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RESEARCH MEMORANDUM

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HOMOTOPY INTERPRETATION OF PRICE ADJUSTMENT PROCESSES

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Abstract

In this paper we consider a number of always converging price adjustment processes, which have been introduced recently, in a homotopy setting. The main feature of these processes is that they memorize the starting vector along the path. Here we show that the paths followed by the processes can be viewed upon as being projections of zero point sets of appropriate homotopies. By doing so we put the processes in a unifying and familiar framework. This makes it easy to derive for example conditions under which the processes converge monotone to an equilibrium. Besides, we propose a new price adjustment process related to a very simple homotopy.

Keywords: homotopy, price adjustment, equilibrium.

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1. Introduction

Recently, some price adjustment processes have been introduced, which always converge to an equilibrium price vector in a pure exchange economy (see [4]). These processes can start at an arbitrary price vector and find equilibrium prices via a sequence of price adaptions. The most important feature of these processes is that they memorize the starting vector. In this paper we consider these processes in a homotopy framework. Solving the equilibrium problem via the homotopy approach means that we start in a solution of a trivial problem on an artificial set. The trivial problem is then continuously deformed to the real problem on the set of interest. By following a path of solutions for these intermediate problems, which starts with the trivial solution, eventually an equilibrium is reached. We show that the price adjustment processes mentioned above can be viewed upon as being projections of such a homotopy path on the price space. By doing so, we put all these processes in a unifying framework. Since the homotopy approach is a standard tool in mathematics and has a large theoretical background it is rather easy to derive conditions under which the processes exist and converge (monotonically) towards an equilibrium. Furthermore, it appears that the homotopy parameter induces a measure for the accuracy of approximation. Another contribution of this paper is the presentation of a new price adjustment process resulting from a very elegant and simple homotopy.

As already indicated, the problem is the search for an equilibrium price vector in a pure exchange economy. In such an economy there are a finite number of commodities, say n+1, and of consumers each having a vector of initial endowments. Exchanges of goods are based on relative prices. All consumers in this economy exchange goods in order to maximize their utility under the constraint imposed by their initial wealth. An equilibrium price vector is a vector of prices at which for all goods demand equals supply while no consumer can improve upon his situation. All relevant information of such an economy can be captured in an excess demand function which relates to each price vector the corresponding vector of excess demands. Thus, an excess demand function, denoted by z, can be seen as a function from the set $\mathbb{R}^{n+1}_{++} := \{p \in \mathbb{R}^{n+1} | p_j > 0,\ j = 1,\dots,n+1\}$ to $\mathbb{R}^{n+1}$. Standard conditions on z are

$$\text{i)}\quad z(\lambda p) = z(p)\ ,\qquad \forall\lambda > 0 \text{ and } \forall p \in \mathbb{R}^{n+1}_{++} \text{ (homogeneity)}$$

$$\text{ii)}\quad p^{\top}z(p) = 0\ ,\qquad \forall p \in \mathbb{R}^{n+1}_{++} \text{ (complementarity)} \tag{1.1}$$

$$\text{iii)}\ \forall p \in \operatorname{bd} \mathbb{R}^{n+1}_{+}\backslash\{0\}\ \exists\delta > 0\ \forall p' \in \{x \in \mathbb{R}^{n+1}_{++} \,|\, \|p-x\| < \delta\}$$

$$[p_j = 0 \Rightarrow z_j(p') > 0]\ ,$$

with $\mathbb{R}^{n+1}_{+} := \{p \in \mathbb{R}^{n+1} | p_j \geq 0,\ j = 1,\dots,n+1\}$ and $\|.\|$ the Euclidean norm. Furthermore, we assume that z is continuously differentiable (C^1) on $\mathbb{R}^{n+1}_{++}$. To guarantee the latter certain conditions on the preference relations of the consumers are needed. The economic interpretation of (1.1) is straightforward. Condition i) indicates that only relative prices matter. Condition ii) is also known as Walras' law and says that all consumers spend their total income. Condition iii) is a desirability condition, roughly stating that if the price of a good is (relatively) small, the demand for it exceeds its supply. Of course, an equilibrium price vector is a vector p^* for which $z(p^*) = 0$, and such an equilibrium always exists (see [1]).

Well-known price adjustment processes for finding an equilibrium are the standard Walras tatonnement process and the Newton-like method of Smale (see [7]), which both follow the path of solutions to a differential equation. However, these methods can fail to converge. This is illustrated by Scarf [6] for the Walras process and by Keenan [2] for the process of Smale. Recently, a class of always converging price adjustment processes has been introduced. These processes do not operate on $\mathbb{R}^{n+1}_{++}$ but on the n-dimensional unit simplex S^n, defined by $S^n := \{p \in \mathbb{R}^{n+1}_{+} | \Sigma_{j=1}^{n+1} p_j = 1\}$. This restriction is allowed because of condition i). Another common feature of these processes is that they memorize the starting vector. Thus, these processes are not completely myopic as the processes of Walras and Smale.

Here we consider three price adjustment processes discussed in van der Laan and Talman [4]. They all follow the set of solutions to a system of complementary equations involving the starting vector v in S^n. For convenience we take v in the relative interior $\operatorname{int}(S^n)$ of S^n. The first process focusses on minimal excess demands and follows a path of vectors p in S^n satisfying for $j \in I_{n+1} := \{1,\dots,n+1\}$,

$$p_j/v_j = \max_h p_h/v_h \quad \text{if } z_j(p) > \min_h z_h(p)$$
$$p_j/v_j \le \max_h p_h/v_h \quad \text{if } z_j(p) = \min_h z_h(p) \ . \tag{1.2}$$

In economic terms, along the path the prices of the commodities not having minimal excess demand are kept maximal relative to their initial values. Initially at v, the price of the commodity with the lowest excess demand is decreased whereas all other prices are increased proportionally and kept relatively equal.
The second process focusses on maximal excess demands and generates from v a path of vectors p in S^n satisfying for $j \in I_{n+1}$,

$$p_j/v_j = \min_h p_h/v_h \quad \text{if } z_j(p) < \max_h z_h(p)$$
$$p_j/v_j \ge \min_h p_h/v_h \quad \text{if } z_j(p) = \max_h z_h(p) \ . \tag{1.3}$$

The third process is slightly more sophisticated and takes the signs of the excess demands into account. It follows a path of price vectors p from v satisfying for $j \in I_{n+1}$,

$$p_j/v_j = \max_h p_h/v_h \quad \text{if } z_j(p) > 0$$
$$\min_h p_h/v_h \le p_j/v_j \le \max_h p_h/v_h \quad \text{if } z_j(p) = 0 \tag{1.4}$$
$$p_j/v_j = \min_h p_h/v_h \quad \text{if } z_j(p) < 0 \ .$$

In economic terms, this process initially increases (decreases) the prices of the commodities having positive (negative) excess demand. In general, the prices of the commodities with positive (negative) excess demand are, relative to their initial values, maximal (minimal), whereas the relative prices of goods in equilibrium vary between these bounds.

The paths originating at v and satisfying (1.2), (1.3), or (1.4) are generically piecewise C^1 and can be followed arbitrarily close by a simplicial algorithm. To apply such an algorithm the simplex S^n is triangulated into simplices. The algorithm then operates with the same system of complementary equations but with z replaced by its piecewise linear approximation. Van der Laan and Talman [3] showed that for the fixed point problem on S^n the piecewise linear path generated by such a simplicial algorithm can be considered as the projection of the zero point set of a homotopy. In this paper we present a homotopy setting for the piecewise C^1 paths of each of the three processes defined above. The underlying homotopy functions are all functions from the convex hull V of $U\times\{0\}$ and $W\times\{1\}$ to $\mathbb{R}^{n+1}$, where U and W are n-dimensional subsets of $\mathbb{R}^{n+1}$ and $\text{int}(S^n)$, respectively. Both U and W contain v and for all $u \in U$, $\Sigma_{i=1}^{n+1}u_i = 1$. The general form of such a homotopy function h is given by

$$h(x,\delta) = (1-\delta)(v-u) + \delta g(p) , \qquad (x,\delta) \in V , \tag{1.5}$$

where (x,δ) uniquely determines $u \in U$ and $p \in W$ such that $x = (1-\delta)u + \delta p$. The C^1-function $g : W \to \{x \in \mathbb{R}^{n+1} | \Sigma_{i=1}^{n+1}x_i = 0\}$ is related to z in such a way that $g(p) = 0$ iff $z(p) = 0$. Then, h will be a (piecewise) C^1-function from an (n+1)-dimensional set in $\mathbb{R}^{n+1}\times[0,1]$ into an n-dimensional subset of $\mathbb{R}^{n+1}$. Furthermore, the set $h^{-1}(0)$ on level zero ($\delta=0$) equals $\{(v,0)\}$ while $h^{-1}(0)$ on level one ($\delta=1$) is the set of points $(p^*,1)$ for which $z(p^*) = 0$. If $0 \in \mathbb{R}^{n+1}$ is a regular value of h then the set $h^{-1}(0)$ in general consists of disjoint piecewise C^1 paths and loops (see Milnor [5]). For each process we prove that the set $h^{-1}(0)$ of the corresponding homotopy contains a path connecting $(v,0)$ and a $(p^*,1)$ with $z(p^*) = 0$. The projection of this path on W then equals the path followed by that process.

In this paper we also introduce a price adjustment process operating on the positive part of the ball in $\mathbb{R}^{n+1}$ with centre $\frac{1}{2}v$ and passing through the starting vector v in $\mathbb{R}^{n+1}_{++}$. In fact this set yields a different normalization of the price space. The homotopy function related to this process is a function $h : \mathbb{R}^{n+1}_{++}\times[0,1] \to \mathbb{R}^{n+1}$, defined by

$$h(p,\delta) = (1-\delta)(v-p) + \delta z(p) , \qquad (p,\delta) \in \mathbb{R}^{n+1}_{++}\times[0,1] . \tag{1.6}$$

A homotopy function as in (1.6) is also known as the homotopy in standard form. Observe the differences between (1.5) and (1.6). The path of zero points induced by (1.6) leading from $(v,0)$ to $(p^*,1)$ with $z(p^*) = 0$ can be followed by a simplicial algorithm operating in $\mathbb{R}^{n+1}_{+}\times[0,1]$.

This paper is organized as follows. In Section 2 we discuss the new adjustment process derived from the standard homotopy. Section 3 considers how the price adjustment processes related to the maximal and minimal excess demands can be put into a homotopy framework. In Section 4 the same is done for the process focussing on the sign pattern of the excess demands.

