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Adjustment processes for finding
economic equilibria

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Adjustment processes for finding economic equilibria

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Abstract

In this paper we deal with adjustment mechanisms which lead to an economic equilibrium starting from an arbitrarily chosen initial point. This problem goes back to Walras, who was concerned with the problem to find for a pure exchange economy a price adjustment mechanism leading from an initial price system to an equilibrium. It should be noticed that convergence of the Walrasian tatonnement process can be proved if certain conditions are satisfied, e.g., Revealed Preferences. Although this condition may be satisfied for many excess demand function in operational economic models, it has been shown that any continuous function on the unit price simplex satisfying Walras' law can be realized as the excess demand function for some pure exchange economy.

A more advanced method of price adjustment is the Global Newton method. However, also for this method convergence may not hold for an arbitrarily chosen initial point.

We give several adjustment processes which can start anywhere and always lead to an equilibrium point. It appears that these processes can serve as a convergent alternative for the classical Walrasian tatonnement process. Along the paths traced by the various processes the components of the excess demand function will satisfy certain conditions. In particular we prove the existence of a path along which all components successively become equal to zero. More precisely, by increasing the prices of the commodities with excess demand, decreasing the prices of the commodities with excess supply, and adjusting the prices of the commodities in equilibrium in order to keep them in equilibrium, all markets successively become in equilibrium. However, to assure convergence, we allow that a market becomes in disequilibrium again when the ratio of the current price and the initial price raises above (falls

below) the ratio of these prices for commodities with excess demand (supply). So, also the initial price system is kept in mind. This protects the process from cycling or leaving the price space.

1. Introduction

In this paper we deal with adjustment mechanisms which lead to an economic equilibrium, starting from an arbitrarily chosen initial point. This problem goes back to Walras, who was concerned with the problem to find for a pure exchange economy a price adjustment mechanism leading from an initial price system to an equilibrium price system. A straightforward choice for such a mechanism is the differential equation in case z is a continuously differentiable function

$$\frac{dp}{dt} = z(p), \tag{1.1}$$

where, for an economy with n+1 commodities indexed $j = 1,\dots,n+1$, $z(p) = (z_1(p),\dots,z_{n+1}(p))^T$ is the excess demand at prices $p = (p_1,\dots,p_{n+1})^T$. From Walras' law we know that the inner product $p^T z(p) = 0$ for all p and hence z(p) is a vector field tangent to the set of prices

$$P = \{p \in R_+^{n+1} \mid \sum_{j=1}^{n+1} p_j^2 = 1\}.$$

When starting at a point p^o in P, the differential equation has a solution curve of points on P. Unfortunately, the solution curve may fail to converge to a vector of equilibrium prices, even when the set of initial price systems is restricted to points near the boundary or, on the contrary, to points close to an equilibrium price system. So, neither global nor local convergence can be guaranteed. Otherwise stated, the mechanism is not effective in the sense of Saari and Simon [18], who defined a mechanism to be effective if the solutions converge to an equilibrium point for almost all initial price systems in some subset of the manifold on which z(p) is given. Counterexamples have been given by Scarf [19]. In these examples each solution curve leads in the limit to a cycle around the unique equilibrium point p^*, except the curve which starts in p^* itself.

It should be noticed that convergence of the Walrasian tatonnement process can be proved if certain conditions are satisfied. For example if z(p) has the property of Revealed Preferences ${p^*}^T z(p) > 0$ for all $p \neq p^*$, then the Lyapunov function $V(p) = \Sigma_{j=1}^{n+1} (p_j - p_j^*)^2$ is monotonically declining along the solution path of (1.1), implying that the path will converge to p^*. Other sufficient conditions for convergency of the tatonnement process are Gross Substitutability or Diagonal Dominance (see e.g. Arrow and Hahn [1]). Although some of these conditions may be satisfied for most of the excess demand functions in operational economic models, in this paper we want to give processes which will converge for any excess demand function. Not only because it has been shown by several authors (see e.g. Sonnenschein [23] and Debreu [2]) that any smooth vector field on P satisfying Walras' law can be realized as the excess demand function for some pure exchange economy, but also because in the last ten years general equilibrium theory has been concerned with the existence of equilibria under price rigidities. For instance, Drèze [3] proved existence for a pure exchange economy with prices between upper and lower bounds. In this proof a vector of variables $q = (q_1,\ldots,q_{n+1})^T$ defines for each j either a price p_j (between the bounds) or a quantity constraint on either the demands or supplies of the j-th commodity. From these prices and quantities induced by q the excess demand $z(q)$ is obtained. Again, an equilibrium point is a vector q^* for which $z(q^*) = 0$. Also in this case we may consider the differential equation

$$\frac{dq}{dt} = z(q)$$

as a mechanism leading from an initial point p^o to an equilibrium point q^*. However, under which conditions will this mechanism converge? For excess demand functions under quantity rationing we can not use the properties of for instance Revealed Preference or Gross Substitutability. Suppose we say that $z(q)$ prevails Revealed Preferences if $q^* z(q) > 0$ for all $q \neq q^*$. Even when this holds, it is not clear whether there exists a Lyapunov function which monotonically declines along the solution path. Because we do not have $q^T z(q) = 0$ as in the case of prices, the function $V(q) = \Sigma_{j=1}^{n+1} (q_j - q_j^*)^2$ does not serve. So, it is hard to derive convergence conditions for tatonnement processes which adjust quantities.

A more advanced method of price adjustment is the so-called Global Newton method of Smale [22], which has the form

$$Dz(p)\frac{dp}{dt} = -\lambda(p)z(p) \tag{1.2}$$

with $Dz(p)$ the $n \times n$ Jacobian matrix of $(z_1,\ldots,z_n)^T$ evaluated at $(p_1,\ldots,p_n)^T$ with $p_{n+1} = 1$ the price of the numeraire commodity. The scalar $\lambda(p)$ is a real valued function depending on the behavior of f near the boundary of R^n_+. A relevant choice is $\lambda(p) = \det Dz(p)$. The Global Newton process (1.2) is effective in the sense that when the eigen values of $Dz(p)$ are non-zero at a zero of z, it converges to a solution point starting from almost all points on the boundary of R^n_+. In a more recent paper, Keenan [6] showed that the Global Newton process also converges locally. However, Keenan argued that convergence may not hold for an arbitrarily chosen starting point. So the question arises whether there are processes which will converge globally, in the sense that they converge to an equilibrium point for any arbitrarily chosen initial price system. In Saari and Simon [18], it is shown that for such processes not only knowledge of the excess demands $z(p)$ is required (as in the tatonnement process) but also knowledge of the gradients of all of its component functions, except for the numeraire commodity. Clearly in (1.2) the Jacobian $Dz(p)$ is used and hence the gradients. On the other hand, the Global Newton process can be rewritten as

$$\frac{dz(p)}{dt} = -\lambda z(p). \tag{1.3}$$

So, along the trajectory of the process the prices are adjusted in such a way that the changes in the excess demands in $z(p)$ are proportional to $z(p)$ itself. This means that along the path traced by the process the excess demands change proportionally. More generally, we conjecture that for any convergent process the components of $z(p)$ must satisfy certain conditions along the trajectory. This allows us to define processes by stating conditions on the components to be hold along the path of the process. In some sense, this is comparable with the observation that for convergent mechanisms knowledge about $z(p)$ and its gradients is required. A well-known example of a path which leads to an equilibrium price is the path traced by the algorithm of Scarf [20, 21]. When starting in the vertex $(0, 0,\ldots,0, 1)^T$ of the n-dimensional unit simplex

$$S^n = \{p \in R^n_+ \mid \sum_{j=1}^{n+1} p_j = 1\}$$

a path on S^n is followed which is characterized by the condition that for all prices p on the path

$$z_j(p) = z_k(p) \quad \text{for all } j, k \neq n+1.$$

So, as in the Global Newton method the changes in the components of z is proportional to z itself. It has been recognized by several authors that the path followed by Scarf's algorithm coincides with the path traced by the Global Newton process when the latter is started in an appropriately chosen point on the boundary.

In this paper we will define several adjustment processes induced by stating conditions on the components of both p and z(p). These processes can start anywhere in (the interior of) S^n and will be shown to converge to an equilibrium if the function z is continuously differentiable on S^n and some regularity condition is satisfied. Much attention will be paid to the economic interpretation of the processes when they are applied to find a Walrasian equilibrium price vector in a pure exchange economy or to reach a supply-contrained equilibrium. One of these processes will have some similarities with the classical Walras tatonnement process in the sense that at the starting point the prices of the commodities with excess demand are increased whereas the prices of the commodities with excess supply are decreased. These price changes will be not proportional to the excess demand but are relative to the initial price system. In this way the starting point is left in one out of $2^{n+1} - 2$ directions depending on the sign pattern of the excess demands at the starting point. In general the process keeps the prices of the commodities with excess demand relatively to the initial price system larger than all other prices and keeps the prices of the commodities with excess supply relatively smaller than all other prices. Other processes to be defined in this paper leave the starting price system by increasing the price of the commodity with the largest excess demand and by decreasing some or all other prices in order to keep the sum of the prices equal to one. In this way the intial price system can be left in n+1 directions. Also the process in which the price of the commodity with the largest excess supply is decreased and some or all other prices are increased will be discussed. All the processes to be defined in this paper can be approximately excecuted by so-called simplicial algorithms (see e.g. [5] and

[15]). These algorithms follow a piecewise linear path in a simplicial subdivision of S^n. The limiting path of these algorithms can be considered as the path of the corresponding adjustment process.

This paper is organized as follows. The notion of an excess demand function and a supply-constrained equilibrium is treated in section 2. In the sections 3 and 4 the adjustment processes are defined. Section 3 discusses the processes in which the initial price system can be left in $n+1$ directions and section 4 treats the process which has similarities with the classical Walras tatonnement process. The existence proofs of the paths of points followed by all these processes can be found in section 5.

2. Excess demand functions

In this paper we deal with excess demand functions on the n-dimensional unit simplex $S^n = \{p \in R_+^{n+1} \mid \Sigma_i p_i = 1\}$. In case of a competitive exchange economy with $n+1$ commodities, S^n is the price simplex with the sum of the prices normalized to one. Suppose we have an economy with m consumers and for each consumer $i = 1,\ldots,m$ holds

a) the consumption set X^i is a compact, convex subset of R_+^{n+1}, containing the set

$$\{x \in R^{n+1} \mid 0 \leqslant x_j \leqslant \Sigma_i w_j^i,\ j = 1,\ldots,n+1\},$$

where $w^i = (w_1^i,\ldots,w_{n+1}^i)$ is the vector of initial endowments of consumer i;

b) $w_j^i > 0$ for all i, j;

c) the preferences $\geqslant_i$ of the consumers are continuous, monotonic and strictly convex.

Let $x^i(p)$ be the demand of consumer i given price $p \in S^n$, i.e. $x^i(p)$ is a maximal element for $\geqslant_i$ subject to $x \in X^i$ and $p^T x \leqslant p^T w^i$. Then the total excess demand $z(p) = \Sigma_{i=1}^{m}(x^i(p) - w^i)$ is a continuous function on S^n and satisfies

i) for all $p \in S^n$, $p^T z(p) = 0$ (Walras' law)

ii) $z_j(p) \geq 0$ if $p \in S_j^n = \{p \in S^n | p_j = 0\}$ (nonnegative excess demand if $p_j = 0$).

In the next sections we allow for more general excess demand functions.

Definition 2.1. A continuous function $z : S^n \to R^{n+1}$ is an excess demand function if

i) for all $p \in S^n$, there exists a nonnegative vector $y(p)$ with $y_j(p) > 0$ if $p_j > 0$, such that $y^T(p)z(p) = 0$

ii) $z_j(p) \geq 0$ if $p \in S_j^n$.

