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RESEARCH MEMORANDUM

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AN EXTENSION OF KÖNIG'S THEOREM TO
GRAPHS WITH NO ODD-K_4

by

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Abstract

We prove the following min-max relations. Let G be an undirected graph, without isolated nodes, not containing an odd-K_4 (a homeomorph of K_4 (the 4-clique) in which the triangles of K_4 have become odd cycles). Then the maximum cardinality of a stable set in G is equal to the minimum cost of a collection of edges and odd circuits in G, covering the nodes of G. Here the cost of an edge is 1 and the cost of a circuit of length 2k+1 equal to k.
Moreover, the minimum cardinality of a node-cover for G is equal to the maximum profit of a collection mutually node disjoint edges and odd circuits in G. Here the profit of an edge is 1 and the profit of a circuit of length 2k+1 is equal to k+1. Also weighted versions of these min-max relations hold. The result extends König's well-known min-max relations for stable sets and node-covers in bipartite graphs. Moreover it extends results of Chvâtal, Boulala, Fonlupt, and Uhry. A weaker, fractional, version of these min-max relations follows from earlier results obtained by Schrijver and the author.

1. Introduction

The subject of this paper is to give an extension of the following well-known result.

(1.1) If G has no odd circuit,
then $\alpha(G) = \rho(G)$ and $\tau(G) = \nu(G)$ (König [1931,1933])

Here, and in the sequel, $G = (V(G), E(G))$ denotes an undirected graph without isolated nodes. As usual, the parameters α, ρ, τ and ν are defined by:

$\alpha(G)$ = the maximum cardinality of a stable set in G. ($S \subset V(G)$ is a stable set if $u,v \in S$ implies $uv \notin E(G)$.)

$\rho(G)$ = the minimum cardinality of an edge-cover for G. ($E' \subset E(G)$ is an edge-cover if for each $u \in V$ there exists an $e \in E'$ covering u.)

$\nu(G)$ = the maximum cardinality of a matching in G. ($M \subset E(G)$ is a matching if $e_1, e_2 \in M$, $e_1 \neq e_2$ implies $e_1 \cap e_2 = \emptyset$.)

$\tau(G)$ = the minimum cardinality of a node-cover for G. ($N \subset V(G)$ is a node-cover if $uv \in E(G)$ implies $u \in N$ or $v \in N$.)

We introduce two new parameters:

$\tilde{\rho}(G)$ = the minimum cost of a collection of edges and odd circuits in G covering the nodes of G. The cost of an edge is equal to 1, and the cost of a circuit with 2k+1 edges is equal to k. The cost of a collection of edges and odd circuits is equal to the sum of the costs of its members.

$\tilde{\nu}(G)$ = the maximum profit of a collection of mutually node disjoint edges and odd circuits in G. The profit of an edge is equal to 1 and the profit of a circuit of length 2k+1 is equal to k+1. The profit of a collection of edges and odd circuits is equal to the sum of the profits of its members.

The following inequalities are obvious:

$$\alpha(G) \leqq \tilde{\rho}(G) \leqq \rho(G),$$
(1.2)
$$\tau(G) \geqq \tilde{\nu}(G) \geqq \nu(G).$$

König's Theorem (1.1) can be extended to the following result. (It follows from the more general Theorem 1.8, which will be proved in section 2.)

Theorem 1.3
Let G be an undirected graph, without isolated nodes. If G does not contain any odd-K_4 as a subgraph, then $\alpha(G) = \tilde{\rho}(G)$ and $\tau(G) = \tilde{\nu}(G)$. □

An odd-K_4 is a homeomorph of K_4 (the 4-clique) in which all triangles have become odd circuits. (See figure 1, wriggled lines stand for pairwise openly disjoint paths; odd indicates that the corresponding faces are odd circuits.)

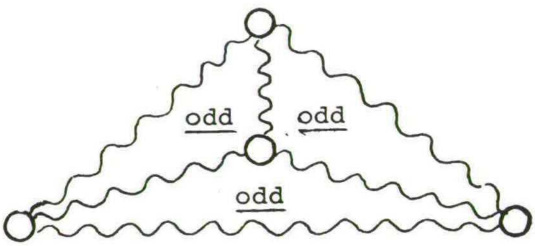


figure 1

To see that Theorem 1.3 extends König's Theorem (1.1), observe that a bipartite graph G has no odd-K_4, and satisfies $\tilde{\rho}(G) = \rho(G)$, $\tilde{\tau}(G) = \tau(G)$ (as G has no odd circuits.)
The two equalities in (1.1) are equivalent, for any graph G. This follows from

(1.4) $\alpha(G) + \tau(G) = |V(G)| = \rho(G) + \nu(G)$ (Gallai [1958,1959]).

A similar equivalence for the equalities $\alpha(G) = \tilde{\rho}(G)$ and $\tau(G) = \tilde{\nu}(G)$ follows from the following result observed by Schrijver, analogous to Gallai's result above.

Theorem 1.5

Let G be an undirected graph without isolated nodes. Then $\tilde{\rho}(G) + \tilde{\nu}(G) = |V(G)|$.

Proof:

First, let $e_1,\ldots,e_m$, $C_1,\ldots,C_n$ be a collection of mutually node disjoint edges and odd circuits such that the profit $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| + 1)$ of the collection is equal to $\tilde{\nu}(G)$.

Let $V_1 := V(G)\setminus \bigcup_{i=1}^{n} V(C_i)$, and let G_1 be the subgraph of G induced by V_1. Then obviously $m = \nu(G_1)$. Let $f_1,\ldots,f_{\rho(G_1)}$ be a minimum edge cover for G_1. Then $f_1,\ldots,f_{\rho(G_1)}$, $C_1,\ldots,C_n$ is a collection of edges an odd circuits covering V(G). The cost of this collection is (using Gallai's identity (1.4)):

$$\rho(G_1) + \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)| - 1) = |V_1| - \nu(G_1) - \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)| + 1) + \sum_{i=1}^{n} |V(C_i)|$$

$$= |V(G)| - \tilde{\nu}(G).$$

Hence $\tilde{\rho}(G) + \tilde{\nu}(G) \leqq |V(G)|$.

The reverse inequality is proved almost identically. However there is a small technical difference, dealt with in the claim below.

Let $e_1,\ldots,e_m$, $C_1,\ldots,C_n$ be a collection of edges and odd circuits covering V(G) such that the cost $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| - 1)$ of the collection is equal to $\tilde{\rho}(G)$, and n is as small as possible.