2. The standard homotopy

In this section we consider the price adjustment process induced by the set of zero points of the homotopy function h in standard form, i.e.,

$$h(p,\delta) = (1-\delta)(v-p) + \delta z(p) , \qquad (p,\delta) \in \mathbb{R}^{n+1}_{++}\times[0,1] ,$$

where z is a continuously differentiable excess demand function and v is an arbitrarily chosen point in $\mathbb{R}^{n+1}_{++}$. Because z is assumed to be a C^1-function, the homotopy function h is a C^1-function from an (n+2)-manifold with boundary into an (n+1)-manifold. Throughout this paper we use the shorthand notation k-manifold for k-dimensional manifold. The boundary of the domain of h is equal to $(\mathbb{R}^{n+1}_{++}\times\{0\}) \cup (\mathbb{R}^{n+1}_{++}\times\{1\})$. So, if $0 \in \mathbb{R}^{n+1}$ is a regular value of h then $h^{-1}(0)$ is a C^1 1-manifold, i.e., a collection of disjoint paths and loops. Moreover, an end point of a path in $h^{-1}(0)$ either lies in $\mathbb{R}^{n+1}_{++}\times\{0\}$ or in $\mathbb{R}^{n+1}_{++}\times\{1\}$. Clearly, $(p,0) \in h^{-1}(0)$ implies $p = v$ and so there is only one end point on level 0. Furthermore, $(p^*,1) \in h^{-1}(0)$ if and only if $z(p^*) = 0$. We will now show that the path having the point $(v,0)$ as end point on level 0 also has an end point in $\mathbb{R}^{n+1}_{++}\times\{1\}$. The projection of this path on $\mathbb{R}^{n+1}_{++}$ then gives the path of an adjustment process from v to a zero point of z. We remark that we cannot take the homotopy function h on the relative interior of the unit simplex S^n, since for any $\delta \in (0,1)$ the set $h^{-1}(0) \cap (S^n\times\{\delta\})$ is in general empty.

In order to show that $h^{-1}(0)$ contains a path from $(v,0)$ to a point in $\mathbb{R}^{n+1}_{++}\times\{1\}$ first notice that $h(p,\delta) = 0$, $0 \le \delta < 1$, implies $p^{\top}(v-p) = 0$, since $p^{\top}z(p) = 0$ and $\delta \ne 1$. Hence, let B be the part of the ball in $\mathbb{R}^{n+1}$ around $\frac{1}{2}v$ and passing through v, which lies in the positive orthant, i.e.,

$$B := \{p \in \mathbb{R}^{n+1}_{++} \,|\, (p-\tfrac{1}{2}v)^{\top}(p-\tfrac{1}{2}v) = \tfrac{1}{4}v^{\top}v\} ,$$

then all zero points of h between the levels 0 and 1 (and on level 0) lie in $B\times[0,1)$. Since the closure of B is a compact subset of $\mathbb{R}^{n+1}_{+}\backslash\{0\}$ and because of the continuity and the desirability condition on z, there exists an ε, $0 < \varepsilon < \min_h v_h$, such that $z_i(p) > 0$ whenever $p_i \le \varepsilon$ and $p \in B$. Now let $B(\varepsilon)$ be the compact subset of $\mathbb{R}^{n+1}_{++}$ defined by

$$B(\varepsilon) := \{p \in B | p_i \geq \varepsilon \text{ for all } i \in I_{n+1}\} ,$$

then the next lemma guarantees that all zero points of h in $B\times[0,1]$ lie in $B(\varepsilon)\times[0,1]$.

Lemma 2.1. Let (p,δ) be a point in $B \times [0,1]$. Then $h_i(p,\delta) > 0$ if $p_i \leq \varepsilon$.

Proof. By definition we have for all $(p,\delta) \in B\times[0,1]$

$$h_j(p,\delta) = (1-\delta)(v_j-p_j) + \delta z_j(p) , \qquad j \in I_{n+1} .$$

If $p_i \leq \varepsilon$ we obtain since $v_i > \varepsilon$ that $v_i - p_i > 0$ whereas from above we know that $z_i(p) > 0$.

□

Consequently, if 0 is a regular value of h, then the path P, having $(v,0)$ as an end point cannot cross the boundary of $B(\varepsilon)$ between the levels 0 and 1 and therefore remains in the compact set $B(\varepsilon)\times[0,1]$. Thus P must have another end point on $B(\varepsilon)\times\{0\}$ or $B(\varepsilon)\times\{1\}$. Since $(v,0)$ is the only zero point of h on level 0, the other end point is a point $(p^*,1)$ on level 1 at which $z(p^*) = 0$. Moreover, all other paths in $h^{-1}(0)$, if any, also lie in $B(\varepsilon)\times[0,1]$ and connect two zero points of z on level 1, whereas all loops in $h^{-1}(0)$ lie in $B(\varepsilon)\times(0,1)$.

For 0 to be a regular value of h, it is required that the Jacobian matrix of h is of full rank at all zero points of h. The first n+1 columns, $D_p h$, of this $(n+1)\times(n+2)$ matrix contain the derivatives of h with respect to p while the last column, $D_\delta h$, is the derivative of h with respect to δ. More precisely, the Jacobian matrix at (p,δ) is equal to

$$Dh(p,\delta) = [-(1-\delta)I + \delta Dz(p) \mid p - v + z(p)] ,$$

where I is the $(n+1)\times(n+1)$ identity matrix and $Dz(p)$ is the $(n+1)\times(n+1)$ matrix of derivatives of z at p. Thus, the value 0 is regular for h if the matrix $Dh(p,\delta)$ is of rank n+1 for all $(p,\delta) \in B(\varepsilon)\times[0,1]$ for which $h(p,\delta) = 0$. If $0 \leq \delta < 1$, in general this holds. However, due to the complementarity condition the rank of $Dh(p,\delta)$ cannot be equal to n+1 if

$h(p,\delta) = 0$ and $\delta = 1$, since $p^T Dz(p) = 0$ and $[p-v+z(p)]^T p = 0$ and hence $p^T Dh(p,1) = 0$. More precisely, if the path P reaches level 1 at the point $(p^*,1)$, then bifurcation takes place, since then $h(\lambda p^*,1) = 0$ for all $\lambda >$ 0. Because $(p,\delta) \in h^{-1}(0)$ lies in $B\times[0,1]$ when $0 \le \delta < 1$, it is sufficient to require the regularity of 0 for h only on the set $B \times [0,1]$. In that case the set B can be seen as a specific normalization of the domain of z and there is no bifurcation in B on level 1.

We conclude this section with an illustration of $h^{-1}(0)$ and provide an interpretation of the path P, when projected on $\mathbb{R}^{n+1}_{++}$, as an adjustment process. In Figure 2.1 the set $h^{-1}(0)$ consists of the path P and a path Q connecting two zero points of z on level 1. Notice that $h^{-1}(0)$ cannot contain loops when n = 1.

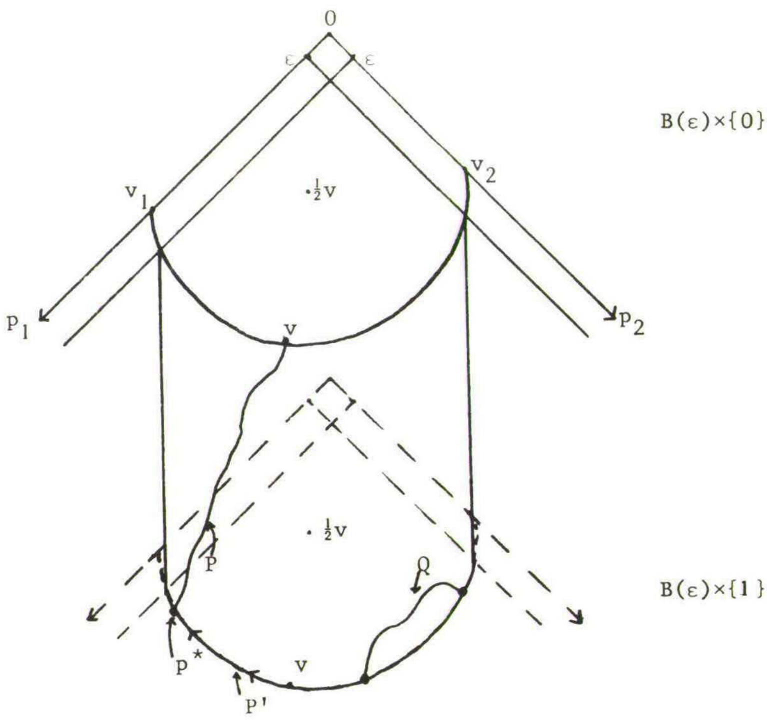

Figure 2.1. Illustration of $h^{-1}(0)$ in $B(\varepsilon)\times[0,1]$, n = 1.

Now let P' be the projection of P on $\mathbb{R}^{n+1}_{++}$, then (assuming $z(v)\neq 0$) P' must be a C^1 curve in $B(\varepsilon)$ connecting v and a zero point p^* of z. This follows from the fact that $(p,\delta_1) \in h^{-1}(0)$ and $(p,\delta_2) \in h^{-1}(0)$ imply $\delta_1 = \delta_2$. More precisely, the derivative $D_\delta h(p,\delta) = p-v+z(p)$ has rank 1 along the path P. Observe that this only does not hold when at $p = v$ we have $z(p) = 0$. But then P is the line segment $\{v\}\times[0,1]$ and $P' = \{v\}$. So, if v is not a zero point of z, the path P' is a C^1 curve in $B(\varepsilon)$. For any point p, $p \neq v$, along the path P', we have that $z(p) = \beta(p-v)$ for some $\beta \geq 0$. Thus, in economic terms, at prices p on the path induced by the set of zero points of the standard homotopy, the excess demand is a multiple of the difference between p and the initial price vector v. Notice the similarity to the classical Walrasian adjustment process when we replace initial by previous (in time). An illustration of this interpretation is given in Figure 2.2.