Defining the continuous function f from S^n into itself by

$$f_j(p) = [p_j + \max\{0, z_j(p)\}]/c(p) \qquad j = 1,\ldots,n+1$$

with $c(p) = 1 + \Sigma_j \max\{0, z_j(p)\}$, it follows from Brouwer's fixed point theorem that any excess demand function $z : S^n \to R^{n+1}$ has a zero point p^*, i.e. $z(p^*) = 0$. In case of the classical Walrasian excess demand funtion p^* is the vector of equilibrium prices. In the next example we consider an economy in which the prices are bounded.

Example. Let $E = (\{X^i, \succ_i, w^i\})_{i=1}^m$ be an exchange economy with m consumers and n+1 commodities. Suppose the conditions a)-c) above hold. Now assume that the set of admissible prices is given by

$$P = \{p \in R_+^{n+1} | 0 < \underline{p}_j \leq p_j \leq \bar{p}_j \text{ for all } j\}.$$

Clearly, P does not necessarily contain a vector p^* such that $z(p^*) = 0$. However, Drèze [3] defined an equilibrium concept with quantity constraints on the excess supplies and excess demands. The existence of an equilibrium with quantity constraints on the supplies only is proven by van der Laan [13, 14] and Kurz [11]. In addition we have that there is a supply-constrained equilibrium with no rationing on at least one commodity.

Definition 2.2. A supply-constrained equilibrium is an allocation x^i, $i = 1,\dots,m$, a price vector $p \in P$ and a rationing scheme $\ell \leq 0$ such that

i) for all i, x^i is a maximal element for $\geq_i$ in the set $B^i(p,\ell) = \{x \in X^i \mid p^\top x \leq p^\top w^i,\ x - w^i \geq \ell\}$

ii) $\Sigma_i x^i = \Sigma_i w^i$

iii) $\ell_j = -\infty$ if $p_j > \underline{p}_j$ $\quad j = 1,\dots,n+1$

iv) $\ell_j = -\infty$ for at least one j.

We construct now an excess demand function such that a zero point yields a supply-constrained equilibrium. For $q \in S^n$, let $p(q)$ and $\ell(q)$ be defined by

$$p_j(q) = \max[\underline{p}_j,\ \tilde{q}_j\bar{p}_j] \qquad j = 1,\dots,n+1$$

$$\ell_j(q) = -\min[\tilde{q}_j\bar{p}_j/\underline{p}_j,\ 1]w_j \quad j = 1,\dots,n+1,$$

where $\tilde{q}_j = q_j/\max_h q_h$ and $w_j = \Sigma_i w_j^i$, $j = 1,\dots,n+1$. Now, let $x^i(q)$ be maximal for $\geq_i$ in the set $B^i(q) = \{x \in X^i \mid p^\top(q)x \leq p^\top(q)w^i$ and $x - w^i \geq \ell(q)\}$, and let $z(q) = \Sigma_i\{x^i(q) - w^i\}$. From the conditions a)-c) it follows that $x^i(q)$ is a continuous function of q and satisfies $p^\top(q)x^i(q) = p^\top(q)w^i$. Hence z is a continuous function from S^n into R^{n+1} satisfying $y^\top(q)z(q) = 0$ for all $q \in S^n$ with $y(q) = p(q) > 0$. Finally, $q_j = 0$ implies $\ell_j(q) = 0$ and hence $z_j(q) \geq 0$. So, z is an excess demand function. Clearly $x^i(q^*)$, $i = 1,\dots,m$, $p(q^*)$ and $\ell(q^*)$ induce a supply-constrained equilibrium iff $z(q^*) = 0$.

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The example shows that definition 2.1 covers excess demand functions z which may arise both from an economy with flexible prices (Walrasian) and an economy with bounded prices. Also the existence of a Drèze equilibrium can be shown by constructing an excess demand function on the unit simplex. In the next sections we deal with convergent processes to reach a zero point of z. These processes allow for an arbitrarily chosen starting point.

3. Convergent adjustment processes I

In this section we describe several adjustment processes to find a zero point of an excess demand function on S^n which can start anywhere and always lead to such an equilibrium point. Except describing the paths of points followed by these processes we also discuss the economic interpretation of them. It will appear that the processes can serve as a convergent alternative for the classical tatonnement process in case of a Walrasian economy. The existence of the paths we describe in this section will be examined in section 5. Let v be the initial price system.

First we consider the homotopy function H from $S^n \times [0,1]$ to $0^n = \{x \in R^{n+1} | \Sigma_i x_i = 0\}$ given by

$$H(p,t) = t\hat{z}(p) + (1-t)(v-p),$$

where $\hat{z}_j(p) = z_j(p) - \Sigma_i z_i(p)/n+1$, $j = 1,\ldots,n+1$. Assuming that 0 is a regular value of H, it can easily be shown that for continuously differentiable excess demand functions z, $H^{-1}(0)$ contains a curve of points (p,t) starting at (v,0). Adapting condition ii) of definition 2.1 for $\hat{z}_j(p) \geqslant 0$ it follows that the curve cannot cross bd $(S^n) \times [0,1]$ and reaches a point $(p^*, 1)$. Clearly, $H(p^*,1) = 0$ implies

$$\hat{z}(p^*) = 0,$$

so that for $j = 1,\ldots,n+1$, $z_j(p^*) = \Sigma_i z_i(p^*)/n+1$. According to $y^T(p)z(p) = 0$ for all p, we have that p^* must be a zero (equilibrium) point of z.

Since for a point (p,t), $0 < t < 1$, in $H^{-1}(0)$ we have

$$\hat{z}(p) = \lambda(p-v)$$

with $\lambda = (1-t)/t > 0$, in case of a Walras economy the path in $H^{-1}(0)$ starting in (v,0) can be economically interpreted as a path of prices along which the difference of the current price system p and the initial price v is proportional to the relative excess demand being the difference between the excess demand and the average excess demands of the goods. The path of points can be followed approximately by the so-called Sandwich method due to Kuhn and MacKinnon [10] and independently proposed for problems on R^n in Merrill [17]. In this algorithm the set $S^n \times [0,1]$ is simplicially subdivided. Starting with the unique simplex containing $(v,0)$, a sequence of adjacent $(n+1)$-dimensional simplices is generated which leads to a simplex yielding an approximate equilibrium point $(\hat{p},1)$. The path is followed more accurate according as the mesh of the subdivision decreases. Therefore, if a more accurate approximation is required, the algorithm can be restarted with a finer subdivision in order to follow a new curve with v being the last found approximate solution $\hat{p}$.

An other so-called simplicial restart algorithm on S^n to find economic equilibria on S^n was proposed by van der Laan and Talman [15]. Instead of a path of $(n+1)$-dimensional simplices in $S^n \times [0,1]$, their algorithm generates a path of simplices in S^n of varying dimension. From the starting point v, being a zero-dimensional simplex, for varying t a path of adjacent t-dimensional simplices is followed, $0 \leq t \leq n$, until an n-dimensional simplex yielding an approximate solution is found. Again the accuracy can be improved by restarting at the approximate solution with a finer subdivision.

The path of points followed approximately by the simplicial algorithm leaves the starting point into one of n+1 different directions or rays. To determine which ray the function value $z(v)$ at v is calculated and the component j is determined for which $z_j(v)$ is minimal. So, in the case that z is an excess demand function arising from a Walras' economy, the good j is determined with lowest excess demand (highest excess supply) at prices v. Then the price of this good is lowered whereas the prices of some other goods are raised in order to keep the sum of the prices equal to one. The price of good j is lowered until for another good, say k, z_k is equal to z_j. Since $\min_h z_h(p) \leq 0$ for all p and $z_j(p) \geq 0$ if $p_j = 0$, by lowering p_j we must reach a price vector p for which $z_k(p) = z_j(p) \leq 0$ for some $k \neq j$. Then from this point the price of good k is also lowered and a path of prices is followed on which $z_k(p) = z_j(p) = \min_h z_h(p)$ until a price vector is reached for which a third good has also minimal excess demand, etc. To be sure that such a price vector indeed will be found we have to protect the procedure against cycling or leaving S^n. To clearify this and the way in which the process proceeds in general we formalize the description. Therefore, let $\mathcal{T}$ be the collection of subsets of $I_{n+1} = \{1,\ldots,n+1\}$ and let $a(1),\ldots,a(n+1)$ be $n+1$ vectors in 0^n such that $\Sigma_i a(i) = 0$ and any set of n vectors $a(j)$, $j \neq i$, $i = 1,\ldots,n+1$, is linearly independent. For simplicity we assume v to be in the interior of S^n.

Definition 3.1. For any $T \in \mathcal{T}$, the subset $A(T)$ of S^n is given by

$$A(T) = \{p \in S^n \mid p = v + \sum_{j \in T} \lambda_j a(j), \quad \lambda_j > 0 \text{ for all } j \in T\}.$$

Observe that $A(\phi) = v$, $A(I_{n+1}) = S^n$ and that for any $T \neq I_{n+1}$, $A(T)$ is a t-dimensional convex polyhedron in S^n, where $t = |T|$ denotes the cardinality of the set T. Moreover, for any $T \in \mathcal{T}$, we define the set $C(T)$ by

$$C(T) = \{p \in S^n \mid z_i(p) = \min_h z_h(p), \ i \in T\}.$$

Clearly $C(\phi) = S^n$ and $C(I_{n+1}) = \{p \in S^n \mid z_i(p) = \min_h z_h(p)$ for all $i \in I_{n+1}\} = \{p^* \in S^n \mid z(p^*) = 0\}$, since $y^T(p)z(p) = 0$ for all p. So, $A(\phi) \cap C(\phi) = \{v\}$ and $A(I_{n+1}) \cap C(I_{n+1})$ is the set of all equilibrium points. In section 5 we show that under some conditions the union B of all sets $B(T) = A(T) \cap C(T)$, $T \in \mathcal{T}$, contains a curve of points in S^n starting in $B(\phi) = \{v\}$ and ending with an element of the set $B(I_{n+1})$ of equilibrium points, if the vectors $a(i)$, $i = 1,\ldots,n+1$, are chosen to be equal to $b(i) - e(i)$ with $e(i)$ the i-th unit vector in R^{n+1} and $b(i)$ some vector in S^n. The variable dimension restart algorithm follows this curve approximately by a sequence of adjacent t-simplices in $A(T)$ for varying $T \in \mathcal{T}$. We consider the curve in B having v as one of its end-points in more detail. Suppose that for some unique j, the starting point v is in the set $C(\{j\})$. Then the curve starting in v leaves v along the one-dimensional set $A(\{j\})$ lowering the price of good j, until for some $k \neq j$ a point in $C(\{j,k\})$ is reached. So, by increasing λ_j from zero, the ray

$$v + \lambda_j(b(j) - e(j)), \qquad \lambda_j \geq 0$$

is followed from v, until a point $\tilde{p}$ is reached such that for some $k \neq j$,

$$z_k(\tilde{p}) = z_j(\tilde{p}) = \min_h z_h(\tilde{p}).$$

Then the region $A(\{j,k\})$ is entered by increasing the coefficient λ_k of $b(k) - e(k)$ from zero and a path of prices p in $A(\{j,k\})$ is followed by the curve such that $z_k(p)$ is kept equal to $z_j(p)$ and less than the excess demand of the other goods, i.e. a path in $B(\{j,k\})$ is followed starting at $\tilde{p}$. Following this path, two cases can occur. Firstly, a price $\hat{p}$ can be reached for which a third good, say ℓ, has also minimal excess demand. Then the process continues along the curve in $B(\{j,k,\ell\})$ starting in $\hat{p}$. Secondly, the curve can hit the boundary of $A(\{j,k\})$. From the definition of A(T) we have that on the boundary of $A(\{j,k\})$ either one of the prices p_j or p_k is equal to zero, or one of the variables λ_j or λ_k is equal to zero. Since $z_i(p) \geq 0$ if $p_i = 0$ it is not possible that the path in $A(\{j,k\})$ on which $z_j(p) = z_k(p) = \min_h z_h(p)$ crosses the boundary $p_j = 0$ or $p_k = 0$ without finding another index ℓ for which $z_\ell(p) = \min_h z_h(p)$. So, if the curve hits the boundary of $A(\{j,k\})$ then either λ_j or λ_k is equal to zero, i.e. the path hits either the ray $A(\{k\})$ or $A(\{j\})$. Suppose λ_j becomes equal to zero. Now, λ_j is not decreased further but is kept equal to zero, and the process continues in $A(\{k\})$ with prices p such that $z_k(p) = \min_h z_h(p)$.