Claim: For each $i,j=1,..,n$ $(i\neq j)$; $k=1,\ldots,m$ we have $V(C_i) \cap V(C_j) = \emptyset$, $V(C_i) \cap e_k = \emptyset$.

Proof of Claim: Suppose, $u \in V(C_i)$ $(i=1,\dots,n)$, such that u is also contained in another odd circuit among $C_1,\dots,C_n$, or in one of the edges $e_1,\dots,e_m$. Let $f_1,\dots,f_p \in E(C_i)$ be the unique maximum cardinality matching in C_i not covering u. Then $p = \frac{1}{2}(|V(C_i)| - 1)$. Obviously $e_1,\dots,e_m$, $f_1,\dots,f_p$, $C_1,\dots,C_{i-1}$, $C_{i+1},\dots,C_n$ is a collection of edges and odd circuits covering $V(G)$. Its cost is $\tilde{\rho}(G)$. However it contains only $n-1$ odd circuits, contradicting the minimality of n.

end of proof of claim.

As before we define $V_1 = V(G) \setminus \bigcup_{i=1}^{n} V(C_i)$ and G_1 as the subgraph of G induced by V_1. By similar arguments as used in the first part of the proof one gets:

$$\tilde{\rho}(G) = \rho(G_1) + \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)| - 1)) = |V_1| - \nu(G_1) - \sum_{i=1}^{n} \tfrac{1}{2}(|V(C_i)| + 1)$$

$$+ \sum_{i=1}^{n} |V(C_i)| \geqq |V(G)| - \tilde{\nu}(G). \qquad \square$$

Corollary 1.6

Let G be an undirected graph without isolated nodes. Then $\alpha(G) = \tilde{\rho}(G)$ if and only if $\tau(G) = \tilde{\nu}(G)$. $\square$

As mentioned, we prove Theorem 1.3 in section 2. In fact we shall prove a more general weighted version of this theorem (Theorem 1.8 below).

Weighted versions

We define weighted versions of the numbers α, ρ, ν, τ, $\tilde{\rho}$, and $\tilde{\nu}$ and state the obvious generalizations of the results mentioned.

Let $w \in \mathbb{Z}^{V(G)}$.

$\alpha_w(G)$ = maximum $\{ \sum_{u \in S} w_u \mid S$ is a stable set in $G\}$,

$\rho_w(G)$ = the minimum cardinality of a w-edge-cover for G. (A w-edge-cover for G is a collection $e_1,\dots,e_m$ in $E(G)$ (repetition allowed) such that for each $u \in V(G)$ there are at least w_u edges among $e_1,\dots,e_m$ incident with u. The cardinality of $e_1,\dots,e_m$ is m.)

$\nu_w(G)$ = the maximum cardinality of a w-matching in G. (A w-matching is a collection $e_1,\ldots,e_m$ in $E(G)$ (repetition allowed) such that for each $u \in E(G)$ there are at most w_u edges among $e_1,\ldots,e_m$ incident with u.)

$\tau_w(G)$ = minimum $\{ \sum_{u\in N} w_u | N$ is node-cover for G$\}$.

Moreover we define:

A w-cover (w-packing respectively) by edges and odd circuits is a collection $e_1,\ldots,e_m$ of edges and $C_1,\ldots,C_m$ of odd circuits (repetition allowed), such that for each $u \in V(G)$:

$|\{i=1,\ldots,m | u$ incident with $e_i\}| + |\{i=1,\ldots,n | u \in V(C_i)\}| \geqq w_u (\leqq w_u$ respectively).

The cost of $e_1,\ldots,e_m$, $C_1,\ldots,C_n$ is $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| - 1)$, its profit is $m + \sum_{i=1}^{n} \frac{1}{2}(|V(C_i)| + 1)$.

$\tilde{\rho}_w(G)$ = the minimum cost of a w-cover by edges and odd circuits for G.

$\tilde{\nu}_w(G)$ = the maximum profit of a w-packing by edges and odd circuits in G.

These numbers satisfy:

If G has no odd circuit, then $\alpha_w(G) = \rho_w(G)$ and $\tau_w(G) = \nu_w(G)$ (Egerváry [1931]),

(1.7) $\alpha_w(G) \leqq \tilde{\rho}_w(G) \leqq \rho_w(G)$,
$\tau_w(G) \geqq \tilde{\nu}_w(G) \geqq \nu_w(G)$,
$\alpha_w(G) + \tau_w(G) = \tilde{\rho}_w(G) + \tilde{\nu}_w(G) = \rho_w(G) + \nu_w(G) = \sum_{u\in V(G)} w_u$.

(1.7) can be proved easily from the cardinality versions stated before (with $w \equiv 1$), using the following construction:

Define G_w by:

$V(G_w) = \{[u,i] \mid u \in V(G);\ i=1,\ldots,w_u\}$,

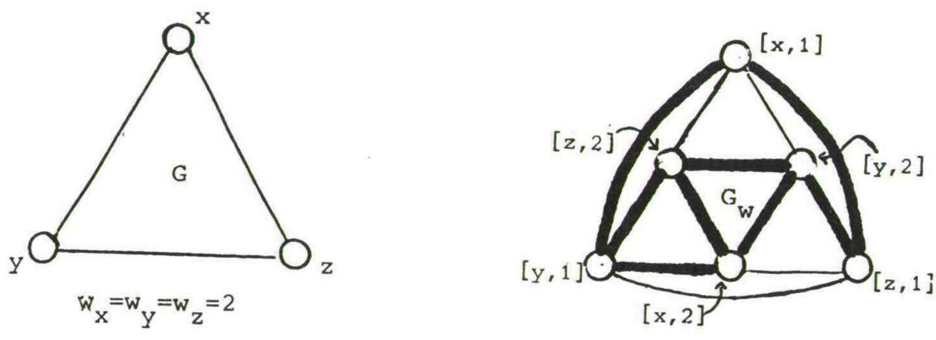
$E(G_w) = \{[u,i][v,j] \mid u,v \in V(G);\ uv \in E(G);\ i=1,\ldots,w_u;\ j=1,\ldots,w_v\}$.

Then one easily proves that $\alpha_w(G) = \alpha(G_w)$, $\rho_w(G) = \rho(G_w)$, $\nu_w(G) = \nu(G_w)$, $\tau_w(G) = \tau(G_w)$, $\tilde{\rho}_w(G) = \tilde{\rho}(G_w)$, $\tilde{\nu}_w(G) = \tilde{\nu}(G_w)$, and $V(G_w) = \sum_{u \in V(G)} w_u$. Moreover G_w is bipartite if and only if G is. These yield (1.7). Theorem 1.3 can be generalized as well.

