generalization, since it is additionally needed that the $(2+\delta)$-th absolute moment of the generating rv's exists.

In Section 3 things are simplified by considering decomposable arrays with bounded residual part satisfying an $(\varepsilon(n), \delta(n))$-condition. A CLT for such arrays is proved. Some remarks are made about asymptotic normality for the partial sums of arrays like $(f_n(X_i^{(n)}))$ or $(f_n(X_i^{(n)})g_n(Y_i^{(n)}))$ if $(X_i^{(n)})$ and $(Y_i^{(n)})$ are decomposable and satisfy an $(\varepsilon(n), \delta(n))$-condition. Here f_n and g_n are functions for which Lipschitz-conditions hold.

Some applications are considered in Section 4. The first example is about a stationary, decomposable sequence (X_i) of Un(0,1) rv's satisfying an $(\varepsilon(n), \delta(n))$-condition. It is assumed that (X_1, X_{1+h}) is positively quadrant dependent in the sense of Lehmann (1966). A central limit result can be derived for the partial sums of an array $(J_n(X_i))$. Here (J_n) is some sequence of (score) functions. Some special, *not* strongly mixing, sequence with Un(0,1) marginals is considered, which satisfies the above conditions. In the second example asymptotic normality of a class of serial rank statistics is studied. Result of Nieuwenhuis & Ruymgaart (1989) are considered within the scope of the present research.

Remark. Throughout this paper A, A', B, B', C, C' $\in (0,\infty)$ will be used as generic constants. They are independent of all the relevant parameters (like e.g. the sample size n). Expressions in n are sometimes valid for $n \geq n_0$ only, without mention. Here n_0 does not depend on the relevant parameters either.

2. Generalization of some classical results

Limit theorems for linear processes are classical and well-known (Marsaglia (1954), Parzen (1957), Anderson (1971; Th. 7.7.8), Brockwell & Davies (1987; Prop. 6.3.10)). They have been a motivation to the present research.

Theorem 2.1. *Let (X_i) be the two-sided moving average*

$$(2.1) \qquad X_i = \sum_{k\in\mathbb{Z}} g_k Z_{i-k}, \quad i \in \mathbb{Z},$$

where (Z_j) is a sequence of independently and identically distributed rv's with $E(Z_j) = 0$ and $E(Z_j^2) = \sigma^2$. Suppose further that $\Sigma_{k\in\mathbb{Z}}|g_k| < \infty$. Then $\Sigma_{i=1}^n X_i/\sqrt{n}$ has a limiting normal distribution with mean 0 and variance $\Sigma_{k\in\mathbb{Z}}\sigma(k)$, where $\sigma(k) = \sigma^2\Sigma_{s\in\mathbb{Z}} g_s g_{s+k}$.

This theorem is usually proved by splitting X_i into two terms (cf. e.g. Anderson (1971)):

$$X_i = \sum_{|k|\le\frac{1}{2}m} g_k Z_{i-k} + \sum_{|k|>\frac{1}{2}m} g_k Z_{i-k} =: X_{i,m} + \bar{X}_{i,m}, \quad i \in \mathbb{Z},$$

where $m \in \mathbb{N}$ is fixed, i.e. does not depend on n, the number of observations. The resulting sequence $(X_{i,m})_{i\in\mathbb{Z}}$ is m-dependent.
Relation (1.1) is a generalization of this idea, apart from the fact that here m depends on n. So, it is natural to consider the limit behavior of arrays $(X_i^{(n)})$ with m(n)-dependent main part. At first we need a limit theorem for m(n)-dependent sequences (Berk (1973)).

Theorem 2.2. *Let $(Y_i^{(n)})$ be a triangular array of random variables with $h(n) \to \infty$ as $n \to \infty$. Suppose that this array is m(n)-dependent and is standardized such that $Var(\Sigma_{i=1}^{h(n)} Y_i^{(n)}) \to 1$. Assume further that*

$$\text{(a)} \qquad \max_{1\le i\le h(n)} \mathbb{E}|Y_i^{(n)}|^{2+\delta} = O\left[\frac{1}{h(n)^{1+\delta/2}}\right] \text{ and } \frac{m(n)^{2+2/\delta}}{h(n)} \to 0$$

for some $\delta > 0$;

(b) $$\max_{i<j\le h(n)} \frac{1}{j-i} Var\left[\sum_{k=i+1}^{j} Y_k^{(n)}\right] = \mathcal{O}\left[\frac{1}{h(n)}\right] .$$

Then

$$\sum_{i=1}^{h(n)} (Y_i^{(n)} - \mathbb{E}Y_i^{(n)}) \to_d N(0,1) \text{ as } n \to \infty.$$

The above theorem can straightforwardly be extended to a limit theorem for sequences with m(n)-dependent main part.