Economically, decreasing the variable λ_j below zero is not very appropriate, since $\lambda_j < 0$ implies an increase of the price of good j whereas the excess demand of good j is negative. So decreasing λ_j below zero should imply that the excess demand of good j is kept on the minimum while increasing its price. Instead of doing so, λ_j is kept equal to zero and the excess demand of good j is forced to become greater than the minimum excess demands. This protects the process from cycling and leaving S^n.

In general, for varying $T \subset I_{n+1}$, the process follows a path of prices p in $B(T) = A(T) \cap C(T)$. Clearly, if $p \in B(T)$ then we have the complementarity condition

$$\lambda_i > 0 \text{ and } z_i(p) = \min_h z_h(p) \quad \text{for all } i \in T$$

and

$$\lambda_i = 0 \text{ and } z_i(p) > \min_h z_h(p) \quad \text{for all } i \notin T,$$

where $p = v + \Sigma_{i=1}^{n+1}\lambda_i(b(i) - e(i))$. Since $b(i) \in S^n$ we have that the sum of the prices of the goods with minimal excess demand is smaller than the sum of the initial prices v_i of these goods. As soon as a price p on the path in B(T) is generated for which a good j, $j \notin T$ has excess demand equal to the excess demand of the goods in T, the process continues along a curve in $B(T \cup \{j\})$, i.e. also λ_j is increased from zero causing a decrease in the price of good j. If on the other hand λ_k becomes zero for some $k \in T$, then the process continues along a curve of $B(T\setminus\{k\})$, keeping λ_k equal to zero and forcing the excess demand of good k to become larger than the minimum excess demand. An example of the path followed by this process is given in figure 1. In this example B contains three curves. One curve is the loop L in $B(\{1,2\})$. The second curve is the path c having two endpoints in $B(I_{n+1}) = C(I_{n+1})$, each being an equilibrium point. The third curve is the path P having an endpoint in $B(\phi) = A(\phi) = \{v\}$ and an endpoint in $B(I_{n+1})$ being an equilibrium point. Starting in $\{v\}$, the latter path is followed by the process until the equilibrium point is reached. Observe that the process cannot cycle or reach bd S^n. Even when v should have been chosen within the loop L, cycling cannot occur because the curves in B depend on the starting point v. If v lies inside L then each ray $A(\{j\})$, $j \in I_{n+1}$, crosses L, which prevents the process from cycling. In figure 2 the path followed by the process is given in case v lies inside L.

The path P from v to p^* can be followed approximately by a sequence of adjacent simplices of varying dimension, where in a t-dimensional region A(T) the path is followed by adjacent t-dimensional simplices. Therefore we need a triangulation or simplicial subdivision of S^n which for each T induces a subdivision of A(T) in t-dimensional simplices. A well-known triangulation of S^n is the so-called Q-triangulation (see e.g. Todd [24]). This triangulation subdivides the sets A(T) if and only if $b(i) = e(j)$ for some $j \neq i$. Since any set of n vectors a(i) must be linearly independent we have to require that $b(i_1) \neq b(i_2)$ for all $i_1 \neq i_2$. A typical choice is $b(i) = e(i+1)$ with $i+1 = 1$ if $i = n+1$ (see [12] and [15]). In this case we have that a decrease of the price of a

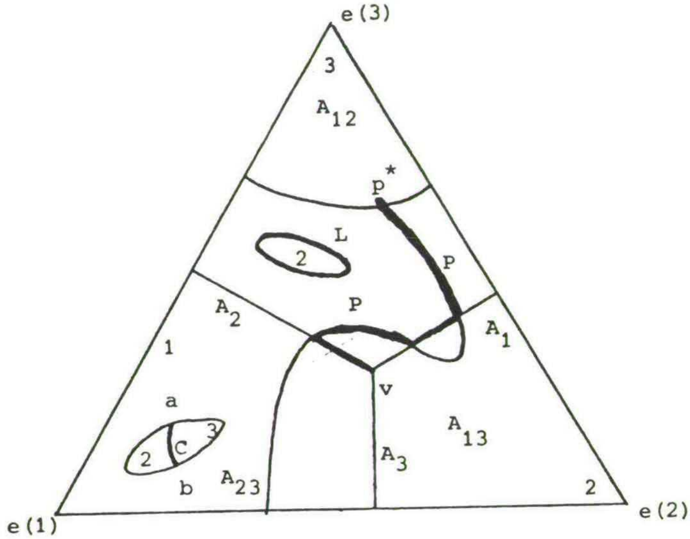

Figure 1. n = 2. The sets $C(\{i\})$ are denoted by i, i = 1,2,3; $C(T) = \bigcap_{i\in T} C(\{i\})$. The sets $A(\{i\})$ and $A\{i,j\}$ are denoted by A_i and A_{ij} resp., $i,j \in \{1,2,3\}$. B consists of a path P from v to p^*, a path C from a to b and the loop L.

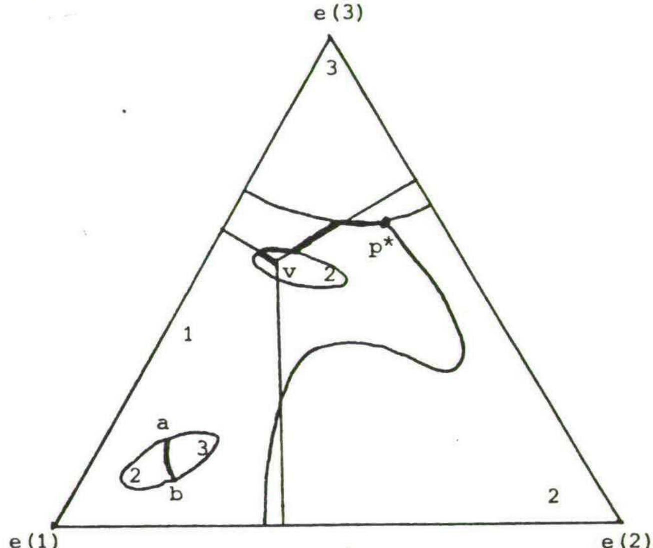

Figure 2. B consists of a path from v to p^* and one from a to b.

good i with the smallest excess demand is compensated by an increase of the price of good i+1 in order to keep the sum of the prices equal to one. Clearly, economically it is rather unsatisfactory that a decrease of a price of one commodity is compensated by an increase of the price of just one other good. However, when defining the sets A(T) with respect to the affine hull of S^n instead of S^n itself, i.e. by taking

$$A(T) = \{p \in U^n | p = v + \sum_{j \in T} \lambda_j (b(j)-e(j)),\ \lambda_j \geq 0,\ j \in T\} \tag{3.1}$$

with $U^n = \{x \in R^{n+1} | \Sigma_{i=1}^{n+1} x_i = 1\}$ we can construct a triangulation of U^n which induces a triangulation in t-simplices of each region A(T) for any admissible choice of the vectors b(j), j = 1,...,n+1. A special choice is given by $b(j) = (n+1)^{-1}e$, where $e = (1,...,1)^T$. In this case the decrease of a price p_i with an amount α is compensated by increasing all other prices with an equal amount α/n. So at the starting point the price of the commodity with the smallest excess demand is decreased and the prices of all other goods are equally increased. Not only this choice makes more sense from an economic viewpoint, also the triangulation of U^n induced by this choice of the vectors b(j), j = 1,...,n+1 is very appropriate for use in a simplicial algorithm, as has been shown in [16].

Recently, Doup and Talman [4] found a simplicial subdivision of S^n itself which gives a triangulation of the t-dimensional sets A(T) in t-simplices when b(j) is chosen to be equal to v for all j. Then the sets A(T) can be written as

$$A(T) = \{p \in S^n | p = v + \Sigma_{j \in T} \lambda_j (v - e(j)),\ \lambda_j \geq 0 \text{ for all } j \in T\}$$

$$= \{p \in S^n | p = (1+b)v - \Sigma_j \lambda_j e(j),\qquad \lambda_j \geq 0 \text{ for all } j \in T\}$$

with $b = \Sigma_{j \in T} \lambda_j$. So, leaving v along the ray A({j}) with j the index of the commodity with the smallest excess demand, v_j is decreased with $\lambda_j(1-v_j)$ whereas all other prices are increased with $\lambda_j v_h$, $h \neq i$, so that the prices of all other goods are increased proportionally with the initial prices and are therefore kept relatively equal to each other. In general, for a price vector $p \in A(T)$

$$p_j \leq (1+b)v_j \qquad \text{if } j \in T$$

$$p_j = (1+b)v_j \qquad \text{if } j \notin T.$$

So, if $p \in B(T) = A(T) \cap C(T)$ we have that for all commodities k not in T

$$p_k/v_k = \max_h p_h/v_h \text{ and } z_k(p) \geq \min_h z_h(p)$$

whereas for all k in T

$$p_k/v_k \leq \max_h p_h/v_h \text{ and } z_k(p) = \min_h z_h(p).$$

That means, relatively to the initial prices, the prices of the goods with minimal excess demand are lower than the prices of the other goods. From an economic viewpoint this seems to be rather attractive and appealing. Doup and Talman [4] showed that this is also computationally efficient. However they followed not the path obtained by decreasing the price of the good with the smallest excess demand but by increasing the price with the highest excess demand (see also e.g. van der Laan [12]). In fact, for all choices discussed in this section the paths can be reversed, redefining A(T) and C(T) by

$$A'(T) = \{p \in S^n \mid p = v + \Sigma_{j \in T} \lambda_j (e(j)-b(j)),\ \lambda_j \geq 0 \text{ for all } j \in T\}$$

and

$$C'(T) = \{p \in S^n \mid z_i(p) = \max_h z_h(p) \text{ for all } i \in T\}$$

where again, $b(j)$ lies in S^n, $j = 1,\ldots,n+1$.