<u>Theorem 1.8</u>

Let G be an undirected graph, without isolated nodes. If G does not contain any odd-K_4 as a subgraph, then $\alpha_w(G) = \tilde{\rho}_w(G)$ and $\tau_w(G) = \tilde{\nu}_w(G)$ for any $w \in \mathbb{Z}^{V(G)}$. □

The proof of Theorem 1.8 is in section 2. It should be noted that Theorem 1.8 does not follow from Theorem 1.3 by using G_w. The reason is that it is possible that G_w contains an odd-K_4 even if G does not. This is illustrated by the graph in figure 2. (The bold edges, in figure 2b form an odd-K_4.)



(a) (b)

figure 2

The statement "$\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$" can be reformulated in terms of integer linear programming.

(1.9) Both optima in the following primal-dual pair of linear programs, are attained by integral vector if w is integer valued.

Primal:

$$\tilde{\rho}^*_w(G) := \max \sum_{u \in V(G)} w_u x_u$$

$$\text{s.t. } x_u + x_v \leqq 1 \qquad (uv \in E(G)),$$

$$\sum_{u \in V(C)} x_u \leqq \tfrac{1}{2}(|V(C)| - 1) \qquad (C \in \mathcal{C}(G)),$$

$$x_u \geqq 0 \qquad (u \in V(G)).$$

Dual:

$$\tilde{\rho}^*_w(G) = \min \sum_{e \in E(G)} y_e + \sum_{C \in \mathcal{C}(G)} \tfrac{1}{2}(|V(C)| - 1) z_C$$

$$\text{s.t. } \sum_{\substack{e \in E(G) \\ e \ni u}} y_e + \sum_{\substack{C \in \mathcal{C}(G) \\ V(C) \ni u}} z_C \geqq w_u \qquad (u \in V(G)),$$

$$y_e \geqq 0 \qquad (e \in E(G)),$$

$$z_C \geqq 0 \qquad (C \in \mathcal{C}(G)).$$

($\mathcal{C}(G)$ denotes the collection of odd circuits $C = (V(C), E(C))$ in G.)

So Theorem (1.8) implies that if G has no odd-K_4, then $\tilde{\rho}_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}_+^{V(G)}$. In other words, the system of linear inequalities in the primal problem of (1.9) is totally dual integral (cf. Edmonds-Giles [1977]). Consequently (Edmonds-Giles [1977], Hoffman [1974]), if G has no odd-K_4, then $\alpha_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}_+^{V(G)}$. This means that the system of linear inequalities in the primal problem of (1.9) describes the stable set polytope of G. (The stable set polytope of G is the convex hull of the characteristic vectors of the stable sets of G, considered as subsets of V(G).)

Obviously, also the statement "$\tau_w(G) = \tilde{\nu}_w(G)$ for each $w \in \mathbb{Z}_+^{V(G)}$" can be formulated in a way similar to (1.9).

We conclude this section with some remarks. Section 2 contains the proof of Theorem 1.3 and 1.8. Finally, in section 3, we consider some algorithmic aspects of the results in this paper.

Remarks

(i) Earlier results on this topic are:

- Chvátal [1975]: If G is series-parallel (i.e. G contains no homemorph of K_4), then $\alpha(G) = \tilde{\rho}(G)$.
- Boulala and Uhry [1979]: If G is series-parallel, then $\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$. (In fact they only emphasize $\alpha_w(G) = \tilde{\rho}^*_w(G)$ (which was conjectured by Chvátal [1975]). But their proof implicitly yields the stronger result. Recently Mahjoub [1985] gave a very short proof of $\alpha_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$ for series-parallel graphs G.)
- Fonlupt and Uhry [1982]: If there exists a $u \in V(G)$ such that $u \in V(C)$ for all $C \in \mathcal{C}(G)$, then $\alpha_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$. Sbihi and Uhry [1984] give a new proof of Fonlupt and Uhry's result. This proof implicitly yields $\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$.

Obviously, the graphs considered by Chvátal, Boulala, Fonlupt, Sbihi, and Uhry do not contain an odd-K_4.

- Gerards and Schrijver [1985]: If G has no odd-K_4 then $\alpha_w(G) = \tilde{\rho}^*_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$.

(ii) Theorem 1.8 (and 1.3) can be refined by allowing w-covers (w-packings) by edges and odd circuits only to use edges not contained in a triangle, and odd circuits not having a chord. In other words, if G has no odd-K_4, then the system:

$$(*)\quad \begin{array}{ll} x_u + x_v \leqq 1 & (uv \in E(G),\ uv \text{ is not contained in a triangle}) \\ \sum\limits_{u \in V(C)} x_u \leqq \tfrac{1}{2}(|V(C)| - 1) & (C \in \mathcal{C}(G),\ C \text{ has no chord}) \\ x_u \geqq 0 & (u \in V(G)) \end{array}$$

is a totally dual integral system defining the stable set polytope of G. In fact the unequalities in (*) are all facets of the polyhedron defined by (*) (for any graph G). So (*) is the unique minimal totally dual integral system (cf. Schrijver [1981]) for the stable set polytope of G, in case G has no odd-K_4.

(iii) Lovász, Schrijver, Seymour, and Truemper [1984] give a constructive characterization of graphs with no odd-K_4: G has no odd-K_4 if and only if one of the following holds:

- There exists a $u \in V(G)$ such that $u \in V(C)$ for all $C \in \mathcal{C}(G)$ (Fonlupt and Uhry's case mentioned in remark (i) above).
- G is planar, and at most two faces of G are odd circuits.
- G is the graph in figure 3.
- G can be decomposed into smaller graphs with no odd-K_4.

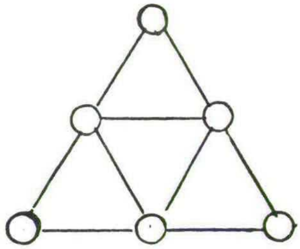


figure 3

2. The proof of Theorem 1.8

We first derive a special case of Theorem 1.8. To state and prove it we need some extra notions and an auxilary result (Theorem 2.1). An odd-K_3^2 is a graph as indicated in figure 4 (wriggles and dotted lines stand for pairwise openly disjoint paths, dotted lines may have length zero, wriggles lines have always positive length, odd indicates that the corresponding faces are odd cycles).

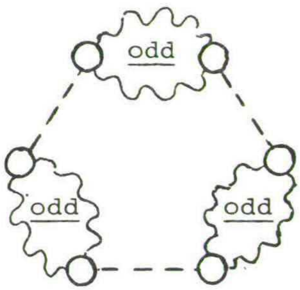


figure 4

An orientation of an undirected graph G is a directed graph obtained from G by directing the edges. We say that a directed graph has discrepancy 1 if in each circuit the number of forwardly directed arcs minus the number of backwardly directed arcs is 0 or ± 1.