Theorem 2.3. *Suppose that the array* $(X_i^{(n)})$ *has an* $m(n)$*-dependent main part* $(X_{i,m(n)})$ *and a residual part* $(\bar{X}_{i,m(n)})$. *Set* $b_n^2 := Var(\Sigma_{i=1}^{h(n)} X_i^{(n)})$. *Assume that*

(a) $$\max_{1\le i\le h(n)} \frac{\mathbb{E}|X_i^{(n)}|^{2+\delta}}{b_n^{2+\delta}} = \mathcal{O}\left[\frac{1}{h(n)^{1+\delta/2}}\right],$$

$$\max_{1\le i\le h(n)} \frac{\mathbb{E}|\bar{X}_{i,m(n)}|^{2+\delta}}{b_n^{2+\delta}} = \mathcal{O}\left[\frac{1}{h(n)^{1+\delta/2}}\right], \text{ and}$$

$$\frac{m(n)^{2+2/\delta}}{h(n)} \to 0 \text{ for some } \delta > 0;$$

(b) $$\max_{i<j\le h(n)} \frac{1}{(j-i)b_n^2} Var\left[\sum_{k=i+1}^{j} X_k^{(n)}\right] = \mathcal{O}\left[\frac{1}{h(n)}\right] \text{ and}$$

$$\max_{i<j\le h(n)} \frac{1}{(j-i)b_n^2} Var\left[\sum_{k=i+1}^{j} \bar{X}_{k,m(n)}\right] = o\left[\frac{1}{h(n)}\right] .$$

Then

(2.2) $$\frac{1}{b_n} \sum_{i=1}^{h(n)} (X_i^{(n)} - \mathbb{E}X_i^{(n)}) \to_d N(0,1) \text{ as } n \to \infty.$$

Proof. Since

$$(2.3)\quad \frac{1}{b_n}\sum_{i=1}^{h(n)}(X_i^{(n)} - \mathbb{E}X_i^{(n)}) = \frac{1}{b_n}\sum_{i=1}^{h(n)}(X_{i,m(n)} - \mathbb{E}X_{i,m(n)}) + $$
$$+ \frac{1}{b_n}\sum_{i=1}^{h(n)}(\bar{X}_{i,m(n)} - \mathbb{E}\bar{X}_{i,m(n)})$$

and

$$\mathbf{P}\left[\sum_{i=1}^{h(n)}(\bar{X}_{i,m(n)} - \mathbb{E}\bar{X}_{i,m(n)}) \geq \varepsilon b_n\right] \leq \frac{\mathrm{Var}\left[\sum_{i=1}^{h(n)}\bar{X}_{i,m(n)}\right]}{\varepsilon^2 b_n^2} \to 0,$$

a consequence of Assumption (b), we only have to prove the asymptotic normality of the first part on the right in (2.3). So, we want to apply Theorem 2.2 to the double sequence $(X_{i,m(n)}/b_n)$. Note that

$$(2.4)\quad \mathrm{Var}\left[\sum_{i=1}^{h(n)}\frac{X_{i,m(n)}}{b_n}\right] = \frac{1}{b_n^2}\mathrm{Var}\left[\sum_{i=1}^{h(n)}X_i^{(n)}\right] + o(1) +$$
$$- \frac{2}{b_n^2}\mathrm{Cov}\left[\sum_{i=1}^{h(n)}X_i^{(n)},\ \sum_{j=1}^{h(n)}\bar{X}_{j,m(n)}\right].$$

The absolute value of the covariance-term in (2.4) is dominated by

$$\left[\frac{\mathrm{Var}\left[\sum_{i=1}^{h(n)}X_i^{(n)}\right]}{b_n^2}\cdot\frac{\mathrm{Var}\left[\sum_{i=1}^{h(n)}\bar{X}_{i,m(n)}\right]}{b_n^2}\right]^{1/2} = \left[\frac{1}{b_n^2}\mathrm{Var}\left[\sum_{i=1}^{h(n)}\bar{X}_{i,m(n)}\right]\right]^{1/2},$$

which tends to o by Assumption (b). Hence (2.4) tends to 1. Condition (b) in Theorem 2.2 for $Y_k^{(n)} = X_{k,m(n)}/b_n$ follows by similar arguments, while (a) is an immediate consequence of Minkovski's inequality:

$$\mathbb{E}|X_{i,m(n)}/b_n|^{2+\delta} \leq \left\{(\mathbb{E}|X_i^{(n)}/b_n|^{2+\delta})^{1/(2+\delta)} + \right.$$

$$+ \left(\mathbb{E}|\bar{X}_{i,m(n)}/b_n|^{2+\delta}\right)^{1/(2+\delta)}\Big]^{2+\delta} .$$