Again, under some conditions the union B' of all sets $B'(T) = A'(T) \cap C'(T)$, $T \subset I_{n+1}$, contains a unique path P' going from v in $B'(\phi)$ to an equilibrium point in $B'(I_{n+1})$. However, we need an extra condition on z to avoid that the curve crosses the boundary of S^n. Taking the sets A(T) and C(T) this cannot happen because if $z_j(p) = \min_h z_h(p)$ and $p_j = 0$ we must have that $z(p) = 0$. In other words, we have that for each subset J of I_{n+1}, $J \neq I_{n+1}$, the set $\{p \in S^n \mid p_j = 0 \text{ for } j \in J\}$ is covered by the union of sets $C(\{i\})$, $i \notin J$. We remember that if all sets $C(\{j\})$ are closed this condition guarantees that the intersection of all sets $C(\{j\})$ is not empty (Lemma of Knaster, Kuratowski and Mazurkiewicz [7]). Along the path P' at v, the price of the good with the highest excess demand is increased while the prices of other commodities are lowered. In general if $p \in B'(T)$ we have the complementarity conditions

$$\lambda_i \geq 0 \quad \text{and } z_i(p) = \max_h z_h(p) \quad \text{for } i \in T$$

and

$$\lambda_i = 0 \quad \text{and } z_i(p) \leq \max_h z_h(p) \quad \text{for } i \notin T$$

where $p = v + \Sigma_{j=1}^{n+1} \lambda_j(e(j) - b(j))$. Now suppose for some $p \in A'(T)$ we have that $p_i = 0$. Then T must contain an index $j \neq i$ with $p_j \neq 0$. Hence, when $p_i = 0$, then $p \in B'(T)$ if $z_i(p) > z_h(p)$ for all h with $p_h \neq 0$. So, $p \in C'(\{i\})$ if $p_i = 0$ is a sufficient condition to guarantee that the curve in B' starting in v does not cross boundary S^n. As has been proved by Scarf [21], this condition is sufficient to guarantee that $C'(I_{n+1}) = \cap_i C'\{i\})$ is not empty.

Again we may take the vector $b(j)$ equal to $e(j+1)$ or $(n+1)^{-1}e$. However, the most interesting choice is $b(j) = v$ for all j. Then the sets $A'(T)$ become

$$A'(T) = \{p \in S^n | p = v + \Sigma_{j \in T} \lambda_j(e(j) - v),\ \lambda_j \geq 0 \text{ for all } j \in T\}$$

$$= \{p \in S^n | p = (1-b)v + \Sigma_{j \in T} \lambda_j e(j),\ \ \lambda_j \geq 0 \text{ for all } j \in T\}$$

with $b = \Sigma_{j \in T} \lambda_j$. So, for any price vector p in some $B'(T)$ we have for some $0 \leq b \leq 1$

$$p_j \geq (1-b)v_j \quad \text{and } z_j(p) = \max_h z_h(p) \quad \text{if } j \in T$$

and (3.2)

$$p_j = (1-b)v_j \quad \text{and } z_j(p) \leq \max_h z_h(p) \quad \text{if } j \notin T,$$

i.e., for a point p on the curve from v to p^* all prices of the commodities with highest excess demand have been increased relative to the initial price system v whereas for all other commodities the relative prices have not been changed compared with the initial price system. As soon as, relative to the starting price, the price of a good with highest excess demand becomes equal to the prices of the commodities not having maximum excess demand, that price is not further decreased but kept relatively equal to these prices. In addition, the excess demand of this good is forced to decrease from the maximal excess demands.

We have seen that in general the condition $z_i(p) \geqslant 0$ if $p_i = 0$ is not sufficient to be sure that the path in $B'(T)$ starting in v does not break down at a point $p \in \text{bd } S^n$. However, when taking $b(j) = v$ for all j this condition is sufficient. For, if for some T a point $p \in B'(T)$ with $p_i = 0$ for some i is reached, then $b = 1$ in (3.2) i.e. $p_j = 0$ for all j not in T. So, if $p_k > 0$ then $k \in T$ and $z_k(p) = \max_h z_h(p)$. However this implies that $\max_h z_h(p) = 0$. Since $z_i(p) \geqslant 0$ for all i with $p_i = 0$ we obtain that $z(p) = 0$. Hence, if a point p on the boundary is reached this point is an equilibrium point and hence an endpoint of the path (see figure 3).

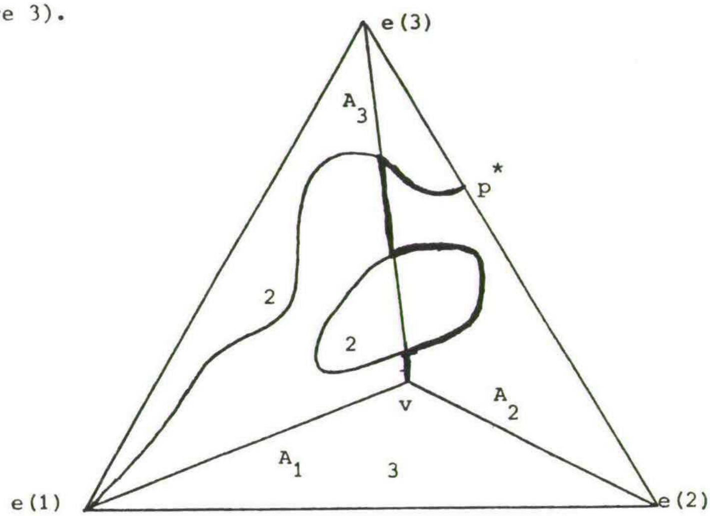

Figure 3. $n = 2$; the path in B' starting at v ends at an equilibrium point p^* in bd S^n, $C(\{1\}) = C(I_{n+1}) = \{p^*\}$

For all the processes discussed in this section we have seen that during the process the index $j \in T$ is deleted from T as soon as λ_j becomes zero, i.e. when the price of good j relative to the initial price v_j raises above (in case of maximal excess demands) or falls below (in case of minimal excess demands) a certain level. That means, during the process the initial price system is kept in mind. This contrasts to both the classical tatonnement process and the Global Newton method in which the initial price system is not kept in mind. In fact, to prevent the process from cycling we have that during the process there are conditions on both the excess demands ($p \in C(T)$) and the prices ($p \in A(T)$). The latter does not hold for the Global Newton method, so that in the Global Newton method the set of admissible starting points is restricted.

A drawback of the processes discussed in this section is that either the prices of the commodities with the highest excess supplies or the prices of the goods with the highest excess demands are adjusted, but not simultaneously.

In the next section we give a process in which all prices are adjusted simultaneously, increasing the prices of the commodities with positive excess demand and decreasing the prices of the goods with negative excess demand.

4. Convergent adjustment processes II

In the previous section we described several adjustment processes to find an equilibrium point. In these processes the starting point can be left into $n+1$ directions, namely the $n+1$ rays $A(\{j\})$ (or $A'(\{j\})$). In this section we describe a process in which the starting point can be left into $2^{n+1} - 2$ directions. Starting with the initial price system all prices are adjusted simultaneously, increasing the prices of the goods with the excess demand positive and decreasing the prices of the goods with the excess demands negative.

To describe the process, let Ω be the set of all sign vectors in R^{n+1} having at least one component equal to +1 and one component equal to -1. Further, for $s \in \Omega$ we define

$$I(s) = \{i \in I_{n+1} \,|\, s_i = 0\}.$$

Each $s \in \Omega$ induces an $(|I(s)|+1)$-dimensional subset of S^n given by

$$A(s) = \{p \in S^n \,|\, p_i/v_i = \min_h p_h/v_h \text{ if } s_i = -1, \text{ and}$$
$$p_i/v_i = \max_h p_h/v_h \text{ if } s_i = 1\}, \tag{4.1}$$

where v is again the initial price system in the interior of S^n. So, A(s) is the set of prices in S^n such that relative to v, p_i is minimal if s_i is negative and p_i is maximal if s_i is positive. When $s_i = 0$, the price p_i may vary between the relative lower and upper bounds. Observe that the number of different sign vectors s in Ω for which I(s) is empty is equal to $2^{n+1} - 2$, implying that there are $2^{n+1} - 2$ rays along which the initial price system can be left. From v there is a ray to each face of S^n. For n = 2 the sets A(s), $s \in \Omega$, are illustrated in figure 4.

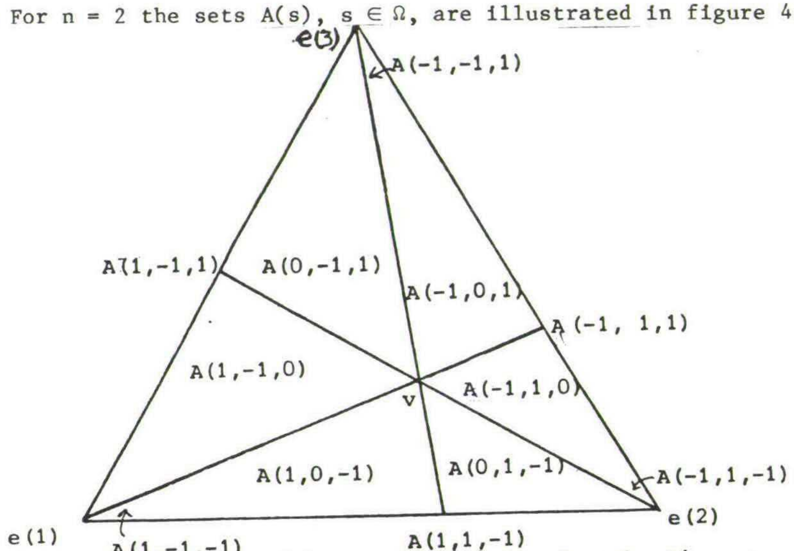

Figure 4. The sets A(s), $s \in \Omega$, n = 2. A(s) is given by A(s_1, s_2, s_3)

The process will leave v along the ray $A(s^o)$ with s^o = sgn z(v), causing a relative decrease of the prices of the commodities with negative excess demand (excess supply) and simultaneously a relative increase of the prices of the commodities with positive excess demand. The process continues along this ray until for one of the commodities, say i, the excess demand becomes equal to zero. Then s_i becomes equal to zero and the process continues in the corresponding region A(s), i.e. the price of the commodity i is not further increased or decreased relative to v, but varies between the relative upper and lower bounds while the excess demand is kept equal to zero. In general, for varying s in Ω, the process traces a path of prices p in A(s) such that

$$p \in C(s) = \mathrm{Cl}\{p' \in S^n \mid \text{sign } z(p') = s\}$$

where Cl(S) is the closure of the set S and sign a = 0 if a = 0. So, for various s in Ω, a path of prices p in B(s) is followed with B(s) = A(s) $\cap$ C(s). Clearly, the set B(s) is equal to

$$\{p \in S^n \,|\, p_i/v_i = \min_h p_h/v_h \text{ and } z_i(p) \leq 0 \text{ if } s_i = -1$$
$$\min_h p_h/v_h \leq p_i/v_i \leq \max_h p_h/v_h \text{ and } z_i(p) = 0 \text{ if } s_i = 0$$
$$p_i/v_i = \max_h p_h/v_h \text{ and } z_i(p) \geq 0 \text{ if } s_i = +1\}.$$

In words, the process follows a path of prices such that relative to the initial price system v, the price of a commodity with negative excess demand is kept minimal and the price of a commodity with positive excess demand is kept maximal while the prices of the commodities in equilibrium may vary between the relative bounds. As soon as the process reaches a price p in B(s) for which the excess demand of a good i becomes zero for some i with $s_i \neq 0$, then the process continues in B(s') with $s'_i = 0$ and $s'_j = s_j$ for all $j \neq i$. On the other hand, when for some p in B(s) the relative price p_i/v_i of a commodity i with zero excess demand ($s_i = 0$) reaches the upper or lower bound, then the process continues in B(s') with $s'_i = 1$ respectively $s'_i = -1$, and $s'_j = s_j$ for all $j \neq i$. As will be proved in the next section, in this way the sets B(s) can be linked together and the union B of B(s) over all sign vectors s in Ω contains a curve leading from the initial price system v to an equilibrium price system p^* (see figures 5 and 6). In the figures the curves along which $z_i = 0$ are drawn for i = 1, 2 and 3. Figure 5 shows the simple case in which B consists of one curve going from v to the equilibrium price p^*. In figure 6, B consists of a curve P from v to p^*, a curve C between the two equilibria a and b in A(+1, −1, 0) and the loop L in A(0, −1, +1). Observe that $z_i(p) > 0$ if $p_i = 0$ and that sign $z_i(p)$ changes if the curve $z_i = 0$ is crossed. So corresponding to the fact that C is in A(+1, −1, 0), along the curve C we have that $z_1 > 0$, $z_2 < 0$ and $z_3 = 0$.