Theorem 2.1 (Gerards and Schrijver [1986])
Let G be an undirected graph. Then G does not contain an odd-K_4 or an odd-K_3^2 if and only if G has an orientation with discrepancy 1. $\square$

Using this theorem we obtain the following special case of Theorem 1.8.

Theorem 2.2
Let G be an undirected graph without isolated nodes. If G does not contain any odd-K_4 or any odd-K_3^2, then $\alpha_w(G) = \tilde{\rho}_w(G)$ and $\tau_w(G) = \tilde{\nu}_w(G)$ for each $w \in \mathbb{Z}^{V(G)}$.

Proof:
According to Theorem 2.1, G has an orientation with discrepancy 1. Let $\vec{A}$ denote the set of arcs in this orientation. For each $\overrightarrow{uv} \in \vec{A}$ we add a reversely directed arc $\overrightarrow{vu}$ too. Denote $\overleftarrow{A} := \{\overrightarrow{vu} \mid \overrightarrow{uv} \in \vec{A}\}$.
Consider the following "circulation" problem:

$$(2.3)\qquad \min \sum_{a\in\vec{A}} f_a$$

$$\text{s.t.} \sum_{\substack{a\in\vec{A}\cup\overleftarrow{A}\\ a \text{ enters } u}} f_a - \sum_{\substack{a\in\vec{A}\cup\overleftarrow{A}\\ a \text{ leaves } u}} f_a = 0 \qquad (u\in V(G))$$

$$\sum_{\substack{a\in\vec{A}\cup\overleftarrow{A}\\ a \text{ enters } u}} f_a \geqq w_u \qquad (u\in V(G))$$

$$f_a \geqq 0 \qquad (a\in \vec{A}\cup\overleftarrow{A}),$$

and its linear programming dual:

$$(2.4)\qquad \max \sum_{u\in V(G)} w_u x_u$$

$$\text{s.t. } \pi_v - \pi_u + x_v \leqq 1 \qquad (\overrightarrow{uv}\in\vec{A})$$

$$\pi_u - \pi_v + x_u \leqq 0 \qquad (\overrightarrow{vu}\in\overleftarrow{A})$$

$$x_u \geqq 0 \qquad (v\in V(G)).$$

The theorem is proved by the following three propositions:

<u>Proposition 1</u>: The constraint matrix of (2.3) is totally unimodular. Consequently both (2.3) and (2.4) have integral optimal solutions (Hoffman and Kruskal [1956]).

<u>Proposition 2</u>: Let $\pi\in\mathbb{Z}^{V(G)}$, $x\in\mathbb{Z}^{V(G)}$ be a feasible solution of (2.4). Then x is a feasible solution of the primal problem of (1.9).

<u>Proposition 3</u>: Let $f\in\mathbb{Z}^{\vec{A}\cup\overleftarrow{A}}$ be a feasible solution of (2.3). Then there exists a $y\in\mathbb{Z}^{E(G)}$ and a $z\in\mathbb{Z}^{\mathcal{C}(G)}$, which form a feasible solution of the dual problem of (1.9), such that:

$$\sum_{e\in E(G)} y_e + \sum_{C\in\mathcal{C}(G)} \tfrac{1}{2}(|V(C)|-1)z_C \leqq \sum_{a\in\vec{A}} f_a.$$

Indeed, the three propositions together prove that $\alpha_w(G) > \tilde{\rho}_w(G)$. By (1.7), this yields $\alpha_w(G) = \tilde{\rho}_w(G)$ and $\tau_w(G) = \tilde{\nu}_w(G)$.
The three propositions above are shown as follows:

Proof of Proposition 1:

If we are given a directed graph $D = (V(D), A(D))$ and a spanning directed tree $T = (V(D), T(D))$ on the same node set (not necessarily $T(D) \subset A(D)$), then the network matrix N of D with respect to T is defined as follows:

$N \in \{0,1,-1\}^{A(T) \times A(D)}$. For $u,v \in V(D)$ let $P(u,v) \subset A(T)$ be the unique path in T from u to v. Then for each $a_1 \in A(T)$, $a_2 = \overrightarrow{uv} \in A(D)$:

$$N_{a_1,a_2} := \begin{cases} 1 & \text{if } a_1 \in P(u,v)\text{, and } a_1 \text{ is passed forwardly going along } P(u,v) \text{ from } u \text{ to } v \\ -1 & \text{if } a_1 \in P(u,v)\text{, and } a_1 \text{ is passed backwardly going along } P(u,v) \text{ from } u \text{ to } v \\ 0 & \text{if } a_1 \notin P(u,v)\}. \end{cases}$$

Network matrices are totally unimodular (Tutte [1965]). We prove Proposition 1 by proving that the constraint matrix of (2.3) is a network matrix.

Indeed, let $V(D) := V(T) := \{v_0\} \cup \{[u,i] \mid u \in V(G),\ i \in \{1,2\}\}$,
$A(D) := \{\overrightarrow{[u,1][v,2]} \mid \overrightarrow{uv} \in \vec{A}\}$, and
$A(T) := \{\overrightarrow{v_0[u,1]} \mid u \in V(G)\} \cup \{\overrightarrow{u_1u_2} \mid u \in V(G)\}$.

Proof of Proposition 2:

Since x is integral we only need to prove that $x_u + x_v \leqq 1$ for $uv \in E(G)$. Indeed $x_v + x_u \leqq (1 - \pi_v + \pi_u) + (\pi_v - \pi_u) = 1$ if $uv \in E(G)$ ($\overrightarrow{uv} \in \vec{A}$).

Proof of Proposition 3:

We can write f as $f = \sum_{D \in \Delta} \lambda_D f^D$, where Δ is a collection of directed circuits in $\vec{A} \cup \overleftarrow{A}$, $f^D \in \{0,1\}^{\vec{A} \cup \overleftarrow{A}}$ with $f^D_a = 1$ if and only if $a \in D$, and

$\lambda_D \in \mathbb{Z}_+$ for each $D \in \Delta$.

For every even circuit $D \in \Delta$, let M_D be an arbitrary maximum cardinality matching in $\{uv \in E(G) \mid \overrightarrow{uv} \in D \text{ or } \overrightarrow{vu} \in D\}$. (In particular if

$D = \{\overrightarrow{uv}, \overrightarrow{vu}\}$. then $M_D = \{uv\}$.) Define $y^D \in \mathbb{Z}^{E(G)}$ by:

$$y_e^D = \begin{cases} \lambda_D & \text{if } e \in M_D \\ 0 & \text{else.} \end{cases}$$

Next $y \in \mathbb{Z}^{E(G)}$ is defined by $y = \sum_{\substack{D \in \Delta \\ D \text{ even}}} y^D$.