Application of Theorem 2.2 yields Relation (2.2). □

It would be nice to find out that Theorem 2.3 is indeed a generalization of Theorem 2.1. Unfortunately, it is only an 'almost' generalization. In fact, Theorem 2.1 is a corollary of Theorem 2.3 if it is additionally assumed that $E(Z_1^{2+\delta}) < \infty$ for some $\delta > 0$. To prove this observation we choose δ as above and $m(n)$ such that $m(n)^{2+2/\delta}/n \to 0$. Set $X_i^{(n)} := n^{-1/2} X_i$, where (X_i) is the observed linear process. Note that $b_n^2 = n^{-1}\mathrm{Var}(\Sigma_{i=1}^n X_i)$

and that

$$\frac{1}{j-i} \mathrm{Var}\left[\sum_{k=1}^{j-i} X_k\right] = \sum_{h=-(j-i-1)}^{j-i-1} \left[1 - \frac{|h|}{j-i}\right]\sigma(h),$$

where

$$\sigma(h) := \mathrm{Cov}(X_1, X_{1+h}) = \sigma^2 \sum_{k\in\mathbb{Z}} g_k g_{k+h}.$$

Hence $(j-i)^{-1}\mathrm{Var}(\Sigma_{k=1}^{j-i} X_k) \to \Sigma_{h\in\mathbb{Z}}\sigma(h) = \sigma^2(\Sigma_{k\in\mathbb{Z}} g_k)^2$ as $j-i \to \infty$.

Write $X_i = Y_{i,m(n)} + \bar{Y}_{i,m(n)}$, where

$$Y_{i,m(n)} := \Sigma_{|k|\le m(n)/2}\, g_k Z_{i-k} \text{ and } \bar{Y}_{i,m(n)} := X_i - Y_{i,m(n)}.$$

Set $\bar{X}_{i,m(n)} := n^{-1/2}\, \bar{Y}_{i,m(n)}$. Then

$$\frac{1}{j-i} \mathrm{Var}\left[\sum_{k=1}^{j-i} \bar{Y}_{k,m(n)}\right] \le \sum_{h\in\mathbb{Z}} |\mathrm{Cov}(\bar{Y}_{1,m(n)}, \bar{Y}_{1+h,m(n)})|$$

$$\le \sigma^2 \sum_{h\in\mathbb{Z}} \sum_{|k|>m(n)/2} \sum_{\substack{|\ell|>m(n)/2 \\ \ell=k+h}} |g_k||g_\ell|$$

$$= \sigma^2\Big(\sum_{|k|>m(n)/2} |g_k|\Big)^2 \to 0 \text{ as } n \to \infty,$$

uniformly in i and j with $i < j \le n$.
From these observations it can easily be deduced that $(X_i^{(n)})$ and $(\bar{X}_{i,m(n)})$ satisfy Assumption (b) in Theorem 2.3. The relations in (a) are also fulfilled, since $n^{-1}\mathrm{Var}(\Sigma_{i=1}^{n} X_i)$ tends to some constant > 0 (see above) and since (Minkovski's inequality):

$$\mathbf{E}|Y_{i,m(n)}|^{2+\delta} \le \Big(\sum_{|k|\le m(n)/2} |g_k|\Big)^{2+\delta}\, \mathbf{E}|Z_1|^{2+\delta},$$

$$\mathbf{E}|\bar{Y}_{i,m(n)}|^{2+\delta} \le [(\mathbf{E}|X_i|^{2+\delta})^{1/(2+\delta)} + (\mathbf{E}|Y_{i,m(n)}|^{2+\delta})^{1/(2+\delta)}]^{2+\delta}$$

and (Fatou's lemma)

$$\mathbf{E}|X_i|^{2+\delta} \le \liminf_{n\to\infty} \mathbf{E}|Y_{i,m(n)}|^{2+\delta}$$

$$\le \Big(\sum_{k\in\mathbb{Z}} |g_k|\Big)^{2+\delta}\, \mathbf{E}|Z_1|^{2+\delta} < \infty.$$

3. CLT's for decomposable arrays with bounded residual part

In Section 2 it was not explicitly assumed that $\bar{X}_{i,m(n)} \to 0$ in probability. From now on we will do so. In fact we even assume that $(X_i^{(n)})$ is decomposable and that it satisfies an $(\varepsilon(n), \delta(n))$-condition (1.4) for some sequences $(\varepsilon(n))$ and $(\delta(n))$. We intend to simplify Theorem 2.3 under additional assumptions on the process $(X_i^{(n)})$.
At first we assume that $s(n) > 0$ exist such that

$$\max_{1\le i\le h(n)} |\bar{X}_{i,m(n)}| \le s(n) \text{ wp1.} \tag{3.1}$$

As a consequence of (3.1) we obtain for $\gamma > 0$:

$$\mathbf{E}|\bar{X}_{i,m(n)}|^{\gamma} = \int_0^{\infty} \mathbf{P}[|\bar{X}_{i,m(n)}|^{\gamma} > x]dx = \int_0^{\varepsilon(n)} + \int_{\varepsilon(n)^{\gamma}}^{s(n)^{\gamma}}$$

$$\le \varepsilon(n)^{\gamma} + s(n)^{\gamma}\delta(n), \tag{3.2}$$

uniformly in $i \in \{1,\dots,h(n)\}$.