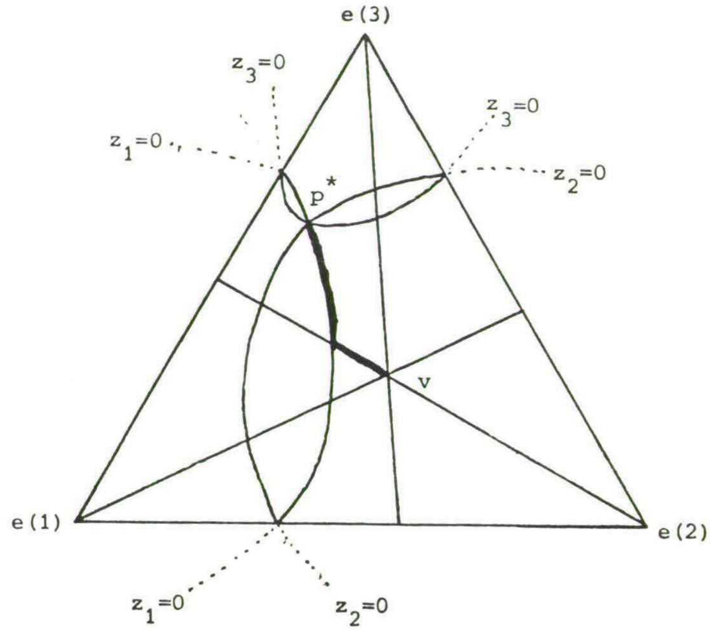

Figure 5. B consists of a curve from v to p^*.

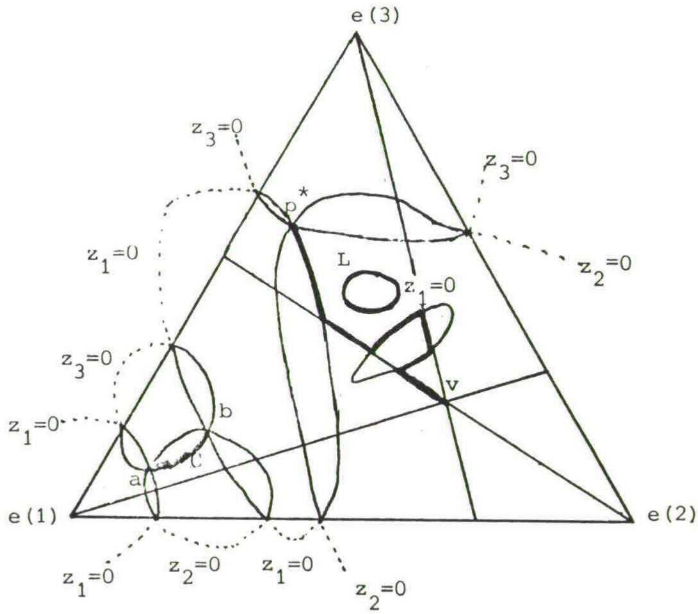

Figure 6. B consists of a curve P from v to p^*, a curve C in $A(+1, -1, 0)$ from a to b and the loop L in $A(0, -1, +1)$.

We show now that a path in $B(s)$, $s \in \Omega$, cannot leave S^n. Suppose that for some s in Ω, p is a point in $B(s)$ on the boundary of S^n, implying that $p_i = 0$ for at least one i. Hence $\min_h p_h/v_h = 0$ and therefore $p_j = 0$ for all indices j with $s_j = -1$. Since $s_j = -1$ implies $z_j(p) \leq 0$ and $p_j = 0$ implies $z_j(p) \geq 0$ we must have $z_j(p) = 0$ for all j with $s_j = -1$. For all h with $s_h = +1$ we have that $z_h(p) \geq 0$ and $p_h > 0$. Hence $z_h(p) = 0$ since $y^T(p)z(p) = 0$ for all p. Finally we also have that $z_k(p) = 0$ for all k with $s_k = 0$. Hence $z(p) = 0$ implying that p is an equilibrium point. So, if the process reaches a point p on boundary S^n, then an equilibrium point is reached.

In these two sections we have described convergent processes to find an equilibrium point of an excess demand function. Along the path traced by such a process the prices and excess demands satisfy certain conditions. In particular the process described in this section is rather interesting. Analogeously to the classical tatonnement process, at the starting point the prices of the commodities with positive excess demand are increased and the prices of the commodities with negative excess demand are decreased. As soon as an excess demand becomes equal to zero, this commodity is kept in equilibrium, unless the price of such a commodity reaches the relative upper or lower bound on which the prices of the commodities with positive respectively negative excess demands are kept. In this case we could have increased (decreased) the price further in order to keep the excess demand equal to zero. Instead of doing that, the price is kept on the relative upper or lower bound enforcing that the excess demand becomes positive respectively negative. However, increasing (decreasing) the price of a commodity with zero excess demand above (below) the relative prices of the commodities with positive (negative) excess demand does not seem to be very satisfactory. Preventing this by keeping in mind the initial price system v the process is protected from cycling or leaving S^n. So, again we have that the starting point v plays a very essential role. In fact, the convergence of the process is assured by memorizing the starting point during the process.

Finally we remark that also the path followed by the process given in this section can be generated approximately by a sequence of simplices of varying dimension. For a detailed description we refer to [5].

Until now we only discussed the economic interpretation of the adjustment processes in case of a Walrasian pure exchange economy. However, the second application mentioned in section 2, the computation of a supply-constrained equilibrium, gives very similar interpretations of the several adjustment processes. For example, in the last process, the accounting prices q_j are increased relatively to the initial accounting price v_j if the corresponding excess demands $z_j(q)$ are positive while the other accounting prices are relatively to the initial prices decreased. When a price $p_j(q)$ becomes equal to $\underline{p}_j$ the price p_j is not further decreased but is kept equal to $\underline{p}_j$ and commodity j becomes rationed. The rationing becomes stronger when the accounting price q_j decreases causing a decrease in the supply of the corresponding good j which has an excess supply. On the other hand, the prices of the goods j with maximal accounting prices are kept equal to $\bar{p}_j$. In general, the accounting prices of the goods with excess demand are during the process kept relatively to the initial price system maximal and the accounting prices of the goods with excess supply minimal whereas the accounting prices of the goods having zero excess demand are allowed to vary between these two bounds. When the accounting price q_j induces a real price equal to $\underline{p}_j$ good j is rationed whereas the goods h with maximal accounting prices have maximal real prices $\bar{p}_h$. Notice the important role of the starting accounting price system v. If for some j q_j is maximal at the initial point, then $p_j(q)$ is set equal to $\bar{p}_j$ although good j might have an excess supply. If so, then q_j is immediately decreased, relatively to the initial accounting price. On the other hand, when at the initial system $p_h(q) = \underline{p}_h$ for some h, so that good h is rationed, this commodity might have an excess demand. The process then will increase q_h immediately in order to relax the rationing and after that to increase the real price $p_h(q)$ from $\underline{p}_h$.

Similar interpretations can be given for the adjustment processes given in section 3. In the process induced by definition 3.1 the accounting price of the commodity j with the smallest excess demand is decreased causing a decrease in the real price p_j if this price is larger than the lower bound $\underline{p}_j$ and causing a stronger rationing if the price $p_j(q)$ is equal to $\underline{p}_j$. All other accounting prices are relatively increased in order to decrease the excess demands of these goods. In general, the process keeps the accounting prices of the commodities with maximal excess supply, relatively to the initial price system, lower than the accounting prices of the goods not having maximal excess supply. The accounting prices of these goods are kept relatively to the initial price system equal to each other. The reverse interpretation holds for the other adjustment process of section 3 in which the accounting prices of the commodities with maximal excess demand are kept relatively larger than those of the goods not having maximal excess demand in order to reach an equilibrium. Increasing the accounting prices q_j of the goods with maximal excess demand causes a relaxation of the rationing if the lower bound of the price is binding and an increase of the prices if there is no rationing (anymore).

5. Existence proofs

In the previous sections we have described sets B and B' which contain a path of points leading from an arbitrarily chosen starting point v in the interior of the unit simplex S^n to an equilibrium point. In this section we present the existence proofs of these paths. We will assume that the function z is continuously differentiable. To give the proofs we need the concept of a primal-dual pair of subdivided manifolds abbreviated PDM. This concept has been introduced in Kojima and Yamamoto [8]. We will give here only the basic tools and some theorems. For a complete discussion of the PDM-theory and the detailed proofs we refer to [8] and [9]. The existence of the paths is obtained by defining an appropriate PDM. An m-cell in R^k is an m-dimensional convex polyhedral set being the intersection of a finite number of closed half spaces in R^k. If a cell D is a face of a cell E we write $D < E$. Letting $\mathcal{M}$ be a (finite) collection of m-dimensional cells in R^k, the collection of faces $\{D \mid D < E, E \in \mathcal{M}\}$ is denoted by $\overline{\mathcal{M}}$ and the union of all m-cells E, $E \in \mathcal{M}$, by $|\mathcal{M}|$. The collection $\mathcal{M}$ of m-cells is called a subdivided m-manifold if

a) for all D, $E \in \mathcal{M}$, $D \cap E = \phi$ or $D \cap E$ is common face of both D and E;
b) each (m-1)-cell in $\overline{\mathcal{M}}$ lies in at most two m-cells of $\mathcal{M}$;
c) $\mathcal{M}$ is locally finite, i.e., each point x in $|\mathcal{M}|$ has a neighborhood which intersects with only a finite number of cells in $\mathcal{M}$.

The boundary of M, denoted by δM, is the collection of all (m-1)-cells of $\bar{M}$ which lie in only one m-cell of M. A 2-manifold with 7 cells is pictured in figure 7. Observe that we allow a cell to be unbounded.

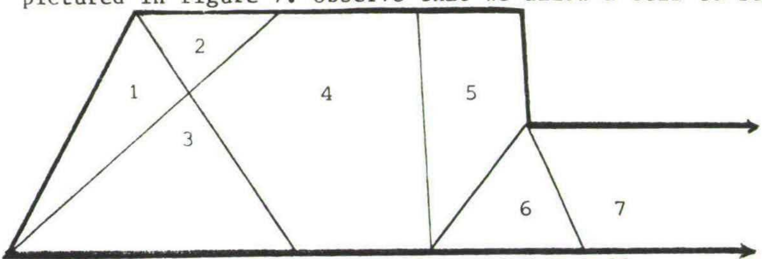

Figure 7. A subdivided m-manifold of 7 m-cells, m = 2. The boundary is heavily drawn

Now let P and D be two subdivided m-manifolds and d a dual operator such that

1) $|D|$ is a bounded polyhedral set;
2) d is an operator from $\bar{P} \times \bar{D}$ into itself such that $X^d \in \bar{D}$ for all $X \in \bar{P}$ and $Y^d \in \bar{P}$ for every $Y \in \bar{P}$;
3) if $Z \in \bar{P} \cup \bar{D}$ then $(Z^d)^d = Z$ and $\dim Z + \dim Z^d = m$;
4) if $X_1, X_2 \in \bar{P}$ and $X_1 < X_2$ then $X_2^d < X_1^d$;
5) if $Y_1, Y_2 \in \bar{D}$ and $Y_1 < Y_2$ then $Y_2^d < Y_1^d$.

Then the triplet (P,D,d) is a primal-dual pair of subdivided manifolds with degree m. P and D are the primal and dual subdivided manifolds respectively of the PDM.
An example of a PDM is given in figure 8.