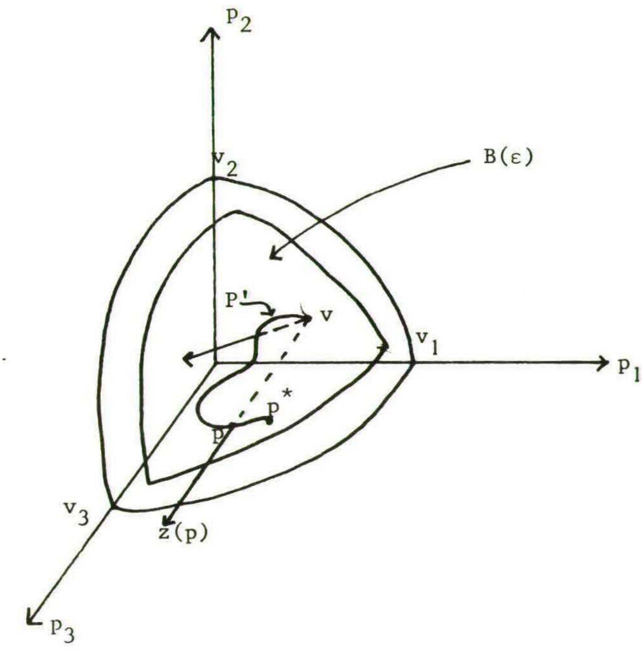

Figure 2.2. Illustration of P' for n = 2.

Notice further that, if $h(p,\delta) = 0$, $(p,\delta) \neq (v,0)$, and so $z(p) = \beta(p-v)$ for some $\beta \geq 0$, we must have $\beta = (1-\delta)/\delta$. Thus, β decreases if δ

increases and δ gives an indication for how close $z(p)$ is equal to zero. We say that the process converges monotone if δ increases monotone from 0 to 1 along the path P. The latter only holds if $D_p h(p,\delta) = -(1-\delta)I + \delta Dz(p)$ always has rank n+1 along P.

Finally, we remark that the path of zero points of h can be followed approximately by a simplicial homotopy method on $\mathbb{R}_+^{n+1} \times [0,1]$.

3. Homotopy functions based on minimal or maximal excess demands

In this section we describe how the paths of points generated by the adjustment processes induced by (1.2) and (1.3) can be interpreted as the projection on S^n of a path of zero points of some homotopy function. Again, the starting vector is an arbitrary point in $\mathrm{int}(S^n)$ and is denoted by v. Furthermore, from the conditions on z we derive the existence of an $\eta > 0$ such that for all p in $\mathrm{int}(S^n)$ and all $i \in I_{n+1}$, $z_i(p) > 0$ if $p_i \le \eta$. For the process related to (1.2) we obtain a homotopy on the convex hull V of $U^n\times\{0\}$ and $S^n(\eta)\times\{1\}$. The set U^n is the convex hull of the n+1 vectors $u(j) := v - (n+1)^{-1}e + e(j)$, $j \in I_{n+1}$, where e is the (n+1)-vector of ones while e(j) is the j-th unit vector in $\mathbb{R}^{n+1}$. The set $S^n(\eta)$ is defined by

$$S^n(\eta) := \{p \in S^n \mid p_i \ge \eta v_i,\ i \in I_{n+1}\} .$$

The set of zero points of this homotopy function will induce a path P' of points in S^n connecting the arbitrarily chosen starting point v and a zero point p* of z. Vectors p along the path P' satisfy (1.2), i.e.,

$$p_j/v_j = \max_h p_h/v_h \qquad \text{if } z_j(p) > \min_h z_h(p) . \tag{3.1}$$

In the second part of this section we consider the homotopy which induces the path followed by the (price) adjustment process focussing on the maximal excess demands. This homotopy is defined on the convex hull $\bar{V}$ of $\bar{U}^n\times\{0\}$ and $S^n(\eta)\times\{1\}$, where $\bar{U}^n$ is the convex hull of the vectors $\bar{u}(j) := v + (n+1)^{-1}e - e(j)$, $j \in I_{n+1}$. The set of zero points of this homotopy induces a path of points $\bar{P}'$ in S^n connecting v and a zero point of z. The points p on this path satisfy (1.3), i.e.,

$$p_j/v_j = \min_h p_h/v_h \qquad \text{if } z_j(p) < \max_h z_h(p) . \tag{3.2}$$

Let us start with the first homotopy. The set U^n is an n-dimensional simplex congruent to S^n, lying in the affine hull of S^n, and having v as its barycentre. In order to define the homotopy h on the convex hull V of $U^n\times\{0\}$ and $S^n(\eta)\times\{1\}$, we have to subdivide V in an appropriate way into cells. In fact, this subdivision is needed to relate each (x,δ) in V to a unique pair $u \in U^n$ on level 0 and $p \in S^n(\eta)$ on level 1 (see (1.5)). The subdivision of V is completely determined by the subdivision of $S^n(\eta)$ on level 1 into subsets $A(T)$, $T \subsetneq I_{n+1}$, defined by

$$A(T) := \{p \in S^n(\eta) \mid p_j/v_j = \max_h p_h/v_h,\ j \notin T\} .$$

This subdivision of $S^n(\eta)$ is illustrated in Figure 3.1 for $n = 2$ and corresponds to the left part of expression (3.1). Notice that $A(\emptyset)$ is equal to $\{v\}$. Moreover, for $i \in I_{n+1}$, the set $A(\{i\})$ is a segment of the line through v and the vertex $e^\eta(i)$ of $S^n(\eta)$, where $e^\eta_j(i) = \eta v_j$ if $j \neq i$ and $e^\eta_j(i) = 1 - (1-v_i)\eta$ if $j = i$. In general, the dimension of $A(T)$ is equal to $t = |T|$, being the number of elements in T.

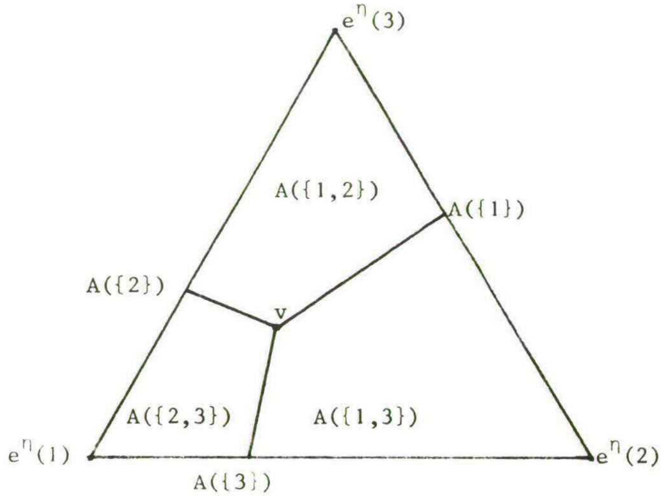

Figure 3.1. Subdivision of $S^n(\eta)$ into subsets $A(T)$, $n = 2$.

Furthermore, let $U^n(T)$, $T \subsetneq I_{n+1}$, be the face of U^n defined by

$$U^n(T) := \mathrm{conv}(\{u(j) \mid j \notin T\}) .$$

The set V is now subdivided into sets $C(T)$, $T \subsetneq I_{n+1}$, where $C(T)$ is the convex hull of $A(T)\times\{1\}$ and $U^n(T)\times\{0\}$. In particular, $C(\emptyset)$ is the convex hull of the point v on level 1 and the set $U^n \times \{0\}$. For general T, the set C(T) is a cell of dimension n+1 (see Figure 3.2).

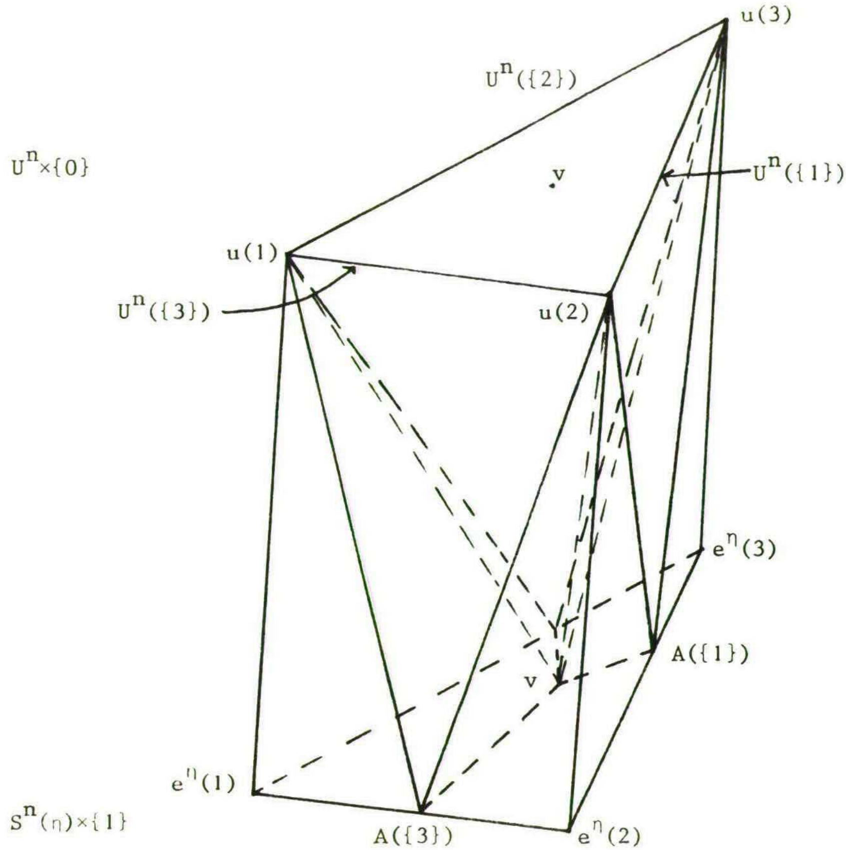

Figure 3.2. Subdivision of V into cells C(T), n = 2.

In Theorem 3.1 we prove that the sets C(T) indeed form a subdivision of V. More precisely, we show that for any $(x,\delta) \in V$ with $0 < \delta < 1$, there exists a unique (possibly empty) proper subset T of I_{n+1} such that for unique vectors p and u

$$x = (1-\delta)u + \delta p, \tag{3.3}$$

with p in A(T) and u in the relative interior $\mathrm{int}(U^n(T))$ of $U^n(T)$. We call p and u the projection of the point (x,δ) on level 1 and level 0, respectively.

Now we are ready to define an appropriate homotopy function on V. This function is defined by

$$h(x,\delta) = (1-\delta)(v-u) + \delta(z(p)-\bar{z}(p)e),\ (x,\delta) \in V ,$$

where $\bar{z}(p) = (n+1)^{-1} \Sigma_{j=1}^{n+1} z_j(p)$ and $x = (1-\delta)u + \delta p$ with u and p as in (3.3). Because u and p are uniquely determined by (x,δ), the function h is well-defined on V. Moreover, h is a piecewise C^1-function from V to $\mathbb{R}^{n+1}$ deforming the trivial function f with $f(u) = v-u$ on level 0 into the function $\hat{z}$ with $\hat{z}(p) = z(p) - \bar{z}(p)e$ on level 1.