For each odd circuit $D \in \Delta$, let $C_D \in \mathcal{C}(G)$ be defined by $C_D = \{uv \mid \overrightarrow{uv} \in D$ or $\overrightarrow{vu} \in D\}$. Define $z \in \mathbb{Z}^{\mathcal{C}(G)}$ by:

$$z_C = \begin{cases} \lambda_D & \text{if } C = C_D \text{ for some } D,\ D \in \Delta,\ |D| \text{ odd} \\ 0 & \text{else.} \end{cases}$$

The vectors $y \in \mathbb{Z}^{E(G)}$ and $z \in \mathbb{Z}^{\mathcal{C}(G)}$ form a feasible solution to the dual problem of (1.9). Moreover

$$\begin{aligned} \sum_{a \in \vec{A}} f_a &= \sum_{D \in \Delta} \lambda_D |\vec{A} \cap D| \\ &\geq \sum_{\substack{D \in \Delta \\ D \text{ even}}} \lambda_D |M_D| + \sum_{\substack{D \in \Delta \\ D \text{ odd}}} \lambda_D \cdot \tfrac{1}{2}(|V(C_D)| - 1) \\ &= \sum_{e \in E(G)} y_e + \sum_{C \in \mathcal{C}(G)} \tfrac{1}{2}(|V(C)| - 1) z_C. \end{aligned}$$ □

Before we prove Theorem 1.8 we state a result of Lovász and Schrijver [1984] (cf. Gerards-Schrijver. [1986, Theorem 2.6]). This result indicates that, in a sense, Theorem 2.2 is the core of Theorem 1.8.

Theorem 2.5

Let G be an undirected graph, containing no odd-K_4. If G contains an odd-K_3^2, then one of the following holds

(i) G is disconnected or has a one node cutset

(ii) G has a two node cutset. Both sides of the cutset are not bipartite. □

Using this we finally prove Theorem 1.8.

Proof of Theorem 1.8

Let G be a graph with no odd-K_4. Assume that all graphs G' with $|E(G')| < |E(G)|$ satisfy Theorem 1.8. We shall prove that then G satisfies Theorem 1.8. Obviously we may assume G to be connected. Let $w \in \mathbb{Z}^{V(G)}$. By the weighted version of Theorem 1.5 we only need to prove that $\alpha_w(G) = \tilde{\rho}_w(G)$. Obviously we may assume that $w_u \geqq 0$ for each $u \in V(G)$. According to Theorem 2.2 and 2.5 we may assume that G satisfies (i) or

(ii) of Theorem 2.5. So we have subsets V_1, V_2 of V(G) such that $|V_1 \cap V_2| \leqq 2$, $V_1 \cup V_2 = V(G)$, and both $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are non empty sets not joined by an edge in E(G). Moreover, in case $|V_1 \cap V_2| = 2$, the subgraphs G_1 and G_2 in G induced by V_1, V_2 respectively are not bipartite. In the sequel we shall use the following notation: For each stable set $U \subset V_1 \cap V_2$ the number s(U) ($s^1(U)$, $s^2(U)$ respectively) denotes the maximum weight $\sum_{u \in S} w_u$ of a stable set in G (G_1, G_2 respectively) satisfying $S \cap V_1 \cap V_2 = U$. Note that: $s(U) = s^1(U) + s^2(U) - \sum_{u \in U} w_u$ for each stable set U in $V_1 \cap V_2$. We consider two cases.

Case I: $V_1 \cap V_2$ induces a clique in G.
Define the following weight functions:

$$w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus V_2 \\ w_u + s^1(\emptyset) - s^1(\{u\}) & \text{if } u \in V_1 \cap V_2; \end{cases}$$

$$w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus V_1 \\ s^1(\{u\}) - s^1(\emptyset) & \text{if } u \in V_1 \cap V_2. \end{cases}$$

Obviously G_1 and G_2 do not contain an odd-K_4. Moreover $|E(G_1)| < |E(G)|$, $|E(G_2)| < |E(G)|$. Hence there exist a w^1- and a w^2-cover by edges and odd circuits in G_1, G_2 respectively, with cost $s^1(\emptyset)$, $\alpha_w(G) - s^2(\emptyset)$ respectively. The union of these two covers is a w-cover with edges and odd circuits in G with cost $\alpha_w(G)$. Hence $\alpha_w(G) = \tilde{\rho}_w(G)$.

Case II: $|V_1 \cap V_2| = 2$, $V_1 \cap V_2 = \{u_1, u_2\}$ say, and $u_1u_2 \notin E(G)$. Define for i=1,2; k=2,3 the graph G_i^k by adding to G_i a path from u_1 to u_2 with k-edges. (See figures 5 and 6.)

Claim 1: We may assume that G_i^k does not contain an odd-K_4 (i=1,2; k=2,3). Moreover $|E(G_i^k)| < |E(G)|$.

Proof of Claim 1: To prove the first assertion (for $i=1$), it is sufficient to prove that in G_2 there exists an odd as well as an even path from u_1 to u_2. Suppose this is not the case. Since G_2 is not bipartite this implies the existence of a cutnode in G_2 separating $\{u_1,u_2\}$ from an odd cycle in G_2. But such a cutnode is also a cutnode of G. In that case we can apply Case I to prove $\alpha_w(G) = \tilde{\rho}_w(G)$. So we may assume that G_i^k has no odd-K_4. If $|E(G_1^k)| \geqq |E(G)|$, then $|E(G_2)| \leqq 3$. Hence, since G_2 is not bipartite, G_2 is a triangle. So $u_1u_2 \in E(G)$, contradicting our assumption that $u_1u_2 \notin E(G)$. end of proof of claim 1

Define $\Delta := s^2(\{u_1\}) + s^2(\{u_2\}) - s^2(\{u_1,u_2\}) - s^2(\emptyset)$. Again we consider two cases.

Case IIa: $\Delta \geq 0$.

Let b_1,b_2 be the new nodes in G_1^3, b the new node in G_2^2. (See figure 5 below.) Moreover, let e_1, e_2, $\tilde{e}$, f_1, and f_2 be the edges indicated in figure 5.