Theorem 3.1. *Suppose that $(X_i^{(n)})$ is decomposable and satisfies an $(\varepsilon(n),\ \delta(n))$-condition with*

$$(3.3)\qquad \frac{h(n)\varepsilon(n)}{b_n} \to 0 \quad \text{and} \quad \frac{h(n)^2 s(n)^2 \delta(n)}{b_n^2} \to 0,$$

where $b_n^2 := Var(\Sigma_{i=1}^{h(n)} X_i^{(n)})$. Assume that (3.1) is fulfilled and that

$$\text{(a)}\qquad \max_{1\le i\le h(n)} \frac{\mathbb{E}|X_i^{(n)}|^{2+\delta}}{b_n^{2+\delta}} = \mathcal{O}\left(\frac{1}{h(n)^{1+\delta/2}}\right), \quad \frac{m(n)^{2+2/\delta}}{h(n)} \to 0 \text{ and}$$

$$\frac{h(n)s(n)^2\delta(n)^{2/(2+\delta)}}{b_n^2} = \mathcal{O}(1) \text{ for some } \delta > 0;$$

$$\text{(b)}\qquad \max_{i<j\le h(n)} \frac{1}{(j-i)b_n^2}\, Var\left[\sum_{k=i+1}^{j} X_k^{(n)}\right] = \mathcal{O}\left(\frac{1}{h(n)}\right).$$

Then

$$\frac{1}{b_n}\sum_{i=1}^{h(n)} (X_i^{(n)} - \mathbb{E}X_i^{(n)}) \to_d N(0,1) \text{ as } n \to \infty.$$

Proof. By Relation (3.2) we have:

$$\frac{h(n)^{1+\delta/2}\mathbb{E}|\bar{X}_{i,m(n)}|^{2+\delta}}{b_n^{2+\delta}} \le C\, \frac{h(n)^{1+\delta/2}(\varepsilon(n)^{2+\delta} + s(n)^{2+\delta}\delta(n))}{b_n^{2+\delta}} = \mathcal{O}(1),$$

$$\frac{h(n)}{(j-i)b_n^2}\, \mathrm{Var}\left[\sum_{k=i+1}^{j} \bar{X}_{k,m(n)}\right] \le \frac{h(n)}{(j-i)b_n^2}\, \mathbb{E}\left[\sum_{k=i+1}^{j} \bar{X}_{k,m(n)}\right]^2,$$

$$\le \frac{h(n)}{b_n^2}\,(j-i)(\varepsilon(n)^2 + s(n)^2\delta(n))$$

$$\leq C\,\frac{h(n)^2}{b_n^2}\,(\varepsilon(n)^2 + s(n)^2\delta(n)) \to 0.$$

Apply Theorem 2.3. □

In many relevant cases liminf $b_n^2/h(n) > 0$, cf. Ibragimov & Linnik (1971; Th. 18.2.1)). Then Condition (3.3) and the last part of (a) are satisfied if $\varepsilon(n) = \mathcal{O}(h(n)^{-\eta})$ and $\delta(n) = \mathcal{O}(h(n)^{-\eta_1})$ with $\eta > \frac{1}{2}$ and η_1 such that $h(n)^{1-\eta_1}s(n)^2 \to 0$.

Next consider a sequence of functions (f_n) for which $\sigma \in \mathbb{R}$ exists such that

$$(3.4)\qquad |f_n(x) - f_n(y)| \leq Ah(n)^{\sigma}|x-y|$$

uniformly in x, y $\in \mathbb{R}$. Note that this inequality is a Lipschitz condition. Continuously differentiable functions f_n with derivative bounded by $Ah(n)^{\sigma}$ satisfy this condition.

Suppose that $(X_i^{(n)})$ is decomposable with $X_i^{(n)} = X_{i,m(n)} + \bar{X}_{i,m(n)}$ and satisfies the $(\varepsilon(n), \delta(n))$-condition. The array $(f_n(X_i^{(n)}))$ can be decomposed as follows:

$$f_n(X_i^{(n)}) = f_n(X_{i,m(n)}) + \bar{Y}_{i,m(n)},$$

where $\bar{Y}_{i,m(n)} := f_n(X_i^{(n)}) - f_n(X_{i,m(n)})$. By (3.4) we obtain:

$$P[|\bar{Y}_{i,m(n)}| \geq \tilde{\varepsilon}(n)] \leq P[|\bar{X}_{i,m(n)}| \geq A^{-1}h(n)^{-\sigma}\tilde{\varepsilon}(n)]$$

$$\leq \delta(n) = \tilde{\delta}(n),$$

if $\tilde{\varepsilon}(n) := A\varepsilon(n)h(n)^{\sigma}$ and $\tilde{\delta}(n) := \delta(n)$. So, $(f_n(X_i^{(n)}))$ is decomposable and satisfies the $(\tilde{\varepsilon}(n), \tilde{\delta}(n))$-condition if $(\varepsilon(n))$ and σ are such that $\varepsilon(n)h(n)^{\sigma} \to 0$ as $n \to \infty$.