Let $h^{-1}(0)$ be the set of zero points of h in V, then $h^{-1}(0) \cap (U^n\times\{0\})$ consists of the unique element $(v,0)$. Because of conditions ii) and iii) on z we also obtain that $h^{-1}(0) \cap (S^n(\eta)\times\{1\})$ is equal to the set of zero points of z. Furthermore, $(x,\delta) \in h^{-1}(0)$, $0 < \delta < 1$, implies according to (3.3) that there is a unique set T such that $x = (1-\delta)u + \delta p$, with $p \in A(T)$ and $u = \Sigma_{h\notin T}\mu_h u(h) \in \mathrm{int}(U^n(T))$, while

$$z(p) = \bar{z}(p)e + (1-\delta)\delta^{-1}\Big(\sum_{h\notin T} \mu_h e(h)-(n+1)^{-1}e\Big) . \tag{3.4}$$

Hence,

$$z(p) = \beta e + \sum_{h\notin T} \mu_h' e(h)$$

for some numbers β and $\mu_h' > 0$, $h \notin T$. Thus, to each (x,δ), $\delta \neq 0,1$, with $h(x,\delta) = 0$, there corresponds a unique p in $S^n(\eta)$ for which (3.1) holds. From (3.4) we also obtain that

$$\min_h z_h(p) = \bar{z}(p) - (1-\delta)/(\delta(n+1)) .$$

Thus, the homotopy parameter δ at $(x,\delta) \in h^{-1}(0)$ induces a measure for the difference between the minimal and the average excess demand at the projection p of (x,δ) on level 1. The difference decreases when δ increases and is equal to zero when $\delta = 1$.

We would like to have that the set $h^{-1}(0)$ contains a path P in V connecting the unique zero point $(v,0)$ of h on level 0 with a zero point p^* of z on level 1. Projected on S^n, this path would then give a path P' of points connecting v and p^* and satisfying (3.1), since according to (3.4) for $p \in P'$ there is only one $(x,\delta) \in V$ such that $h(x,\delta) = 0$ and $x = (1-\delta)u + \delta p$.

We first prove that a path in $h^{-1}(0)$ cannot cross the boundary of V between the levels 0 and 1. So, let (x,δ), $0 < \delta < 1$, with $h(x,\delta) = 0$, be a point in the boundary bd V of V, i.e., $x_i = (1-\delta)(v_i-(n+1)^{-1}) + \delta\eta v_i$, for some $i \in I_{n+1}$. Let T be such that $x = (1-\delta)u + \delta p$ with $p \in A(T)$ and $u \in \text{int}(U^n(T))$. Clearly, $p_i = \eta v_i$ and $u_i = v_i - (n+1)^{-1}$, and hence $i \in T$. However, $i \in T$ implies that $z_i(p) = \min_h z_h(p) \leq 0$, whereas $p_i = \eta v_i < \eta$ implies $z_i(p) > 0$, yielding a contradiction. Consequently, if $(x,\delta) \in h^{-1}(0) \cap \text{bd } V$, then either $x = v$ and $\delta = 0$ or $x = p^*$ with $z(p^*) = 0$ and $\delta = 1$.

What remains to be proved is that the set of zero points of h indeed consists of paths and loops, one path therefore connecting $(v,0)$ and a $(p^*,1)$ with $z(p^*) = 0$. For each $T \subsetneq I_{n+1}$ the function h is C^1 on $C(T)$. Furthermore, h is a function from the $(n+1)$-dimensional set V to the n-dimensional manifold 0^n defined by

$$0^n = \{q \in \mathbb{R}^{n+1} | \Sigma_{i=1}^{n+1} q_i = 0\} .$$

Hence, if 0 is a regular value of h restricted to $C(T)$, then $h^{-1}(0) \cap C(T)$ is a C^1 1-manifold, i.e., a collection of paths and loops. An end point of a path lies in the boundary of $C(T)$. It is a generic property that a path in $h^{-1}(0) \cap C(T)$ intersects the boundary transversally in the interior of a facet. The latter guarantees that an end point of a path in $h^{-1}(0) \cap C(T)$ is also an end point of a path in $h^{-1}(0) \cap C(\bar{T})$ for some unique $\bar{T} \neq T$. More precisely, since a facet of $C(T)$ not in the boundary of V is equal to either the convex hull of $U^n(T\cup\{k\})\times\{0\}$ and $A(T)\times\{1\}$ for some $k \notin T$ or to the convex hull of $A(T\backslash\{h\})\times\{1\}$ and $U^n(T)\times\{0\}$ for some $h \in T$, the set $\bar{T}$ is equal to either $T \cup \{k\}$ or $T\backslash\{h\}$. Linking the paths in $h^{-1}(0) \cap C(T)$ for different T, $T \subsetneq I_{n+1}$, in this way, we obtain that if 0 is a regular value of h on each $C(T)$ and nondegeneracy on the boundaries is assumed then $h^{-1}(0)$ consists of piecewise C^1 loops and paths. One path, P, has

$(v,0)$ as end point on level 0 in $C(\emptyset)$. The other end point of P lies on level 1 and induces a zero point of z. The path P is C^1 on each $C(T)$ it intersects. As argued above, the projection P' of P on $S^n(\eta)\times\{1\}$ yields the path of points of the adjustment process induced by (3.1). Notice that $h^{-1}(0) \cap C(\emptyset)$ is a line segment connecting $(v,0)$ and a point in bd $C(\{k\})$, where k is the (unique) index for which $z_k(v) = \min_j z_j(v)$.

We conclude the description of the homotopy h on V by proving the following theorem.

Theorem 3.1. Given $(x,\delta) \in V$, $0 < \delta < 1$, there exist unique $T \subsetneq I_{n+1}$, $p \in A(T)$, $u \in \operatorname{int}(U^n(T))$ such that $x = (1-\delta)u + \delta p$.

Proof. Given an (x,δ) in V with $0 < \delta < 1$, we construct a unique set T of indices such that there is a vector p in $S^n(\eta)$ and a vector u in U^n with

$$x_j = \delta p_j + (1-\delta)u_j\ , \qquad u_j = v_j - (n+1)^{-1} \qquad \text{if } j \in T$$

and (3.5)

$$x_j = \delta p_j + (1-\delta)u_j\ , \qquad p_j/v_j = \max_h p_h/v_h\ ,$$

$$\text{and} \qquad u_j > v_j - (n+1)^{-1} \qquad \text{if } j \notin T\ .$$

We first order the elements $(x_i+(1-\delta)(n+1)^{-1})/v_i$, $i \in I_{n+1}$, in decreasing magnitude. Without loss of generality we assume

$$(x_1+(1-\delta)(n+1)^{-1})/v_1 \geq \ldots \geq (x_{n+1}+(1-\delta)(n+1)^{-1})/v_{n+1}\ . \tag{3.6}$$

In the sequel of this proof we denote the expression $(x_i+(1-\delta)(n+1)^{-1})/v_i$ by a_i, $i \in I_{n+1}$. Note that a_j equals $\delta p_j/v_j + (1-\delta)$ if $j \in T$, while $a_j > \delta p_j/v_j + (1-\delta)$ if $j \notin T$. Together with $p_j/v_j = \max_h p_h/v_h$, $j \notin T$, we obtain from (3.6) that either $T = \emptyset$ or $T = \{h+1,\ldots,n+1\}$ for some $h \in I_n$. Clearly, $T = \emptyset$, $p = v$ and $u \in \operatorname{int}(U^n)$ if and only if $a_{n+1} > 1$. If $a_{n+1} \leq 1$, we successively check for $h = 1,\ldots,n$ whether for $T = \{h+1,\ldots,n+1\}$ there exist $p \in S^n(\eta)$ and $u \in \operatorname{int}(U^n)$ such that (3.5) holds. If in step h, $a_h = a_{h+1}$ we may proceed with $T = \{h+2,\ldots,n+1\}$. This because $a_h = a_{h+1}$ with

$h \not\in T$ and $h+1 \in T$ is impossible. If $a_h > a_{h+1}$, we deduce from (3.5) that $p_j = (a_j-(1-\delta))v_j/\delta$, $j = h+1,\dots,n+1$, and together with $\Sigma_{k=1}^{h} p_k = 1 - \Sigma_{j\in T} p_j$ and $p_k = v_k . p_h/v_h$, $k = 1,\dots,h-1$, we obtain a vector p. If $p_h/v_h < p_{h+1}/v_{h+1}$ we proceed with step h + 1, otherwise we can stop with $p \in A(T)$ and $u = (x-\delta p)/(1-\delta) \in int(U^n(T))$.

We now show that this procedure terminates. Suppose that we reach step $h = n$, i.e., $T = \{n+1\}$, and that $a_n > a_{n+1}$. Hence, from (3.5) we obtain $p_{n+1} = (a_{n+1}-(1-\delta))v_{n+1}/\delta$ and because of $a_{n+1} \le 1$ (step 0) also $p_{n+1} \le v_{n+1}$. Furthermore, we have that $\Sigma_j p_j = 1$ and $p_1/v_1 = \dots = p_n/v_n$. Now suppose that $p_n/v_n < p_{n+1}/v_{n+1}$, implying that $p \not\in A\{(n+1)\}$. But then $\Sigma_j p_j < \Sigma_j v_j = 1$, which contradicts $\Sigma_j p_j = 1$. Next, suppose that $a_n = a_{n+1}$. Then the procedure would have been terminated earlier as can be seen as follows. If all a_j's are equal then $a_{n+1} = 2 - \delta > 1$ and therefore $T = \emptyset$. If not, then let j be the index for which $a_{j-1} > a_j$ while $a_k = a_{k+1}$ for $k = j,\dots,n$. In step j - 1 of the procedure we then obtain that $p_k/v_k \leqq 1$, $k = j,\dots,n+1$, since $a_{n+1} \le 1$. Moreover, $\Sigma_\ell p_\ell = 1$ and $p_1/v_1 = \dots = p_{j-1}/v_{j-1}$. If $p_{j-1}/v_{j-1} < p_j/v_j$, implying that $p \not\in A(\{j,\dots,n+1\})$, then $\Sigma_h p_h < \Sigma_h v_h = 1$, yielding a contradiction. From the construction it immediately follows that T is uniquely determined.