Next we define the collection of m-cells L by $L = \langle P, D, d\rangle$ where

$$\langle P, D, d\rangle = \{X \times X^d \,|\, X \in \bar{P}\} = \{Y^d \times Y \,|\, Y \in \bar{D}\}. \tag{5.1}$$

Lemma 5.1. If (P,D,d) is a PDM with degree m, then L is a subdivided m-manifold with empty boundary. Moreover, $|L|$ is a closed subset.

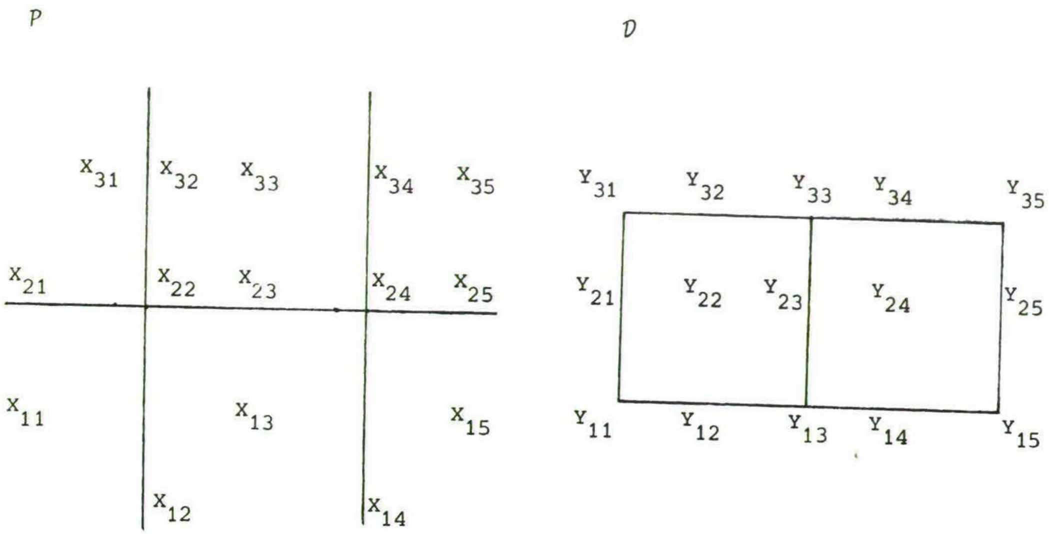

Figure 8. A PDM with degree 2.

$$\bar{P} = \{X_{i,j} \mid 0 < i,\ j < 5\},\ \bar{D} = \{Y_{i,j} \mid 0 < i,\ j < 5\} \text{ and}$$
$$X^d_{ij} = Y_{ij},\ 0 < i,\ j < 5.$$

If $D = X \times Y$ is an $(m-1)$ cell of L with $X \in \bar{P}$ and $Y \in \bar{D}$ and E is an m-cell of L having D as one of its faces, then either $E = X \times X^d$ or $E = Y^d \times Y$. With respect to the m-manifold $L = \langle P,D,d\rangle$ defined in (5.1) we define the subdivided $(m+1)$-manifold K by

$$K = \{Z \times R_+ \mid Z \in L\}. \tag{5.2}$$

Since $\delta L = \phi$, we must have $|\delta K| = |L| \times \{0\}$. More precisely,

$$\delta K = \{Z \times \{0\} \mid Z \in L\}.$$

Finally, let h be a piecewise continuously differentiable, abbreviated PC^1, function from K to an m-dimensional linear subspace L^m of R^k such that the restriction of h to each $(m+1)$-cell of K can be extended to a continuously differentiable function on an open neighbourhood of the cell. A point c in L^m is called a regular value of the PC^1 function h if the dimension of the set $h(E) = \{h(x) \mid x \in E\}$ equals m for all cells

E in $\bar{K}$ for which $c \in h(E)$. If c in L^m is a regular value of h, then the set

$$h^{-1}(c) = \{x \in |K| \,|\, h(x) = c\}$$

does not intersect with any face D in $\bar{K}$ of dimension less than m. From Sard's theorem we know that almost every c in L^m is a regular value.

Theorem 5.2. Let K be a subdivided (m+1)-manifold as defined above and let $h: |K| \to L^m$ be a PC^1 function on K. If c is a regular value of h, then $h^{-1}(c)$ is a disjoint union of piecewise smooth paths and loops satisfying

i) if $E \in K$ and $h^{-1}(0) \cap E \neq \emptyset$ then $h^{-1}(0) \cap E$ is a disjoint union of smooth 1-manifolds;

ii) each loop has no intersection with $|\delta K|$;

iii) $x \in h^{-1}(0)$ is an endpoint of a path if and only if $x \in |\delta K|$;

iv) every open or semiclosed path is unbounded.

We will apply theorem 5.2 for an appropriately chosen subdivided (n+1)-manifold K and PC^1 function h to deduce that $h^{-1}(0)$ corresponds to a set B (or B') defined in the sections 3 and 4 and that $h^{-1}(0)$ contains a path in $|K|$ corresponding to a path from v to an equilibrium point. First, let us consider the process described in section 3 where the starting point v is left by only decreasing the price of the commodity with the highest excess supply. The primal of the corresponding PDM is completely determined by the sets A(T) defined in (3.1) being subsets of U^n whereas the dual is induced by the sets C(T), $T \subset I_{n+1}$. More precisely, the subdivided n-manifold P is defined by

$$P = \{A(T) \,|\, T \subset I_{n+1} \text{ and } |T| = n\},$$

i.e. P is the collection of the n+1 n-dimensional cones A(T) in U^n. Clearly, $\bar{P} = \{A(T) \,|\, T \subset I_{n+1}\}$ is the collection of all cones A(T), $|T| \leq n$, in U^n. The dual subdivided n-manifold D consists of the n-cell Y_0 defined by

$$Y_0 = \{y \in 0^n \,|\, y_j \leq 1 \text{ for all } j \in I_{n+1}\}.$$

Defining the sets $Y(T)$, $T \subset I_{n+1}$, $T \neq I_{n+1}$, by

$$Y(T) = \{y \in O^n \mid y_j = 1,\ j \in T, \text{ and } y_j < 1,\ j \notin T\}$$

we obtain that $\bar{D} = \{Y(T) \mid T \subset I_{n+1} \text{ and } |T| \leq n\}$. Observe that $Y_0 = Y(\emptyset)$. The dual operator between P and D is defined by

$$A^d(T) = Y(T) \text{ and } Y^d(T) = A(T) \text{ for all } T \neq I_{n+1}.$$

Notice that $\dim A(T) = |T|$ and that $\dim Y(T) = n - |T|$ so that $\dim A(T) + \dim Y(T) = n$ for all $T \subset I_{n+1}$, $|T| \leq n$. By verifying all the conditions of a PDM we immediately get the next corollary.

<u>Corollary 5.3</u>. The triplet (P,D,d) is a PDM with degree n.

The above defined PDM is illustrated in figure 9 for $n = 2$.

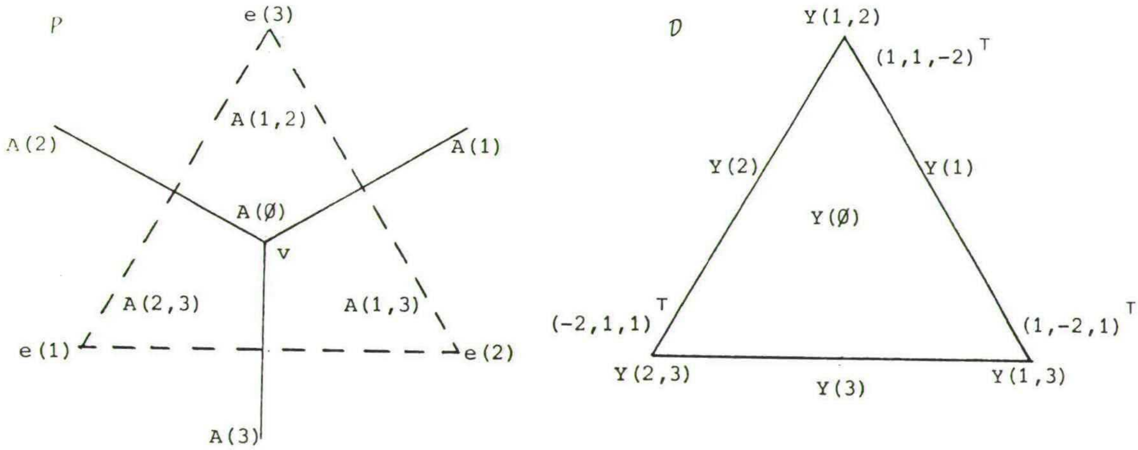

Figure 9. The PDM induced by the (n+1)-ray algorithm on minimal excess demands.

Now let $\langle P,D,d \rangle$ be the n-manifold corresponding to (P,D,d) and let K be defined as in (5.2), i.e.,

$$K = \{A(T) \times Y(T) \times R_+ \mid T \subset I_{n+1},\ |T| \leq n\}.$$

Notice that $\delta K = \{A(T) \times Y(T) \times \{0\} | T \subset I_{n+1}, |T| \leq n\}$. To define a PC^1 function h on K we extend the function z from S^n to R^{n+1} to a continuously differentiable function from U^n to R^{n+1}. Recall that we assume that z is a continuously differentiable function on S^n. Then we define the function $h: |K| \to R^{n+1}$ by

$$h(p,y,t) = y - t\bar{z}(p) \qquad (p,y,t) \in |K| \tag{5.3}$$

where $\bar{z}_j(p) = z^m(p) - z_j(p)$, $j = 1,\ldots,n+1$, with $z^m(p) = \Sigma_{j=1}^{n+1} z_j(p)/(n+1)$ being the averge excess demand. Since y lies in Y_0 and therefore in 0^n and since by definition $\Sigma_{j=1}^{n+1} \bar{z}_j(p) = 0$ implying that also $t\bar{z}(p)$ lies in 0^n, we must have that h is a PC^1 function from the $(n+1)$-manifold $|K|$ to the n-dimensional linear subspace 0^n. Therefore, we may apply theorem 5.2 to obtain the next corollary.

Corollary 5.4. If z is a continuously differentiable function and if 0 is a regular value of the function h defined in (5.3), then $h^{-1}(0)$ consists of piecewise smooth loops and paths having 0, 1 or 2 endpoints each of them lying in $|\delta K|$.

Lemma 5.5. The only endpoint of a path in $h^{-1}(0)$ is the point $(v,0,0)$.

Proof. Since a point (p,y,t) in $h^{-1}(0)$ is an endpoint of a path if and only if (p,y,t) lies in $|\delta K| = |\{A(T) \times Y(T) \times \{0\} | T \subset I_{n+1}\}|$ we must have $t = 0$ for any endpoint (p,y,t). However, (5.3) implies $y = t\bar{z}(p) = 0$ so that y lies in the interior of $Y(\phi)$. Hence, $p \in A(\phi) = \{v\}$, i.e., $p = v$, which proves the lemma.

[]

The lemma says that the point $(v,0,0)$ is the only endpoint of a path in $h^{-1}(0)$ so that this path is a semi-closed unbounded path whereas all other paths in $h^{-1}(0)$, if any, are open and unbounded, according to lemma 5.1. The path in $h^{-1}(0)$ having the point $(v,0,0)$ as endpoint will be denoted by P. Observe that $|K|$ is closed. Consequently, the path P goes to infinity in at least one of the $2(n+1)+1$ components of (p,y,t).

Lemma 5.6. If the point (p,y,t) belongs to $h^{-1}(0)$ then $p \notin \mathrm{bd}S^n$.