The following corollary follows immediately from Theorem 3.1.

Corollary 3.2. *Let (f_n) be a sequence of functions for which (3.4) holds. Let $(X_i^{(n)})$ be decomposable, satisfying an $(\varepsilon(n), \delta(n))$-condition with $(\varepsilon(n))$ and $(\delta(n))$ such that $h(n)^\sigma\varepsilon(n) \to 0$,*

$$(3.5) \qquad \frac{h(n)^{1+\sigma}\varepsilon(n)}{b_n} \to 0 \quad \text{and} \quad \frac{h(n)^2 t(n)^2 \delta(n)}{b_n^2} \to 0,$$

where $b_n^2 := Var(\Sigma_{i=1}^{h(n)} f_n(X_i^{(n)})$ and $(t(n))$ is such that

$$(3.6) \qquad |f_n(X_i^{(n)}) - f_n(X_{i,m(n)})| \le t(n) \text{ wp1}.$$

Assume also that

$$\text{(a)} \qquad \max_{1\le i\le h(n)} \frac{\mathbb{E}|f_n(X_i^{(n)})|^{2+\delta}}{b_n^{2+\delta}} = \mathcal{O}\left[\frac{1}{h(n)^{1+\delta/2}}\right], \qquad \frac{m(n)^{2+2/\delta}}{h(n)} \to 0 \text{ and}$$

$$\frac{h(n)t(n)^2\delta(n)^{2/(2+\delta)}}{b_n^2} = \mathcal{O}(1) \text{ for some } \delta > 0;$$

$$\text{(b)} \qquad \max_{i<j\le h(n)} \frac{1}{(j-i)b_n^2} Var\left[\sum_{k=i+1}^{j} f_n(X_k^{(n)})\right] = \mathcal{O}\left[\frac{1}{h(n)}\right].$$

Then

$$\frac{1}{b_n}\sum_{i=1}^{h(n)} (f_n(X_i^{(n)}) - \mathbb{E}f_n(X_i^{(n)})) \to_d N(0,1) \text{ as } n \to \infty.$$

In view of Condition (3.5) and the last part of (a) it is desirable to choose σ and $t(n)$ as small as possible.
If the residual part of $(X_i^{(n)})$ satisfies (3.1), then (3.6) is fulfilled with $t(n) = Ah(n)^\sigma s(n)$. If each of the functions $|f_n|$ is bounded by a positive number v_n, then (3.6) is also fulfilled (with $t(n) = 2v_n$).

Theorem 3.2 can be generalized in several ways by considering more decomposable arrays and/or more sequences of functions satisfying (3.4). For instance, suppose that apart from (f_n) there is another sequence (g_n) of functions satisfying (3.4) (with σ and A replaced by τ and A'). Suppose also that $|f_n(x)| \le Bh(n)^{\sigma_0}$ and $|g_n(y)| \le B'h(n)^{\tau_0}$ for all $x,y \in \mathbb{R}$. Then we have for $x,y,a,b \in \mathbb{R}$:

$$(3.7)\quad \begin{aligned} |f_n(x)g_n(y) - f_n(a)g_n(b)| &\le |f_n(x)g_n(y) - f_n(x)g_n(b)| + \\ &\quad + |f_n(x)g_n(b) - f_n(a)g_n(b)| \\ &\le Ch(n)^{\mu}(|y-b| + |x-a|), \end{aligned}$$

where $\mu := \max\{\sigma_0 + \tau, \tau_0 + \sigma\}$. Consequently, $(f_n(X_i^{(n)})g_n(X_{i+\ell}^{(n)}))$ is again decomposable, and satisfies (3.1) and an $(\tilde{\varepsilon}(n), \tilde{\delta}(n))$-condition if $h(n)^{\mu}\varepsilon(n) \to 0$. Here $\ell \in \mathbb{N}$ is fixed. Set

$$b_n^2 := \mathrm{Var}(\Sigma_{i=1}^{h(n)-\ell} f_n(X_i^{(n)})g_n(X_{i+\ell}^{(n)})).$$

Application of Theorem 3.1 yields a CLT for $(f_n(X_i^{(n)})g_n(X_{i+\ell}^{(n)}))$, provided that Condition (b) and the moment condition of (a) in Theorem 3.1 are satisfied with $(X_i^{(n)})$ replaced by $(f_n(X_i^{(n)})g_n(X_{i+\ell}^{(n)}))$, and

$$\frac{h(n)^{1+\mu}\varepsilon(n)}{b_n} \to 0, \qquad \frac{h(n)^{2+2(\sigma_0+\tau_0)}\delta(n)}{b_n^2} \to 0,$$

$$\frac{h(n)^{1+2(\sigma_0+\tau_0)}\delta(n)^{2/(2+\delta)}}{b_n^2} \to 0 .$$

Similar results can be obtained if more decomposable arrays are involved. Asymptotic normality is always derived as a corollary of Theorem 3.1.