□

The theorem implies that the collection of cells C(T) forms a subdivision of V into cells. The intersection of C(T) and $S^n(\eta)\times\{1\}$ is equal to $A(T)\times\{1\}$, and the one of C(T) and $U^n\times\{0\}$ is equal to $U^n(T)\times\{0\}$. For n = 2, these sets have already been illustrated in Figure 3.2. More generally, we can consider the sets $C_\delta(T) := \{x \in \mathbb{R}^{n+1} | (x,\delta) \in C(T)\}$, $0 < \delta < 1$. The union V_δ of $C_\delta(T)$ over all T is equal to the convex hull of the vectors $e^{\delta\eta}(j) := (1-\delta)u(j) + \delta e^{\eta}(j)$, $j \in I_{n+1}$. In Figure 3.3 the sets $C_\delta(T)$ are illustrated for three different δ's.

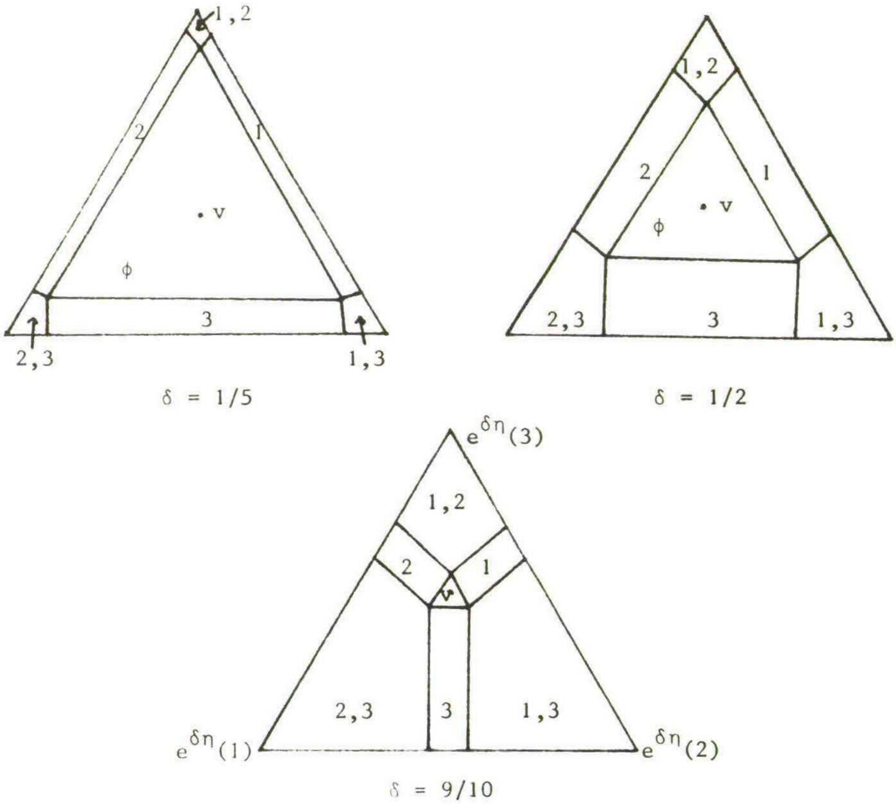

Figure 3.3. Subdivision of V_δ into subsets $C_\delta(T)$ for $\delta = 1/5,\ 1/2,\ 9/10$, $v = (1/4,1/4,1/2)^T$, and $n = 2$. The set $C_\delta(T)$ with $T = \{t_1,\dots,t_t\}$ is denoted by $t_1,\dots,t_t$.

Now we turn to the homotopy inducing the adjustment process on S^n focussing on the maximal excess demands, as described in (3.2). The idea and reasoning are similar to those for the first adjustment process. This homotopy is defined on the convex hull $\bar{V}$ of $\bar{U}^n\times\{0\}$ and $S^n(\eta)\times\{1\}$, where $\bar{U}^n$ is the convex hull of $\bar{u}(j) := v + (n+1)^{-1}e - e(j)$, $j \in I_{n+1}$. In order to define the homotopy on $\bar{V}$ we have to subdivide $\bar{V}$ into sets $\bar{C}(T)$ for $T \subsetneq I_{n+1}$. First we subdivide the set $S^n(\eta)$ on level 1 into subsets $\bar{A}(T)$, $T \subsetneq I_{n+1}$, defined by

$$\bar{A}(T) := \{p \in S^n(\eta) \mid p_j/v_j = \min_h p_h/v_h,\ j \not\in T\}\ .$$

Observe that $\bar{A}(T)$ is the set of vectors in $S^n(\eta)$ satisfying the left part of (3.2). In particular, $\bar{A}(\emptyset) = \{v\}$ and $\bar{A}(\{i\})$, $i \in I_{n+1}$, is equal to $[v, e^{\eta}(i)]$. For n = 2 the sets $\bar{A}(T)$ are illustrated in Figure 3.4.

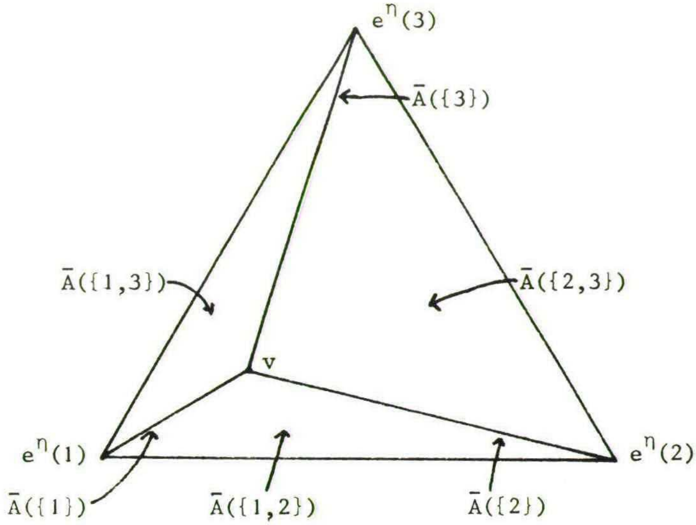

Figure 3.4. Subdivision of $S^n(\eta)$ into subsets $\bar{A}(T)$, n = 2.

Next, let $\bar{U}^n(T)$ be the face of $\bar{U}^n$ defined as the convex hull of $\bar{u}(j)$, $j \notin T$. Then $\bar{V}$ is subdivided by the (n+1)-dimensional cells $\bar{C}(T)$, $T \subsetneq I_{n+1}$, where $\bar{C}(T)$ is the convex hull of $\bar{A}(T)\times\{1\}$ and $\bar{U}^n(T)\times\{0\}$. In particular, $\bar{C}(\emptyset)$ is the convex hull of the point v on level 1 and $\bar{U}^n$ on level 0. In Theorem 3.2 we prove that if $(x,\delta) \in \bar{V}$ then there is a unique subset T such that

$$x = (1-\delta)u + \delta p \qquad (3.7)$$

holds for some unique $p \in \bar{A}(T)$ and $u \in \text{int}(\bar{U}^n(T))$. For n = 2, the subdivision of $\bar{V}$ into subsets $\bar{C}(T)$ is illustrated in Figure 3.5.

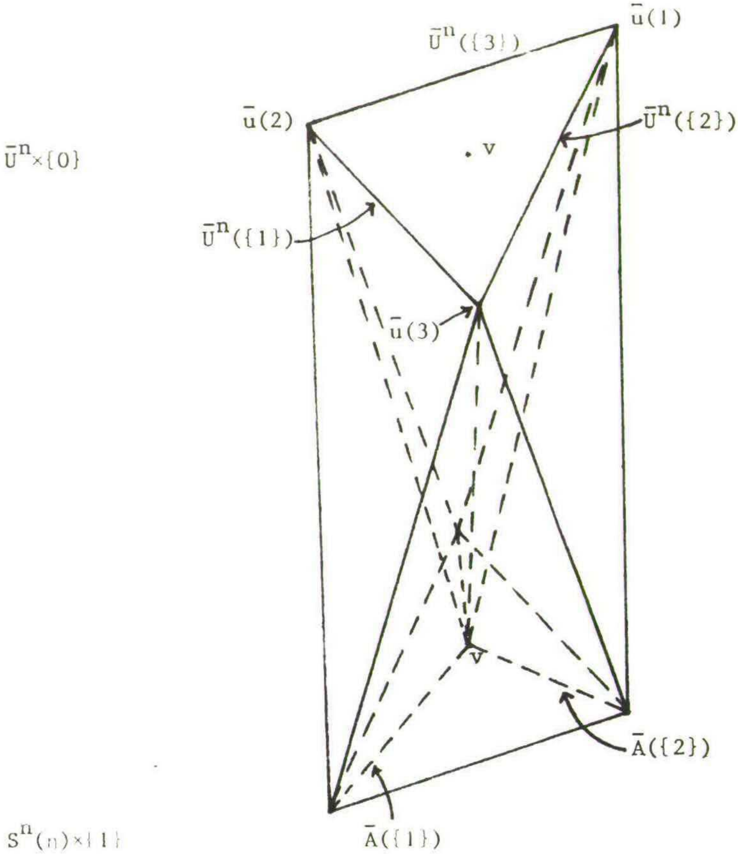

Figure 3.5. Subdivision of $\bar{V}$ into cells $\bar{C}(T)$, n = 2.

The homotopy function h from $\bar{V}$ to $\mathbb{R}^{n+1}$ is now defined by

$$h(x,\delta) = (1-\delta)(v-u) + \delta(z(p) - \bar{z}(p)e) \;, \qquad (x,\delta) \in \bar{V} \;,$$

where $u \in \operatorname{int}(\bar{U}^n(T))$ and $p \in \bar{A}(T)$ are defined as in (3.7). Notice that the homotopy function is identical to the one on V. Let $h^{-1}(0)$ again be the set of zero points of h in $\bar{V}$. Clearly, if $(x,\delta) \in h^{-1}(0)$, then $x = v$ if $\delta = 0$ and $x = p^*$ with $z(p^*) = 0$ if $\delta = 1$. Moreover, when $0 < \delta < 1$, we obtain from $h(x,\delta) = 0$ that there is a unique subset T such that $x = (1-\delta)u + \delta p$ for some $p \in \bar{A}(T)$ and $u = \Sigma_{h \notin T}\mu_h \bar{u}(h) \in \operatorname{int}(\bar{U}^n(T))$ whereas

$$z(p) = \bar{z}(p)e + (1-\delta)\delta^{-1}[(n+1)^{-1}e - \Sigma_{h\notin T}\mu_h e(h)] . \tag{3.8}$$

Thus, $h(x,\delta) = 0$, $0 < \delta < 1$, if and only if there is a set T and a $p \in \bar{A}(T)$ such that $x = (1-\delta)u + \delta p$ for some $u \in \mathrm{int}(\bar{U}^n(T))$ and

$$z_j(p) = \max_h z_h(p) = \bar{z}(p) + (1-\delta)/(\delta(n+1)) \qquad \text{if } j \in T .$$

From this it follows that p satisfies (3.2) and hence that the projection of $h^{-1}(0)$ on $S^n(\eta)$ is equal to the set of vectors p satisfying (3.2). Moreover, for any p satisfying (3.2) there is, according to (3.8), only one $(x,\delta) \in h^{-1}(0)$ such that $x = (1-\delta)u + \delta p$ with $u \in \bar{U}^n$. This implies that when $h^{-1}(0)$ contains a path $\bar{P}$ from the unique zero point on level 0, $(v,0)$, to a point $(p^*,1)$ on level 1, then its projection on $S^n(\eta)$ yields the path $\bar{P}'$ from v to p^* in S^n followed by the adjustment process induced by (3.2). Again, at (x,δ) along the path $\bar{P}$ the homotopy parameter δ induces a measure for the difference between the maximal and the average excess demand at p, the latter being the projection of x on level 1.