Proof. Suppose that p lies in $\mathrm{bd}S^n$ for some (p,y,t) in $h^{-1}(0)$. Since v lies in the interior of S^n and $A(\phi) = \{v\}$, there must exists a nonempty set T in I_{n+1} such that $p \in A(T)$. By definition $p_j = 0$ only for some $j \in T$. Since $y \in Y(T)$ we have according to (5.3)

$$z_h(p) = z^m(p) - 1/t \quad \text{for all } h \in T$$

and

$$z_h(p) \geqslant z^m(p) - 1/t \quad \text{for all } h \notin T$$

so that, for all $h \in T$, $z_h(p) = \min_k z_k(p) \leqslant 0$. However, $p_j = 0$ for some $j \in I_{n+1}$ implies $z_j(p) \geqslant 0$, i.e., $z_h(p) = \min_k z_k(p) = 0$, for all $h \in T$. Since $p_h > 0$ for all $h \notin T$, this implies $z(p) = 0$ and so $y = t\bar{z}(p) = 0$, which contradicts the fact that T is nonempty and therefore that at least one y_j is equal to one.

[]

From this lemma it follows that no path in $h^{-1}(0)$ can cross $\mathrm{bd}S^n$ in p. Therefore since the point v lies inside S^n, the path P in $h^{-1}(0)$ which originates in the point $(v,0,0)$ must stay in the compact S^n in the components of p and in the compact Y_0 in the components of y. Consequently, along the unbounded semi-closed path P the variable t must go to infinity whereas (p,y) converges to a limit point (p^*, y^*). However, $h(p,y,t) = 0$ implies

$$\bar{z}_j(p) = y_j/t \leqslant 1/t \quad \text{for all } j$$

so that if t goes to infinity $\bar{z}_j(p^*) \leqslant 0$, $j \in I_{n+1}$ and hence $z(p^*) = 0$. The path P therefore approaches a limit point (p^*, y^*) in $S^n \times Y_0$ with $z(p^*) = 0$ when t goes to infinity, i.e., P leads from the point $(v,0,0)$ to an equilibrium point.
We now prove that if the point (p,y,t) lies in $h^{-1}(0)$ and $p \in S^n$, then p lies in the set B defined in section 3.

Theorem 5.7. Let (p,y,t) be a point in $h^{-1}(0)$ with $p \in S^n$, then there is a $T \subset I_{n+1}$, $|T| \leqslant n$, such that

$p \in B(T) = A(T) \cap C(T)$.

Proof. Since $(p,y,t) \in |K|$, there is a $T \subset I_{n+1}$, $|T| \leq n$, such that $p \in A(T)$ and $y \in Y(T)$. Consequently, since $y = t\bar{z}(p)$,

$$z_j(p) = z^m(p) - 1/t \quad \text{for all } j \in T$$

and

$$z_j(p) \geq z^m(p) - 1/t \quad \text{for all } j \notin T$$

implying that $z_j(p) = \min_h z_h(p)$ for all $j \in T$. Hence $p \in C(T) \cap A(T)$.

[]

We remark that we allow T to be empty in which case $p = v \in B(\emptyset)$, $0 \leq t \leq (\min_h \bar{z}_h(p))^{-1}$ and $y = t\bar{z}(v) \in Y_0$.

From theorem 5.7 it follows that along the path P in $h^{-1}(0)$ starting in $(v,0,0)$ a path of prices p in B is traced. The latter path starts in v, leads to an equilibrium price p^* and is the primal projection of the path P in P . With the primal and dual projection of (a path in) $h^{-1}(0)$ we mean the set of points $\{p \in S^n | (p,y,t) \in h^{-1}(0)\}$ and $\{y \in Y_0 | (p,y,t) \in h^{-1}(0)\}$ respectively.

Corollary 5.8. The set B is the primal projection of $h^{-1}(0)$ and contains a path of prices leading from v to an equilibrium point p^*. This path is the primal projection of the path P. Moreover, any path or loop in $h^{-1}(0)$ corresponds to a path with two endpoints or a loop in B, being its primal projection.

Notice that we implicitely assume that the point v is not an equilibrium. For, since v lies in the interior of S^n, the regularity assumption on h implies that $z(v) \neq 0$. If, however, $z(v) = 0$, then the path P still exists and is equal to the ray $\{(v,0,t) | t \geq 0\}$ having the point v as primal projection.

In the case that $z(v) \neq 0$, along the path P starting in $(v,0,0)$ first y_j is increased from 0 to 1 with j the index for which the excess demand is minimal. Simultaneously t is increased from 0 to $(\bar{z}_j(v))^{-1}$ to keep $y_j - t\bar{z}_j(v)$ equal to zero while for $i \neq j$ the component y_i is kept equal to $t\bar{z}_i(v)$. In this way, a (linear) path in $A(\emptyset) \times Y(\emptyset) \times R_+$ is traced from $(v,0,0)$ to the point $(v,\bar{z}(v)/\bar{z}_j(v),\ 1/\bar{z}_j(v))$ in $A(\{\emptyset\}) \times Y(\{j\}) \times R_+$. Then the path P smoothly continues in $A(\{j\}) \times Y(\{j\}) \times R_+$ keeping y_j equal to 1 and t equal to $(\bar{z}_j(p))^{-1}$, until a point (p,y,t) is reached for which $z_i(p)$ becomes equal to $z_j(p) = \min_h z_h(p)$ for some $i \neq j$ and so $y_i = y_j = 1$. Then the path P continues in $A(\{i,j\}) \times Y(\{i,j\}) \times R_+$ keeping y_i and y_j equal to one and t equal $(\bar{z}_j(p))^{-1} = (\bar{z}_i(p))^{-1}$. In general, the path P in $h^{-1}(0)$ traces in $A(T) \times Y(T) \times R_+$ for various $T \subset I_{n+1}$ a smooth path of points (p,y,t) such that $y_j = 1$ for all $j \in T$ and so $z_j(p) = \min_h z_h(p)$, $j \in T$, whereas t is equal to $(\min_h \bar{z}_h(p))^{-1}$. Moreover, $y_i = \bar{z}_i(p)/\min_h \bar{z}_h(p) < 1$ for all $i \notin T$. When $|T| < n$, an endpoint in $A(T) \times Y(T) \times R_+$ is reached if either y_h becomes equal to 1 for some $h \notin T$ or p lies in $A(T\setminus\{k\})$ for some $k \in T$. In the first case the path P continues in $A(T \cup \{h\}) \times Y(T \cup \{h\}) \times R_+$ keeping y_h equal to 1 whereas in the second case P continues in $A(T\setminus\{k\}) \times Y(T\setminus\{k\}) \times R_+$ by decreasing y_k away from 1. The latter case can also occur when $|T|$ is equal to n. If $|T| = n$ the parameter t can go to infinity yielding an equilibrium point. Notice that if $|T| = n$, then $y_j = -n$ for the unique index j not in T whereas $y_i = 1$ for all $i \neq j$.

The path P in $h^{-1}(0)$ is illustrated in figure 10 for n = 2. In this figure the path P lies in $A(T) \times Y(T) \times R_+$ for subsequently $T = \emptyset$, $\{2\}$, $\{1,2\}$, $\{1\}$ and $\{1,3\}$. When $T = \{1,2\}$ or $\{1,3\}$ the vector y on the path P is equal to $(1,\ 1,\ -2)^T$ and $(1,\ -2,\ 1)^T$ respectively.

The proof of the existence of the path from v to an equilibrium in the set B' is very similar to the proof given above. We only need to replace the function h from $|K|$ to 0^n by

$$h'(p,y,t) = y + t\bar{z}(p) \qquad (p,y,t) \in |K| .$$

Then again if z is a continuous differentiable function and 0 a regular value of h', there exists a piecewise smooth path P' in $h'^{-1}(0)$ starting in $(v,0,0)$, which approaches an equilibrium point for t going to infinity (under appropriate properness conditions on the boundary of S^n). Moreover, the set B' is the primal projection of the set $h'^{-1}(0)$ and the primal projection of the path P' on S^n is the path in B' which connects the point v with an equilibrium point.

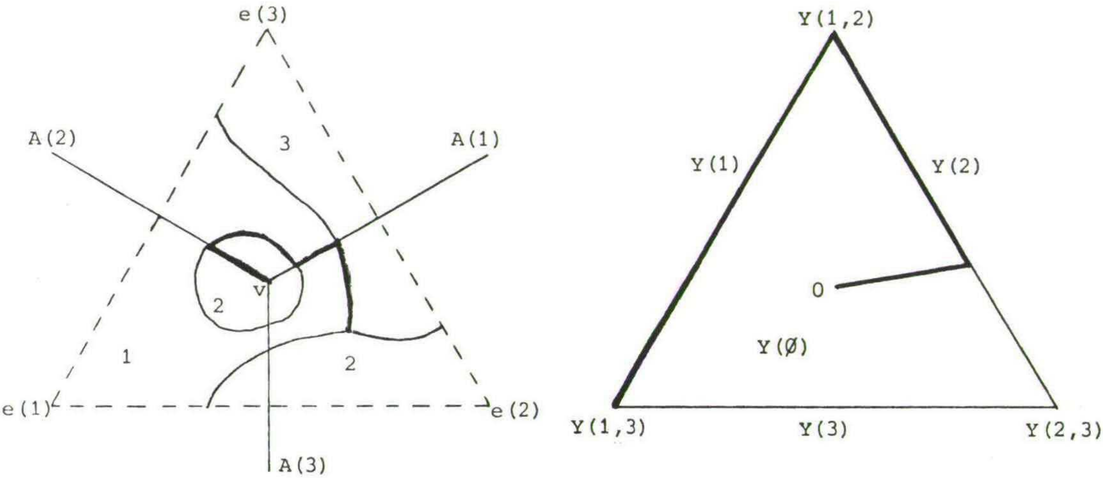

Figure 10. The primal and dual projection of P, n = 2.

It remains to prove the existence of a path of points from v to an equilibrium point in S^n as described in section 4. First we define again the appropriate PDM and then a PC^1 mapping whose zero points yield the path. The primal subdivided n-manifold P of the PDM is completely determined by the n-dimensional cones A(s), $s \in \Omega$. More precisely,

$$P = \{A(s)\,|\,s \in \Omega \text{ and } |I(s)| = n-1\}$$

where

$$A(s) = \{p \in U^n\,|\,p_i/v_i = \min_h p_h/v_h \text{ for all } i \text{ with } s_i = -1 \text{ and}$$
$$p_i/v_i = \max_h p_h/v_h \text{ for all } i \text{ with } s_i = +1\}.$$

Notice that $s \in \Omega$ and $|I(s)| = n-1$ imply that s is a signvector in R^{n+1} with n-1 zero elements, one element equal to -1 and one element equal to -1, so that indeed dim A(s) = n.
Clearly, the collection $\bar{P}$ of faces of P is equal to

$$\bar{P} = \{A(s) \mid s \in \Omega\} \cup A(0)$$

where $A(0) = \{v\}$. The dual subdivided n-manifold D consists of the n-cell $Y(0)$ defined by

$$Y(0) = \{y \in O^n \mid \Sigma_j y_j^+ \leq 1\},$$

where, for $a \in R$, $a^+ = \max(0,a)$. Defining the sets $Y(s)$, $s \in \Omega$, by

$$Y(s) = \{y \in O^n \mid y_j \geq 0 \text{ if } s_j = +1,\ y_j = 0 \text{ if } s_j = 0,\ y_j \leq 0 \text{ if } s_j = -1,\ j = 1,\dots,n+1 \text{ and } \sum_{\{j \mid s_j = +1\}} y_j = +1\}$$

we have that $\bar{D} = \{Y(s) \mid s \in \Omega^o\}$ where $\Omega^o = \Omega \cup \{0\}$.
Each $Y(s)$, $s \in \Omega$, is an $(n-|I(s)|-1)$-dimensional face of $Y(0)$, whereas each $A(s)$, $s \in \Omega$, is an $(|I(s)|+1)$-dimensional cone in U^n so that, for all $s \in \Omega^o$, $\dim A(s) + \dim Y(s) = n$.
Clearly, the triplet (P,D,d) is a PDM with degree n with the dual operator d defined by

$$A^d(s) = Y(s) \quad \text{and} \quad Y^d(s) = A(s) \quad \text{for all } s \in \Omega^o.$$

We call this (P,D,d) the PDM with respect to the $(2^{n+1}-2)$-ray algorithm on S^n. The PDM is pictured in figure 11. Recall that the number of one-dimensional cones $A(s)$ is equal to $2^{n+1}-2$.