4. Some applications

In this section we will apply the results of Section 3 when some special decomposable processes are considered.

Example 4.1. Let (X_i) be a stationary, decomposable sequence of Un(0,1) rv's satisfying an $(\varepsilon(n), \delta(n))$-condition with $m(n) = [cn^{\rho}]$, $\varepsilon(n) = \mathcal{O}(n^{-\eta})$

and $\delta(n) = \mathcal{O}(n^{-\eta_1})$. Here $c \in (0,\infty)$, $\rho \in (0,1/2)$ and $\eta, \eta_1 > 0$. It is assumed that for all $h \in \mathbb{N}$ (X_1, X_{1+h}) is *positively quadrant dependent*, i.e.,

$$G(x,y) := \mathbb{P}[X_1 \le x; X_{1+h} \le y] - \mathbb{P}[X_1 \le x]\mathbb{P}[X_{1+h} \le y] \ge 0$$

for all $x,y \in (0,1)$ (cf. Lehmann (1966; p. 1137)). The score function J is defined by

$$(4.1) \qquad J(x) := \frac{1}{(1-x)^{\alpha}}, \qquad x \in (0,1),$$

where $0 < \alpha < \frac{1}{2}$. Since $\mathbb{E}J'(X_i)$ does not exist, we will consider $(J_n(X_i))$ instead of $(J(X_i))$, where

$$(4.2) \qquad J_n(t) := J(\ell_n(t)) \quad \text{and} \quad \ell_n(t) := n^{-\zeta} + (1-2n^{-\zeta})t, \qquad t \in [0,1],$$

for $\zeta > 0$. We want to apply Corollary 3.2 to $(J_n(X_i))$. First we note that

$$(4.3) \qquad J_n'(t) \le Cn^{\zeta(\alpha+1)}, \qquad t \in [0,1].$$

By the mean value theorem it is clear that (J_n) satisfies (3.4) with $\sigma = \zeta(\alpha+1)$. Set $b_n^2 := \mathrm{Var}(\Sigma_{i=1}^n J_n(X_i))$ and $\sigma_n(h) := \mathrm{Cov}(J_n(X_1), J_n(X_{1+h}))$, $h \in \{1,\ldots,n-1\}$. Because of stationarity we have:

$$b_n^2 = n \,\mathrm{Var}\, J_n(X_1) + 2\sum_{h=1}^{n-1} (n-h)\sigma_n(h).$$

Since J_n is non-decreasing, $(J_n(X_1), J_n(X_{1+h}))$ is also positively quadrant dependent. By Relation (3.1) of Lehmann (1966) we obtain:

$$\sigma_n(h) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x,y)dx\, dy.$$

Consequently, $\sigma_n(h) \ge 0$ and

$$(4.4) \qquad \frac{b_n^2}{n} \ge \mathrm{Var}\, J_n(X_1).$$

Since $1 - \ell_n(X_1) \sim \mathrm{Un}[n^{-\zeta}, 1-n^{-\zeta}]$, we obtain for $o < \gamma < 1/\alpha$:

$$\mathbb{E}J_n^{\gamma}(X_1) = (1-2n^{-\zeta})^{-1} \int_{n^{-\zeta}}^{1-n^{-\zeta}} y^{-\alpha\gamma} dy$$

(4.5) $$= \frac{1}{(1-2n^{-\zeta})(1-\alpha\gamma)} \left((1-n^{-\zeta})^{1-\alpha\gamma} - n^{-\zeta(1-\alpha\gamma)}\right).$$

Hence

$$\mathrm{Var}\ J_n(X_1) \to \frac{1}{1-2\alpha} - \frac{1}{(1-\alpha)^2} \quad \text{as } n \to \infty$$

and (cf. Relation (4.4)) $b_n^2 \geq nC > 0$ for $n \geq n_0$. We need a further assumption about the parameters. Suppose that ρ, ζ, η and η_1 can be chosen such that

(4.6) $$0 < \rho < \tfrac{1}{2} - \tfrac{1}{2}\frac{\alpha}{1-\alpha}, \quad \eta > \tfrac{1}{2} + \zeta(\alpha+1) \quad \text{and}$$

$$\eta_1 > \max\{1 + 2\zeta\alpha,\ \zeta\alpha\frac{2-2\rho}{1-2\rho}\}.$$

Then (3.5) and (3.6) are fulfilled (note that $\sigma = \zeta(\alpha+1)$, that $t(n) = Cn^{\zeta\alpha}$, and that $h(n) = n$). Choose δ such that

$$\frac{2\rho}{1-2\rho} < \delta < \min\{\frac{1}{\alpha} - 2,\ \frac{\eta_1 - 2\zeta\alpha}{\zeta\alpha}\}.$$