That $h^{-1}(0)$ is indeed a 1-manifold containing a path connecting level 0 and 1 can be argued in a way similar to the reasoning for the homotopy on V. Again, we first show that a path in $h^{-1}(0)$ cannot cross the boundary of $\bar{V}$ between the levels 0 and 1. So, let (x,δ), $0 < \delta < 1$, be a point in the boundary bd $\bar{V}$ of $\bar{V}$, i.e., for some i, $i \in I_{n+1}$, either

$$x_i = (1-\delta)(v_i+(n+1)^{-1}) + \delta(1-(1-v_i)\eta)$$

or (3.9)

$$x_i = (1-\delta)(v_i+(n+1)^{-1}-1) + \delta\eta v_i .$$

Let T, $p \in \bar{A}(T)$, and $u \in \mathrm{int}(\bar{U}^n(T))$ be such that $x = (1-\delta)u + \delta p$. In the first case, $p_j = \eta v_j$ for all $j \neq i$ and $i \in T$ since $u_i = v_i + (n+1)^{-1}$ and $p_i = (1-(1-v_i)\eta)$. However, $i \in T$ implies $z_i(p) = \max_h z_h(p) > 0$, whereas $p_j = \eta v_j < \eta$, $j \neq i$, implies $z_j(p) > 0$ also for $j \neq i$, contradicting condition ii) in (1.1). In the second case of (3.9), $p_i = \eta v_i$ and $i \notin T$, and hence $p_j = \eta v_j < \eta$ for all $j \notin T$. Therefore, $j \notin T$ implies $z_j(p) > 0$, whereas $j \in T$ implies $z_j(p) = \max_h z_h(p) > 0$. This also contradicts condition ii) on z. Consequently, if $(x,\delta) \in h^{-1}(0) \cap \text{bd } \bar{V}$, then either $x = v$ and $\delta = 0$ or $x = p^*$ with $z(p^*) = 0$ and $\delta = 1$.

Since h is a C^1-function from each $(n+1)$-dimensional cell $\bar{C}(T)$ to the n-dimensional set 0^n, $h^{-1}(0) \cap \bar{C}(T)$ consists of C^1 paths and loops if 0 is a regular value of h on $\bar{C}(T)$. Again, assuming that each path in $h^{-1}(0) \cap \bar{C}(T)$ hits the boundary transversally in a facet, the paths in $h^{-1}(0) \cap \bar{C}(T)$ for different T can be linked to piecewise C^1 paths and loops in $\bar{V}$. Each path has exactly two end points. Each end point lies either on level 0 or on level 1 since a path cannot have points in common with the boundary of $\bar{V}$ between the two levels. Hence there is one path, $\bar{P}$, in $h^{-1}(0)$ which connects $(v,0)$ and a $(p^*,1)$ for which $z(p^*) = 0$.

We still have to prove (3.7). This is done similar to the proof of Theorem 3.1. Therefore we only state the theorem.

Theorem 3.2. Given $(x,\delta) \in \bar{V}$, $0 < \delta < 1$, there exist unique $T \subsetneq I_{n+1}$, $p \in \bar{A}(T)$ and $u \in \mathrm{int}(\bar{U}^n(T))$ such that $x = (1-\delta)u + \delta p$.

Finally, in Figure 3.6 the sets $\bar{C}_\delta(T) := \{x \in \mathbb{R}^{n+1} | (x,\delta) \in \bar{C}(T)\}$ are illustrated for three different δ's. Notice that $\bar{C}_\delta(T)$ is equal to $\bar{A}(T)$ if $\delta = 1$ and equal to $\bar{U}^n(T)$ if $\delta = 0$. The union of $\bar{C}_\delta(T)$ over all T is equal to the set $\bar{V}_\delta := \{x \in \mathbb{R}^{n+1} | (x,\delta) \in \bar{V}\}$. Observe that for $n = 2$ and $0 < \delta < 1$, $\bar{V}_\delta$ is a hexagone.

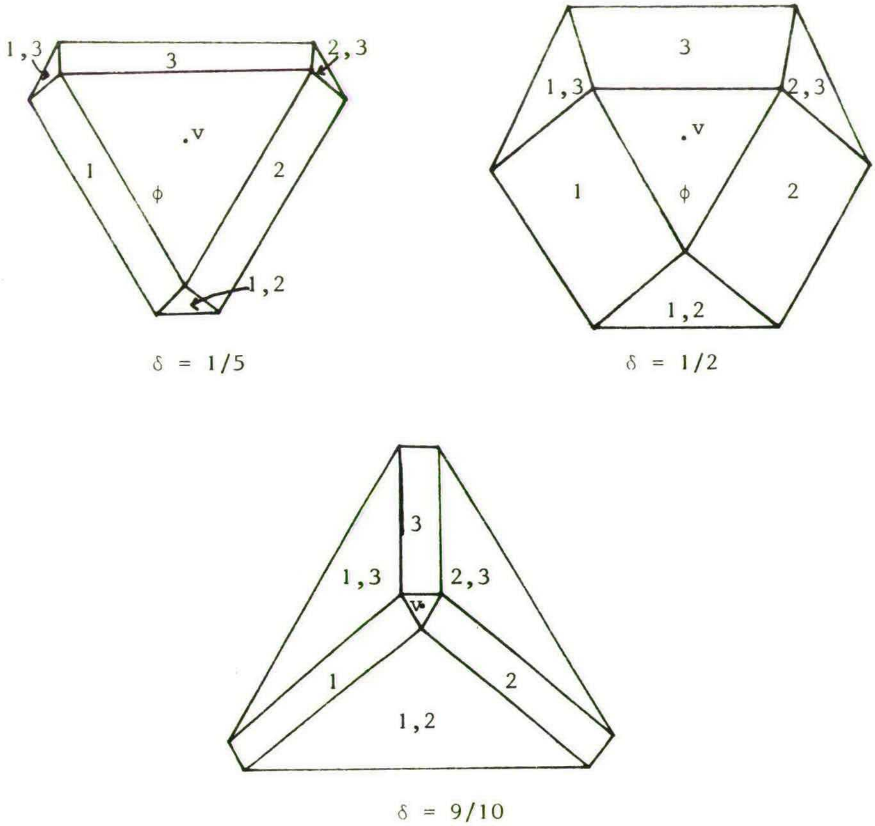

Figure 3.6. Subdivision of $\bar{V}_\delta$ into subsets $\bar{C}_\delta(T)$ for $\delta = 1/5,\ 1/2,\ 9/10$, $v = (1/4, 1/4, 1/2)^T$, and $n = 2$. The set $\bar{C}_\delta(T)$ with $T = \{t_1, \ldots, t_t\}$ is denoted by $t_1, \ldots, t_t$.

4. A homotopy function based on the sign pattern of the excess demands

In this section we consider a homotopy function related to the price adjustment process on S^n which focusses on the sign pattern of the excess demand function. More precisely, the process connects an arbitrary starting vector $v \in \operatorname{int}(S^n)$ with an equilibrium price vector p^* through a sequence of prices p satisfying

$$
\begin{aligned}
p_i/v_i &= \max_h p_h/v_h && \text{if } z_i(p) > 0 \\
\min_h p_h/v_h &= p_i/v_i && \text{if } z_i(p) < 0 .
\end{aligned}
\tag{4.1}
$$

For the process given by (4.1) we obtain a homotopy function on the convex hull V of $U\times\{0\}$ and $S^n(\eta)\times\{1\}$. The set U is given by

$$U = \operatorname{conv}(\{v+e(i)-e(j)\,|\,i,j\in I_{n+1},\ i\neq j\}) .$$

The set of zero points of the homotopy function induces a path P' of points in S^n satisfying (4.1). This path connects the point v with a zero point p^* of the function z.

In order to define the homotopy function h on the set V we have to subdivide V in an appropriate way into cells. Again, this subdivision is needed to relate each (x,δ) in V to a unique pair u in U on level 0 and p in $S^n(\eta)$ on level 1. The subdivision of V is completely determined by the subdivision of $S^n(\eta)$ on level 1 into subsets A(s), where $s = (s_1,\ldots,s_{n+1})^\top \in \{-1,0,+1\}^{n+1}$ is a sign vector in $\mathbb{R}^{n+1}$ containing at least one +1 and one -1. For such a <u>feasible</u> sign vector s the set A(s) is given by

$$
\begin{aligned}
A(s) := \{p \in S^n(\eta)\,|\,\min_h p_h/v_h &= p_i/v_i && \text{if } s_i = -1 \\
p_i/v_i &= \max_h p_h/v_h && \text{if } s_i = +1\} .
\end{aligned}
$$

Observe that the definition of A(s) corresponds to the left part of expression (4.1). The subdivision of $S^n(\eta)$ into sets A(s) is illustrated in Figure 4.1.

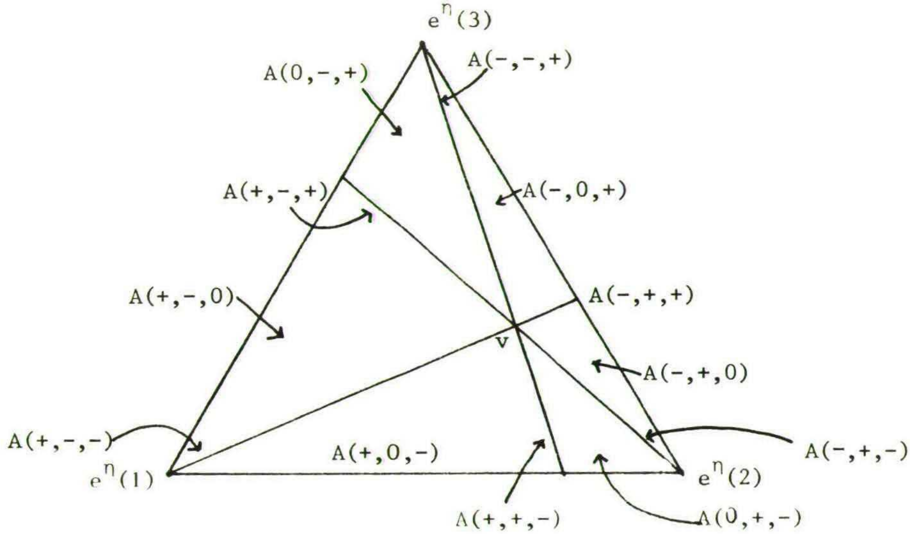

Figure 4.1. Subdivision of $S^n(\eta)$ into sets A(s), n = 2. The set A(-1,-1, +1) is denoted by A(-,-,+), etc.