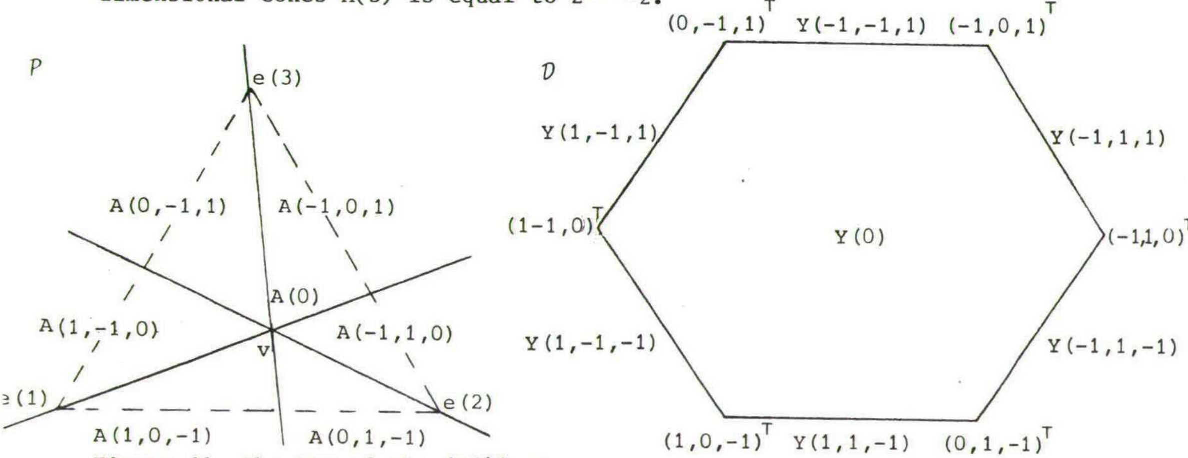

Figure 11. The PDM of the $(2^{n+1}-2)$-ray algorithm, n = 2

Again the collection $L = \{X \times X^d | X \in \bar{P}\}$ of n-cells is a subdivided n-manifold with empty boundary and $|L|$ closed. Furthermore, the collection $K = \{Z \times R_+ | Z \in L\}$ is a subdivided (n+1)-manifold with

$$\delta K = \{A(s) \times Y(s) \times \{0\} | s \in \Omega^o\}.$$

With z again being extended to a continuously differentiable function from U^n to R^{n+1} we define the function $g : |K| \to R^{n+1}$ by

$$g(p,y,t) = y - t[p.z(p)] \qquad (p,y,t) \in |K| \tag{5.4}$$

where $p.z(p) = [p_1z_1(p), p_2z_2(p),\dots,p_{n+1}z_{n+1}(p)]^T$, $p \in U^n$. Since $p^Tz(p) = 0$ we obtain that $\Sigma_j[p.z(p)]_j = \Sigma_j p_j z_j(p) = 0$. Hence, g is a PC^1 mapping from $|K|$ to 0^n if z is a continuously differentiable function from U^n to R^{n+1}. If 0 is also a regular value of g, $g^{-1}(0)$ consists according to theorem 5.2 of a disjoint union of piecewise smooth loops and paths. Furthermore, analogously to lemma 5.5, the point (v,0,0) is the only endpoint of a path in $g^{-1}(0)$ so that this path, denoted G, is a semi-closed unbounded path whereas all other paths in $g^{-1}(0)$ are unbounded and open.

Lemma 5.9. For all (p,y,t) in $g^{-1}(0)$ holds that $p \notin bdS^n$.

Proof. Suppose that p lies in the boundary of S^n. Then there is an s in Ω^o such that $p \in A(s)$ and $y \in Y(s)$. Since $p \neq v$ we must have $s \neq 0$. Moreover, $p \in bdS^n$ and $p \in A(s)$ imply that $p_j = 0$ for all j with $s_j = -1$ and that $p_j > 0$ for all j with $s_j = +1$. However, this implies $z_j(p) = 0$ for all j since $y^T(p)\,z(p) = 0$, sign $z(p) = s$ and $z_j(p) \geq 0$ if $p_j = 0$. Therefore, since $g(p,y,t) = y-tp.z(p) = 0$, we also have $y = 0$, i.e. y lies in the interior of Y(0) and in no other $Y(\bar{s})$, $\bar{s} \in \Omega$. This contradicts the fact that $s \neq 0$.

[]

The lemma says that the paths in $g^{-1}(0)$ cannot cross bdS^n in p. Since both S^n and Y(0) are compact we must have that along the path G in

$g^{-1}(0)$ originating in $(v,0,0)$ the variable t goes to infinity and that the path G approaches a limit point (p^*, y^*) in $|L|$. According to (5.4)

$$p_j^* z_j(p^*) = 0 \quad \text{for } j = 1,\ldots,n+1.$$

This implies that for all j $p_j^* = 0$ or $z_j(p^*) = 0$. Letting s^* be the sign vector in Ω with the smallest number of nonzero elements such that p^* lies in $A(s^*)$, then $p_j^* = 0$ implies that $s_j^* = -1$ and therefore $z_j(p^*) < 0$. However, $p_j^* = 0$ also implies $z_j(p^*) > 0$. Hence, $z_j(p^*) = 0$ for all $j = 1,\ldots,n+1$. Consequently, the path G in $g^{-1}(0)$ starting at $(v,0,0)$ leads for t going to infinity to an equilibrium point p^* in S^n. Moreover, as will be shown in the next theorem the set B being the union of all sets $B(s)$, $s \in \Omega$, is the primal projection of $g^{-1}(0)$.

Theorem 5.10. Let (p,y,t) be a point in $g^{-1}(0)$ such that p lies in S^n. There is an $s \in \Omega^o$ such that

$$p \in B(s) = A(s) \cap C(s).$$

Proof. Since $(p,y,t) \in g^{-1}(0)$ there is a signvector s in Ω^o such that

$$p \in A(s) \quad \text{and } y \in A^d(s) = Y(s).$$

Hence, $p_j z_j(p) = 0$ if $s_j = 0$ and

$$p_j z_j(p) = y_j/t > 0 \qquad \text{when } s_j = +1,$$

and

$$p_j z_j(p) = y_j/t < 0 \qquad \text{when } s_j = -1.$$

From lemma 5.9 we know that $p \notin \text{bd}S^n$ so that $p_j > 0$ for all j. Therefore, $\text{sgn } p_j z_j(p) = \text{sgn } z_j(p)$, $j = 1,\ldots,n+1$, and so p lies in $C(s)$.

[]

Corollary 5.11. The set $B = \cup_s B(s)$, where the union is over all s in Ω^o, contains a path of points in S^n going from v to an equilibrium point p^*. This path is the primal projection of the path G in $g^{-1}(0)$.

The homotopy-parameter t is determined from the fact that if $p \neq v$

$$t \sum_{j=1}^{n+1} p_j z_j^+(p) = \sum_{j=1}^{n+1} y_j^+ = 1$$

for all points $(p,y,t) \in g^{-1}(0)$, so that

$$t = (\Sigma_{j=1}^{n+1} p_j z_j^+(p))^{-1}.$$

When $p = v$, then $(p,y,t) \in G$ for all t, $0 \leq t \leq (\Sigma_{j=1}^{n+1} v_j z_j^+(v))^{-1}$, and y $= tv.z(v) \in Y(0)$. If $t = (\Sigma_{j+1}^{n+1} v_j z_j^+(v))^{-1}$, then $(v,y,t) \in G$ with

$$y = v.z(v)/\Sigma_{j=1}^{n+1} v_j z_j^+(v) \in Y(s^o)$$

where s^o is equal to sgn z(v). Notice that we assume that s^o does not contain any zero. In case $z(v) = 0$ the path G is the ray $\{(v,0,t) | t \geq 0\}$. The primal and dual projection of the path G are illustrated in figure 12. The sequence of signvectors s for which G passes $A(s) \times Y(s) \times R_+$ succesively is $(0,0,0)^T$, $(1,-1,1)^T$, $(1,-1,0)^T$, $(1,-1,-1)^T$ and $(1,0,-1)^T$. When p is equal to v, y goes from the point 0 to b, while p goes from v to w if y goes from b to c. Notice that the point b is equal to

$$b = v.z(v)/(v_1 z_1(v) + v_3 z_3(v)) \in Y(1,-1,1)$$

and that $c = (1,-1,0)$. When p goes from w to u, y is equal to c and so $z_3(p) = 0$. Finally, p goes from u to q if y goes from c to d and y = d if p goes from q to the equilibrium point p^*.

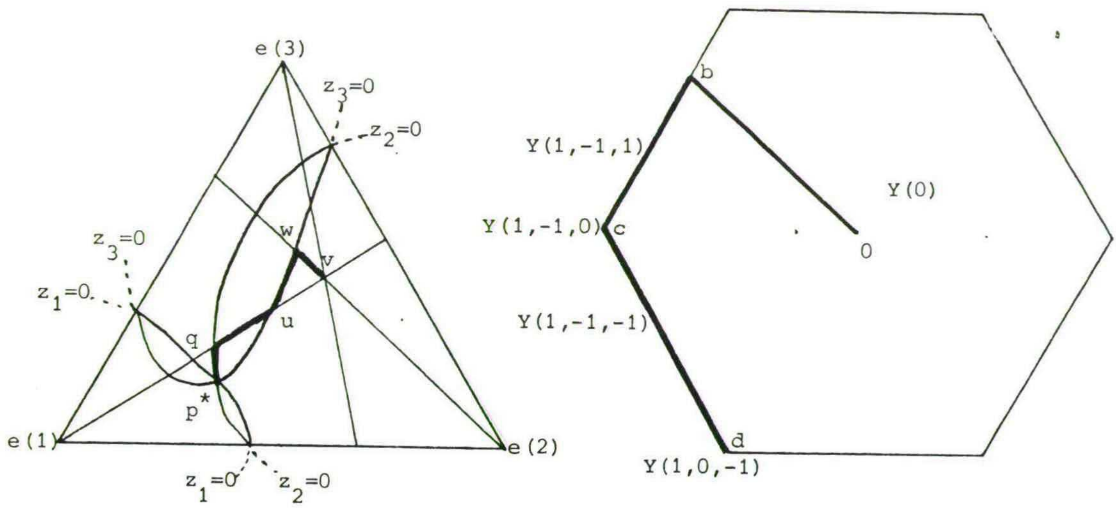

Figure 12. The primal and dual projection of the path G, n = 2.

Finally we remark that the theory above can be easily generalized when we allow v to lie on the boundary of S^n or when we drop the condition that $p_j = 0$ implies $z_j(p) \geq 0$. In the latter case a path of points can be shown to exist from v to a point p^* for which $z(p^*) \leq 0$.

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