Then

$$n^{1+\delta/2}\frac{\mathbb{E}(J_n(X_1))^{2+\delta}}{b_n^{2+\delta}} \leq C\mathbb{E}(J_n(X_1))^{2+\delta} \leq C',$$

since (4.5) holds for $\gamma = 2+\delta$ as well. So, the conditions in (a) of Corollary 3.2 are also satisfied. Further we note that for $1 \leq i < j \leq n$:

$$\frac{1}{j-i}\mathrm{Var}\left[\sum_{k=i+1}^{j} J_n(X_k)\right] = \mathrm{Var}\ J_n(X_1) + 2\sum_{h=1}^{j-i-1}\left(1 - \frac{h}{j-i}\right)\sigma_n(h)$$

$$\le \text{Var } J_n(X_1) + 2 \sum_{h=1}^{n-1} (1 - \frac{h}{n})\sigma_n(h) = \frac{b_n^2}{n},$$

which implies (b). Consequently,

(4.7) $$\frac{1}{b_n} \sum_{i=1}^{n} (J_n(X_i) - \mathbf{E}J_n(X_1)) \to_d N(0,1) \quad \text{as} \quad n \to \infty. \ \square$$

Let us next construct a sequence which satisfies the conditions of Example 4.1. Let $(Z_k)_{k\in\mathbf{Z}}$ be an iid sequence with $\mathbf{P}[Z_k = 0] = \mathbf{P}[Z_k = 1] = \frac{1}{2}$. Consider the sequence $(X_i)_{i\in\mathbf{Z}}$ defined by

$$X_i := \sum_{k=0}^{\infty} 2^{-(k+1)} Z_{i-k}, \quad i \in \mathbf{Z},$$

cf. Bradley (1986; p. 180) and Nieuwenhuis & Ruymgaart (1990). This sequence is a, *not* strongly mixing, strictly stationary AR(1) process with Un(0,1) marignals. Let $(X_j)_{j=1}^n$ be n subsequent observations from this time series. Define

$$X_{j,m(n)} := \sum_{k=0}^{m(n)} 2^{-(k+1)} Z_{j-k} \quad \text{and} \quad \bar{X}_{j,m(n)} := \sum_{k=m(n)+1}^{\infty} 2^{-(k+1)} Z_{j-k},$$

$j \in \{1,\dots,n\}$, where $m(n) = [cn^{\rho}]$ for some $c \in (0,\infty)$ and $0 < \rho < \frac{1}{2}$. Then $X_j = X_{j,m(n)} + \bar{X}_{j,m(n)}$ and $(X_{j,m(n)})$ is m(n)-dependent. For $\varepsilon(n) = \mathcal{O}(n^{-\eta})$ and $\delta(n) = \mathcal{O}(n^{-\eta_1})$, η and η_1 arbitrary but positive, we have:

$$\mathbf{P}[|\bar{X}_{i,m(n)}| \ge \varepsilon(n)] = 0 \le \delta(n) \text{ for } n \ge n_0.$$

Hence the sequence (X_i) is decomposable and satisfies an $(\varepsilon(n), \delta(n))$-condition. So, the parameters ρ, ζ, η and η_1 can be chosen as in (4.6). Since

$$X_{1+h} = 2^{-h}X_1 + \sum_{k=0}^{h-1} 2^{-(k+1)} Z_{1+h-k} =: 2^{-h}X_1 + U_h$$

and X_1 and U_h are independent, it can easily be proved that (X_1, X_{1+h}) is positively quadrant dependent. Consequently, Relation (4.7) follows.

Another sequence with Un(0,1) marginals was already mentioned in Example 1.1. Some additional conditions can be formulated such that (4.7) is valid.

The approach presented in Sections 2 and 3 can be used to prove central limit theorems for a special type of serial rank statistics. The next example reflects the ideas of Nieuwenhuis & Ruymgaart (1989), now presented in the light of the results of the present research.

Example 4.2. Consider the following statistic:

$$T_n := \frac{1}{n-h} \sum_{i=1}^{n-h} J_n\left(\frac{R_{i,n}}{n}\right) K_n\left(\frac{R_{i+h,n}}{n}\right).$$

Here $R_{1,n},\ldots,R_{n,n}$ are the ranks of a sample $X_1,\ldots,X_n$ of n successive observations from a general linear process. So, (X_i) has the form (2.1) and is decomposable. It is assumed that it satisfies an $(\varepsilon(n), \delta(n))$-condition for some $(\varepsilon(n))$ and $(\delta(n))$. For J_n and K_n we assume that $J_n(t) := J(\ell_n(t))$ and $K_n(t) := K(\ell_n(t))$, $t \in [0,1]$, with $\ell_n(t)$ as in (4.2) and $J,K : (0,1) \to \mathbb{R}$ twice continuously differentiable functions such that

$$|J^{(i)}(x)| \le \frac{C}{(x(1-x))^{\alpha+i}} \quad \text{and} \quad |K^{(i)}(x)| \le \frac{C}{(x(1-x))^{\tilde{\alpha}+i}},$$