Next, let the set $C(\emptyset)$ be given by

$$C(\emptyset) := \text{conv}(U\times\{0\},\{v\}\times\{1\}) ,$$

i.e., $C(\emptyset)$ is equal to the set $\{(x,\delta) \in \mathbb{R}^{n+1}\text{x}[0,1] \mid \Sigma_{i=1}^{n+1} x_i = 1, -(1-\delta) \le x_i - v_i \le 1-\delta, i\in I_{n+1}\}$. Furthermore, for a feasible sign vector s let the face U(s) of U be given by

$$U(s) := \text{conv}(\{v+e(i)-e(j) \mid s_i = +1, s_j = -1, i,j\in I_{n+1}\}).$$

It is easy to show that U(s) is equal to the set $\{u \in \mathbb{R}^{n+1} | u = v + \Sigma_{h=1}^{n+1} \mu_h s_h e(h),\ \mu_h \geq 0 \text{ for } h \in I_{n+1}, \text{ and } \Sigma_{s_h=+1}\mu_h = \Sigma_{s_h=-1}\mu_h = 1\}$. The set V is now subdivided into C(∅) and sets C(s), s a feasible sign vector, where C(s) is the convex hull of U(s)×{0} and A(s)×{1}. Observe that the cells C(s) all have dimension n+1 and that also C(∅) is a cell of dimension n+1. These cells are illustrated in Figure 4.2.

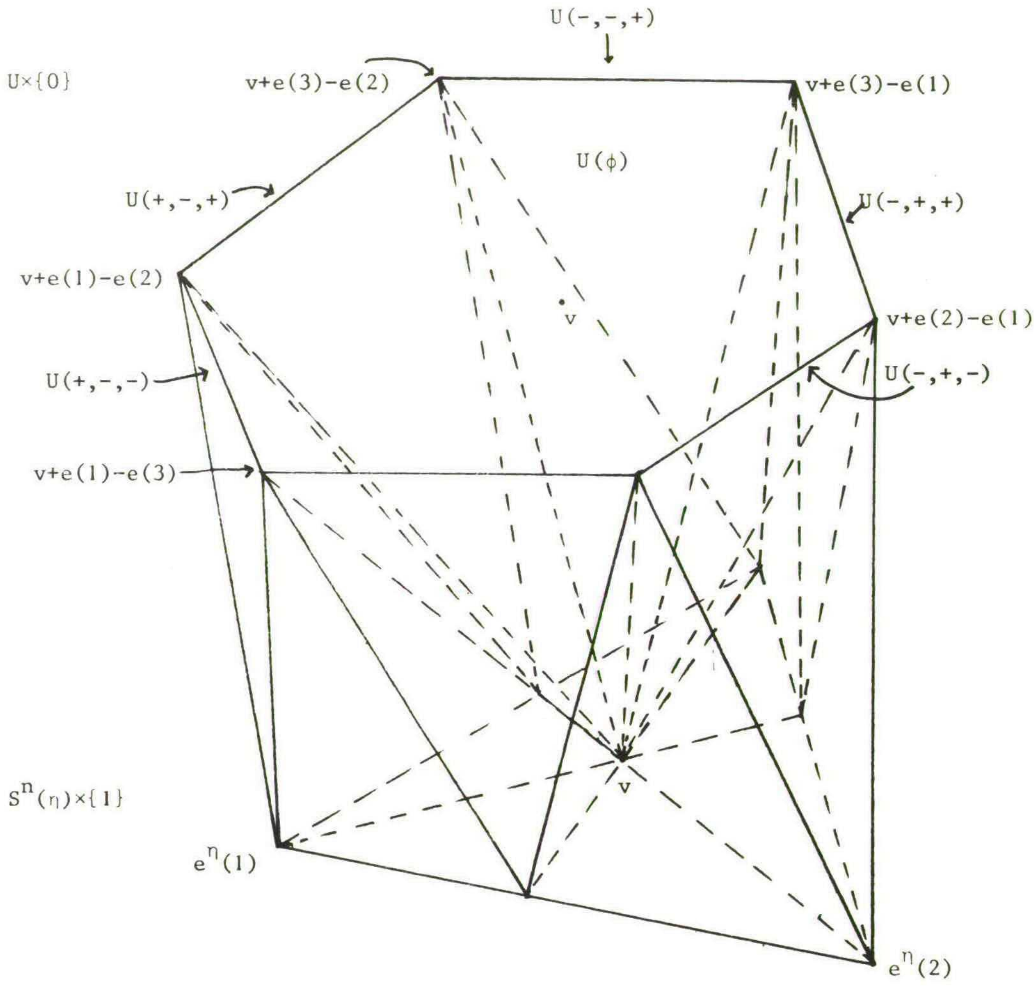

Figure 4.2. Subdivision of V into cells C(s), n = 2.

In Theorem 4.1 we prove that $C(\emptyset)$ together with all the $C(s)$ indeed form a subdivision of V. More precisely, we show that for any $(x,\delta) \in V$ with $0 < \delta < 1$, there either exists a unique sign vector s such that for unique vectors p in A(s) and u in int(U(s))

$$x = (1-\delta)u + \delta p \ , \tag{4.2}$$

or (4.2) holds with $p = v$ and $u \in \text{int}(U)$ uniquely defined by $u = (1-\delta)^{-1}(x-\delta v)$. We call p and u the projection of the point (x,δ) on level 1 and level 0, respectively.

Now, we are ready to define an appropriate homotopy function on V. This function is given by

$$h(x,\delta) = (1-\delta)(v-u) + \delta\tilde{z}(p) \ ,$$

where $\tilde{z}_i(p) = p_i z_i(p)$, $i \in I_{n+1}$, and u and p are given by (4.2). Because u and p are uniquely determined by (x,δ), the function h is well-defined on V. Moreover, h is a piecewise C^1-function from V to $\mathbb{R}^{n+1}$ deforming the trivial function f with $f(u) = v - u$ on level 0 into the function $\tilde{z}$ on level 1. Again, we are interested in the set $h^{-1}(0)$, i.e., the set of points (x,δ) in V such that for some feasible sign vector s, $x = (1-\delta)u + \delta p$ with $p \in A(s)$ and $u = v + \Sigma_h \mu_h s_h e(h) \in \text{int}(U(s))$ (or $p = v$ and $u \in \text{int}(U)$), while $\tilde{z}(p) = (1-\delta)\delta^{-1}\Sigma_h \mu_h s_h e(h)$. Hence, if $p \neq v$, we have $z_i(p) = (1-\delta)\delta^{-1}\mu_i s_i/p_i$, $i \in I_{n+1}$, i.e., p satisfies (4.1). Observe that $z(p) = 0$ when $\delta = 1$ and that $x = v$ when $\delta = 0$.

We would like to have that the set $h^{-1}(0)$ contains a path P in V connecting the unique zero point (v,0) of h on level 0 with a zero point p^* of z on level 1. The projection of the path P on $S^n(\eta)$ then yields a path P' of points connecting v and p^* such that for all points p on P' (4.1) holds. We first prove that a path in $h^{-1}(0)$ cannot cross the boundary of V between the levels 0 and 1. It is easy to see that the boundary of V consists of $U\times\{0\}$, $S^n(\eta)\times\{1\}$, and $\cup_s(\text{conv}(U(s)\times\{0\}, \{p\in A(s) | p_i=\eta v_i \text{ if } s_i=-1\}\times\{1\}))$. So, let (x,δ), $0 < \delta < 1$, be a point in the boundary of V, i.e., $x = (1-\delta)u + \delta p$ with u in int(U(s)) for some sign vector s and with p such that $p_i = \eta v_i$ if $s_i = -1$ and $p_i \geq \eta v_i$ if $s_i \neq -1$. The point u can be written as $u = v + \Sigma_{h=1}^{n+1}\mu_h s_h e(h)$ for some positive numbers μ_h such that $\Sigma_{s_h=+1}\mu_h = \Sigma_{s_h=-1}\mu_h = 1$. Let i be an index with $s_i = -1$. Hence, $u_i = v_i-\mu_i$ and therefore $v_i-u_i = \mu_i > 0$. Since $p_i = \eta v_i$, we also have that $z_i(p) > 0$. Consequently, $h_i(x,\delta) = (1-\delta)(v_i-u_i) + \delta p_i z_i(p) > 0$, so that $(x,\delta) \notin h^{-1}(0)$.

What remains to be shown is that the set of zero points of h indeed consists of paths and loops, one path, P, connecting $(v,0)$ and a $(p^*,1)$ with $z(p^*) = 0$. Along the same line of arguing as described in Section 3, we can prove that $h^{-1}(0)$ consists of piecewise C^1 loops and paths. The projection P' of P on $S^n(\eta)\times\{1\}$ yields the path of points of the adjustment process induced by (4.1). Notice that in $C(\emptyset)$, $h^{-1}(0)$ is a line segment connecting $(v,0)$ with a point in bd $C(s^0)$ where $s_i^0 = \text{sgn } z_i(v)$, $i \in I_{n+1}$.

We conclude the description of the homotopy h on V by proving the following theorem.

<u>Theorem 4.1</u>. Given $(x,\delta) \in V$, $0 < \delta < 1$, there either exists a unique feasible sign vector s and vectors $p \in A(s)$ and $u \in \text{int}(U(s))$ such that $x = (1-\delta)u + \delta p$, or there exists a unique $u \in \text{int}(U)$ such that $x = (1-\delta)u + \delta v$, i.e., $(x,\delta) \in \text{int}(C(\emptyset))$.

<u>Proof</u>. First we verify when $(x,\delta) \in \text{int}(C(\emptyset))$. This is clearly the case if and only if $-(1-\delta) < x_i-v_i < (1-\delta)$ for all $i \in I_{n+1}$. The corresponding $u \in \text{int}(U)$ and p are given by $u = (1-\delta)^{-1}(x-\delta v)$ and $p = v$.