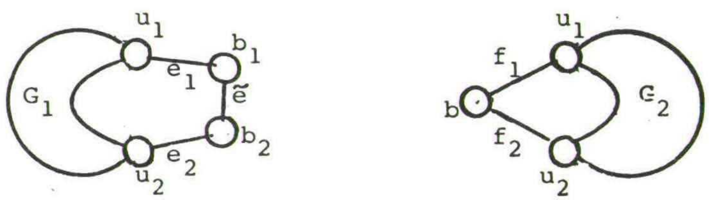


figure 5

We define the following weight functions:

$$w^1 \in \mathbb{Z}^{V(G_1^3)} \text{ by } w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus \{u_1,u_2\} \\ s^2(\{u\}) - s^2(\emptyset) & \text{if } u \in \{u_1,u_2\} \\ \Delta & \text{if } u \in \{b_1,b_2\}; \end{cases}$$

$$w^2 \in \mathbb{Z}^{V(G_2^2)} \text{ by } w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus \{u_1,u_2\} \\ w_u + s^2(\emptyset) - s^2(\{u\}) + \Delta & \text{if } u \in \{u_1,u_2\} \\ \Delta & \text{if } u \in \{b\}. \end{cases}$$

Claim 2: $\alpha_{w^2}(G_1^3) = \alpha_w(G) + \Delta - s^2(\emptyset)$ and $\alpha_{w^2}(G_2^2) = S^2(\emptyset) + \Delta$. Moreover, for i=1,2 there exists a stable set S in G_2^2 with $\sum_{u\in S} w_u^2 = \alpha_{w^2}(G_2^2)$, $u_i \notin S$, and $b \notin S$.

Proof of Claim 2: Straightforward case checking.

end of proof of claim 2

By claim 1 there exists a w^1-cover E^1, $\mathcal{C}^1$ by edges and odd circuits G_1^3 with cost $\alpha_{w^1}(G_1^3) = \alpha_w(G) + \Delta - S^2(\emptyset)$. Let γ_1, γ_2 and $\tilde{\gamma}$ denote the multiplicity of $e_1, e_2, \tilde{e}$ respectively in E^1. Let β denote the sum of the multiplicities of the odd cycles in $\mathcal{C}^1$ containing b_1 (and b_2). Assume E^1 and $\mathcal{C}^1$ are such that $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$ is minimal.

Claim 3: $\gamma_i + \tilde{\gamma} + \beta = \Delta$ for i=1,2. Consequently, $\gamma_1 = \gamma_2$.

Proof of Claim 3: $\gamma_i + \tilde{\gamma} + \beta \geqq \Delta$ since E^1, $\mathcal{C}^1$ is a w^1-cover. Suppose $\gamma_1 + \tilde{\gamma} + \beta > \Delta$. Then $\gamma = 0$. Indeed, if not, then increasing γ_2 by 1 and decreasing $\tilde{\gamma}$ by 1 would yield a w^1-cover with cost $\alpha_{w^1}(G_1^3)$, and smaller $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$. Moreover, $\gamma_1 = 0$. Otherwise, take some $u_1v \in E(G^1)$. Adding u_1v to E^1 (or increasing its multiplicity in E^1) and decreasing γ_1 by 1, again yields a w^1-cover with cost $\alpha_{w^1}(G_1^3)$, and smaller $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$. Finally $\beta = 0$, contradicting the fact that $\Delta \geqq 0$. Indeed if $\beta > 0$ remove an odd circuit C with $b_1 \in V(C)$ from $\mathcal{C}^1$, and add the edges in the unique maximum cardinality matching $M \subset E(C)$ not covering b_1 to E^1. Since $M = \frac{1}{2}(|V(C)| - 1)$ this again yields a w^1-cover with cost $a_{w^1}(G_1^3)$, and smaller $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$.

end of proof of claim 3

By claim 1, there also exists a w^2-cover E^2, $\mathcal{C}^2$ by edges and odd circuits in G_2^2 with cost $\alpha_{w^2}(G_2^2) = S^2(\emptyset) + \Delta$. Let E^2 and $\mathcal{C}^2$ be such that the sum, δ, of the multiplicities of the odd cycles in $\mathcal{C}^2$ containing b is minimal.

Claim 4: f_1 and f_2 do not occur (i.e. have multiplicity 0) in E^2. Moreover $\delta = \Delta$.

Proof of Claim 4: Since the cost of E^2, $\mathcal{C}^2$ is $\alpha_{w^2}(G_2^2)$ and there exists a stable set S in G_2^2 with $\sum_{u\in S} w_u^2 = \alpha_{w^2}(G_2^2)$ and $u_1, b \notin S$ (Claim 2), the edge f_1 does not occur in E^2 ("complementary slackness"), Equivalently f_2 does not occur in E^2. The proof that $\delta = \Delta$ is similar to the proof of claim 3. end of proof of claim 4

Using E^1, $\mathcal{C}^1$ and E^2, $\mathcal{C}^2$ we are now able to construct a w-cover $\tilde{E}$, $\tilde{\mathcal{C}}$ in G by edges and odd circuits, and with cost $\alpha_w(G)$. Thus proving $\alpha_w(G) = \tilde{\rho}_w(G)$. The construction goes as follows:

Step 1: The edges in E^1 and E^2, except e_1, e_2 and $\tilde{e}$ are added to $\tilde{E}$ (with the same multiplicity). The odd circuits in $\mathcal{C}^1$ and $\mathcal{C}^2$ not containing b_1 (b_2), or b are added to $\tilde{\mathcal{C}}$.

Step 2: Let $C_1^2, \ldots, C_\Delta^2$ be the odd circuit in $\mathcal{C}^2$ containing b. (Remind that some of them may be equal.)

(i) Let $C_1^1, \ldots, C_\beta^1$ be the odd circuits in $\mathcal{C}^1$ containing b_1, define for each $i=1,\ldots,\beta$ the odd circuit $C_i \in \mathcal{C}(G)$ by $E(C_i) = E(C_i^1) \cup E(C_i^2) \setminus \{e_1, e_2, \tilde{e}, f_1, f_2\}$. Add all the odd circuits $C_1, \ldots, C_\beta$ to $\tilde{\mathcal{C}}$.
Note that, for each $i=1,\ldots,\beta$: $\frac{1}{2}|V(C_i)| - 1 = \frac{1}{2}(|V(C_i^1)| - 1) + \frac{1}{2}(|V(C_i^2)| - 1) - 2$.

(ii) Define for each $i=\beta+1,\ldots,\beta+\gamma_1$ the collection of edges M_i as the unique maximum cardinality matching in $E(C_i^2)$ not covering b. Each edge occuring in M_i ($i=\beta+1,\ldots,\beta+\gamma_1$) is added to $\tilde{E}$ (as often as it occurs in an M_i).
Note that, for each $i=\beta+1,\ldots,\beta+\gamma_1$: $|M_i| = \frac{1}{2}(|V(C_i^2)| - 1)$.