$x \in (0,1)$, $i \in \{0,1,2\}$. Here $\alpha,\tilde{\alpha} > 0$. In the above reference it is observed that T_n is a natural rank estimator of $\tau_n := \mathbb{E}[J_n(\xi_1)K_n(\xi_{1+h})]$, where $\xi_i := F(X_i)$ with F the distribution function of X_i which is assumed to be continuously differentiable with bounded derivative.
From the arguments of the authors it can be derived that

$$(4.8) \qquad \sqrt{n}(T_n - \tau_n) = A_n + B_n,$$

where B_n is some residual term tending to 0 if some conditions about m(n), $\varepsilon(n)$, $\delta(n)$, α, $\tilde{\alpha}$ and ζ are fulfilled. The main term A_n equals

$$\sqrt{n}\ \Sigma_{i=1}^{n-h}(\psi_n(\xi_i, \xi_{i+h}) - \tau_n)/(n-h),$$

where

$$\psi_n(\xi_i, \xi_{i+h}) := J_n(\xi_i)K_n(\xi_{i+h}) + \frac{n-h}{n}\varphi_n(\xi_i).$$

Here φ_n is the sum of the functions $\varphi_{1,n}$ and $\varphi_{2,n}$ mentioned in the reference; $\mathbf{E}\varphi_n(\xi_i) = 0$. Although φ_n is not differentiable it can be proved that it satisfies (3.4) with $\sigma = \zeta(\alpha + \tilde{\alpha} + 1)$. Relation (3.4) is also valid for J_n (with $\sigma = \zeta(\alpha+1)$) and K_n (with $\sigma = \zeta(\tilde{\alpha}+1)$).
We want to apply Theorem 3.1 to $(\psi_n(\xi_i, \xi_{i+h}))$. The sequence (ξ_i) is decomposable and satisfies an $(\varepsilon(n), \delta(n))$-condition, see Example 1.1.

By arguments as in (3.7) it can be proved easily that $(\psi_n(\xi_i, \xi_{i+h}))$ is also decomposable and satisfies the $(\tilde{\varepsilon}(n), \tilde{\delta}(n))$-condition with

$$\tilde{\varepsilon}(n) := Cn^{\zeta(\alpha+\tilde{\alpha}+1)}\varepsilon(n) \quad \text{and} \quad \tilde{\delta}(n) := C'\delta(n),$$

if $\varepsilon(n) \to 0$ fast enough. Hence, the assumptions in Theorem 3.1 can be formulated in terms of $(\psi_n(\xi_i, \xi_{i+h}))$ such that as a conclusion:

$$\text{(4.9)} \qquad \frac{1}{b_n}\sum_{i=1}^{n-h}(\psi_n(\xi_i, \xi_{i+h}) - \tau_n) \to_d N(0,1),$$

provided that the resulting conditions are fulfilled. Here $b_n^2 := \mathrm{Var}(\Sigma_{i=1}^{n-h}\psi_n(\xi_i, \xi_{i+h}))$. If, moreover, $\sqrt{n}\, B_n/b_n \to 0$ with B_n as in (4.8), then (4.9) is equivalent to

$$\frac{\sqrt{n}(T_n - \tau_n)}{b_n/\sqrt{n}} \to_d N(0,1) \text{ as } n \to \infty. \ \square$$

References

[1] ANDERSON, T.W. (1971): *The Statistical Analysis of Time Series*. Wiley.

[2] BERK, K.N. (1973): A central limit theorem for m-dependent random variables with unbounded *m*. *Ann. Probab.* **2**, 352-354.

[3] BRADLEY, R.C. (1986): Basic properties of strong mixing conditions. In: *Dependence in Probability and Statistics* (eds. Eberlein & Taqqu) p. 165-192, Birkhäuser, Boston.

[4] BROCKWELL, P.J. & R.A. DAVIES (1987): *Time Series: Theory and Methods*. Springer.

[5] CHANDA, K.C., M.L. PURI & F.H. RUYMGAART (1989): Asymptotic normality of *L*-statistics based on decomposable time series. Report 8922. Dep. of Math., Catholic University of Nijmegen.

[6] IBRAGIMOV, I.A. & YU.V. LINNIK (1971): *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhof.

[7] LEHMANN, E.L. (1966): Some concepts of dependence. *Ann. Math. Stat.* II, 1137-1153.

[8] MARSAGLIA, G. (1954): Iterated limits and the central limit theorem for dependent variables. *Proc. Amer. Math. Soc.* **5**, 987-991.

[9] NIEUWENHUIS, G. & F.H. RUYMGAART (1989): Some stochastic inequalities and asymptotic normality of serial rank statistics in general linear processes. *J. Stat. Planning and Inf.* To appear.

[10] PARZEN, E. (1957): A central limit theorem for multilinear stochastic processes. *Ann. Math. Stat.* **28**, 252-256.

[11] PRIESTLEY, M.B. (1981): *Spectral Analysis and Time Series*. Vol. 1, Ac. Press, New York.

[12] SHORACK, G.R. & J.A. WELLNER (1986): *Empirical Processes with Applications in Statistics*. Wiley, New York.

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