In the remaining of the proof we show that when $(x,\delta) \notin \text{int}(C(\emptyset))$ we can find an s as stated in the theorem. First we rank the numbers $(x_i-(1-\delta)v_i)/\delta v_i$, $i \in I_{n+1}$, in increasing order. Without loss of generality we may assume that

$$(x_1-(1-\delta)v_1)/\delta v_1 \leq (x_2-(1-\delta)v_2)/\delta v_2 \leq\ldots\leq (x_{n+1}-(1-\delta)v_{n+1})/\delta v_{n+1}.$$

In the sequel we often replace the expression $(x_i-(1-\delta)v_i)/\delta v_i$ by a_i, $i \in I_{n+1}$. From the definition of C(s) we derive that if $(x,\delta) \in C(s)$ then $x_i = (1-\delta)(v_i+\mu_i s_i) + \delta p_i$, $i \in I_{n+1}$, with $p \in A(s)$ and the μ_i's defined as above. Thus, $a_i = p_i/v_i + (1-\delta)\mu_i s_i/\delta v_i$, $i \in I_{n+1}$. Combined with the definition of A(s) we get that for all $i \in I_{n+1}$,

$$a_i \le p_i/v_i = \min_h p_h/v_h \qquad \text{if } s_i = -1$$

$$a_i = p_i/v_i \qquad \text{if } s_i = 0 \qquad (4.3)$$

$$a_i \ge p_i/v_i = \max_h p_h/v_h \qquad \text{if } s_i = +1 .$$

Since $a_1 \le \ldots \le a_{n+1}$, the sign vector s must be such that there are two indices k, $\ell \in I_{n+1}$ with $k < \ell$ such that $s_i = -1$ if $i \le k$, $s_i = 0$ if $k < i < \ell$, and $s_i = +1$ if $i \ge \ell$.

After this first observation we determine the index k and the value f of $\min_h p_h/v_h$. The values f and k have to be such that $\Sigma_{i=1}^{k}\mu_i = 1$ and $a_k < f \le a_{k+1}$. We find f by gradually increasing $\min_h p_h/v_h$ from a_1 and therefore increasing $\Sigma_{i=1}^{k}\mu_i$, for $k = 1,2,\ldots,$ from 0. Let us suppose that we can not find such an f. Then we must meet the situation that $p_i/v_i = \min_h p_h/v_h$ for $i = 1,\ldots,n$ and $a_{n+1} = \min_h p_h/v_h = p_{n+1}/v_{n+1}$. Therefore $p = v$ and $a_{n+1} = 1$. We argue that in this situation $x_i - v_i < 1-\delta$ for all $i \in I_{n+1}$. Let us suppose that $x_k - v_k \ge 1-\delta$ for some $k \in I_{n+1}$. Then $a_k = (x_k-(1-\delta)v_k)/\delta v_k \ge 1 + (1-\delta)/\delta v_k > 1$. But this is in contradiction with $1 = a_{n+1} \ge a_i$, $i = 1,\ldots,n$. Thus, we obtain that we can not find an appropriate $\min_h p_h/v_h$ if and only if $x_i - v_i < 1-\delta$ for all $i \in I_{n+1}$. Similarly, we can search for the value of $\max_h p_h/v_h$ by decreasing the maximum from a_{n+1}. We then get that this procedure does not succeed if and only if $x_i - v_i > -(1-\delta)$, $i \in I_{n+1}$.

Since we consider the case that $(x,\delta) \notin \mathrm{int}(C(\emptyset))$, there must exist an index $i \in I_{n+1}$ such that $x_i - v_i \ge 1-\delta$ or $x_i - v_i \le -(1-\delta)$. Let us consider the first case. From above we obtain that we can find an index k and a value f of $\min_h p_h/v_h$ such that $a_k < f \le a_{k+1}$. We now show that we always find an appropriate value g of $\max_h p_h/v_h$ by increasing the maximum from f. Indeed, g must be such that for some $\ell > k$, $\Sigma_{i=\ell}^{n+1}\mu_i = 1$ and $a_{\ell-1} \le g < a_\ell$. Because this sum decreases if we increase $\max_h p_h/v_h$ and equals zero if $\max_h p_h/v_h = a_{n+1}$, it suffices to show that $\Sigma_{i=\ell}^{n+1}\mu_i \ge 1$ when $\max_h p_h/v_h = f$. Since $\Sigma_{i=1}^{k}\mu_i = 1$, we obtain in case $\max_h p_h/v_h = f$,

$$\Sigma_{i=k+1}^{n+1}\mu_i = \Sigma_{i=k+1}^{n+1}(a_i-f)\delta v_i/(1-\delta) = \Sigma_{i=k+1}^{n+1}(x_i-(1-\delta)v_i-\delta f v_i)/(1-\delta)$$

$$= (1-(1-\delta)-\delta f)/(1-\delta) + \Sigma_{i=1}^{k}(-x_i+(1-\delta)v_i+\delta f v_i)/(1-\delta)$$

$$= \delta(1-f)/(1-\delta) + \Sigma_{i=1}^{k}\mu_i = \delta(1-f)/(1-\delta) + 1 \ .$$

Now suppose $\Sigma_{i=k+1}^{n+1}\mu_i < 1$. But then $f > 1$. Because $f = \min_h p_h/v_h$ this is in contradiction with the fact that both p and $v \in S^n(\eta)$.

Along the same lines we can treat the case in which $x_i - v_i \leq -(1-\delta)$ for some $i \in I_{n+1}$. Then we know that there exists a suitable maximum and from that we can show the existence of an appropriate minimum.

Thus, in the foregoing we proved, given an $(x,\delta) \notin \mathrm{int}(C(\emptyset))$, the existence of a feasible sign vector s such that $(x,\delta) \in C(s)$. Besides, we found a related $p \in A(s)$. From the construction above it follows that the point $u := (x-\delta p)(1-\delta)^{-1}$ lies in $\mathrm{int}(U(s))$. Consequently, $x = (1-\delta)u + \delta p$ with $p \in A(s)$ and $u \in \mathrm{int}(U(s))$ uniquely determined.

□

The theorem implies that the collection of cells C(s), s a feasible sign vector in $\mathbb{R}^{n+1}$, and $C(\emptyset)$ form a subdivision of V into cells. The intersection of C(s) with $S^n(\eta)\times\{1\}$ is equal to $A(s)\times\{1\}$ and the intersection of C(s) with $U\times\{0\}$ is equal to $U(s)\times\{0\}$. The intersection of $C(\emptyset)$ with $S^n(\eta)\times\{1\}$ is equal to $\{(v,1)\}$ and the intersection of $C(\emptyset)$ with $U\times\{0\}$ is equal to $U\times\{0\}$. These sets have been illustrated in Figure 4.2. More generally, we can consider on level δ, $0 \leq \delta \leq 1$, sets $C_\delta(s) := \{x \in \mathbb{R}^{n+1} | (x,\delta) \in C(s)\}$ and $C_\delta(\emptyset) := \{x \in \mathbb{R}^{n+1} | (x,\delta) \in C(\emptyset)\}$. The union of $C_\delta(\emptyset)$ and the sets $C_\delta(s)$, s a feasible sign vector, equals the set $V_\delta := \{x \in \mathbb{R}^{n+1} | (x,\delta) \in V\}$. In Figure 4.3 the set V_δ is illustrated for three different δ's.

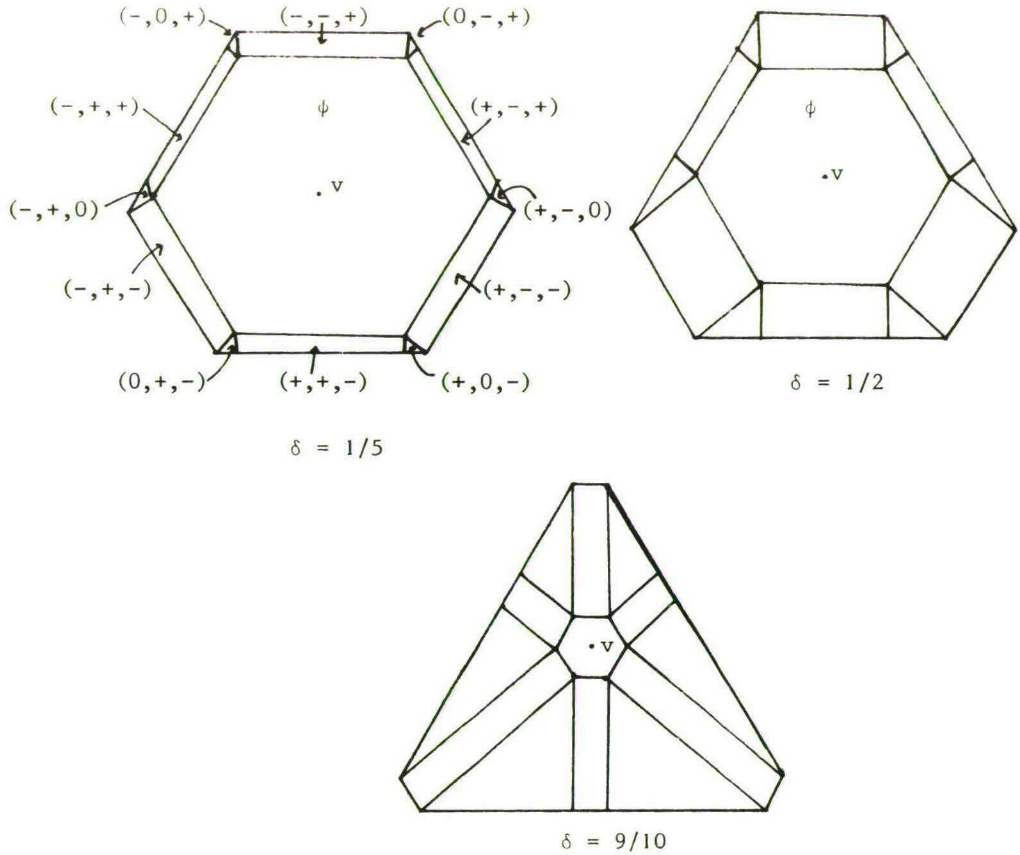

Figure 4.3. Subdivision of V_δ into subsets $C_\delta(s)$ and $C_\delta(\emptyset)$ for $\delta = 1/5$, $1/2$, $9/10$, $v = (1/4, 1/4, 1/2)^T$, and $n = 2$. The set $C_\delta(s)$ is denoted by s.

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