(iii) Define for each $i=\beta+\gamma_1+1,\ldots,\beta+\gamma_1+\tilde{\gamma} = \Delta$ the collection of edges N_i as the unique maximum cardinality matching in $E(C_i^2)$ not covering u_1 and not covering u_2. All the edges occuring in an N_i are added to $\tilde{E}$ (as often as they occur in an N_i).
Note that, for each $i=\beta+\gamma_1+1,\ldots,\Delta$, $|N_i| = \frac{1}{2}(|V(C_i^2)| - 1) - 1$.

Claim 5: The collections $\tilde{E}$, $\tilde{\mathcal{C}}$ form a w-cover by edges and odd circuits in G.

Proof of Claim 5: It is not hard to see that each $u \in (V_1 \setminus V_2) \cup (V_2 \setminus V_1)$ is covered w_u times by $\tilde{E}$, $\tilde{\mathcal{C}}$. (The matchings in step 2(ii) and in step 2(iii) of the construction do not decrease the number of times that a node in $V_2 \setminus V_1$ is covered.) The node u_1 is covered at least $s^2(\{u\}) - s^2(\emptyset)$ times by E^2, $\mathcal{C}^2$, and at least $w_u + s^2(\emptyset) - s^2(\{u\}) + \Delta$ times by E^1, $\mathcal{C}^1$. So u_1 is covered at least $w_u + \Delta$ times by E^1, $\mathcal{C}^1$ and E^2, $\mathcal{C}^2$ together. During the construction this amount is decreased with β by step 2(i), with γ_1 by step 2(ii), and with $\tilde{\gamma}$ by step 2(iii). Since $\beta + \gamma_1 + \tilde{\gamma} = \Delta$, $\tilde{E}$ and $\tilde{\mathcal{C}}$ cover u_1 at least w_u times. Similarly one deals with u_2, as $\gamma_1 = \gamma_2$. end of proof of claim 5

Claim 6: The cost of $\tilde{E}$, $\tilde{\mathcal{C}}$ is $\alpha_w(G)$.

Proof of Claim 6: The cost of E^1, $\mathcal{C}^1$ plus the cost of E^2, $\mathcal{C}^2$ is equal to $\alpha_{w^1}(G_1^3) + \alpha_{w^2}(G_2^2) = \alpha_w(G) + \Delta - s^2(\emptyset) + s^2(\emptyset) + \Delta = \alpha_w(G) + 2\Delta$. During the construction we lost exactly: 2β in step 2(i), $\tilde{\gamma}$ in step 2(iii), and $2\gamma_1 + \tilde{\gamma}$ by ignoring the edges e_1, e_2, $\tilde{e}$. So the cost of $\tilde{E}$, $\tilde{\mathcal{C}}$ is $\alpha_w(G) + 2\Delta - 2\beta - \tilde{\gamma} - (2\gamma_1+\tilde{\gamma}) = \alpha_w(G)$. end of proof of claim 6

Claim 5 and 6 together yield that $\alpha_w(G) = \tilde{\rho}_w(G)$.

Case IIb: $\Delta \leqq 0$.
The proof of this case is similar to the proof of case IIa. Therefore we shall only give the beginning of it.
Let b be the new node in G_1^2 and let b_1 and b_2 be the new nodes in G_2^3 (see figure 6).

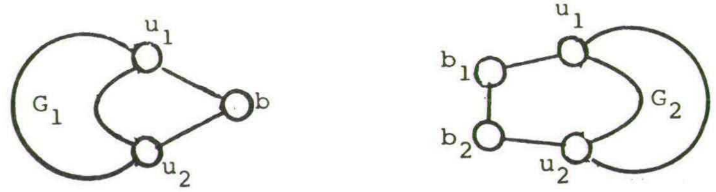


figure 6

Define the following weight functions:

$$w^1 \in \mathbb{Z}^{V(G_1^2)} \text{ by } w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus V_2 \\ s^2(\{u\}) - s^2(\emptyset) - \Delta & \text{if } u \in \{u_1, u_2\} \\ -\Delta & \text{if } u = b; \end{cases}$$

$$w^2 \in \mathbb{Z}^{V(G_2^3)} \text{ by } w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus V_1 \\ w_u + s^2(\emptyset) - s^2(\{u\}) & \text{if } u \in \{u_1, u_2\} \\ -\Delta & \text{if } u \in \{b_1, b_2\}. \end{cases}$$

The first thing to be proved now is

Claim 7: $\alpha_{w^1}(G_1^2) = \alpha_w(G) - \Delta - s^2(\emptyset)$ and $\alpha_{w^2}(G_2^3) = -\Delta + s^2(\emptyset)$. Moreover, for each $U \in \{\{u_1, b_1\}, \{b_1, b_2\}, \{u_2, b_2\}\}$ there exists a stable set S in G_2^3 with $\sum_{u \in S} w_u = \alpha_{w^2}(G_2^2)$, and $S \cap U = \emptyset$.

From this point it is not hard to see how arguements similar to those used in Case IIa prove that $\alpha_w(G) = \tilde{\rho}_w(G)$. □

Remarks:

The proof of Case I of the proof above is identical with the proof of Theorem 4.1 in Chvátal [1975]. The techniques used in Case IIa and Case IIb of the proof are similar to the techniques used by Boulala and Uhry [1979]. However they restrict G_2 to paths and odd cycles. Sbihi and Uhry [1984] also use the decompositions of Case II. In their case G_2 is always bipartite. Recently, Barahona and Mahjoub [1986] derived a construction to derive all facets of the stable polytope of G, in case G has a two node cutset $\{u_1, u_2\}$, from the facets of the stable set polytopes of G_1^+, and G_2^+. (Here G_1 and G_2 are as in the proof above, G_i^+ is derived from G_i by adding a five cycle $\{u_1, b, u_2, b_1, b_2\}$).

3. Computational Aspects

In this final section we give some attention to the computational complexity of the problems: Given G and $w \in \mathbb{Z}^{V(G)}$, determine $\alpha_w(G)$, $\tilde{\rho}_w(G)$, $\rho_w(G)$, $\tau_w(G)$, $\tilde{\nu}_w(G)$, and $\nu_w(G)$. Well known results are:

It is NP-hard to determine $\alpha_w(G)$, $\tau_w(G)$, even if $w \equiv 1$ (Karp [1972]). There exists a polynomial time algorithm to determine a maximum cardinality w-matching, or a minimum cardinality w-edge-cover (Edmonds [1965] for $w \equiv 1$, Cunningham and Marsh [1978] for general w).

Pulleyblank observed that determining $\tilde{\rho}_w(G)$, or $\tilde{\nu}_w(G)$ is NP-hard, even is $w \equiv 1$. There is a reduction from PARTITION INTO TRIANGLES (cf. Garey and Johnson [1979]).
Indeed, given a graph G there is partition of V(G) into triangles in G if and only if $|\tilde{\rho}(G)| \leqq \frac{1}{3}|V(G)|$. Since PARTITION INTO TRIANGLES remains NP-complete for planar graphs (Dyer and Frieze [1986]), determining $\tilde{\rho}(G)$, or $\tilde{\nu}(G)$ remains NP-hard even if G is planar.
If G has no odd-K_4 $\tilde{\rho}_w(G)$ and $\tilde{\nu}_w(G)$ can be found efficiently (i.e. in polynomial time). Indeed, an algorithm can be obtained from the proofs in section 2 (proof of Theorem 2.2, proof of Theorem 1.8). The only difficulty is finding an orientation $\vec{A}$ of descrepancy 1, and solving (2.3) and (2.4).

<u>Finding $\vec{A}$</u>: Using a constructive characterization of graphs with no odd-K_4 and no odd-K_3 (Lovasz, Schrijver, Seymour, Truemper [1984], cf. Gerards-Schrijver [1986]) similar to the result in remark (iii) of section 1, one easily derives a polynomial time algorithm to find $\vec{A}$, or to decide that $\vec{A}$ does not exist (i.e. that G has an odd-K_4 or an odd-K_3^2, Theorem 2.1).

<u>Solving (2.3) and (2.4)</u>: Define the directed graph $D = (V(D), A(D))$ by: $V(D) := \{u_i \mid u \in V(G);\ i=1,2\}$; $A(D) := \{\overrightarrow{u_1u_2} \mid u \in V(G)\} \cup \{\overrightarrow{u_2v_1} \mid \overrightarrow{uv} \in \vec{A}\}$.
Then (2.3) is equivalent to the min-cost-circulation problem:

$$(3.2) \qquad \min \sum_{\overrightarrow{uv} \in \vec{A}} g_{\overrightarrow{u_2v_1}}$$
$$\text{s.t. } g \text{ is a non-negative circulation in } D,$$
$$g_{\overrightarrow{u_1u_2}} \geqq w_u \quad (u \in V(D)).$$

(3.2) can be efficiently solved by the out-of-kilter method of Ford and Fulkerson [1962]. (Note that since the cost function is $\{0,1\}$-valued, there is no need to appeal to more sophisticated techniques as used by Edmonds and Karp [1972], Röck [1980] or Tardos [1985].)

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182 Cristina Pennavaja
Periodization approaches of capitalist development.
A critical survey

183 J.P.C. Kleijnen, G.L.J. Kloppenburg and F.L. Meeuwsen
Testing the mean of an asymmetric population: Johnson's modified T test revisited

184 M.O. Nijkamp, A.M. van Nunen
Freia versus Vintaf, een analyse

185 A.H.M. Gerards
Homomorphisms of graphs to odd cycles

186 P. Bekker, A. Kapteyn, T. Wansbeek
Consistent sets of estimates for regressions with correlated or uncorrelated measurement errors in arbitrary subsets of all variables

187 P. Bekker, J. de Leeuw
The rank of reduced dispersion matrices

188 A.J. de Zeeuw, F. van der Ploeg
Consistency of conjectures and reactions: a critique

189 E.N. Kertzman
Belastingstructuur en privatisering

190 J.P.C. Kleijnen
Simulation with too many factors: review of random and group-screening designs

191 J.P.C. Kleijnen
A Scenario for Sequential Experimentation

192 A. Dortmans
De loonvergelijking
Afwenteling van collectieve lasten door loontrekkers?

193 R. Heuts, J. van Lieshout, K. Baken
The quality of some approximation formulas in a continuous review inventory model

194 J.P.C. Kleijnen
Analyzing simulation experiments with common random numbers

195 P.M. Kort
Optimal dynamic investment policy under financial restrictions and adjustment costs

196 A.H. van den Elzen, G. van der Laan, A.J.J. Talman
Adjustment processes for finding equilibria on the simplotope

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Variance heterogeneity in experimental design

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Selecting random number seeds in practice

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Regression analysis of simulation experiments: functional software specification

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An algorithm for the linear complementarity problem with upper and lower bounds

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Alternative specification tests for Tobit and related models

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202 J.H.F. Schilderinck
Interregional Structure of the European Community. Part III

203 Antoon van den Elzen and Dolf Talman
A new strategy-adjustment process for computing a Nash equilibrium in a noncooperative more-person game

204 Jan Vingerhoets
Fabrication of copper and copper semis in developing countries. A review of evidence and opportunities.

205 R. Heuts, J. v. Lieshout, K. Baken
An inventory model: what is the influence of the shape of the lead time demand distribution?

206 A. v. Soest, P. Kooreman
A Microeconometric Analysis of Vacation Behavior

207 F. Boekema, A. Nagelkerke
Labour Relations, Networks, Job-creation and Regional Development
A view to the consequences of technological change

208 R. Alessie, A. Kapteyn
Habit Formation and Interdependent Preferences in the Almost Ideal Demand System

209 T. Wansbeek, A. Kapteyn
Estimation of the error components model with incomplete panels

210 A.L. Hempenius
The relation between dividends and profits

211 J. Kriens, J.Th. van Lieshout
A generalisation and some properties of Markowitz' portfolio selection method

212 Jack P.C. Kleijnen and Charles R. Standridge
Experimental design and regression analysis in simulation: an FMS case study

213 T.M. Doup, A.H. van den Elzen and A.J.J. Talman
Simplicial algorithms for solving the non-linear complementarity problem on the simplotope

214 A.J.W. van de Gevel
The theory of wage differentials: a correction

215 J.P.C. Kleijnen, W. van Groenendaal
Regression analysis of factorial designs with sequential replication

231 J.P.C. Kleijnen
Analyzing simulation experiments with common random numbers

232 A.B.T.M. van Schaik, R.J. Mulder
On Superimposed Recurrent Cycles

233 M.H.C. Paardekooper
Sameh's parallel eigenvalue algorithm revisited

234 Pieter H.M. Ruys and Ton J.A. Storcken
Preferences revealed by the choice of friends

235 C.J.J. Huys en E.N. Kertzman
Effectieve belastingtarieven en kapitaalkosten